RÉDACTION NON NUMÉROTÉE

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TITRE SUPPLEMENT TO THE LINEAR ALGEBRA

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NOMBRE DE PAGES NUMÉRISÉES 17

NOMBRE DE FEUILLES PRISES EN COMPTE 17

Supplement to the Linear Algebra

1) Extension of multilinear functions.

Let the E_i be vector-spaces over a field k; let f be a multilinear mapping of $\prod E_i$ into a vector-space F. Let A be an algebra over k. Then f can be canonically extended to a mapping f_A (also denoted simply by f) of $\prod (A \otimes E_i)$ into $A \otimes F$, by putting

$$f_A(\alpha_1 \otimes x_1, \ldots, \alpha_n \otimes x_n) = (\alpha_1 \ldots \alpha_n) \otimes f(x_1, \ldots, x_n),$$

for $\alpha_i \in A$, $x_i \in E_i$.

N.B. If E is a vector-space over k, one can give, to the tensor-product $A \otimes E$, not only the left-A-module structure defined in Alg. Chap. III, but a structure of A-bimodule, by putting $\lambda(\alpha \otimes x) = (\lambda\alpha) \otimes x$, $(\alpha \otimes x)\mu = (\alpha\mu) \otimes x$; by $E_{(A)}$, one will frequently denote this bimodule (rather than the left-module, as in Chap. III). If f is a bilinear mapping of $E \times E'$ into F, f_A is an A-bilinear mapping (left-linear on $E_{(A)}$, right-linear on $E'_{(A)}$) of $E_{(A)} \times E'_{(A)}$ into the bimodule $F_{(A)}$.

2) Theory of polynomial algebras.

The theory of Grassmann algebras (Chap.III), and of polynomials (Chap. IV), have to be combined together as follows.

Let E be a vector-space over a field k ; let σ

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be an involutory automorphism of E (i.e. $\sigma^2 = I$, with I the identity automorphism of E ; if the characteristic of k is 2, we assume $\sigma = I$). Then E decomposes into a direct sum of subspaces $E^{(\nu)}$, $\nu = 0, 1$, such that, if $x \in E^{(\nu)}$, $x^\sigma = (-1)^\nu x$. There is, then, one and (up to isomorphism) only one algebra $P = P(E, \sigma)$ over k, containing E and generated by E (and by a unit-element 1), such that $E^{(0)}$ is in the center of P and that $(E^{(1)})^2 = 0$, and such that, if A is another algebra with these same properties, the identical automorphism of E can be extended to a representation of P on A. If the X_ι are a basis for $E^{(0)}$, P can also be described as the tensor-product of the algebra of (ordinary) polynomials in the X_ι, and of the Grassmann algebra over $E^{(1)}$. If (e_λ) is the union of a basis for $E^{(0)}$ and of a basis for $E^{(1)}$, then P has a basis consisting of monomials in the e_λ.

Usually, E is given as the direct sum of vector-spaces $(E_\iota)_{\iota \in I}$, and there is a partition $I = J \cup J'$ into two disjoint sets J, J' (either of which may be empty) so that, for $x \in E_\iota$, $x^\sigma = \varepsilon_\iota x$, with $\varepsilon_\iota = +1$ for $\iota \in J$, $\varepsilon_\iota = -1$ for $\iota \in J'$. Then P is made into a multigraded algebra by its decomposition into the direct sum of the subspaces $P_{(n_\iota)}$ respectively generated by the monomials $x_1 x_2 \ldots x_N$, where each x_i is an element of some E_ι, and, for each ι, exactly n_ι of the x_i are in

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E_ι. This will frequently be made into a graded algebra by defining, for each ι, the elements of E_ι to be of degree e_ι, where e_ι is an integer ≥ 0; then the degree of the elements of $P_{(n_\iota)}$ is $\sum_\iota n_\iota e_\iota$; the most usual case is that in which all e_ι are > 0, $\varepsilon_\iota = (-1)^{e_\iota}$, and, for each $N > 0$, the direct sum of those E_ι for which $e_\iota \leq N$ is of finite dimension (which implies that each E_ι is of finite dimension, and that I is finite or countably infinite). When that is so, we also denote the algebra $P(E, \sigma)$ by $P(E_\iota, e_\iota)$. In particular, $P(E,1)$ is the Grassman algebra over E, with the usual grading; if each E_ι is of dimension 1, and generated by an element X_ι, then $P(E_\iota, 2)$ is the algebra of (ordinary) polynomials in the X_ι, with twice the usual grading; etc.

