RÉDACTION N° 119

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TITRE : Functional Notation and General Associativity

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Functional notation and general associativity. (Eilenberg)

Let $E, A_1, A_2, \ldots$ be mutually disjoint sets. The elements of $E$ will be denoted by $x, y, \ldots$; the elements of $A_n$ by $F^n$. Using the elements of these sets we define words. The product $W_1W_2$ of two words is defined by juxtaposition.

**Meaningful words** are defined recursively as follows. The word $w$ for $x \in E$ is meaningful. If $W_1, \ldots, W_n$ are meaningful and $F^n \in A_n$ then $F^nW_1\ldots W_n$ is meaningful.

The **weight** $p(W)$ is defined as follows

$$p(x) = -1, \quad p(F^n) = n^{-1}, \quad p(W_1W_2) = p(W_1) + p(W_2)$$

**Theorem.** A word $W$ is meaningful if and only if the following three conditions hold

1. the last letter of $W$ is in $E$
2. $p(W) = -1$
3. if $W = W_1W_2$ with $W_2$ non-empty then $p(W_1) \geq 0$.

**Corollary.** If $W_1, \ldots, W_n$, $V_1, \ldots, V_n$ are meaningful and $F^nW_1\ldots W_n = F^nV_1\ldots V_n$ then $W_i = V_i$ for $i = 1, \ldots, n$.

Now specialize to the case when $A_n$ is a subset of the set of all functions of $n$-variables in $E$ with values in $E$

$$F^n : E^n \to E.$$ 

Then we may define the **value** $v(W)$ of a meaningful word $W$ as follows

$$v(x) = x, \quad v(F^nW_1\ldots W_n) = F^n(v(W_1), \ldots, v(W_n))$$

It follows from the corollary that $v(W)$ is a uniquely defined element of $E$.

Let now $E$ be a set with an internal law of composition defined for all pairs

$$F : E^2 \to E$$
Let the set \( A_2 \) consist of \( F \) alone while \( A_n = 0 \) for \( n \neq 2 \). Each meaningful word \( W \) has then a value \( v(W) \) in \( E \).

We shall say that two words are similar if they become equal after all \( F \)'s are omitted.

**General associativity theorem.** If the composition law \( F \) is associative

\[
F(F(x,y),z) = F(x,F(y,z))
\]

then any two similar meaningful words have the same value.

**Proof.** Define normal words as follows: \( x \) is normal, if \( W \) is normal then \( FxW \) is normal. Clearly every meaningful word \( W \) is similar to a unique normal word \( W^n \). It therefore suffices to prove that \( W \) and \( W^n \) have the same value. Suppose that this has been proved for shorter words. Let \( W = FW_1W_2 \) with \( W_1 \) and \( W_2 \) meaningful. Then \( W \) has the same value and is similar to \( V = FW_1W_2^n \). If \( W_1 \) is a single symbol \( x \) then \( V \) is normal and \( V = W^n \). Therefore assume \( W_1 = FxW_3 \) with \( W_3 \) normal. Then \( V = FW_1W_2^n \) and \( V \) is similar to \( V' = FW_3W_2^n \). By the ordinary associativity \( V \) and \( V' \) have the same value. Let \( W_4 \) be the normal word similar to \( FW_3W_2^n \); then \( W_4 \) has the same value as \( FW_3W_2^n \) since \( FW_3W_2^n \) is shorter than \( W \). Thus \( W \) is similar to and has the same value as \( FxW_4 \), which is a normal word.