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ON THE THEORY AND NUMERICAL ANALYSIS OF THE NAVIER-STOKES EQUATIONS

by

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INTRODUCTION

These lecture notes develop some results related to the theory and numerical analysis of the Navier-Stokes equations of viscous incompressible fluids.

We successively consider the linearized stationary case (Chapter I), the nonlinear stationary case (Chapter II), and the full nonlinear evolution case (Chapter III).

In Chapter I, after a short presentation of existence and uniqueness results, we describe the approximation of the Stokes problem by several finite difference and finite element methods. This is an opportunity to introduce various methods of approximation of the divergence free vector functions which are also essential tools for the numerical aspects of the problems studied in Chapters II and III.

In Chapter II, we introduce compactness tools in both the continuous and discrete cases. Otherwise this chapter is just an extension to the nonlinear case of the results obtained for the linear case in the preceding chapter.

Chapter III deals with the full nonlinear evolution case. We first describe some typical results of the present state of the mathematical theory of the Navier-Stokes equations (existence and uniqueness results). We then give a brief introduction to the numerical aspects, combining the discretization in the space variables discussed in Chapter I with the usual discretization techniques in the time variable. The stability and convergence of the schemes are studied by energy methods. The numerical analysis of the full Navier-Stokes equations will be more fully developed in an extended version of this course to appear elsewhere.

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CONTENTS

CHAPTER I

THE STATIONARY STOKES EQUATIONS

INTRODUCTION

In this chapter we study the stationary Stokes equations; that is, the stationary linearized form of the Navier-Stokes equations. The study of the Stokes equations is useful by itself; it also gives us an opportunity to introduce several tools necessary for a treatment of the full Navier-Stokes equations.

In Section 1 we consider some function spaces (spaces of divergence free vector functions with L^2-components). In Section 2 we give the variational formulation of the Stokes equations and prove existence and uniqueness of solutions by the projection theorem. In Sections 3 and 4 we recall a few definitions and results on the approximation of a normed space and of a variational linear equation (Section 3). We propose then several types of approximation of a certain fundamental space V of divergence free vector functions; this includes an approximation by the finite differences method (Section 3), and by conforming and non-conforming finite element methods (Section 4). In Section 5 we discuss some algorithms of approximation for the Stokes equations and the corresponding discretized equations. The purpose of these algorithms is to overcome the difficulty caused by the condition $\text{div}\, u = 0$. As it will be shown, this difficulty, sometimes, is not merely solved by discretization.

Finally in Section 6 we study the linearized equations of slightly compressible fluids and their asymptotic convergence to the linear equations of incompressible fluids (i.e., Stokes equations).

§1. SOME FUNCTION SPACES

In this section we introduce and study some fundamental function spaces. The results are important for what follows, but the methods used in this section will not reappear so that the reader can skim through the proofs and retain only the general notations described in Sec. 1.1 and the results summarized in Remark 1.6.

1.1 Notation

In euclidian space $\mathbb{R}^n$ we write $e_1 = (1,0,\cdots,0)$, $e_2 = (0,1,0,\cdots,0),\cdots,$ $e_n = (0,\cdots,0,1)$, the canonical basis, and $x = (x_1,\cdots,x_n)$, $y = (y_1,\cdots,y_n)$, $z = (z_1,\cdots,z_n),\cdots,$ will denote points of the space.

The differentiation operator $\frac{\partial}{\partial x_i}$ $(1 \leq i \leq n)$, will be written D_i and if $j = (j_1,\cdots j_n)$ is a multi-index, D^j will be the differentiation operator

(1.1) $$D^j = D_1^{j_1} \cdots D_n^{j_n} = \frac{\partial^{[j]}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$$

where

(1.2) $$[j] = j_1 + \cdots + j_n$$

When $j_i = 0$ for some i, $D_i^{j_i}$ is the identity operator; in particular if $[j] = 0$, D^j is the identity.

<u>The Set</u> Ω

Let Ω be an open set of $\mathbb{R}^n$ with boundary Γ. In general we will need some smoothness property for Ω. Sometimes we will assume that Ω is smooth in the following sense:

(1.3) $\begin{cases} \text{The boundary } \Gamma \text{ is a (n-1)-dimensional manifold of class } \mathscr{C}^r \ (r \geq 1 \\ \text{which must be specified) and } \Omega \text{ is locally located on one side of } \Gamma. \end{cases}$

We will say that a set Ω satisfying (1.3) is of class $\mathscr{C}^r$. However this hypothesis is too strong for practical situations (such as a flow in a square) and all the main results will be proved under a weaker condition:

(1.4) The boundary of Ω is locally Lipschitzian.

This means that in a neighborhood of any point $x \in \Gamma$, Γ admits a representation as a hypersurface $y_n = \theta(y_1, \cdots, y_{n-1})$ where θ is a lipschitzian function, and $(y_1, \cdots y_n)$ are rectangular coordinates in $\mathbb{R}^n$ in a basis that may be different from the canonical basis $e_1, \cdots, e_n$.

Of course if Ω is of class $\mathscr{C}^1$, then Ω is locally lipschitzian.

It is useful for the sequel of this section, to notice that a set r satisfying (1.4) is "locally star-shaped" This means that each point $x_j \in \Gamma$, has an open neighborhood $\mathcal{O}_j$, such that $\mathcal{O}'_j = \Omega \cap \mathcal{O}_j$ is star-shaped with respect to one of its points. According to (1.4) we can moreover suppose that the boundary of $\mathcal{O}'_j$ is lipschitzian.

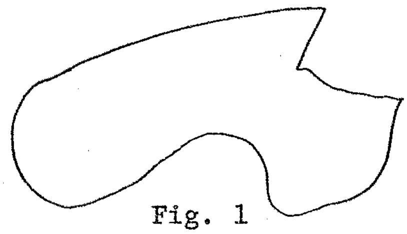

Fig. 1

If Γ is bounded, it can be covered by a finite family of such sets $\mathcal{O}_j$, $j \in J$; if Γ is not bounded, the family $(\mathcal{O}_j)_{j \in J}$ can be chosen to be locally finite.

<u>It is understood that</u> Ω <u>will always satisfy</u> (1.4), unless we mention explicitly that Ω is any open set in $\mathbb{R}^n$ or that some other smoothness property is required.

<u>L^p and Sobolev Spaces</u>

Let Ω be any open set in $\mathbb{R}^n$. We denote by $L^p(\Omega)$, $1 \leq p < +\infty$ (resp. $L^\infty(\Omega)$) the space of real functions defined on Ω with the p^{th} power absolutely integrable (resp. essentially bounded real functions) for the Lebesgue measure $dx = dx_1 \cdots dx_n$. This is a Banach space with the norm

$$\|u\|_{L^p(\Omega)} = \{\int_\Omega |u(x)|^p dx\}^{\frac{1}{p}} \tag{1.5}$$

(resp.

$$\|u\|_{L^\infty(\Omega)} = \text{ess.}\sup_\Omega |u(x)|).$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u,v) = \int_\Omega u(x)v(x)dx \tag{1.6}$$

The Sobolev space $W^{m,p}(\Omega)$ is the space of functions in $L^p(\Omega)$ with derivatives of order less than or equal to m in $L^p(\Omega)$ (m = integer, $1 \le p \le +\infty$). This is a Banach space with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \{\sum_{|j|\le m} \|D^j u\|^p_{L^p(\Omega)}\}^{\frac{1}{p}}. \tag{1.7}$$

When $p = 2$, $W^{m,2}(\Omega) = H^m(\Omega)$ is a Hilbert space with the scalar product

$$((u,v))_{H^m(\Omega)} = \sum_{|j|\le m} (D^j u, D^j v). \tag{1.8}$$

Let $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(\overline{\Omega})$) be the space of $\mathcal{C}^\infty$ functions with compact support contained in Ω (resp. $\overline{\Omega}$). The closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ ($H_0^m(\Omega)$ when $p = 2$).

We will recall when needed the classical properties of these spaces such as the density or trace theorems (assuming regularity properties for Ω).

We will often be concerned with n-dimensional vector functions with components in one of these spaces. We will use the notation

$$\mathbb{L}^p(\Omega) = \{L^p(\Omega)\}^n, \quad \mathbb{W}^{m,p}(\Omega) = \{W^{m,p}(\Omega)\}^n$$

$$\mathbb{H}^m(\Omega) = \{H^m(\Omega)\}^n, \quad \mathbb{D}(\Omega) = \{\mathcal{D}(\Omega)\}^n,$$

and we suppose that these product spaces are equipped with the usual product norm or an equivalent norm (except $\mathcal{D}(\Omega)$ and $\mathbb{D}(\Omega)$ which are not normed spaces.)

The following spaces will appear very frequently

$$L^2(\Omega), \quad \mathbb{L}^2(\Omega), \quad H_0^1(\Omega), \quad \mathbb{H}_0^1(\Omega).$$

The scalar product and the norm are denoted by $(\cdot,\cdot)$ and $|\cdot|$ on $L^2(\Omega)$ or $\mathbb{L}^2(\Omega)$

(resp. $[\![\cdot,\cdot]\!]$ and $[\![\cdot]\!]$ on $H_0^1(\Omega)$ or $\mathbb{H}_0^1(\Omega)$).

We recall that if Ω is <u>bounded in some direction</u>[1] then the Poincaré inequality holds:

$$(1.9)\qquad \|u\|_{L^2(\Omega)} \leq c(\Omega)\|Du\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega),$$

where D is the derivative in that direction and $c(\Omega)$ is a constant depending only on Ω. In this case the norm $[\![\cdot]\!]$ on $H_0^1(\Omega)$ (resp. $\mathbb{H}_0^1(\Omega)$) is equivalent to the norm:

$$(1.10)\qquad \|u\| = \Big(\sum_{i=1}^{n} |D_i u|^2\Big)^{\frac{1}{2}}$$

The space $H_0^1(\Omega)$ (resp. $\mathbb{H}_0^1(\Omega)$) is also a Hilbert space with the associated scalar product

$$(1.11)\qquad ((u,v)) = \sum_{i=1}^{n} (D_i u, D_i v).$$

This scalar product and this norm are denoted by $((\cdot,\cdot))$ and $\|\cdot\|$ on $H_0^1(\Omega)$ and $\mathbb{H}_0^1(\Omega)$ (Ω bounded in some direction).

