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Banach algebras and absolutely summing operators

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§1. Introduction.

If R is a Banach algebra and $\varphi \in R'$, the dual space, then we may define a bounded linear map $\tilde{\varphi} : R \to R'$ by

$$\langle \tilde{\varphi}(x), y \rangle = \langle \varphi, xy \rangle \qquad \forall x, y \in R.$$

We shall show that for suitable p the requirement that each $\tilde{\varphi}$ be p-absolutely summing constrains R to be an operator algebra, or even, in certain cases, a uniform algebra. In this way we are able to give generalisations of results of Varopoulos [12] and Kaijser [4].

The apparently artificial conditions imposed on R may be seen to have very natural interpretations in terms of the continuity of the multiplication map $M : R \otimes R \to R$ when $R \otimes R$ is equipped with certain $\otimes$-norms of Grothendieck [3] and Saphar [9]. We shall go into this in more detail in the next section.

First, let us give precise definitions of the notions with which we work.

DEFINITION. (a) A uniform algebra is a closed subalgebra of the usual Banach algebra $C(X)$ of continuous functions on some compact Hausdorff space X.

(b) A Q-algebra is a Banach algebra (algebraically) isomorphic with a quotient of a uniform algebra.

(c) An operator algebra is a Banach algebra (algebraically) isomorphic with a closed subalgebra of L(H), the usual Banach algebra of bounded linear operators on some Hilbert space H.

If E and F are Banach spaces and $1 \leq p < \infty$, then the linear mapping $u : E \to F$ is said to be p-absolutely summing if there is a positive number K such that

$$\Sigma_{j=1}^{J} \|u(e_j)\|^p \leq K^p \sup\{\Sigma_{j=1}^{J} |\langle e_j, e' \rangle|^p : \quad e' \in \text{ball}(E')\}$$

for every finite set $e_1, \ldots, e_J$ in E. The least such constant K is written $\pi_p(u)$. π_p defines a complete norm on the vector space $\pi_p(E,F)$ of all p-absolutely summing operators $E \to F$.

DEFINITION. The Banach algebra R is a p-summing algebra $(1 \leq p < \infty)$ if there is a positive K such that for each $\varphi \in R'$ the mapping $\tilde{\varphi}$ defined above is p-absolutely summing and satisfies $\pi_p(\tilde{\varphi}) \leq K\|\tilde{\varphi}\|$. If K may be taken to be 1, then R is said to be an isometrically p-summing algebra.

Charpentier [1] has proved that every commutative 1-summing algebra is a Q-algebra. On the other hand, Cole [13] has shown that every Q-algebra is an operator algebra. We work in this wider context, but it is perhaps worth noting that Charpentier's result could be obtained by much the same method.

THEOREM 1. Every 2-summing algebra is an operator algebra.

As an immediate consequence, we have a simpler proof of a striking result of Varopoulos [12].

COROLLARY 2. If an $\mathcal{L}_\infty$-space (in the sense of [6]) has a Banach algebra structure, then it must be an operator algebra.

It would be of interest to know whether one can replace "$\mathscr{L}_\infty$-space" by "the disc algebra A(D)" in corollary 2. Indeed, any non-trivial replacement would be welcome.

In the case of algebras with an identity (always of norm 1) we are able to generalise a result of Kaijser [4] to show

THEOREM 3. <u>Every isometrically p-summing algebra</u> ($1 \leq p < \infty$) <u>with (normalised) identity is a uniform algebra</u>.

In fact, in theorem 3, the weaker hypothesis that the Banach algebra R has an approximate identity whose elements have norm ≤ 1 will ensure that $\|x\|^2 = \|x^2\|$ $\forall x \in R$. The example of ℓ^1 with pointwise multiplication shows however that some such hypothesis is necessary.

§ 2. The approach via tensor products.