Let f be a k-linear mapping of E into an algebra A (with 1) over k, such that, for all $x \in E^{(0)}$, $f(x)$ is in the center of A, and, for all $x \in E^{(1)}$, $f(x)^2 = 0$; then f can be extended, in one and only one way, to a representation of P into A; for $p \in P$, the image of p in this representation will be denoted by $p(f)$; in particular, if I is the identical automorphism of E, $p(I) = p$. Usually A will be a graded algebra, satisfying $uv = (-1)^{pq} vu$ for u, v homogeneous of degrees p, q; and f maps $E^{(0)}$ into the subspace $A^{(0)}$ of A generated by the elements of even degree, and $E^{(1)}$ into the subspace $A^{(1)}$ of A generated by the elements of odd degree. In particular, for $P = P(E_\iota, e_\iota)$, f will usually be a "homogeneous" mapping,

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i.e., for each ι, one has a linear mapping f_ι of E_ι into the subspace A_{e_ι} of elements of degree e_ι of A, and f is the mapping of E which induces f_ι on E_ι for each ι; then, for $p(f)$, we also write $p(f_\iota)$.

In particular, the automorphism σ of E can thus be extended to an automorphism $p \rightarrow p(\sigma)$ of $P(E, \sigma)$; P decomposes into a direct sum of subspaces $P^{(0)}$, $P^{(1)}$, consisting of the elements $p \in P$ such that $p(\sigma) = p$ and $p(\sigma) = -p$, respectively; if P is graded by taking elements of $E^{(0)}$, $E^{(1)}$ to be of degrees 0, 1 respectively, then $P^{(0)}$, $P^{(1)}$ are the subspaces generated by the elements of even and of odd degree, respectively; the same assertion holds for the graded algebra $P(E_\iota, e_\iota)$. By induction on the degree, one sees that, if $p \in P^{(\mu)}$, $q \in P^{(\nu)}$, with $\mu, \nu = 0$ or 1, then $pq = (-1)^{\mu\nu} qp$; in particular, $P^{(0)}$ is contained in the center of P, and is a commutative subalgebra of P.

More generally, for every graded algebra A satisfying $uv = (-1)^{mn} vu$ for u, v homogeneous of degrees m, n we define an automorphism, usually denoted by σ, by putting $u^\sigma = (-1)^m u$ for u of degree m. Then a mapping f of E into A will be extendable to a representation of $P = P(E, \sigma)$ into A whenever $f(x^\sigma) = f(x)^\sigma$ (this is sufficient, but not necessary).

Let $\bar{E}$ be the direct sum of E and of a space E' isomorphic to E; let f be the identical automorphism of E, f' an isomorphism of E onto E', and define σ on E'

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as $f' \circ \sigma \circ f'^{-1}$, and on $\bar{E}$ so as to induce σ on E and on E'. Then the algebra $\bar{P} = P(\bar{E}, \sigma)$ is canonically isomorphic to the "skew-tensor-product" of $P = P(E, \sigma)$ and $P' = P(E', \sigma)$; P' is isomorphic to P ; we have three representations of P into $\bar{P}$, viz.

$p \to p = p(f)$, $p \to p' = p(f')$, and $p \to \bar{p} = p(f + f')$.

If k is of characteristic 0 , it can be shown that $\bar{p} = p + p'$ if and only if $p \in L$(the "if" is obviously always true).