Let $\mathcal{V}$ be the space (without topology)

$$(1.12)\qquad \mathcal{V} = \{u \in \mathcal{D}(\Omega), \quad \operatorname{div} u = 0\}.$$

The closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ and in $\mathbb{H}_0^1(\Omega)$ are two basic spaces in the study of the Navier-Stokes equations; we denote them by H and V. The results of this section will allow us to give a characterization of V and H.

1.2 <u>A Density Theorem</u>

Let $E(\Omega)$ be the following auxiliary space:

$$E(\Omega) = \{u \in \mathbb{L}^2(\Omega), \quad \operatorname{div} u \in L^2(\Omega)\}.$$

This is a Hilbert space when equipped with the scalar product

$$(1.13)\qquad ((u,v))_{E(\Omega)} = (u,v) + (\operatorname{div} u, \operatorname{div} v).$$

It is clear that (1.13) is a scalar product on $E(\Omega)$; it is easy to see that $E(\Omega)$ is

(1) i.e., Ω is included in a slab whose boundary is two hyperplanes which are orthogonal to this direction.

complete for the associated norm

$$\|u\|_{E(\Omega)} = \{((u,u))_{E(\Omega)}\}^{\frac{1}{2}} \text{ (1)}.$$

Our goal is to prove a trace theorem: for $u \in E(\Omega)$ one can define the value on Γ of the normal component $u \cdot \nu$, ν = the unit vector normal to the boundary. The method for that is a classical method of Lions-Magenes [1]. We first start by proving the

Theorem 1.1.

Let Ω be a locally lipschitzian open set in $\mathbb{R}^n$. Then the set of vector functions in $\mathcal{D}(\bar{\Omega})$ is dense in $E(\Omega)$.

Proof.

Let u be some element of $E(\Omega)$. We have to prove that u is limit in $E(\Omega)$ of vector functions of $\mathcal{D}(\bar{\Omega})$.

i) We first approximate u by functions of $E(\Omega)$ with compact support in Ω.

Let $\phi \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \phi \leq 1$, $\phi = 1$ for $|x| \leq 1$, and $\phi = 0$ for $|x| \geq 2$. For $a > 0$ let ϕ_a be the restriction to Ω of the function $x \mapsto \phi(\frac{x}{a})$. It is easy to check that $\phi_a u \in E(\Omega)$ and that $\phi_a u$ converges to u in this space as $a \to \infty$.

The functions with compact support are a dense subspace of $E(\Omega)$ and we can suppose that u has compact support.

ii) Let us consider first the case $\Omega = \mathbb{R}^n$; hence $u \in E(\mathbb{R}^n)$ and u has compact support.

The result is then proved by regularization. Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a smooth $\mathcal{C}^\infty$ function with compact support, such that $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho(x)dx = 1$. For $\varepsilon \in (0,1]$, let ρ_ε denote the function $x \mapsto \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$. When $\varepsilon \to 0$, ρ_ε converges in the distribution sense to the Dirac distribution and it is classical that

$$\rho_\varepsilon * v \to v \quad \text{in } L^2(\mathbb{R}^n),\ \forall v \in L^2(\mathbb{R}^n). \text{ (2)} \tag{1.14}$$

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(1) For if u_m is a Cauchy sequence in $E(\Omega)$, then u_m is also a Cauchy sequence in $\mathbb{L}^2(\Omega)$ and also $\operatorname{div} u_m$ is a Cauchy sequence in $L^2(\Omega)$; u_m converges to some limit u in $\mathbb{L}^2(\Omega)$, and $\operatorname{div} u_m$ converges to some limit g in $L^2(\Omega)$, necessarily $g = \operatorname{div} u$, and so $u \in E(\Omega)$ and u_m converges to u in $E(\Omega)$.

(2) $*$ is the convolution operator

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

If $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $f*g$ makes sense and belongs to $L^p(\mathbb{R}^n)$.

Now $\rho_\varepsilon * u$ belongs to $\mathcal{D}(\mathbb{R}^n)$ since this function has a compact support ($\subset$(support ρ_ε)+ support u) and components which are $\mathcal{C}^\infty$. According to (1.14) $\rho_\varepsilon * u$ converges to u in $\mathbb{L}^2(\mathbb{R}^n)$ as $\varepsilon \to 0$, and

$$\operatorname{div}(\rho_\varepsilon * u) = \rho_\varepsilon * \operatorname{div} u \quad \text{converges to } \operatorname{div} u \quad \text{in } \mathbb{L}^2(\mathbb{R}^n),$$

as $\varepsilon \to 0$. Hence u is the limit in $E(\mathbb{R}^n)$ of functions of $\mathcal{D}(\mathbb{R}^n)$.

iii) For the general case, $\Omega \neq \mathbb{R}^n$, we use the remark after (1.4): Ω is locally star-shaped. The sets $\Omega, (\mathcal{O}_j)_{j \in J}$, form an open covering of $\bar{\Omega}$. Let us consider a partition of unity subordinated to this covering

$$(1.15) \qquad 1 = \phi + \sum_{j \in J} \phi_j, \quad \phi \in \mathcal{D}(\Omega), \quad \phi_j \in \mathcal{D}(\mathcal{O}_j).$$

We can write

$$u = \phi u + \sum_{j \in J} \phi_j u,$$

the sum $\sum_j$ is actually finite since the support of u is compact.

Since the function ϕu has compact support in Ω it can be shown as in ii) that ϕu is limit in $E(\Omega)$ of functions belonging to $\mathcal{D}(\Omega)$ (the function ϕu extended by 0 outside Ω belongs to $E(\mathbb{R}^n)$ and for ε sufficiently small, $\rho_\varepsilon*(\phi u)$ has compact support in Ω).

Let us consider now one of the functions $u_j = \phi_j u$ not identically equal to zero. The set $\mathcal{O}'_j = \mathcal{O}_j \cap \Omega$ is star-shaped with respect to one of its points; after a translation in $\mathbb{R}^n$ we can suppose this point is 0. Let σ_λ, $\lambda \neq 0$, be the linear (homothetic) transformation $x \mapsto \lambda x$. It is clear that

$$\begin{cases} \mathcal{O}'_j \subset \overline{\mathcal{O}'_j} \subset \sigma_\lambda \mathcal{O}'_j & \text{for } \lambda > 1 \\ \sigma_\lambda \mathcal{O}'_j \subset \overline{\sigma_\lambda \mathcal{O}'_j} \subset \mathcal{O}'_j & \text{for } 0 < \lambda < 1. \end{cases}$$

Let $\sigma_\lambda \circ v$ denote the function $x \mapsto v(\sigma_\lambda(x))$; because of Proposition 1.1 below, the restriction to $\mathcal{O}'_j$ of the function $\sigma_\lambda \circ u_j$, $\lambda > 1$, converges to u_j in $E(\mathcal{O}'_j)$ (or $E(\Omega)$) as $\lambda \mapsto 1$. But if $\psi_j \in \mathcal{D}(\sigma_\lambda(\mathcal{O}'_j))$ and $\psi_j = 1$ on $\mathcal{O}'_j$, the function $\psi_j(\sigma_\lambda \circ u)$ clearly belongs to $E(\mathbb{R}^n)$. Hence we must only approximate in place of the function u_j, a function $v_j \in E(\Omega)$ which is the restriction to Ω of a function $w_j \in E(\mathbb{R}^n)$ with compact support (take $w_j = \psi_j(\sigma_\lambda \circ u)$). The result follows then from point ii).

It remains only to prove the Proposition 1.1 giving some results needed before, and some results we will need later.

Proposition 1.1.

Let $\mathcal{O}$ be an open set which is star-shaped with respect to 0.

(i) If $p \in \mathcal{D}'(\mathcal{O})$ is a distribution in $\mathcal{O}$, then a distribution $\sigma_\lambda \circ p$ can be defined in $\mathcal{D}'(\sigma_\lambda \mathcal{O})$ by

$$(1.16) \qquad \langle \sigma_\lambda \circ p, \phi \rangle = \frac{1}{\lambda^n} \langle p, \sigma_{\frac{1}{\lambda}} \circ \phi \rangle, \quad \forall \phi \in \mathcal{D}(\sigma_\lambda \mathcal{O}) \, (\lambda > 0).$$

The derivatives of $\sigma_\lambda \circ p$ are related to the derivatives of p by the formula

$$(1.17) \qquad D_i(\sigma_\lambda \circ p) = \lambda \sigma_\lambda \circ (D_i p), \quad 1 \leq i \leq n.$$

If $\lambda > 1$, $\lambda \longrightarrow 1$, the restriction to $\mathcal{O}$ of $\sigma_\lambda \circ p$ converges in the distribution sense to p.

(ii) If $p \in L^\alpha(\mathcal{O})$, $1 \leq \alpha < +\infty$, then $\sigma_\lambda \circ p \in L^\alpha(\sigma_\lambda \mathcal{O})$. For $\lambda > 1$, $\lambda \longrightarrow 1$, the restriction to $\mathcal{O}$ of $\sigma_\lambda \circ p$ converges to p in $L^\alpha(\mathcal{O})$.

Proof.

It is clear that the mapping $\phi \longmapsto \frac{1}{\lambda^n} \langle p, \sigma_{\frac{1}{\lambda}} \circ \phi \rangle$ is linear and continuous on $\mathcal{D}(\sigma_\lambda \mathcal{O})$ and hence defines a distribution which is denoted by $\sigma_\lambda \circ p$.