The multiplication on a Banach algebra R may be thought of as a linear map $M : R \otimes R \to R$. In the usual definition, R is given the projective tensor product norm and M is required to be a contraction. As this norm is the greatest of the natural $\otimes$-norms of Grothendieck [3], it is of interest to consider what happens if M is supposed continuous even when $R \otimes R$ is equipped with a smaller $\otimes$-norm. Saphar's paper [9] contains a useful summary of the properties of $\otimes$-norms ; we shall use his notation, except that the norms L and ε are sometimes written $\vee\!/$ and $\vee$, resp.

DEFINITION. Let α be a $\otimes$-norm. The Banach algebra R is said to be an <u>α-algebra</u> if the multiplication $M : R \otimes_\alpha R \to R$ is continuous. If M is a contraction, then R is said to be an <u>α-algebra</u>.

Varopoulos [10] was the first to give significant results about $\underline{\alpha}$-algebras. He was concerned with $\underline{\varepsilon}$-algebras (ε is the familiar injective tensor product norm) and showed that the commutative $\underline{\varepsilon}$-algebras are precisely the "direct" Q-algebras [11]. Kaijser [4] specialised this to prove that unital ε-algebras are uniform algebras. On the other hand, Charpentier [1] generalised Varopoulos' results, showing that commutative $\underline{\nu}$-algebras (which are in fact the commutative 1-summing algebras) are Q-algebras.

The most interesting $\otimes$-norms from our point of view are the norms d_q $(1 < q \le \infty)$ introduced by Saphar [9] and Chevet [2]. Their crucial property is that if E and F are Banach spaces, then $(E \hat{\otimes}_{d_q} F)'$ may be identified isometrically with $\Pi_p(E,F')$ under the norm π_p $(\frac{1}{p} + \frac{1}{q} = 1)$. It is now an immediate consequence of the definitions that the p-summing algebras are exactly the $\underline{d_q}$-algebras and that the isometrically p-summing algebras are exactly the d_q-algebras. We may thus rephrase theorems 1 and 3.

THEOREM 1'. Every $\underline{d_2}$-algebra is an operator algebra.

THEOREM 3'. Every $\underline{d_q}$-algebra $(1 < q \le \infty)$ with (normalized) identity is a uniform algebra.

On the other hand, the natural $\otimes$-norms of Grothendieck [3] are of basic importance. The ones which interest us in this paper are ε, ν, H, H/ and H'. In view of the results of [6], H, H/ and H'-algebras may be defined quite simply in terms of factorisations of the mappings $\tilde{\varphi}$ introduced in §1. If C denotes a C(K) space, L^2 denotes a Hilbert space and L^1 denotes an $L^1(\mu)$-space (as defined in [6]), we have :

R is an $\underline{H}$-algebra if, for each $\varphi \in R'$, the mapping $\tilde{\varphi}$ factorises as follows :

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\varphi}} & R' \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\gamma} \\ C & \xrightarrow[\beta]{} & L^1 \end{array}$$

R is an H/ -algebra if, for each $\varphi \in R'$, the mapping $\tilde{\varphi}$ factorises as follows :

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\varphi}} & R' \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\gamma} \\ C & \xrightarrow[\beta]{} & L^2 \end{array}$$

R is an H'-algebra if, for each $\varphi \in R'$, the mapping $\tilde{\varphi}$ factorises as follows :

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\varphi}} & R' \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & L^2 & \end{array}$$

In each of these cases α, β and γ are bounded linear mappings, the product of whose norms does not exceed a fixed multiple of $\|\tilde{\varphi}\|$.

As Saphar has observed, the norms d_2 and H/ are (uniformly) equivalent (since every bounded linear mapping $C \to L^2$ is 2-absolutely summing), and d_∞ and v/ are equal. Thus, yet another formulation of theorem 1 is

THEOREM 1". Every H/ -algebra is an operator algebra.

This complements Charpentier's result that every operator algebra is an H'-algebra. Since H/ and H' are adjacent in Grothendieck's table of natural $\otimes$-norms, we see that there are operator algebras which are not H/ -algebras, but I do not know of an H'-algebra which is not an operator algebra.