Let $\bar{P}$ be graded by taking elements of E and E' to be of degrees 0, 1 respectively ; then the decomposition of $\bar{p}$ into its homogeneous components is the Taylor formula for the algebra P . We shall consider only the components of degrees 0, 1 ; the former is p ; call the latter Δp . The subspace $\bar{P}_1$ of elements of $\bar{P}$ of degree 1 is canonically isomorphic to $E \otimes P'$, hence to $E_{(P)} = P \otimes E$; $p \to \Delta p$ is a mapping of P into that space, which satisfies $\Delta(pq) = (\Delta p)q + p(\Delta q)$ (where multiplication is understood in the sense of the algebra $\bar{P}$). It should be observed that the (canonical) isomorphism φ of $E_{(P)} = P \otimes E$ onto $\bar{P}_1$ is given by $\varphi(p \otimes x) = x'p$, for $p \in P$, $x \in E$, and is right-P-linear but not left-P-linear; more precisely, this mapping is P-bilinear on the subspace $P^{(0)} \otimes E$, and left-semilinear of the subspace $P^{(1)} \otimes E$ in the sense that $\varphi\{q(p \otimes x)\} = q^{\sigma}\varphi(p \otimes x)$; hence, if $\Delta_0 p$, $\Delta_1 p$ are the components of $\varphi^{-1}(\Delta p)$ in those two spaces, we have $\Delta_0(pq) =$

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$(\Delta_0 p)q + q(\Delta_0 p)$, $\Delta_1(pq) = (\Delta_1 p)q + p^{\sigma}(\Delta_1 q)$.
For $x \in E$, we have $\Delta x = x'$.

Let now f be a k-linear mapping of E into P, satisfying $f(x^\sigma) = (-1)^\nu f(x)^\sigma$, with $\nu = 0$ or 1; in other words, f maps $E_{(0)}$, $E_{(1)}$ respectively into $P_{(0)}$, $P_{(1)}$ if $\nu = 0$, and into $P_{(1)}$, $P_{(0)}$ if $\nu = 1$. We extend f to a right-P-linear mapping F of $\bar{P}_1$ into P by putting $F(x'p) = f(x)p$, and put $D = F \circ \Delta_1$. Then D is a k-linear mapping of P into P, satisfying $Dx = f(x)$ for $x \in E$, and:

(1) $$D(pq) = (Dp)q + p^{\sigma^\nu}(Dq),$$
$$D(p^\sigma) = (-1)^\nu (Dp)^\sigma .$$

Conversely, let D satisfy these conditions ; the first implies $D(1) = 0$; let f be the mapping induced by D on E, and, F being defined as above by $F(x'p) = f(x)p$, put $D_0 = D - F \circ \Delta$; D_0 also satisfies (1) and is 0 on E, whence, using (1), one concludes by induction on the degree (for the grading of P in which all elements of E are of degree 1) that $D_0 = 0$. Therefore the most general D satisfying (1) is of the form $D = F \circ \Delta$. Usually P will be defined as $P = P(E_\iota, e_\iota)$, and f is "homogeneous" of a certain degree d (which may be > 0, 0 or < 0); i.e. one gives, for each ι, a k-linear mapping f_ι of E_ι into the subspace of elements of P of degree $e_\iota + d$; then the operator D is such that Dp is of degree $m + d$ when p is of degree m ; and $\nu \equiv d \pmod{2}$.

Let D be an endomorphism of P, satisfying (1);

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let D' satisfy (1) with ν' instead of ν. Then $D'' = DD' - (-1)^{\nu\nu'} D'D$ satisfies (1) with $\nu'' = \nu + \nu'$ instead of ν. In particular, if D, D' induce linear forms on E, i.e. if they map E into k = k.1 (which implies that D is 0 on $E^{(1)}$ if $\nu = 0$, and on $E^{(0)}$ if $\nu = 1$, and similarly for D'), then DD' and D'D are 0 on E, hence D'' = 0, i.e. $DD' = (-1)^{\nu\nu'} D'D$. Furthermore, if D satisfies (1) with $\nu = 1$, then D^2 satisfies (1) with $\nu = 0$: it follows that, for such a D, D^2 is 0 on P provided it is 0 on E.