The formula (1.17) is easy,

$$\begin{aligned} \langle D_i(\sigma_\lambda \circ p), \phi \rangle &= - \langle (\sigma_\lambda \circ p), D_i \phi \rangle \\ &= - \frac{1}{\lambda^n} \langle p, \sigma_{\frac{1}{\lambda}} \circ (D_i \phi) \rangle \\ &= - \frac{1}{\lambda^{n-1}} \langle p, D_i(\sigma_{\frac{1}{\lambda}} \circ \phi) \rangle \\ &= + \frac{1}{\lambda^{n-1}} \langle D_i p, \sigma_{\frac{1}{\lambda}} \circ \phi \rangle \\ &= \lambda \langle \sigma_\lambda \circ D_i p, \phi \rangle \end{aligned}$$

When $\lambda > 1$, $\lambda \longrightarrow 1$, the functions $\sigma_{\frac{1}{\lambda}} \circ \phi$ have compact support in $\mathcal{O}$ for $\lambda - 1$ small enough, and converge to ϕ in $\mathcal{D}(\mathcal{O})$ as $\lambda \longrightarrow 1$.

(ii) It is clear that

$$\int_{\sigma_\lambda \mathcal{O}} |(\sigma_\lambda p)(y)|^\alpha dy = \lambda^n \int_{\mathcal{O}} |p(x)|^\alpha dx$$

and

$$\|\sigma_\lambda \circ p\|_{L^\alpha(\sigma_\lambda \mathcal{O})} = \|p\|_{L^\alpha(\mathcal{O})}$$

It is then sufficient to prove that $\sigma_\lambda \circ p$ restricted to $\mathcal{O}$ converge to p, for the p's belonging to some dense subspace of $L^\alpha(\Omega)$. But $\mathcal{D}(\mathcal{O})$ is dense in $L^\alpha(\mathcal{O})$, and the result is obvious if $p \in \mathcal{D}(\mathcal{O})$.

1.3. A Trace Theorem

We suppose here that Ω is an open bounded set of class $\mathcal{C}^2$. It is known that there exists a linear continuous operator $\gamma_0 \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ (the trace operator), such that $\gamma_0 u$ = the restriction of u to Γ for every function $u \in H^1(\Omega)$ which is twice continuously differentiable in $\overline{\Omega}$. The space $H_0^1(\Omega)$ is equal to the kernel of γ_0. The image space $\gamma_0(H^1(\Omega))$ is a dense subspace of $L^2(\Gamma)$ denoted $H^{\frac{1}{2}}(\Gamma)$; the space $H^{\frac{1}{2}}(\Gamma)$ can be equipped with the norm carried from $H^1(\Gamma)$ by γ_0. There exists moreover a linear continuous operator $\ell_\Omega \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$ (which is called a lifting operator), such that $\gamma_0 \circ \ell_\Omega$ = the identity operator in $H^{\frac{1}{2}}(\Gamma)$. All these results are given in Lions [1], Lions-Magenes [1].

We want to prove a similar result for the vector functions in $E(\Omega)$.

Let $H^{-\frac{1}{2}}(\Gamma)$ be the dual space of $H^{\frac{1}{2}}(\Gamma)$; since $H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$ with a stronger topology, $L^2(\Gamma)$ is contained in $H^{-\frac{1}{2}}(\Gamma)$ with a stronger topology. We have the following trace theorem (which means that one can define $u \cdot \nu|_\Gamma$ when $u \in E$):

Theorem 1.2.

Let Ω be an open bounded set class $\mathcal{C}^2$. Then there exists a linear continuous operator $\gamma_\nu \in \mathcal{L}(E(\Omega), H^{-\frac{1}{2}}(\Gamma))$, such that

(1.18) $\gamma_\nu u$ = *the restriction of* $u \cdot \nu$ *to* Γ, *for every* $u \in \mathcal{D}(\overline{\Omega})$.

The following generalized Stokes formula is true, for each $u \in E(\Omega)$ *and* $w \in H^1(\Omega)$

(1.19) $$(u, \operatorname{grad} w) + (\operatorname{div} u, w) = \langle \gamma_\nu u, \gamma_0 w \rangle.$$

Proof.

Let $\phi \in H^{\frac{1}{2}}(\Gamma)$ and let $w \in H^1(\Omega)$ with $\gamma_0 w = \phi$. For $u \in E(\Omega)$, let us set

$$X_u(\phi) = \int_\Omega [\operatorname{div} u(x) w(x) + u(x) \operatorname{grad} w(x)] dx$$

$$= (\operatorname{div} u, w) + (u, \operatorname{grad} w).$$

Lemma 1.1.

$X_u(\phi)$ is independent of the choice of w, as long as $w \in H^1(\Omega)$ and $\gamma_0 w = \phi$.

Proof.

Let w_1 and w_2 belong to $H^1(\Omega)$, with

$$\gamma_0 w_1 = \gamma_0 w_2 = \phi,$$

and let $w = w_1 - w_2$.

One has to prove that

$$(\text{div } u, w_1) + (u, \text{grad } w_1) = (\text{div } u, w_2) + (u, \text{grad } w_2),$$

that is to say

(1.20) $$(\text{div } u, w) + (u, \text{grad } w) = 0.$$

But since $w \in H^1(\Omega)$ and $\gamma_0 w = 0$, w belongs to $H_0^1(\Omega)$ and is the limit in $H^1(\Omega)$ of smooth functions with compact support: $w = \lim w_m$, $w_m \in \mathcal{D}(\Omega)$. It is obvious that

$$(\text{div } u, w_m) + (u, \text{grad } w_m) = 0, \ \forall w_m \in \mathcal{D}(\Omega)$$

and (1.20) follows as $m \longrightarrow \infty$.

Let us take now, $w = \ell_\Omega \phi$ (see above). Then by the Schwarz inequality

$$|X_u(\phi)| \leq \|u\|_{E(\Omega)} \|w\|_{H^1(\Omega)},$$

and since $\ell_\Omega \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$

(1.21) $$|X_u(\phi)| \leq c_0 \|u\|_{E(\Omega)} \|\phi\|_{H^{\frac{1}{2}}(\Gamma)},$$

where c_0= the norm of the linear operator ℓ_Ω.

Therefore the mapping $\phi \mapsto X_u(\phi)$ is a linear continuous mapping from $H^{\frac{1}{2}}(\Gamma)$ into $\mathbb{R}$. Thus there exists $g = g(u) \in H^{-\frac{1}{2}}(\Gamma)$ such that

(1.22) $$X_u(\phi) = \langle g, \phi \rangle.$$

It is clear that the mapping $u \mapsto g(u)$ is linear and, by (1.21),

(1.23) $$\|g\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c_0 \|u\|_{E(\Omega)};$$

this proves that the mapping $u \longmapsto g(u) = \gamma_\nu u$ is continuous from $E(\Omega)$ into $H^{-\frac{1}{2}}(\Gamma)$.

The last point is to prove (1.18), since (1.19) follows directly from the definition of $\gamma_\nu u$.

Lemma 1.2.

If $u \in \mathscr{D}(\bar{\Omega})$, then

$$\gamma_\nu \cdot u = \text{the restriction of } u \cdot \nu \text{ on } \Gamma.$$

Proof.

For such a smooth u and for any $w \in \mathscr{D}(\bar{\Omega})$ (or if u and w are twice continuously differentiable in $\bar{\Omega}$),

$$X_u(\gamma_0 w) = \int_\Omega \operatorname{div}(u\, w)\, dx$$

$$= \text{(by the Stokes formula)}$$

$$= \int_\Gamma w(u\cdot\nu)\, d\Gamma = \int_\Gamma (u\cdot\nu)(\gamma_0 w)\, d\Gamma.$$

$$= \langle u\cdot\nu,\ \gamma_0 w\rangle.$$

Since for these functions w, the traces $\gamma_0 w$ form a dense subset of $H^{\frac{1}{2}}(\Gamma)$, the formula

$$X_u(\phi) = \langle u.\nu,\ \phi\rangle$$

is also true by continuity for every $\phi \in H^{\frac{1}{2}}(\Gamma)$. By comparison with (1.22), we get, $\gamma_\nu u = u\cdot\nu|_\Gamma$.

Remark 1.1.

Theorem 1.1 is not explicitly used in the proof of Theorem 1.2, but the density theorem combined with Lemma 1.2 shows that the operator γ_ν is unique since its value on a dense subset is known.

Remark 1.2.

The operator γ_ν actually maps $E(\Omega)$ onto $H^{-\frac{1}{2}}(\Gamma)$.

Let ϕ be given in $H^{-\frac{1}{2}}(\Gamma)$, such that

$$\langle \phi, 1\rangle = 0.$$

Then the Neumann problem

$$\text{(1.24)} \qquad \begin{cases} -\Delta p = 0 & \text{in } \Omega \\ \dfrac{\partial p}{\partial \nu} = \phi & \text{on } \Gamma \end{cases}$$

has a generalized solution $p = p(\phi) \in H^1(\Omega)$ which is unique up to an additive constant (See Lions-Magenes [1]). For one of these solutions p let

$$u = \text{grad } p.$$

It is clear that $u \in E(\Omega)$ and $\gamma_\nu u = \phi$. In addition it is clear that there exists a vector function u_0 with components in $\mathcal{C}^1(\bar{\Omega})$ such that $\gamma_\nu \cdot u_0 = 1$. Then for any ψ in $H^{\frac{1}{2}}(\Gamma)$, writing

$$\psi = \phi + \frac{<\psi,1>}{\text{mes } \Gamma}, \quad \phi = \psi - \frac{<\psi,1>}{\text{mes } \Gamma}, \tag{1.25}$$

one can define a $u = u(\psi)$ such that $\gamma_\nu u = \psi$ by setting

$$u = \text{grad } p(\psi) + \frac{<\psi,1>}{\text{mes } \Gamma} u_0. \tag{1.26}$$

Moreover the mapping $\psi \longmapsto u(\psi)$ is a linear continuous mapping from $H^{\frac{1}{2}}(\Gamma)$ into $E(\Omega)$ (i.e., a <u>lifting</u> operator as ℓ_Ω).

