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Let \( k \) be a finite algebraic number field, \( J \) the idele group of \( k \), topologized as in a recent paper of Weil. \( J \) is a locally compact abelian group containing the principal idele group \( \mathcal{P} \) as a discrete subgroup. We denote by \( J_0 \) the subgroup of \( J \) consisting of ideles \( \alpha = (\alpha_p) \) such that \( \alpha_p = 1 \) for all infinite (i.e. archimedean) primes \( P \). We call \( J_0 \) the finite part of \( J \) and define the infinite part \( J_\infty \) similarly, so that we have

\[
J = J_0 \times J_\infty, \quad \alpha = \alpha_0 \alpha_\infty, \quad \alpha_0 \in J_0, \quad \alpha_\infty \in J_\infty.
\]

We also denote by \( U \) the compact subgroup of \( J \) consisting of ideles \( \alpha = (\alpha_p) \) such that the absolute value \( |\alpha_p|_p = 1 \) for every prime \( P \). \( U_0 = U \cap J_0 \) is then an open, compact subgroup of \( J_0 \) and \( J_0/U_0 \) is canonically isomorphic to the ideal group \( I \) of \( k \). According to Artin-Whaples, we can choose the absolute values \( |\alpha_p|_p \) so that the volume function

\[
V(\alpha) = \prod_p |\alpha_p|_p \quad (\alpha = (\alpha_p))
\]

has the value \( 1 \) at every principal idele \( \alpha \in \mathcal{P} \) (the product formula) and that \( V(\alpha_0)^{-1} \) is equal to the absolute norm \( \mathcal{N}(\alpha_0) \) of the ideal \( \mathcal{I}_0 \), which corresponds to \( \alpha_0 \) by the above isomorphism between \( J_0/U_0 \) and \( I \).

We now define a function \( \varphi(\alpha) \) by

\[
\varphi(\alpha) = \varphi(\alpha_0) \varphi(\alpha_\infty), \quad \alpha = \alpha_0 \alpha_\infty, \\
\varphi(\alpha_0) = \begin{cases} &1, \text{ if } \mathcal{I}_0 \text{ is an integral ideal;} \\
&0, \text{ otherwise,}
\end{cases}
\]
$$\varphi(\mathcal{O}_{\infty}) = \exp\left(-\frac{\pi}{\sqrt{\Delta}} \sum_{i=1}^{\infty} e_{i} \left| a_{\mathcal{O}_{\infty}, i} \right|^{2}\right),$$

where $n$ is the absolute degree of $k$, $\Delta$ is the discriminant of $k$, $a_{\mathcal{O}_{\infty}, i}$ are the components of $\mathcal{O}$ at the infinite primes $\mathcal{P}_{\infty, i}$ and $e_{i}=1$ or 2 according as $\mathcal{P}_{\infty, i}$ is real or complex. Since $U_{0}$ is open in $J_{0}$, $\varphi(\mathcal{O})$ is a continuous function on $J$ and we define a function $\xi(s)$ by

$$(1) \quad \xi(s) = \int_{J} \varphi(\mathcal{O}) V(\mathcal{O})^{s} d\mu(\mathcal{O}), \quad \text{for } s>1.$$ 

Here $\mu(\mathcal{O})$ denotes a Haar measure of the locally compact group $J$.

We shall calculate this integral in two different ways.

First, using $J = J_{0} \times J_{\infty}$, $\varphi(\mathcal{O}) = \varphi(\mathcal{O}_{0}) \varphi(\mathcal{O}_{\infty})$ and $V(\mathcal{O}) = V(\mathcal{O}_{0}) V(\mathcal{O}_{\infty})$, we have

$$\xi(s) = \int_{J_{0}} \varphi(\mathcal{O}_{0}) V(\mathcal{O}_{0})^{3} d\mu(\mathcal{O}_{0}) \int_{J_{\infty}} \varphi(\mathcal{O}_{\infty}) V(\mathcal{O}_{\infty})^{s} d\mu(\mathcal{O}_{\infty}).$$

If we note that $U_{0}$ is an open, compact subgroup of $J_{0}$ and $J_{0}/U_{0} \cong \Gamma$, we see immediately that the first integral on the right-hand side is equal to (up to a positive constant) the zeta-function $\zeta(s) = \sum \nu(\mathcal{O})^{-s}$ ($\mathcal{O}$ = integral ideal) of $k$. On the other hand, $J_{\infty}$ being the direct product of $r$ copies of the multiplicative group $K^{\ast}$ of the real or complex number-field $K$, the second integral is the product of integrals of the form

$$\int_{K^{\ast}} \exp\left(-\frac{\pi}{\sqrt{\Delta}} e_{i} |t|^{2}\right) t^{s} d\mu_{K}(t), \quad e_{i} = 1 \text{ or } 2,$$

which can be easily calculated to be equal to

$$\Delta^{-\frac{r+1}{2}} \pi^{-\frac{r}{2}} \Gamma\left(\frac{r}{2}\right) \text{ or } \Delta^{-\frac{r}{2}} 2^{-s} \pi^{-s} \Gamma(s),$$

according as $K$ is real or complex. We have therefore

$$(2) \quad \xi(s) = \text{const. } 2^{-\frac{r}{2}} \Delta^{-\frac{r+1}{2}} \pi^{-\frac{r}{2}} \Gamma\left(\frac{r}{2}\right) \Gamma(s) \xi(s).$$