Endomorphisms D of P, satisfying (1), will be called <u>derivations</u>; a derivation is <u>even</u> or <u>odd</u> according as $\nu = 0$ or 1; an odd derivation D such that $D^2 = 0$ will be called a <u>differentiation</u>; a <u>differential</u> P-algebra is one on which one has given a differentiation (which is then part of the structure).

We now confine ourselves to the <u>special</u> P-algebras, of the form $P = P(E_i, e_i)$, where $e_i > 0$, the E_i are of finite dimension, and those for which $e_i \leq n$ are in finite number; P being graded by defining the elements of E_i to be of degree e_i, the subspace P_n of elements of degree n is of finite dimension, and $P_o = k = k.1$. For finite-dimensional vector-spaces E, F, we frequently identify (when no misunderstanding is likely) the four spaces $E \otimes F$, $F \otimes E$, $\mathcal{L}(E', F)$, $\mathcal{L}(F', E)$, where E', F' are the duals of E, F; in particular, the identical automorphism of E is to be identified with the corresponding element of $E \otimes E'$

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(or of $E' \otimes E$), which is called the δ-element of $E \otimes E'$, or δ_E, or simply δ (since its components, for any choice of dual bases in E, E', are δ_{ij}). We shall only consider homogeneous derivations D on P, of a certain degree d (D is even or odd according as d is even or odd); such a D is defined by giving its value on E_i, which is a linear mapping of E_i into $P_{e_i + d}$ or an element of $P_{e_i + d} \otimes E_i'$ (where E_i' is the dual of E_i), or, as we also say (by abuse of language) an element of $P \otimes E_i'$ of degree $e_i + d$.

It is also convenient to express this by the following notation. Let first E, F be arbitrary, as above; let f be a linear mapping of E into F, i.e. $f \in \mathcal{L}(E,F)$; let f' be the tensor-product of f and of the identical automorphism of E', this being a linear mapping of $E \otimes E'$ into $F \otimes E'$; if δ is the δ-element of $E \otimes E'$, $f'(\delta)$ is then the element of $F \otimes E'$ which corresponds to f in the canonical isomorphism between $\mathcal{L}(E,F)$ and $F \otimes E'$.

Now let D be a derivation on P; we extend this canonically to $P \otimes H = H_{(P)}$, for any vector-space H over k, by defining $D(p \otimes u) = (Dp) \otimes u$ for $p \in P$, $u \in H$ (in other words, we agree to denote again by D the tensor-product of D with the identical automorphism of H). This applies in particular to $P \otimes E'_i$; as $E_i \subset P$, we have $E_i \otimes E_i' \subset P \otimes E_i'$, so that the δ - element δ_i of $E_i \otimes E_i'$ can be considered as an element of $P \otimes E_i'$, and, with

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the above convention, $D(\delta_i)$ is defined, is an element of $P \otimes E_i'$, and is no other than the element of $P_{e_i+d} \otimes E_i'$ which corresponds, in the canonical isomorphism between this and $\mathcal{L}(E_i, P_{e_i+d})$, to the element D_i of the latter space which is the mapping induced by D on E_i. This shows that D is completely defined if, for each i, the element $D(\delta_i)$ of $P \otimes E_i'$ is given, as an element of degree $e_i + d$ if D is to be of degree d. Also, D will be a differentiation if and only if d is odd and $D[D(\delta_i)] = 0$ for all i.