Let $E_0(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $E(\Omega)$. We have

<u>Theorem 1.3</u>.

<u>The kernel of</u> γ_ν <u>is equal to</u> $E_0(\Omega)$.

<u>Proof</u>.

If $u \in E_0(\Omega)$, then bv the definition of this space, there exists a sequence of functions $u_m \in \mathcal{D}(\Omega)$ which converges to u in $E(\Omega)$ as $m \to \infty$. Theorem 1.2 implies that $\gamma_\nu u_m = 0$ and hence, $\gamma_\nu \cdot u = \lim_{m \to \infty} \gamma_\nu \cdot u_m = 0$.

Conversely let us prove that if $u \in E(\Omega)$ and $\gamma_\nu \cdot u = 0$, then u is the limit in $E(\Omega)$ of vector functions in $\mathcal{D}(\Omega)^n$.

Let Φ be any function in $\mathcal{D}(\mathbb{R}^n)$, and ϕ the restriction of Φ to Ω. Since $\gamma_\nu \cdot u = 0$, we have $< \gamma_\nu \cdot u, \gamma_0 \phi > = 0$ which means

$$\int_\Omega [\text{div } u \cdot \phi + u \cdot \text{grad } \phi] dx = 0.$$

Hence

$$\int_{\mathbb{R}^n} [\widetilde{\text{div } u} \cdot \Phi + \tilde{u} \cdot \text{grad } \Phi] dx = 0, \forall \Phi \in \mathcal{D}(\mathbb{R}^n)$$

and so

$$\text{div } \tilde{u} = \widetilde{\text{div } u}, \tag{1.27}$$

where $\tilde{v}$ denotes the function equal to v in Ω and to 0 in $\complement\Omega$. This implies that $\tilde{u} \in E(\mathbb{R}^n)$.

Following exactly the steps in proving Theorem 1.1 (in particular points i) and ii)) it is possible to reduce the general case to the case where the function u has support included in one of the sets $\mathcal{O}_j \cap \bar{\Omega}$.

For such a function u, we remark that $\tilde{u} \in E(\mathbb{R}^n)$ and that $\sigma_\lambda \circ \tilde{u}$ has a compact support in $\mathcal{O}'_j$, for $0 < \lambda < 1$ ($\mathcal{O}'_j$ is supposed star-shaped with respect to 0). According to Proposition 1.1, $\sigma_\lambda \circ \tilde{u}$ restricted to $\mathcal{O}'_j$ (or Ω) converges to u in $E(\mathcal{O}'_j)$ (or $E(\Omega)$) as $\lambda \longrightarrow 1$. The problem is then reduced to that of approximating by elements of $\mathcal{D}(\mathbb{R}^n)$ a function u with compact support in Ω; this is obvious by regularization (as in point ii) in the proof of Theorem 1.1).

Remark 1.3.

If the set Ω is unbounded or if its boundary is not smooth, some partial results remain true: for example, if $u \in E(\Omega)$, we can define $\lambda_\nu u$ on each bounded part Γ_0 of Γ of class $\mathcal{C}^2$, and $\gamma_\nu u \in H^1(\Gamma_0)$. If Ω is smooth but unbounded or if its boundary is the union of a finite number of bounded (n-1) dimensional manifolds of class $\mathcal{C}^2$, then $\gamma_\nu u$ is defined, in this way, on all Γ. Nevertheless the generalized Stokes formula (1.19) is not available.

The results will be more precise if we know more about the trace of functions in $H^1(\Omega)$. Let us suppose that the following results are available:

- there exists $\gamma_0 \in \mathcal{L}(H^1(\Omega), L^2(\Gamma))$ such that $\gamma_0 u = u|_\Gamma$ for every $u \in \mathcal{D}(\Omega)$. We denote $\gamma_0(H^1(\Gamma))$ by $\mathcal{H}^{\frac{1}{2}}(\Gamma)$.
- there exists a lifting operator $\ell_\Omega \in \mathcal{L}(\mathcal{H}^{\frac{1}{2}}(\Gamma), H^1(\Omega))$, such that $\gamma_0 \circ \ell_\Omega$ = the identity; $\mathcal{H}^{\frac{1}{2}}(\Gamma)$ is equipped with the norm carried by γ_0.
- Ω is a locally lipschitzian set.

Then all the preceding results can be extended to this case. Theorems 1.1 and 1.3 are true. The proof of Theorem 1.2 leads to a definition of $\gamma_\nu \cdot u$ as an element of $\mathcal{H}^{-\frac{1}{2}}(\Gamma)$ = the dual space of $\mathcal{H}^{\frac{1}{2}}(\Gamma)$. The generalized Stokes formula (1.19) is valid.

1.4 Characterization of the Space H.

We recall the notation introduced at the end of Section 1.1:

$$\mathcal{V} = \{u \in \mathcal{D}(\Omega),\ \operatorname{div} u = 0\},$$
$$H = \text{the closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\Omega),$$
$$V = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega).$$

We can give now the following characterization of H and $H^\perp$ (the orthogonal complement of $\mathbb{L}^2(\Omega)$).

Theorem 1.4.

Let Ω be a locally lipschitzian, open bounded set in $\mathbb{R}^n$. Then:

(1.28) $$H^\perp = \{u \in \mathbb{L}^2(\Omega),\ u = \operatorname{grad} p,\ p \in H^1(\Omega)\}$$

(1.29) $$H = \{u \in \mathbb{L}^2(\Omega),\ \operatorname{div} u = 0,\ \gamma_\nu u = 0\}.$$

Proof of (1.28).

i) Let u belong to the space in the right-hand member of (1.28). Then for each $v \in \mathcal{V}$,

$$(u,v) = (\text{grad } p,\ v) = -(p,\ \text{div } v) = 0. \tag{1.30}$$

Hence $u \in H^{\perp}$.

ii) We will prove conversely that $H^{\perp}$ is contained in the space in the right-hand member of (1.28), but this inclusion is less easy to show.

Let u be an element of $H^{\perp}$. We write first that u is orthogonal to some particular elements of $\mathcal{V}$: for every i,j, $1 \leq i,\ j \leq n$, for each $\psi \in \mathcal{D}(\Omega)$, let $v \in \mathcal{V}$ be defined by

$$v_i = D_j\psi,\quad v_j = -D_i\psi,\quad v_\alpha = 0 \ \text{ if } \ \alpha \neq i,\ \alpha \neq j. \tag{1.31}$$

Writing that $(u,v) = 0$, we obtain

$$(u,v) = (u_i, D_j\psi) - (u_j, D_i\psi) = 0,\quad \forall \psi \in \mathcal{D}(\Omega),$$

which means that

$$D_j u_i = D_i u_j \tag{1.32}$$

in the distribution sense. If $u_1, \cdots u_n$, were smooth functions, it would follow from a classical result that (1.32) implies that u is the gradient of some function g on each simply connected open subset of Ω. The same result is true for distributions by the duality theorem of de Rham [1]; u is the gradient of some distribution p on every simply connected open subset of Ω.

Of course if Ω is simply connected, $u = \text{grad } p$ in all Ω, and it follows from a result of Deny-Lions [1] (see Proposition 1.2 in Section 1.5) that $p \in L^2(\Omega)$; (1.28) is proved in this case.

iii) When Ω is not simply connected, let us make some smooth cuts in Ω, $\Omega = \Sigma \cup \dot{\Omega}$, where $\dot{\Omega}$ is open, simply connected, and locally lipschitzian; Σ is a $\mathcal{C}^2$, $(n-1)$-dimensional manifold and $\dot{\Omega} \cap \Sigma = \phi$.(1)

Applying point ii) $u = \text{grad } p$ in $\dot{\Omega}$, where p belongs to $L^2(\dot{\Omega})$. Hence

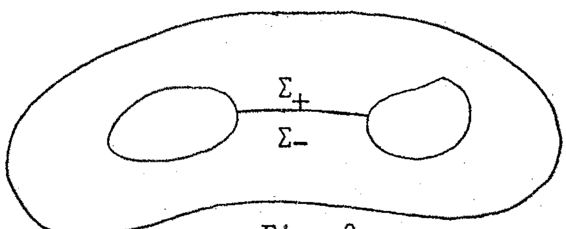


Fig. 2

(1) We assume that we can make such smooth cuts, but this is a very weak hypothesis on Ω; in practice it is always satisfied.

$p \in H^1(\dot{\Omega})$ and we can define the traces of p on both sides of Σ, but these traces may be <u>different</u>. If we prove that p has the same trace on both sides of Σ, then p will be a single valued function and $u = \text{grad } p$ in all Ω.

Let $\gamma_+ p$ and $\gamma_- p$ be the traces of p on the two sides Σ_+ and Σ_- of Σ. In order to show that $\gamma_+ p = \gamma_- p$, it is sufficient to show that

$$\int_\Sigma \phi(\gamma_+ p - \gamma_- p)\, d\Gamma = 0, \tag{1.33}$$

for every smooth function ϕ defined on Σ, with support contained in a "small" ball centered on Σ.

By Lemma 1.3 which will follow, for each function ϕ there exists $w \in H$, $\text{div } w = 0$, with compact support in Ω such that

$$\gamma_{\nu_+} w = \phi \quad (1). \tag{1.34}$$

We write now that

$$0 = (u,w) = \int_\Omega u \cdot w\, dx$$

$$= \int_{\dot{\Omega}} \text{grad } p \cdot w\, dx = (\text{by Stokes formula } (1.19))$$

$$= \langle \gamma_\nu w, \gamma_0 p \rangle_{\partial\dot{\Omega}} - \int_{\dot{\Omega}} p \cdot \text{div } w\, dx \quad (2)$$

$$= \langle \gamma_{\nu_+} w, \gamma_+ p - \gamma_- p \rangle_\Sigma = \int_\Sigma (\gamma_+ p - \gamma_- p)\phi\, d\Gamma,$$

and (1.33) follows.