Finally, in the spirit of corollary 2, we may combine our results with theorems of

Kwapien [5] and Lindenstrauss-Pelczynski [6] to show that the possible Banach algebra structures on the $\mathcal{L}_p$-spaces of [6] are rather limited. This gives, in 4b and 5b, a partial answer to a question in [12], and provides a stronger version of corollary 2.

THEOREM 4.

(a) *An $\mathcal{L}_\infty$-space with a Banach algebra structure is an H-algebra*.

(b) *An $\mathcal{L}_p$-space* $(2 \leq p < \infty)$ *with a Banach algebra structure is an H'-algebra*.

(c) *An $\mathcal{L}_p$-space* $(2 \leq p < \infty)$ *with an r-summing algebra structure* $(1 \leq r < \infty)$ *is a 2-summing algebra*.

(d) *An $\mathcal{L}_p$-space* $(1 < p \leq 2)$ *with a 2-summing algebra structure is a 1-summing algebra*.

(e) *An $\mathcal{L}_1$-space with an H'-algebra structure is an ε-algebra*.

As interesting special cases, we have

COROLLARY 5.

(a) *An $\mathcal{L}_p$-space* $(1 \leq p \leq 2)$ *with a commutative 2-summing algebra structure is a Q-algebra*.

(b) *An $\mathcal{L}_p$-space* $(2 < p < \infty)$ *with an r-summing algebra structure* $(1 \leq r < \infty)$ *is an operator algebra*.

(c) *An $\mathcal{L}_1$-space which is an operator algebra must be an ε-algebra*.

An $\mathcal{L}_2$-space which is a Banach algebra is always an operator algebra [12].

§ 3. The tools.

To prove theorems 1 and 3, we rely on results of Pietsch and of Varopoulos.

THEOREM P [8]. *Suppose that* u *is a p-absolutely summing operator from the Banach space* E *to the Banach space* F. *Write* S *for the unit ball of* E' *provided with the weak * topology. Then there is a probability measure* μ *on the compact set* S *such that* $\|u(e)\| \leq \pi_p(u)\left[\int_S |\langle s,e\rangle|^p \, d\mu(s)\right]^{1/p}$ $\forall e \in E$.

THEOREM V [12]. *The Banach algebra* R *is an operator algebra if there is a positive* K *such that for any* $\varphi \in \mathrm{ball}(R')$ *and any positive integer* N, *there are a Hilbert space* H, *linear mappings* $L_n : R \to L(H)$ $(1 \leq n \leq N)$ *each of norm* $\leq K$, *and* $h, k \in \mathrm{ball}(H)$ *for which*

$$\langle \varphi, x_1 \ldots x_N \rangle = \langle L_1(x_1) \circ \ldots \circ L_N(x_N)\, h, k \rangle$$

for every choice of $x_1, \ldots, x_N$ *in* R.

Here $\langle . , . \rangle$ denotes both the duality between R and R', and the inner product on H.

§ 4. The proofs.

Proof of theorem 1. Suppose that the Banach algebra R satisfies $\pi_2(\tilde{\varphi}) \leq K\|\varphi\|$ $\forall \varphi \in R'$. We shall verify the condition of theorem V for $N = 3$. The same procedure clearly works for arbitrary N. Fix $\varphi \in \mathrm{ball}(R')$. We first construct the associated Hilbert space. By theorem P, $\tilde{\varphi}$ may be factorized as follows :

$$\begin{array}{ccc} R & \xrightarrow{\ \tilde{\varphi}\ } & R' \\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle \Phi} \\ C(S) & \xrightarrow[\ J\ ]{} & L^2(\mu_\varphi) \end{array} .$$

Here S is the unit ball of R' under the weak* topology, μ_φ is the probability measure corresponding to $\tilde{\varphi}$ as in theorem P, I is the natural map $x \longmapsto f_x$ where $f_x(s) = \langle s,x\rangle$ $(s \in S)$, J is the formal inclusion and Φ is a bounded linear map with $\|\Phi\| = \pi_2(\tilde{\varphi})$.