The above calculation also shows that the integral (1) actually converges for $s>1$.\]
We now transform the same integral (1) in another way. Namely, we first integrate the function \( f(\alpha) = \varphi(\alpha) V(\alpha)^3 \) on the subgroup \( \mathfrak{H} \) and then on the factor group \( \overline{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H} \):

\[
\int_{\mathfrak{H}} f(\alpha) \, d\mu(\alpha) = \int_{\overline{\mathfrak{H}}} \left\{ \int_{\mathfrak{H}} f(\alpha \alpha) \, d\mu(\alpha) \right\} \, d\mu(\overline{\alpha}).
\]

However, since \( \mathfrak{H} \) is discrete and \( V(\alpha \alpha) = V(\alpha) V(\alpha) = V(\alpha) V(\overline{\alpha}) \), we have

\[
\int_{\mathfrak{H}} f(\alpha \alpha) \, d\mu(\alpha) = \left( \sum_{\alpha \in \mathfrak{H}} \varphi(\alpha \alpha) \right) V(\overline{\alpha})^3,
\]

and if we put

\[
\varphi(\overline{\alpha}) = \sum_{\alpha \in \mathfrak{H}} \varphi(\alpha \alpha),
\]

the theta-formula

\[
\Theta(\overline{\alpha}) = V(\overline{\alpha})^{-1} \Theta(\overline{\alpha}^{-1}) \quad \text{or} \quad \varphi(\overline{\alpha}) = V(\overline{\alpha})^{-1} \varphi(\overline{\alpha}^{-1}) V(\overline{\alpha})^{-1}
\]

holds, where \( \Theta \) is an idèle of volume 1 such that \( \Theta_0 \) is the different of \( \mathfrak{h} \) and its infinite components are all equal to \( \sqrt{\Delta} \). We have now

\[
\xi(s) = \int_{\mathfrak{H}} \varphi(\overline{\alpha}) V(\overline{\alpha})^3 \, d\mu(\overline{\alpha}) = \int_{V(\overline{\alpha}) \leq 1} 1 + \int_{V(\overline{\alpha}) > 1} \varphi(\overline{\alpha}) V(\overline{\alpha})^3 \, d\mu(\overline{\alpha})
\]

and here the first integral on the right-hand side

\[
\psi(s) = \int_{V(\overline{\alpha}) \leq 1} \varphi(\overline{\alpha}) V(\overline{\alpha})^3 \, d\mu(\overline{\alpha})
\]

gives an integral function of \( s \), for this integral converges absolutely for every complex value \( s \), because of the convergence of (1) for \( s > 1 \) and because of \( V(\overline{\alpha}) \leq 1 \). Using the theta-formula and the invariance of Haar measures, we can transform the second integral as follows:

\[
\int_{V(\overline{\alpha}) > 1} = \int_{V(\overline{\alpha}) \geq 1} (V(\overline{\alpha})^{-1} \varphi(\overline{\alpha}^{-1}) V(\overline{\alpha})^{-1} - 1) V(\overline{\alpha})^3 \, d\mu(\overline{\alpha})
\]

\[
= \int_{V(\overline{\alpha}) \geq 1} (\varphi(\overline{\alpha} \overline{\alpha}) V(\overline{\alpha})^{-1 - s} + V(\overline{\alpha})^{-1 - s} - V(\overline{\alpha})^{-s} \, d\mu(\overline{\alpha}) \quad \text{(by} \overline{\alpha} \rightarrow \overline{\alpha}^{-1} \text{)}
\]

\[
= \int_{V(\overline{\alpha}) \geq 1} \varphi(\overline{\alpha} \overline{\alpha}) V(\overline{\alpha})^{-1 - s} \, d\mu(\overline{\alpha}) + \int_{V(\overline{\alpha}) \geq 1} (V(\overline{\alpha})^{-1 - s} - V(\overline{\alpha})^{-s}) \, d\mu(\overline{\alpha})
\]

\[
= \int_{V(\overline{\alpha}) \geq 1} \varphi(\overline{\alpha} \overline{\alpha}) V(\overline{\alpha})^{-1 - s} \, d\mu(\overline{\alpha}) + \int_{V(\overline{\alpha}) \geq 1} (V(\overline{\alpha})^{-1 - s} - V(\overline{\alpha})^{-s}) \, d\mu(\overline{\alpha})
\]

\[
(\text{by} \overline{\alpha} \rightarrow \overline{\alpha}^{-2} \overline{\alpha} \quad \text{and} \quad V(\overline{\alpha}) = 1)
\]