<u>Lemma 1.3</u>.

<u>Let</u> ϕ <u>be a smooth function on</u> Σ, <u>with compact support in</u> $\Sigma_0 \subset \overline{\Sigma}_0 \subset \Sigma$. <u>There exists</u> $w \in H$, <u>with compact support in</u> Ω <u>such that</u> $\text{div } w = 0$, <u>and</u> $\gamma_{\nu_+} w = \phi$ <u>on</u> Σ.

---

(1) If $w \in H$, $\text{div } w = 0$, then $w \in E(\Omega)$ and we can define $\gamma_{\nu_+} w$ = the trace of $w \cdot \nu_+$ on Σ; here ν_+ stands for the unit normal to Σ pointing in the direction of Σ_-.

(2) Since w has a compact support in Ω, an extended form of (1.19) is valid without needing any regularity property for $\partial\Omega$ (but Σ is of class $\mathcal{C}^2$).

<u>Proof</u>.

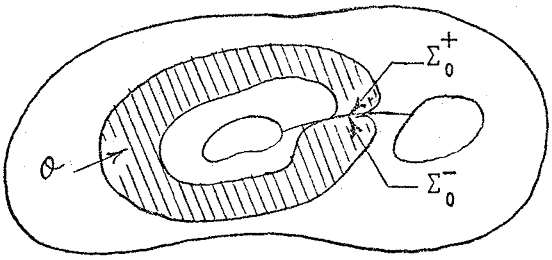


Fig. 3

Let $\mathcal{O}$ be an open tube in Ω with a smooth lateral boundary and cross sections Σ_0^+ and Σ_0^-; we suppose that the lateral boundary of $\mathcal{O}$ matches Σ_0^+ and Σ_0^- smoothly so that the whole boundary of $\mathcal{O}$ is of class $\mathcal{C}^2$.

Let g be the solution of the following Neumann problem in $\mathcal{O}$

$$
(1.35) \qquad \begin{cases} \Delta g = 0 \quad \text{in} \quad \mathcal{O} \\[2ex] \dfrac{\partial g}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{O}-\Sigma_0 \\[2ex] \dfrac{\partial g}{\partial \gamma_+} = \phi \quad \text{on} \quad \Sigma_0^+ , \quad \dfrac{\partial g}{\partial \nu_-} = -\phi \quad \text{on} \quad \Sigma_0^-. \end{cases}
$$

Since the necessary condition $\int_{\partial\mathcal{O}} \frac{\partial g}{\partial \nu}\, d\Gamma = 0$ is satisfied, the problem (1.35) has an infinity of solutions, any two of them differing by a constant. Let then w be the vector defined almost everywhere in Ω (or $\dot\Omega$) by

$$
(1.36) \qquad w = \begin{cases} \text{grad } g \quad \text{in} \quad \mathcal{O} \\ 0 \quad \text{in} \quad \Omega-\mathcal{O}. \end{cases}
$$

We will prove that w satisfies all the conditions.

It is clear that $w \in \mathbb{L}^2(\Omega)$ and $w \in E(\dot\Omega)$; let us show that $w \in E(\Omega)$ and moreover

$$
(1.37) \qquad \text{div } w = 0 \quad \text{in} \quad \Omega.
$$

Let ψ be any test function in $\mathcal{D}(\Omega)$; by definition,

$$
\begin{aligned}
\langle \text{div } w, \psi \rangle &= - (w, \text{grad } \psi) \\
&= - \int_{\mathcal{O}} w \cdot \text{grad } \psi \, dx = \text{(by (1.19))} \\
&= - \langle \gamma_\nu w, \psi \rangle_{\partial\mathcal{O}} + \int_{\mathcal{O}} \text{div } w . \psi dx
\end{aligned}
$$

The last integral equals 0 since $\text{div } w = \Delta g = 0$ in $\dot\Omega$; $\gamma_\nu w = \frac{\partial g}{\partial \nu}$ vanishes on the lateral boundary of $\mathcal{O}$ and with (1.35), the sum of the integrals on Σ_0^+ and Σ_0^- vanishes. There remains

$$
\langle \text{div } w, \psi \rangle = 0, \quad \forall \psi \in \mathcal{D}(\Omega),
$$

and (1.37) follows.

In order to prove that w is the limit in $\mathbb{L}^2(\Omega)$ of functions in $\mathcal{V}$, let us consider a regularization of w, $w_\varepsilon = \rho_\varepsilon * w$, where as usual $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, $\rho \in \mathcal{D}(\mathbb{R}^n)$, $0 \leq \rho \leq 1$, $\rho(x) = 0$ for $|x| \geq 1$, $\int_{\mathbb{R}^n} \rho(x)dx = 1$. For ε less than the distance between $\mathcal{O}$ and Γ, w_ε has compact support in Ω and thus $w_\varepsilon \in \mathcal{D}(\Omega)$; we have also

$$\operatorname{div} w_\varepsilon = \operatorname{div}(\rho_\varepsilon * w) = \rho_\varepsilon * \operatorname{div} w = 0,$$

so that $w_\varepsilon \in \mathcal{V}$.

It is clear that, for $\varepsilon \longrightarrow 0$, w_ε converges to w in $\mathbb{L}^2(\Omega)$, thus $w \in H$.

<u>Proof of</u> (1.29).

Let $H^\bullet$ be the space in the right-hand member of (1.29) and let us prove that $H \subset H^\bullet$. If $u \in H$, then $u = \lim_{m\to\infty} u_m$, $u_m \in \mathcal{V}$; this convergence in $\mathbb{L}^2(\Omega)$ implies convergence in the distribution sense; then since differentiation is a continuous operator in the distribution space and $\operatorname{div} u_m = 0$, we see that $\operatorname{div} u = 0$. Since $\operatorname{div} u_m = \operatorname{div} u = 0$, u_m and u belong to $E(\Omega)$ and

$$\|u-u_m\|_{E(\Omega)} = \|u-u_m\|_{\mathbb{L}^2(\Omega)}$$

so that u_m converges to u in $E(\Omega)$, and $\gamma_\nu u = \lim_{m\to\infty} \gamma_\nu u_m = 0$ $(\gamma_\nu u_m = u_m \cdot \nu|_\Gamma = 0$, $\forall u_m \in \mathcal{V})$. Hence $u \in H^\bullet$.

Let us suppose that H is not the whole space $H^\bullet$ and let $H^{\bullet\bullet}$ be the orthogonal complement of H in $H^\bullet$. By (1.28), every $u \in H^{\bullet\bullet}$ is the gradient of some $p \in H^1(\Omega)$; other properties of p are:

$$(1.37) \qquad \Delta p - \operatorname{div} u = 0, \quad u.\nu|_\Gamma = \frac{\partial p}{\partial \nu}\Big|_\Gamma = 0,$$

and this implies that p is a constant and $u = 0$; therefore $H^{\bullet\bullet} = \{0\}$ and $H = H^\bullet$

<u>Remark 1.4</u>.

If Ω is any open set in $\mathbb{R}^n$, the proof of (1.28) with very slight modifications shows that

$$(1.38) \qquad H^\perp = \{u \in \mathbb{L}^2(\Omega), \ u = \operatorname{grad} p, \ p \in L^2_{loc}(\Omega)\}.$$

If Ω is unbounded but satisfies the condition (1.4) then

(1.39) $$H^{\perp} = \{u \in \mathbb{L}^2(\Omega),\ u = \text{grad } p,\ p \in L^2_{loc}(\bar{\Omega})\}.$$

Theorem 1.5.

Let Ω be an open bounded set of class $\mathcal{C}^2$. Then

(1.40) $$\mathbb{L}^2(\Omega) = H \oplus H_1 \oplus H_2,$$

where H, H_1, H_2 are mutually orthogonal spaces

(1.41) $$H_1 = \{u \in \mathbb{L}^2(\Omega),\ u = \text{grad } p,\ p \in H^1(\Omega),\ \Delta p = 0\},$$

(1.42) $$H_2 = \{u \in \mathbb{L}^2(\Omega),\ u = \text{grad } p,\ p \in H_0^1(\Omega)\}.$$

Proof.

It is clear that H_1 and H_2 are included in $H^{\perp}$, and that the intersection of any two of the spaces H, H_1, H_2, reduces to $\{0\}$.

The spaces H_1 and H_2 are orthogonal; if $u = \text{grad } p \in H_1$, $v = \text{grad } q \in H_2$, then $u \in E(\Omega)$ and by using the generalized Stokes formula (1.19),

$$(u,v) = (u, \text{grad } q) = \langle \gamma_\nu u, \gamma_0 q \rangle - (\text{div } u, q)$$

and this vanishes since $\gamma_0 q = 0$ and $\text{div } u = \Delta p = 0$.

It remains to prove that any element u of $\mathbb{L}^2(\Omega)$ can be written as the sum of elements u_0, u_1, u_2 in H, H_1, H_2. For such a u let p be the unique solution of the Dirichlet problem

$$\Delta p = \text{div } u \in H^{-1}(\Omega), \quad p \in H_0^1(\Omega).$$

We take

(1.43) $$u_2 = \text{grad } p.$$

Let then q be the solution of the Neumann problem

(1.44) $$\Delta q = 0, \quad \frac{\partial q}{\partial \nu}\Big|_\Gamma = \gamma_\nu(u - \text{grad } p).$$

We notice that $\text{div}(u - \text{grad } p) = 0$, so that $u - \text{grad } p \in E(\Omega)$, and $\gamma_\nu(u - \text{grad } p)$ is defined as an element of $H^{-\frac{1}{2}}(\Gamma)$, and by the Stokes fromula (1.19)

$$\langle \gamma_\nu(u - \text{grad } p), 1 \rangle = \int_\Omega \text{div}(u - \text{grad } p)dx = 0.$$

According to a result of Lions-Magenes [1], the Neuman problem (1.44) possesses a solution which is unique up to some additive constant.

