RÉDACTION N° 119

COTE : NBR 026

TITRE : FUNCTIONAL NOTATION AND GENERAL ASSOCIATIVITY

ASSOCIATION DES COLLABORATEURS DE NICOLAS BOURBAKI

NOMBRE DE PAGES : 3

NOMBRE DE FEUILLES : 3

Functional notation and general associativity. (Eilenberg)

Let E, A_1, A_2,... be mutually disjoint sets. The elements of E will be denoted by $x, y, \ldots,$; the elements of A_n by F^n. Using the elements of these sets we define <u>words</u>. The <u>product</u> W_1W_2 of two words is defined by juxtaposition.

<u>Meaningful words</u> are defined recursively as follows. The word w for $x \in E$ is meaningful. If $W_1, \ldots, W_n$ are meaningful and $F^n \in A_n$ then $F^n W_1 \ldots W_n$ is meaningful.

The <u>weight</u> $p(W)$ is defined as follows

$$p(x) = -1, \quad p(F^n) = n-1, \quad p(W_1W_2) = p(W_1) + p(W_2)$$

<u>Theorem</u>. A word W is meaningful if and only if the following three conditions hold

(1) the last letter of W is in E

(2) $p(W) = -1$

(3) if $W = W_1W_2$ with W_2 non-empty then $p(W_1) \geqq 0$.

<u>Corollary</u>. If $W_1, \ldots, W_n$, $V_1, \ldots, V_n$ are meaningful and

$$F^n W_1 \ldots W_n = F^n V_1 \ldots V_n \quad \text{then} \quad W_i = V_i \quad \text{for} \quad i = 1, \ldots, n .$$

Now specialize to the case when A_n is a subset of the set of all functions of n-variables in E with values in E

$$F^n : E^n \to E .$$

Then we may define the <u>value</u> $v(W)$ of a meaningful word W as follows

$$v(x) = x \quad , \quad v(F^n W_1 \ldots W_n) = F^n(v(W_1), \ldots, v(W_n))$$

It follows from the corollary that $v(W)$ is a uniquely defined element of E.

Let now E be a set with an internal law of composition defined for all pairs

$$F : E^2 \to E$$

....

Let the set A_2 consist of F alone while $A_n = 0$ for $n \neq 2$. Each meaningful word W has then a value $v(W)$ in E.

We shall say that two words are <u>similar</u> if they become equal after all F's are omitted.

<u>General associativity theorem</u>. If the composition law F is associative

$$F(F(x,y),z) \quad F(x,F(y,z))$$

then any two similar meaningful words have the same value.

Proof. Define <u>normal</u> words as follows : x is normal, if W is normal then FxW is normal. Clearly every meaningful word W is similar to a unique normal word W^n. It therefore suffices to prove that W and W^n have the same value. Suppose that this has been proved for shorter words. Let $W = FW_1W_2$ with W_1 and W_2 meaningful. Then W has the same value and is similar to $V = FW_1^nW_2^n$. If W_1^n is a single symbol x then V is normal and $V = W^n$. Therefore assume $W_1^n = FxW_3$ with W_3 normal. Then $V = FFxW_3W_2^n$ and V is similar to $V' = FxFW_3W_2^n$. By the ordinary associativity V and V' have the same value. Let W_4 be the normal word similar to $FW_3W_2^n$; then W_4 has the same value as $FW_3W_2^n$ since $FW_3W_2^n$ is shorter than W. Thus W is similar to and has the same value as FxW_4, which is a normal word.