Thus, if $x,y,z \in R$, we have

$$\langle \varphi, xyz\rangle = \langle \tilde{\varphi}(xy), z\rangle = \langle \Phi JI(xy), z\rangle = \langle JI(xy), {}^t\Phi(z)\rangle$$

where ${}^t\Phi : R'' \longrightarrow L^2(\mu_\varphi)$ is the transpose of Φ. Consequently, if we write $Z_\varphi = {}^t\Phi(z) \in L^2(\mu_\varphi)$, we have

$$\langle \varphi, xyz\rangle = \int_S \langle \psi, xy\rangle Z_\varphi(\psi)\, d\mu_\varphi(\psi).$$

Now, applying the same process to $\tilde{\psi}$ and using the natural notation

$$\langle \varphi, xyz\rangle = \int_S \left[\int_S \langle \xi, x\rangle Y_\psi(\xi)\, d\mu_\psi(\xi)\right] Z_\varphi(\psi)\, d\mu_\varphi(\psi).$$

We may thus define a probability measure $\mu_\varphi^{(2)}$ on $S \times S$ such that

$$\langle \varphi, xyz\rangle = \int_{S\times S} \langle \xi, x\rangle Y_\psi(\xi)\, Z_\varphi(\psi)\, d\mu_\varphi^{(2)}(\psi, \xi).$$

Let us now take H to be the Hilbert direct sum $\mathbb{C} \oplus L^2(\mu_\varphi) \oplus L^2(\mu_\varphi^{(2)})$, and define three operators $L_1(x)$, $L_2(y)$, $L_3(z)$ in $L(H)$ by

$L_1(x)(\alpha, f, g) = (0, 0, G_x)$ where $G_x(\psi, \xi) = \langle \xi, x\rangle g(\psi, \xi)$,

$L_2(y)(\alpha, f, g) = (0, 0, F_y)$ where $F_y(\psi, \xi) = Y_\psi(\xi)\, f(\psi)$, and

$L_3(z)(\alpha, f, g) = (0, A_z, 0)$ where $A_z(\psi) = \alpha\, Z_\varphi(\psi)$.

It is easy to see that $\|L_1(x)\| \le \|x\|$, $\|L_2(y)\| \le K\|y\|$ and $\|L_3(z)\| \le K\|z\|$. Thus, we may produce linear operators $L_1, L_2, L_3 : R \to L(H)$ which are bounded in the right way. A simple calculation shows that

$$\langle \varphi, xyz\rangle = \langle L_1(x) \circ L_2(y) \circ L_3(z) 1_{\mathbb{C}},\ 1_{L^2(\mu_\varphi^{(2)})}\rangle$$

and so the condition of theorem V is indeed satisfied for $N = 3$.

Proof of corollary 2. If R is an $\mathcal{L}_\infty$-space, then R' is an $\mathcal{L}_1$-space [7]. Hence, [6], there is a $K > 0$ such that every bounded linear operator $u : R \to R'$ must be2-absolutely summing with $\pi_2(u) \leq K\|u\|$. The conclusion follows immediately from theorem 1.

Proof of theorem 3. If R is our algebra, we have $\pi_p(\tilde{\varphi}) \leq \|\varphi\| \quad \forall \varphi \in R'$. The main idea of the proof (used by Drury and Kaijser in the case of ε-algebras) is to show that every extreme point of the unit ball S of R' must be a scalar multiple of a multiplicative linear functional. Once again, we use theorem P to factorise $\tilde{\varphi}$:

$$\begin{array}{ccc} R & \xrightarrow{\ \tilde{\varphi}\ } & R' \\ & {\scriptstyle I}\searrow \quad \nearrow{\scriptstyle \Phi} & \\ & \Lambda_p & \end{array}$$

Writing μ_φ for the probability measure on S corresponding to $\tilde{\varphi}$, Λ_p is the subspace of $L^p(\mu_\varphi)$ formed by taking the closure of the natural image of R in $C(S)$ under the $L^p(\mu_\varphi)$ norm. I is the canonical map $x \longmapsto f_x$, with $f_x(s) = \langle s,x \rangle$ $(s \in S)$, and Φ is a linear map of norm $\pi_p(\tilde{\varphi})$.