We take

$$u_1 = \text{grad } q \tag{1.45}$$

$$u_0 = u - u_1 - u_2. \tag{1.46}$$

It remains to see that $u_0 \in H$. But $\text{div } u_0 = \text{div}(u - u_1 - u_2) = \text{div } u - \Delta p = 0$, and $\gamma_\nu u_0 = \gamma_\nu(u-u_1-u_2) = \gamma_\nu(u - \text{grad } p) - \frac{\partial q}{\partial \nu} = 0$.

1.5 Characterization of the Space V.

For the sake of completeness we recall first the following result:

Lemma 1.4.

If $\ell \in H^{-1}(\Omega)$ (i.e., the dual space of $H_0^1(\Omega)$), then there exists $(n+1)$ functions $f_0, \cdots, f_n$, belonging to $L^2(\Omega)$ such that

$$\ell = f_0 - \sum_{i=1}^{n} D_i f_i, \tag{1.47}$$

that is to say

$$<\ell,u> = (f_0,u) + \sum_{i=1}^{n} (f_i, D_i u), \quad \forall u \in H_0^1(\Omega). \tag{1.48}$$

The functions $f_0, \cdots f_n$ satisfying (1.47) are not unique.

Proof.

The linear operator $u \longmapsto \bar{\omega} u = (u, D_1 u, \cdots, D_n u)$ is an isomorphism of $H_0^1(\Omega)$ into $L^2(\Omega)^{n+1}$ The linear form $\ell_0 \bar{\omega}^{-1}$ is continuous on the image space of $H_0^1(\Omega)$; by the Hahn-Banach theorem this form can be extended as a linear continuous form on all the space $L^2(\Omega)^{n+1}$; of course the extended form can be written as

$$g \longmapsto (f,g) = \sum_{i=0}^{n+1} (f_i, g_i), \quad f_i \in L^2(\Omega),$$

and (1.48) follows.

Before giving the characterization of V we will state two propositions containing results interesting in themselves.

Proposition 1.2.

Let Ω be a bounded locally lipschitzian open set in $\mathbb{R}^n$.

(i) If a distribution p has all its first order derivatives $D_i p$, $1 \leq i \leq n$, in $L^2(\Omega)$, then $p \in L^2(\Omega)$ and

(1.49) $$\|p\|_{L^2(\Omega)/R} \leq c(\Omega)\ \|\text{grad } p\|_{\mathbb{L}^2(\Omega)}.$$

(ii) *If a distribution* p *has all its first order derivatives* $D_i p$, $1 \leq i \leq n$, in $\mathbb{H}^{-1}(\Omega)$, *then* $p \in L^2(\Omega)$ *and*

(1.50) $$\|p\|_{L^2(\Omega)/R} \leq c(\Omega)\ \|\text{grad } p\|_{\mathbb{H}^{-1}(\Omega)}.$$

In both cases, if Ω *is any open set in* $\mathbb{R}^n$, $p \in L^2_{loc}(\Omega)$.

Proof.

Point (i) and (1.49) are proved in Deny-Lions [1] for a bounded star-shaped open set Ω. In our case, because of this result, p is L^2 on every ball contained in Ω with its closure, and on all the sets $\mathcal{O}'_j$ defined after (1.4). Since a finite number of these balls and sets $\mathcal{O}'_j$ covers Ω the result follows.

When Ω is a set of class $\mathcal{C}^2$, a proof of point (ii) due to Lions is partly reported in Magenes-Stampacchia [1]. For Ω satisfying only (1.4), the proof of (ii) is given in Temam [1] but is too long and technical to be reproduced here.

For a set Ω without any regularity property we apply the preceding results on each ball contained in Ω with its closure, and we get $p \in L^2_{loc}(\Omega)$.

Proposition 1.3.

Let Ω *be any open set in* $\mathbb{R}^n$, *let* $u = (u_1, \cdots, u_n)$ *be a vector distribution with components* u_i *belonging to* $H^{-1}(\Omega)$ *such that*

(1.51) $$\langle u, v \rangle = 0, \quad \forall v \in \mathcal{V}.$$

Then u *is the gradient of some function* p *belonging to* $L^2_{loc}(\Omega)$. *If* Ω *is locally star-shaped and bounded*, $p \in L^2(\Omega)$.

Proof.

According to Lemma 1.4, for each $i = 1, \cdots, n$,

(1.52) $$u_i = f_i - \text{div } g_i, \quad f_i \in L^2(\Omega),\ g_i \in \mathbb{L}^2(\Omega).$$

Let ρ_ε be a sequence of regularizing functions (as in Lemma 1.3 for instance); then for every $w \in \mathcal{V}$, $\rho_\varepsilon * w$ is a vector with components in $\mathcal{D}(\mathbb{R}^n)$; if ε is strictly less than the distance between Γ and the support of w, $\rho_\varepsilon * w$ belongs to $\mathcal{D}(\Omega)$ and moreover to $\mathcal{V}$ since

$$\text{div}(\rho_\varepsilon * w) = \rho_\varepsilon * \text{div } w = 0.$$

Let us suppose that support $w \subset \Omega' \subset \overline{\Omega'} \subset \Omega$, Ω' bounded and $\varepsilon < d(\Omega',\Gamma)$, so that $w_\varepsilon = \rho_\varepsilon * w \in \mathcal{V}$.

By (1.51) and (1.52)

$$\langle u, w_\varepsilon \rangle = 0.$$

$$\sum_{i=1}^{n} \langle f_i - \operatorname{div} g_i, \rho_\varepsilon * w_i \rangle = 0$$

$$\sum_{i=1}^{n} (f_i, \rho_\varepsilon * w_i) + \sum_{i=1}^{n} (g_i, \operatorname{grad}(\rho_\varepsilon * w_i)) = 0.$$

$$\sum_{i=1}^{n} (\tilde{f}_i, \rho_\varepsilon * w_i) + \sum_{i=1}^{n} (\tilde{g}_i, \rho_\varepsilon * \operatorname{grad} w_i) = 0,$$

where $\tilde{f}_i(x) = f_i(x)$, $x \in \Omega$, and $= 0$ $x \notin \Omega$ (same definition for $\tilde{g}_i$).

It is easy to see that

$$(f, \rho_\varepsilon * g) = (\check{\rho}_\varepsilon * f, g), \quad \forall f, g \in L^2(R^n)$$

where

$$\check{\rho}_\varepsilon(x) = \rho_\varepsilon(-x), \quad \forall x.$$

We then have

$$\sum_{i=1}^{n} (\check{\rho}_\varepsilon * \tilde{f}_i, w_i) + \sum_{i=1}^{n} (\check{\rho}_\varepsilon * \tilde{g}_i, \operatorname{grad} w_i) = 0,$$

(1.53) $$\sum_{i=1}^{n} (\check{\rho}_\varepsilon * (\tilde{f}_i - \operatorname{div} \tilde{g}_i), w_i) = 0.$$

Let $\sigma_{\varepsilon i}$ be the restriction to Ω' of $\check{\rho}_\varepsilon * (\tilde{f}_i - \operatorname{div} \tilde{g}_i)$, and $\sigma_\varepsilon = (\sigma_{\varepsilon 1}, \cdots \sigma_{\varepsilon n})$. By (1.53), $(\sigma_\varepsilon, w) = 0$, $\forall w \in \mathcal{V}$ with support in Ω'. Hence σ_ε belongs to the space $H^\perp$ corresponding to Ω' and by (1.28) and Remark 1.5,

(1.54) $$\sigma_\varepsilon = \operatorname{grad} p_\varepsilon \quad \text{in } \Omega', \quad p_\varepsilon \in L^2(\Omega').$$

As $\varepsilon \to 0$, σ_ε converges to u in $H^{-1}(\Omega')$ and using again the de Rham theorem [1] and (1.54), we see that u is the gradient of some distribution p:

(1.55) $$u = \operatorname{grad} p \quad \text{in } \Omega'.$$

Since Ω' is an arbitrary bounded subset of Ω, with $\Omega' \subset \overline{\Omega'} \subset \Omega$, it is clear that u is the gradient of some distribution p in all Ω.

Since the derivatives of p are in $H^{-1}(\Omega)$, it follows from Proposition 1.2 that $p \in L^2_{loc}(\Omega)$ in the general case and $p \in L^2(\Omega)$ if Ω is bounded and locally lipschitzian.

Remark 1.5.

It is clear that the proof of Proposition 1.3 can be extended to more general vector distributions u: it suffices for this proof that the u_i are of type (1.52) with $f_i \in L^{\alpha_0}(\Omega)$, $g_i \in L^{\alpha_1}(\Omega) \times \cdots \times L^{\alpha_n}(\Omega)$, where $1 < \alpha_j$, $j = 0, \cdots, n$. Moreover, the α_j need not be the same for different components u_i of u, and each u_i can be the sum of terms $f_i - \operatorname{div} g_i$ of preceding type with different α_j for each term.

Of course we can only show in this case that u is the gradient of some distribution p, and with regularity results similar to those of Proposition 1.2, that p is a function in $L^1_{loc}(\Omega)$.

The space V can now be characterized as follows:

Theorem 1.6.

Let Ω be an open bounded set which is locally lipschitzian. Then

$$V = \{u \in \mathbb{H}^1_0(\Omega), \quad \operatorname{div} u = 0\}. \tag{1.56}$$

Proof.

Let V' be the space in the right-hand member of (1.56). It is clear that $V \subset V'$, for if $u \in V$, $u = \lim_{m \to \infty} u_m$, $u_m \in \mathcal{V}$. This convergence in $\mathbb{H}^1_0(\Omega)$ implies that $\operatorname{div} u_m$ converges to $\operatorname{div} u$ as $m \to \infty$, and since $\operatorname{div} u_m = 0$, $\operatorname{div} u = 0$.