Finally, D being extended as above, and multilinear functions being extended as explained in 1), the formula (1) can be generalized thus: let f be a multilinear mapping of $\prod_{i=1}^{n} M_i$ into N, where the M_i and N are vector-spaces over k. Then, if $u_i \in P \otimes M_i$ for each i, we have

$$Df(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} f(u_1^{\sim\nu}, \ldots, u_{i-1}^{\sim\nu}, Du_i, u_{i+1}, \ldots, u_n).$$

<u>Examples: differential algebras associated with a Lie algebra.</u> The best known of these is the "algebra of cochains". Let A be a Lie algebra (of finite dimension over k). We take $P = P(A', 1)$, i.e., the Grassmann algebra over the dual vector-space to A. We can extend the bilinear mapping $(x, y) \longrightarrow [x, y]$ of $A \times A$ into A to a mapping of $A_{(P)} \times A_{(P)}$ into $A_{(P)}$. Thus, if X denotes the δ-element of $A' \otimes A \subset P \otimes A = A_{(P)}$, $[X, X]$ will be an element of $A_{(P)}$, of degree 2. We define a

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derivation D, of degree 1, in P, by $D(X) = c[X, X]$, with any $c \in k$. This is a differentiation; for $D^2(X) = c^2([[X, X], X] + [X^\sigma, [X, X]])$; but $X^\sigma = -X$; also, from the definition of the extension of the multilinear function $[[x, y], z]$ to $A_{(P)}$, and the Jacobi identity, one verifies immediately that $[[X, X], X] = 0$ (the latter relation is, in fact, equivalent to the Jacobi identity, once one assumes that $[x, y]$ is alternating and that the characteristic of k is $\neq 2$). Thus $P = P(A', 1)$, with the differentiation D, becomes a differential P-algebra, the <u>algebra of cochains of</u> A (for $c \neq 0$; usually one takes $c = \pm 1/2$).

Similarly, take two spaces A'_1, A'_2, isomorphic to A'; consider the algebra $P = P(A'_1, 1; A'_2, 2)$. When A' is identified with A'_1 by means of the given isomorphism, the δ-element of $A' \otimes A$ becomes an element X_1 of $A'_1 \otimes A$; let X_2 be similarly defined as an element of $A'_2 \otimes A$; for $i = 1, 2$, X_i is an element of degree i of $P \otimes A$. Then put $D(X_1) = X_2 - c[X_1, X_1]$, $D(X_2) = 2c[X_2, X_1]$; this is a derivation of degree 1; it is a differentiation, as can be verified by using the identities $[X_2, X_2] = 0$, $[[X_1, X_1], X_2] = 2[[X_1, X_2], X_1]$. This differential P-algebra is called the <u>universal algebra of</u> A. The definition of D shows that the ideal generated by the elements of A'_2 is a homogeneous differential ideal I; therefore the quotient-algebra P/I is a graded differential algebra, isomorphic with the algebra of cochains of A. Usually $c = 1/2$.

<u>Note on the extension of multilinear functions by means of P-algebras</u>: let P be any P-algebra, decomposed as before into $P^{(0)}$, $P^{(1)}$ by means of the automorphism σ; let E, F

be vector-spaces over k, and f a bilinear mapping of $E \times E$ into F, either symmetric or alternating, so that $f(x, y) = (-1)^{\mu} f(x, y)$, and $f(x, x) = 0$ for $\mu = 1$; then, if each one of the elements u, v is either in $P^{(0)} \otimes E$ or in $P^{(1)} \otimes E$, we have $f(u, v) = \pm f(v, u)$ (with a suitable sign); if f is symmetric and u is in $P^{(1)} \otimes E$, or if f is alternating and u is in $P^{(0)} \otimes E$, then $f(u, u) = 0$. Furthermore, let k be of characteristic 0; let f be a multilinear mapping of E^n into F; let f_o, f_1 be the symmetric resp. alternating mapping of E^n into F obtained from f by symmetrization resp. antisymmetrization. Then, if $u \in P^{(\nu)} \otimes E$, we have $f(u, u, \ldots, u) = 1/n! \; f_{\nu}(u, u, \ldots, u)$.