\[
= \psi(1 - s) + \int_{V(\overline{\alpha}) \geq 1} (V(\overline{\alpha})^{-1 - s} - V(\overline{\alpha})^{-s}) \, d\mu(\overline{\alpha}).
\]
Now, the set of all ideles \( \mathfrak{A} \) such that \( V(\mathfrak{A}) = 1 \) forms a closed subgroup \( \overline{J}_1 \) of \( J \) and it can be seen easily that \( J \) is the direct product of \( \overline{J}_1 = J_1 / P \) and a subgroup \( S \) which is canonically isomorphic to the multiplicative group \( T = \{ t = V(\mathfrak{A}) \} \) of positive real numbers. Hence we have

\[
\int_{V(\mathfrak{A}) > 1} (V(\mathfrak{A})^1 - s - V(\mathfrak{A})^{-s}) \, d\mu(\mathfrak{A}) = \int_{\mathfrak{A}_1} \times \int_{S, V(\mathfrak{A}) > 1} \, \\
= \mu(\overline{J}_1) \int_{t \geq 1} (t^{1-s} - t^{-s}) \, \frac{dt}{t} \\
= \mu(\overline{J}_1) \left( \frac{1}{s-1} - \frac{1}{s} \right).
\]

We have, therefore, the formula

(3) \( \xi(s) = \psi(s) + \psi(1-s) + \mu(\overline{J}_1) \left( \frac{1}{s-1} - \frac{1}{s} \right), \quad (s > 1) \).

It then follows immediately that \( \xi(s) \) is a regular analytic function of \( s \) on the whole \( s \)-plane except for simple poles at \( s = 0, 1 \) and it satisfies the equation

\[
\xi(s) = \xi(1-s),
\]

which is nothing but the functional equation of the zeta-function \( \xi(s) \) (cf. (2)).

The formula (3) also shows that the measure \( \mu(\overline{J}_1) \) of \( \overline{J}_1 \) is finite. Since \( \overline{J}_1 \) is a locally compact group, this means that \( \overline{J}_1 \) is compact.

Now, we put \( H = (U_0 \times J_{\infty}) \cap J_1 \) and consider the sequence of groups

\[
J_1 \supset H P \supset U P \supset P.
\]

Since \( U \) is compact \( U P \) is closed in \( J_1 \), and, since \( U_0 \times J_{\infty} \) is open in \( J \), \( H \) and \( H P \) are open subgroups of \( J_1 \). It then follows from the compactness of \( \overline{J}_1 = J_1 / P \) that \( J_1 / H P \) and \( H P / U P \) are both compact groups. But, as \( H P \) is open and \( J_1 / H P \) is discrete, \( J_1 / H P \) must be finite. Consequently, the group \( \frac{J}{(U_0 \times J_{\infty})P} \), which is easily seen to be isomorphic to \( J_1 / H P \), is a finite group and this proves the finiteness
of the ideal classes of \( k \). Now, \( \Pi/U \) is isomorphic to \( (J_1 \cap J_{\infty})/(U \cap J_{\infty}) \) and hence is an \((r-1)\)-dimensional vector group. On the other hand, we see from the isomorphisms

\[
\Pi/\Pi^P = \Pi/U(\Pi \cap P), \quad U(\Pi \cap P)/U = \Pi \cap P/\Pi \cap P,
\]

that \( \Pi/U(\Pi \cap P) \) is compact and \( U(\Pi \cap P)/U \) is discrete. Since \( \Pi/U \) is a vector group, this implies that \( U(\Pi \cap P)/U \) is an \((r-1)\)-dimensional lattice in \( \Pi/U \) and, consequently, that \( \Pi \cap P/\Pi \cap P \) is a free abelian group with \( r-1 \) generators. However, as is readily seen, \( \Pi \cap P \) and \( U \cap P \) are the unit group and the group of roots of unity in \( k \). Hence the classical Dirichlet's unit theorem has been proved.

The above method of proving the functional equation can be also applied to Hecke's \( \zeta \)-functions with "Grossencharakteren", for such a character \( \chi \) is a continuous character of \( J \) which is trivial on \( S \).

The integrand of (1) must be then replaced by

\[
\chi(\varpi) \wp(\varphi, \chi) \zeta(\alpha),
\]

where \( \wp(\varphi, \chi) \) is a similar function to \( \varphi(\chi) \), depending on \( \chi \).

The zeta-function (or \( \zeta \)-functions) of a division algebra over a finite algebraic number-field can also be treated in a similar way, though here integrations over linear groups appear and calculations are more complicated.

For the above proof of the functional equation of \( \zeta(s) \), two group-theoretical facts seem to be essential. One is the topological structure of the group \( J \), that of its subgroups and factor groups, together with the invariance of Haar measures on them, and the other is the theta-formula, which is an analytical expression for the self-duality of the additive group of the ring \( R \) of valuation vectors (= additive idèles) of \( k \). \( J \) being exactly the multiplicative group of \( R \), here the additive and multiplicative properties of \( R \) are subtly mixed up and it seems to me likely that something essential to the arithmetic of \( k \) is still hidden in this connection, though I only know that the usual topology of \( J \) coincides with the one which is obtained by considering \( J \) as a group of automorphisms of the additive group of \( R \) in the sense of Braconnier.