Now, if $x,y \in R$,

$$\langle \varphi, xy \rangle = \langle \tilde{\varphi}(x), y \rangle = \langle \Phi I(x), y \rangle = \langle I(x)\ ,\ {}^t\Phi(y) \rangle$$

where ${}^t\Phi : R'' \longrightarrow (\Lambda_p)'$ is the transpose of Φ. But, $(\Lambda_p)' = L^q(\mu_\varphi)/(\Lambda_p)^o$ $(\frac{1}{p} + \frac{1}{q} = 1)$, where $(\Lambda_p)^o$ is the annihilator of Λ_p in $L^q(\mu_\varphi)$. A weak* limit argument shows that ${}^t\Phi(y)$ has a representative function $B_y \in L^q(\mu_\varphi)$ of norm $\|{}^t\Phi(y)\|$.

Hence $\langle \varphi, xy \rangle = \int_S \langle \psi, x \rangle B_y(\psi)\, d\mu_\varphi(\psi)$.

In particular, if e is the identity of R,

$$\langle \varphi, x \rangle = \int_S \langle \psi, x \rangle B_e(\psi)\, d\mu_\varphi(\psi),$$

or, symbolically, $\varphi = \int_S \psi B_e(\psi)\, d\mu_\varphi(\psi)$.

Suppose now that φ is an extreme point of S and that $S = S_1 \cup S_2$, where S_1 and S_2 are disjoint measurable sets. Define

$$\varphi_i = \int_{S_i} \psi\, B_e(\psi)\, d\mu_\varphi(\psi) \qquad (i = 1\,,\, 2).$$

Then $\varphi = \varphi_1 + \varphi_2$ and $\|\varphi_i\| \le \int_{S_i} |B_e(\psi)|\, d\mu_\varphi(\psi)$. In fact, $\|\varphi_i\| = \int_{S_i} d\mu_\varphi(\psi)$, for

$$1 = \|\varphi\| \le \|\varphi_1\| + \|\varphi_2\| \le \|B_e\|_{L^1(\mu_\varphi)} \le \|B_e\|_{L^q(\mu_\varphi)} \le \|e\| = 1.$$

Since $q \ne 1$, the equality $\|B_e\|_{L^1} = \|B_e\|_{L^q} = 1$ gives $|B_e(\psi)| = 1$ μ_φ-a. e..

Thus $\|\varphi_i\| = \int_{S_i} |B_e(\psi)|\, d\mu_\varphi(\psi) = \int_{S_i} d\mu_\varphi(\psi)$. If $\mu_\varphi(S_1) \ne 0$, the fact that φ is extreme now gives $\varphi = \varphi_1/\|\varphi_1\|$,

i.e. $$\int_{S_1} \varphi\, d\mu_\varphi(\psi) = \int_{S_1} \psi\, B_e(\psi)\, d\mu_\varphi(\psi).$$

This equality is thus valid for every measurable subset of S, whence

$$\varphi = \psi B_e(\psi) \qquad \mu_\varphi\text{-a.e.}$$

Consequently,
$$\begin{aligned}\langle\varphi,e\rangle\langle\varphi,xy\rangle &= \langle\varphi,e\rangle \int_S \langle\psi,x\rangle B_y(\psi)\, d\mu_\varphi(\psi) \\ &= \int_S \langle\psi,e\rangle\, B_e(\psi)\langle\psi,x\rangle B_y(\psi)\, d\mu_\varphi(\psi) \\ &= \int_S \langle\psi,e\rangle\langle\varphi,x\rangle B_y(\psi)\, d\mu_\varphi(\psi) \\ &= \langle\varphi,x\rangle\langle\varphi,y\rangle.\end{aligned}$$