It is also useful to make the following remark in which, in order to simplify notations, we confine ourselves to the case of an algebra $P = P(E_1, 1; E_2, 2)$; the extension to any algebra $P(E_i, e_i)$ will be obvious (at least provided the E_i are in finite number). Let X_i, for i = 1, 2, be the δ-element of $E'_i \otimes E_i$. Consider the bigrading on P for which the elements of E_1, E_2 are of bidegree (1, 0) and (0, 1), respectively; let P_{mn} be the subspace of P consisting of the elements of bidegree (m, n). Then, if f is any multilinear form on $(E'_1)^m \times (E'_2)^n$, $f(X_1, \ldots, X_1; X_2, \ldots, X_2)$ is an element of P_{mn}, and every element of P_{mn} can be so written. Naturally, such an f is nothing else than an element of $(\overset{m}{\otimes} E_1) \otimes (\overset{n}{\otimes} E_2)$, and the mapping $f \longrightarrow f(X_1, \ldots, X_2)$ is nothing but the restriction, to the latter tensor-product, of the canonical homomorphism of the tensor-algebra of the direct sum $E_1 + E_2$ onto P by which the latter is defined.

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But the notation $f(X_1, \ldots, X_2)$, to denote an arbitrary element of P_{mn}, is particularly convenient in all calculations involving differentiations in P. Furthermore, if the characteristic of k is 0, then every element of P_{mn} can be written in one and only one way as $f(X_1, \ldots, X_1; X_2, \ldots, X_2)$, where f is a multilinear form on $(E_1')^m \times (E_2')^n$, alternating with respect to the first m variables (those in E_1'), and symmetric with respect to the last n variables (those in E_2'). If f is such a form, and D is a derivation on P, we have

$$Df(X_1, \ldots, X_1; X_2, \ldots, X_2) = m.f(DX_1, \ldots, X_1; X_2, \ldots, X_2) + (-1)^m n.f(X_1, \ldots, X_1; DX_2, X_2, \ldots, X_2),$$

a useful explicit formula for the extension to P of a derivation which is given on E_1 and E_2.

<u>Duality of P-algebras</u>. Take $P = P(E_i, e_i)$; let E_i^* be the dual space to E_i; put $P^* = P(E_i^*, e_i)$. For each $x^* \in E_i^*$, let $D(x^*) = D_{x^*}$ be the derivation on P which induces on E_i the linear form $< x^*, x >$, and is 0 on E_j for all $j \neq i$; this is a derivation of degree $-e_i$ on P. It has been proved above that, if $x^* \in E_i^*$ and $y^* \in E_j^*$ (with $i = j$ or $i \neq j$), then $D(x^*)D(y^*) = (-1)^{e_i e_j} D(y^*)D(x^*)$, and that $D(x^*)^2 = 0$ if $e_i \equiv 1 \pmod{2}$. Therefore the mappings $x^* \longrightarrow D(x^*)$ of the E_i^* into the algebra of endomorphisms of the vector-space P can be extended to a <u>representation</u> $p^* \longrightarrow D(p^*) = D_{p^*}$ of the algebra P^* into the algebra of endomorphisms of P. The algebra P^* will also be called the <u>dual</u> algebra to P; if $p^* \in P^*$, $p \in P$, we also write $D_{p^*}(p) = p^* \lrcorner\, p$, and call it the <u>interior product</u> of p^* and p; this