To prove that $V = V'$, we will show that any continuous linear form L on V' which vanishes on V is identically equal to 0. With almost exactly the same proof as in Lemma 1.4, noticing that V' is a closed subspace of $\mathbb{H}^1_0(\Omega)$, one can prove that L admits a (non unique) representation of type

$$L(v) = \sum_{i=1}^{n} \langle \ell_i, v_i \rangle, \quad \ell_i \in H^{-1}(\Omega). \tag{1.57}$$

The vector distribution $\ell = (\ell_1, \cdots, \ell_n)$ belongs to $\mathbb{H}^{-1}(\Omega)$, and $\langle \ell, v \rangle = 0$, $\forall v \in \mathcal{D}$. Proposition 1.3 is applicable and shows that

$$\ell = \operatorname{grad} p, \quad p \in L^2(\Omega),$$

thus

$$<\ell_i, v_i> = <D_i p, v_i> = -(p, D_i v_i), \quad \forall v_i \in H_0^1(\Omega).$$

For $v \in V^{\bullet}$,

$$L(v) = \sum_{i=1}^{n} <\ell_i, v_i> = -(p, \operatorname{div} v) = 0,$$

and L vanishes on all of $V^{\bullet}$

Remark 1.6.

We can give a more direct and much simpler proof of (1.56) for a set Ω which is globally star-shaped with respect to one of its points.

Let us suppose that Ω is star-shaped with respect to 0 and let σ_λ denote as before the linear transformation $x \longmapsto \lambda x$.

Let $u \in V^{\bullet}$ Then the function $\sigma_\lambda \circ u$ belongs to $\mathbb{H}_0^1(\sigma_\lambda\Omega)$ and $\operatorname{div} \sigma_\lambda u = 0$. The function u_λ equal to $\sigma_\lambda \circ u$ in $\sigma_\lambda\Omega$ and to 0 in $\Omega - \sigma_\lambda\Omega$ $(0 < \lambda < 1)$ is in $\mathbb{H}_0^1(\Omega)$ and $\operatorname{div} u_\lambda$ equals $\lambda\sigma_\lambda(\operatorname{div} u)$ in $\sigma_\lambda\Omega$ and 0 in $\Omega - \sigma_\lambda\Omega$; hence $\operatorname{div} u_\lambda = 0$, $u_\lambda \in V^{\bullet}$ and has a compact support in Ω. In this case it is easy to check by regularization that $u_\lambda \in V$ and since u_λ converges to u in $\mathbb{H}_0^1(\Omega)$ as $\lambda \longrightarrow 1$, $u \in V$, and $V = V^{\bullet}$.

Remark 1.7.

The results which will be used in the sequel are all the results given in this Section 1.5: Lemma 1.4, Propositions 1.2 and 1.3, Theorem 1.6.

§2. EXISTENCE AND UNIQUENESS FOR THE STOKES EQUATIONS

The Stokes equations are the linearized stationary form of the full Navier-Stokes equations. We give here the variational formulation of Stokes problem, an existence and uniqueness result using the projection theorem, and some other remarks concerning the case of an unbounded domain and the regularity of solutions.

2.1 Variational Formulation of the Problem.

Let Ω be an open bounded set in $\mathbb{R}^n$ with boundary Γ, and let $f \in L^2(\Omega)$ be a given vector function in Ω. We seek a vector function $u = (u_1, \cdots, u_n)$ representing the velocity of the fluid, and a scalar function p representing the pressure, which are defined in Ω and satisfy the following equations and boundary conditions (ν is the coefficient of kinematic viscosity, a constant):

(2.1) $$-\nu \, \Delta u + \operatorname{grad} p = f \quad \text{in} \quad \Omega, \ (\nu > 0)$$

(2.2) $$\operatorname{div} u = 0 \quad \text{in} \quad \Omega,$$

(2.3) $$u = 0 \quad \text{on} \quad \Gamma.$$

If f, u, p are smooth functions satisfying (2.1) - (2.3), then taking the scalar product of (2.1) with a function $v \in \mathcal{V}$, we obtain,

$$(-\nu\Delta u + \operatorname{grad} p, v) = (f, v)$$

and, integrating by parts, the term $(-\Delta u, v)$ gives

(2.4) $$\sum_{i=1}^{n} (\operatorname{grad} u_i, \operatorname{grad} v_i) = ((u,v))$$ [(1)]

and the term $(\operatorname{grad} p, v)$ gives

$$-(p, \operatorname{div} v) = 0$$

and there results

(2.5) $$\nu \, ((u,v)) = (f,v), \quad \forall v \in \mathcal{V}.$$

Since each side of (2.5) depends linearly and continuously on v for the $H_0^1(\Omega)$ topology, the equality (2.5) is still valid by continuity for each $v \in V$,

---

(1) See the notation at the end of Section 1.1.

the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega)$. If the set Ω is of class $\mathcal{C}^2$, then due to (2.3) the (smooth) function u belongs to $\mathbb{H}_0^1(\Omega)$, and because of (2.2) and Theorem 1.6, $u \in V$. We arrive then at the following conclusion:

$$(2.6) \qquad \begin{cases} u \text{ belongs to } V \text{ and satisfies} \\ \nu((u, v)) = (f, v), \forall v \in V. \end{cases}$$

Conversely, let us suppose that u satisfies (2.6), and let us then show that u satisfies (2.1) - (2.3) in some sense. Since u belongs only to $\mathbb{H}_0^1(\Omega)$, we have less regularity than before and we can only expect u to satisfy (2.1) - (2.3) in a sense weaker than the classical sense. Actually, $u \in \mathbb{H}_0^1(\Omega)$ implies that the traces $\gamma_0 u_i$ of its components are zero in $H^{\frac{1}{2}}(\Gamma)$; $u \in V$ implies (using Theorem 1.6) that $\operatorname{div} u = 0$ in the distribution sense; and using (2.6) we have

$$< -\nu\Delta u - f, v > = 0, \quad \forall v \in \mathcal{V}$$

where, according to Lemma 1.4, $-\nu\Delta u - f$ belongs to $\mathbb{H}^{-1}(\Omega)$. Then because of Proposition 1.3, there exists some distribution $p \in L^2(\Omega)$, such that

$$-\nu\Delta u - f = -\operatorname{grad} p$$

in the distribution sense in Ω.

We have thus proved

Lemma 2.1.

Let Ω be an open bounded set of class $\mathcal{C}^2$.

The following conditions are equivalent

i) $u \in V$ *and satisfies* (2.6)

ii) u *belongs to* $\mathbb{H}^1(\Omega)$ *and satisfies* (2.1) - (2.3) *in the following weak sense:*

$$(2.7) \qquad \begin{cases} \text{there exists } p \in L^2(\Omega) \text{ such that} \\ -\nu\Delta u + \operatorname{grad} p = f \\ \text{in the distribution sense in } \Omega \end{cases}$$

$$(2.8) \qquad \operatorname{div} u = 0 \text{ in distribution sense in } \Omega$$

$$(2.9) \qquad \gamma_0 u = 0.$$

Definition 2.1.

The problem: find $u \in V$ *satisfying* (2.6) *is called the variational formulation of problem* (2.1) - (2.3).

Remark 2.1.

Before studying existence and uniqueness problems for (2.6), let us make some remarks.

(i) The variational formulation of problem (2.1) - (2.3) was introduced by J. Leray [1][2][3]. It reduces the classical problem (2.1) - (2.3) to the problem of finding only u; then the existence of p is a consequence of Lemma 1.5.

(ii) When the set Ω is not smooth, we have two spaces which we called V and V' in the proof of Theorem 1.6 and which may be differnt:

$$V = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega)$$

$$V' = \{u \in \mathbb{H}_0^1(\Omega), \quad \text{div } u = 0\}$$

$$V \subset V'.$$

We can then pose two (perhaps different) variational formulations: either (2.6) exactly, or the same problem with V replaced by V'.

Sets Ω such that $V \neq V'$ are not known and, a fortiori, the relation between these two problems is not known. For technical reasons, particularly important in the non linear case, we will always work with the space V and consider only the variational problem (2.6).

Let us remark as a complement of Lemma 2.1, that for any set Ω, if u satisfies (2.6) (or (2.6) with V replaced by V'), then it satisfies (2.7) with the restriction that $p \in L^2_{loc}(\Omega)$ only; it satisfies (2.8) without any modification; and it satisfies (2.9) in the sense that $u \in \mathbb{H}_0^1(\Omega)$, a more precise meaning depends on the trace theorems available for Ω.

2.2. The Projection Theorem.

Let Ω be any open set of $\mathbb{R}^n$ such that

(2.10) $\qquad\Omega$ is bounded in some direction.

According to (1.10), $\mathbb{H}_0^1(\Omega)$ is a Hilbert space for the scalar product (2.4); $\mathcal{V}$ is defined in (1.12) and V is the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega)$.

Theorem 2.1.

For any open set $\Omega \subset \mathbb{R}^n$ which is bounded in some direction, and for every $f \in \mathbb{L}^2(\Omega)$, the problem (2.6) has a unique solution u.

Moreover, there exists a function $p \in L^2_{loc}(\Omega)$ such that (2.7) - (2.8) are satisfied.

If Ω is an open bounded set of class $\mathscr{C}^2$, then $p \in L^2(\Omega)$ and (2.7) - (2.9) are satisfied by u and p.

This theorem is a very easy consequence of the preceding lemma and the following classical projection theorem.

Theorem 2.2.