Easily, $|\langle\varphi,e\rangle| = 1$, and so $\varphi/\langle\varphi,e\rangle$ is a multiplicative linear functional. Now, for $x \in R$,
$$\begin{aligned}\|x^2\| &= \sup\left\{ |\langle\varphi,x^2\rangle| : \varphi \text{ extreme point of } S\right\} \\ &= \sup\left\{ \left|\left\langle\frac{\varphi}{\langle\varphi,e\rangle}, x^2\right\rangle\right| : \varphi \text{ extreme point of } S\right\} \\ &= \sup\left\{ \left|\left\langle\frac{\varphi}{\langle\varphi,e\rangle}, x\right\rangle\right|^2 : \varphi \text{ extreme point of } S\right\} \\ &= \|x\|^2.\end{aligned}$$

The result follows at once.

Proof of theorem 4. We shall only prove (e). The rest is a straightforward consequence of the fact that the dual of an $\mathcal{L}_p$-space is an $\mathcal{L}_q$-space $(\frac{1}{p}+\frac{1}{q}=1)$ (see [7]) and of the results in [5] and [6]. Note that for (a) we need the fact [6] that an $\mathcal{L}_1$-space which is a dual space - and so complemented in its bidual - is a complemented subspace of an $L^1(\mu)$-space.

Suppose now that R is an $\mathcal{L}_1$-space with an H′-algebra structure. In all that follows, the constants K (with or without a subscript) will be independent of the algebra structure of R. If $\varphi \in R'$, it follows from §2 that $\tilde{\varphi}$ factors through a Hilbert space H. But by [6, p. 286] a bounded linear operator $f: R \to H$ is 1-absolutely summing and satisfies $\pi_1(f) \le K\|f\|$. Hence R is a d_∞-algebra. To show that it is an ε-algebra, choose $\{x_1, \ldots, x_J, y_1, \ldots, y_J\} \subseteq R$ and consider the closed subspace generated by these elements and $\{x_1y_1, \ldots, x_Jy_J\}$. This is contained in some subspace E of R of finite dimension n for which there are isomorphisms $u: E \to \ell^1_n$ and $v: \ell^1_n \to E$ such that $\|u\| \le K_1$, $\|v\| \le 1$ and $v \circ u = Id_E$. Since R is a d_∞-algebra,

$$\|\Sigma_{j=1}^J x_jy_j\|_R \le K.\|\Sigma_{j=1}^J x_j \otimes y_j\|_{R\hat{\otimes}_{d_\infty} R} \le K\|\Sigma_{j=1}^J x_j \otimes y_j\|_{E\hat{\otimes}_{d_\infty} E}$$

$$= K\|\sum_{j=1}^J v(u(x_j)) \otimes v(u(y_j))\|_{E\check{\otimes}E}$$

$$\le K\|\sum_{j=1}^J u(x_j) \otimes u(y_j)\|_{\ell^1_n \check{\otimes} \ell^1_n}$$

$$= K\|\sum_{j=1}^J u(x_j) \otimes u(y_j)\|_{\ell^1_n \check{\otimes} \ell^1_n} \quad \text{by definition of } v$$

$$\le K.K_1^2 \|\sum_{j=1}^J x_j \otimes y_j\|_{E\check{\otimes}E}$$

$$= K.K_1^2\|\Sigma_{j=1}^J x_j \otimes y_j\|_{R\check{\otimes}R}.$$

Since our choice of the x_j's and y_j's was arbitrary, R is an $\underline{\varepsilon}$-algebra.

Finally, corollary 5 needs no proof.

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