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operation defines P as a P^*-module; by definition, we have $D_{p^*} \circ D_{q^*} = D_{p^*q^*}$, i.e., $p^* \rfloor (q^* \rfloor p) = (p^* q^*) \rfloor p$. If $p \in P$ is homogeneous of degree n, and $p^* \in P^*$ is homogeneous of degree m, then $p^* \rfloor p$ is homogeneous of degree n-m, hence is 0 for $m > n$. If P_n, P^*_n are the spaces of elements of degree n in P, P^* (with P_o, P^*_o identified with k as usual), then, for $p^* \in P^*_n$, D_{p^*} induces on P_n a linear form. Let $(a_\lambda)_{\lambda \in L}$ be the union of bases for all the E_i, so that there is a partition (L_i) of L such that $(a_\lambda)_{\lambda \in L_i}$ is a basis for E_i for each i; let $(a^*_\lambda)_{\lambda \in L_i}$ be the dual basis to $(a_\lambda)_{\lambda \in L_i}$ for E^*_i: we can take for P_n a basis consisting of monomials $b_\rho = M_\rho(a_\lambda)$ in the a_λ, and for P^*_n a basis consisting of the corresponding monomials $b^*_\rho = M_\rho(a^*_\lambda)$ in the a^*_λ. Then it is easily seen that $b^*_\rho \rfloor b_\sigma = 0$ for $\rho \neq \sigma$ and that $b^*_\rho \rfloor b_\rho$ is equal to $\prod_\lambda (n_\lambda !)$ if $b_\rho = \prod_\lambda a_\lambda^{n_\lambda}$ (as usual, $0! = 1$). It follows from this that, <u>if</u> k is of characteristic 0 <u>or</u> <u>if</u> all e_i are odd (P being, in the latter case, isomorphic to a Grassmann algebra), the bilinear form $< p^*, p > = p^* \rfloor p$ defines P^*_n and P_n as the dual spaces to each other, and that the representation $p^* \longrightarrow D_{p^*}$ is then faithful. In that case, we have, by the above formulas, $< p^*q^*, p > = < p^*, q^* \rfloor p >$ for $p^* \in P^*_m$, $q^* \in P^*_n$, $p \in P_{m+n}$, which shows that P^*-multiplication and interior multiplication are then the transpose of each other (the same cannot be said for general P-algebras and arbitrary characteristic, since P_n and P^*_n need not then be dual to each other).

As before, let, for each i, $\bar{E}_i$ be the direct sum of E_i and of a vector-space E'_i isomorphic to E_i; for each i, let f_i

be the identical automorphism of E_i, and f'_i a fixed isomorphism of E_i onto E'_i. Put $P = P(E_i, e_i)$, $P' = P(E'_i, e_i)$, $\overline{P} = P(\overline{E}_i, e_i)$. Then for $p \in P$, we have again the three representations $p \longrightarrow p = p(f_i)$, $p \longrightarrow p' = p(f'_i)$, $p \longrightarrow \overline{p} = p(f_i + f'_i)$ of P into $\overline{P}$. Furthermore, we can define an isomorphism F of the vector-space $P \otimes P$ onto $\overline{P}$ by putting $F(p \otimes q) = pq'$; if we put on $P \otimes P$ the structure of the "skew-tensor-product" of the graded algebra P with itself, then F is an isomorphism of the algebra $P \otimes P$ onto $\overline{P}$. Let P'^*, $\overline{P}^*$, F^* be similarly defined from $P^* = P(E^*_i, e_i)$. Then, for $p^* \in P^*_m$, $q^* \in P^*_n$, $p \in P_{m+n}$, we have $F^*(p^* \otimes q^*) \lrcorner \overline{p} = (p^*q^*) \lrcorner p$ (proof: take bases as before, and prove the formula for monomials), hence, for characteristic 0 (or for Grassmann algebras of arbitrary characteristic) the fundamental formula $< F^*(p^* \otimes q^*), \overline{p} > = < p^*q^*, p >$. This is essentially Taylor's formula for the P-algebra P (for an ordinary polynomial algebra, written either as $P = P(E, 2)$ or as $P = P(E_i, 2)$ with all the E_i of dimension 1, Taylor's formula for polynomials follows from this at once). It also shows that the multiplication $(p^*, q^*) \longrightarrow p^*q^*$ in P^* is the transpose of the representation $p \longrightarrow \overline{p}$ of P into $\overline{P}$ (for characteristic 0 <u>or</u> Grassman algebras). The converse of this is Hopf's (generalized) theorem, which we shall merely state here (proof in letters to H.C., 3.8.49 and 11.8.49):

Let A be a <u>normal graded algebra</u> over a field k (this means that a) the subspace A_n of elements of degree n is of

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finite dimension for all $n \geq 0$; b) $xy = (-1)^{mn} yx$ for $x \in A_m$, $y \in A_n$; c) A_0 consists of the scalar multiples of the unit-element 1 of A, and is identified with k). Let $\bar{A}$ be the skew-tensor-product $A \hat{\otimes} A$ (i.e. it is the normal graded algebra defined by defining $x \hat{\otimes} y$ to be of degree $m + n$ for $x \in A_m$, $y \in A_n$, and by putting $(x \hat{\otimes} y).(x' \hat{\otimes} y') = (-1)^{mn}(xx') \hat{\otimes} (yy')$ for $x' \in A_m$, $y \in A_n$). Let f be a representation of A into $\bar{A}$ (for their structure as graded algebras, i.e. $f(x)$ is homogeneous of degree n for $x \in A_n$). Let A^* be the direct sum of the duals A^*_n to the A_n; as $\bar{A}_n$ (space of elements of $\bar{A}$ of degree n) is the direct sum of the spaces $A_i \hat{\otimes} A_{n-i}$ for $0 \leq i \leq n$, we can identify the dual $\bar{A}^*_n$ of $\bar{A}_n$ with the direct sum of the $A^*_i \hat{\otimes} A^*_{n-i}$, so that, for $x^* \in A^*_i$, $y^* \in A^*_{n-i}$, $x^* \hat{\otimes} y^*$ defines a linear form on $\bar{A}_n$. Let then $\varphi(x^*, y^*)$ be the element of A^*_n defined by $< f(x), x^* \hat{\otimes} y^* > = < x, \varphi(x^*, y^*) >$ for $x \in A_n$; in other words, φ is defined by the condition that $< f(x), x^* \hat{\otimes} y >$ $= < x, \varphi(x^*, y^*) >$ whenever $x^* \in A^*_m$, $y^* \in A^*_n$, $x \in A_{m+n}$, and may be called the transposed mapping to f. Assume now that A^* is made into a normal graded algebra by taking as multiplication $(x^*, y^*) \longrightarrow x^*y^* = \varphi(x^*, y^*)$ (this being extended by linearity to all elements of A^*; this includes the assumption that the basis-element of A^*_0 which is dual to the basis-element 1 for A_0 is the unit-element in A^*). Then, if k is of characteristic 0, A is a P-algebra; more precisely, let E be the subspace of A consisting of all elements x such that $f(x) = x \hat{\otimes} 1 + 1 \hat{\otimes} x$; let (e_i) be the

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sequence of all the distinct values of n for which $E \cap A_n \neq \{0\}$, and put $E_i = E \cap A_{e_i}$; then we have $A = P(E_i, e_i)$, and, with the above multiplication-law, $A^* = P(E_i^*, e_i)$, with E_i^* the dual space to E_i; also, if $\bar{A}$ is identified with the space $P(\bar{E}_i, e_i)$, where for each i $\bar{E}_i$ is the direct sum of E_i and of a space E_i' isomorphic to E_i, then the mapping f is the same as the mapping $p \longrightarrow \bar{p} = p(f_i + f_i')$ defined above. If we call "Hopf algebras" the normal graded algebras satisfying the conditions of our theorem, it is therefore seen that, for characteristic 0, the theory of Hopf algebras is identical with the theory of P-algebras. The importance of Hopf algebras is partly due to the fact that the cohomology algebra of a compact connected Lie group (over any <u>field</u> of coefficients) is a Hopf algebra of finite dimension; in particular, for a field of characteristic 0, it is therefore a P-algebra $P(E_i, e_i)$, where all the e_i are <u>odd</u> (hence isomorphic to the Grassmann algebra over the direct sum of the E_i). As has been shown, P-algebras over fields of characteristic $\neq 0$ have some unpleasant features (P^* not dual to P_n, etc.), so that possibly Hopf algebras might give, for characteristic $\neq 0$, a more satisfactory analogue of the P-algebras of characteristic 0 than the P-algebras themselves.