Let W be a separable real Hilbert space (norm $\|\cdot\|$) and let $a(u,v)$ be a bilinear continuous form on $W \times W$, which is coercive, i.e., there exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_W^2, \quad \forall u \in W. \tag{2.11}$$

Then for each ℓ in W', the dual space of W, there exists one and only one $u \in W$ such that

$$a(u,v) = \langle \ell, v \rangle, \quad \forall v \in W. \tag{2.12}$$

Of course to apply this theorem to (2.6), we take W = the space V equipped with the norm associated with (2.4), $a(u,v) = \nu((u,v))$, and for $v \mapsto \langle \ell, v \rangle$ the form $v \mapsto (f,v)$ which is obviously linear and continuous on V. The space V is separable as a closed subspace of the separable space $\mathbb{H}_0^1(\Omega)$.

Proof of theorem 2.2.

Uniqueness. Let u_1 and u_2 be two solutions of (2.12) and let $u = u_1 - u_2$. We have

$$a(u_1,v) = a(u_2,v) = \langle \ell, v \rangle, \quad \forall v \in W,$$

and

$$a(u_1 - u_2, v) = 0 \quad \forall v \in W.$$

Taking $v = u$ in this equality, we see with (2.11) that

$$\alpha \|u\|^2 \leq a(u,u) = 0,$$

and hence $u = 0$.

Existence. Since W is separable, there exists a sequence of elements $w_1, \cdots w_m, \cdots$, of W which is free and total in W. Let W_m be the space spanned by $w_1, \cdots, w_m$. For each fixed integer m we define an approximate solution of (2.12) in W_m; that is, a vector $u_m \in W_m$

$$u_m = \sum_{i=1}^{m} \xi_{i,m} w_i, \quad \xi_{i,m} \in \mathbb{R}, \tag{2.13}$$

satisfying

$$a(u_m,v) = \langle \ell,v \rangle, \quad \forall v \in W_m. \tag{2.14}$$

Let us show that there exists one and only one u_m such that (2.14) holds. Equation (2.14) is equivalent to the set of m equations

$$a(u_m,w_j) = \langle \ell,w_j \rangle, \quad j = 1,\cdots,m, \tag{2.15}$$

and (2.15) is a linear system of m equations for the m components $\xi_{i,m}$ of u_m:

$$\sum_{i=1}^{m} \xi_{i,m}\, a(w_i,w_j) = \langle \ell,w_j \rangle, \quad j = 1,\cdots,m. \tag{2.16}$$

The existence and uniqueness of u_m will be proved if we show that the linear system (2.16) is regular. In order to show that it is sufficient to prove that the homogeneous linear system associated with (2.16), i.e.,

$$\sum_{i=1}^{m} \xi_i\, a(w_i,w_j) = 0, \quad j = 1,\cdots,m, \tag{2.17}$$

has only one solution $\xi_1 = \cdots = \xi_m = 0$. But if $\xi_1,\cdots,\xi_m$, satisfy (2.17), then by multiplying each equation (2.17) by the corresponding ξ_j and adding these equations, we obtain

$$\sum_{i,j=1}^{m} \xi_i\xi_j\, a(w_i,w_j) = 0$$

or, because of the bilinearity of a,

$$a\Big(\sum_{i=1}^{m} \xi_i w_i, \sum_{j=1}^{m} \xi_j w_j\Big) = 0;$$

using (2.11) we find

$$\sum_{i=1}^{m} \xi_i w_i = 0$$

and finally $\xi_1 = \cdots = \xi_m = 0$ since $w_1,\cdots,w_m$ are linearly independent.

Passage to the Limit.

When we put $v = u_m$ in (2.14), we obtain

$$a(u_m,u_m) = \langle \ell,u_m \rangle \tag{2.18}$$

from which, by (2.11), it follows that

$$\alpha\|u_m\|_W^2 \leq a(u_m,u_m) = \langle \ell,u_m\rangle \leq \|\ell\|_{W'}\|u_m\|$$

(2.19) $$\|u_m\|_W \leq \frac{1}{\alpha}\|\ell\|_{W'}$$

which proves that the sequence u_m is bounded independently of m in W. Since the closed balls of a Hilbert space are weakly compact, there exists an element u of W and a sequence $u_{m'}$, $m'\longrightarrow\infty$, extracted from u_m, such that

(2.20) $u_{m'}\longrightarrow u$ in the weak topology of W, as $m'\longrightarrow\infty$.

Let v be a fixed element of W_j for some j. As soon as $m' \geq j$, $v\in W_{m'}$, and according to (2.14) we have

$$a(u_{m'},v) = \langle \ell,v\rangle.$$

By using the following lemma, we can take the limit in this equality as $m'\longrightarrow\infty$, and we obtain:

(2.21) $$a(u,v) = \langle \ell,v\rangle.$$

Equality (2.21) holds for each $v\in\bigcup_{j=1}^{\infty} W_j$, and since this set is dense in W, equality (2.21) still holds by continuity for v in W. This proves that u is a solution of (2.12).

Lemma 2.2.

Let $a(u,v)$ be a bilinear continuous form on a Hilbert space W. Let ϕ_m (resp. ψ_m) be a sequence of elements of W which converges to ϕ (resp. ψ) in the weak (resp. strong) topology of W. Then

(2.22) $$\lim_{m\longrightarrow\infty} a(\psi_m,\phi_m) = a(\psi,\phi)$$

(2.23) $$\lim_{m\longrightarrow\infty} a(\phi_m,\psi_m) = a(\phi,\psi).$$

Proof.

We write

$$a(\psi_m,\phi_m) - a(\psi,\phi) = a(\psi_m-\psi,\phi_m) + a(\psi,\phi_m-\phi).$$

Since the form a is continuous and the sequence ϕ_m is bounded,

$$|a(\psi_m-\psi,\phi_m)| \leq c\|\psi_m-\psi\|_W \|\phi_m\|_W \leq c'\|\psi_m-\psi\|_W ,$$

and this term converges to 0 as $m \to \infty$.

We notice next that the linear operator $v \mapsto a(\psi,v)$ is continuous on W and hence there exists some element of W', depending on ψ and denoted by $A(\psi)$, such that

$$a(\psi,v) = \langle A(\psi),v\rangle, \quad \forall v \in W. \tag{2.24}$$

We can now write

$$a(\psi,\phi_m-\phi) = \langle A(\psi),\phi_m-\phi\rangle$$

and this converges to 0 as $m \to \infty$, as a consequence of the weak convergence of ϕ_m.

This proves (2.22). For (2.23) we only have to apply (2.22) to the bilinear form

$$a^*(u,v) = a(v,u).$$

Remark 2.2.

(i) Theorem 2.1 is also true if f is given $H^{-1}(\Omega)$.

(ii) It can be proved that the sequence $\{u_m\}$ constructed in the proof of Theorem 2.2, as a whole, converges to the solution u of (2.12) in the strong topology of W. We do not prove this result here; it will appear as a consequence of Theorem 3.1.

(iii) Using the form $A(\psi)$ introduced in the proof of Lemma 2.2 (cf. (2.24)), one can write equation (2.12) in the form

$$\langle A(u),v\rangle = \langle \ell,v\rangle$$

which is equivalent to

$$A(u) = \ell \quad \text{in } W'. \tag{2.25}$$

An alternate classical proof of the projection theorem is to show that the operator $u \mapsto A(u)$ is an isomorphism from W onto W'; see for instance Temam [2].

A Variational Property.

Proposition 2.1.

The solution u of (2.6) is also the unique element of V such that

$$E(u) \leq E(v), \quad \forall v \in V \tag{2.26}$$

where

$$E(v) = \nu\|v\|^2 - 2(f,v). \tag{2.27}$$

Proof.

Let u be the solution of (2.6). Then as

$$\|u - v\|^2 \geq 0, \quad \forall v \in V,$$

we have

$$\nu\|u\|^2 + \nu\|v\|^2 - 2\nu((u,v)) \geq 0. \tag{2.28}$$

Because of (2.6), we have

$$-\nu\|u\|^2 = \nu\|u\|^2 - 2(f,u) = E(u),$$

$$-2\nu((u,v)) = -2(f,v)$$

and thus (2.28) gives exactly (2.26).

Conversely, if $u \in V$ satisfies (2.26), then for any $v \in V$ and $\lambda \in \mathbb{R}$, one has

$$E(u) \leq E(u+\lambda v).$$

After working this out and simplifying, one finds

$$\nu\lambda^2\|v\|^2 + 2\lambda\nu((u,v)) - 2\lambda(f,v) \geq 0, \quad \forall \lambda \in \mathbb{R}. \tag{2.29}$$

This inequality can hold for each $\lambda \in \mathbb{R}$ only if

$$\nu((u,v)) = (f,v),$$

and thus u is indeed a solution of (2.6).

Remark 2.3.

If the spaces V and V^* of Remark 2.1 are different, Theorem 2.2 gives as well the existence and uniqueness of a $\tilde{u} \in V^*$ such that

$$\nu((\tilde{u},v)) = (f,v), \quad \forall v \in V^*. \tag{2.30}$$

Proposition 2.1 is also valid with u and V replaced by $\tilde{u}$ and V'.

2.3 The Unbounded Case.

We consider here the case where Ω is unbounded, without satisfying (2.10).

If Ω is of class $\mathcal{C}^2$, problem (2.1) - (2.3) is not equivalent to (2.6) as in Lemma 2.1. The reason is that if u is a classical solution of (2.1) - (2.3), it is not clear without further information about the behavior of u at infinity, that $u \in \mathbb{H}^1(\Omega)$; hence perhaps $u \notin V$ and equation (2.5) cannot be extended by continuity to the closure V of $\mathcal{V}$. Besides that, if one would try to solve directly problem (2.6), using Theorem 2.2, the difficulty is that hypothesis (2.11) is not satisfied: V is a Hilbert space for the norm

$$[\![u]\!] = \sqrt{|u|^2 + \|u\|^2},$$

which is not equivalent to the norm $\|u\|$ since we lose the Poincaré inequality.

In order to pose and solve a variational problem in the general case, let us introduce the space

$$\text{(2.31)} \qquad W = \text{the completion of } \mathcal{V} \text{ under the norm } \|\cdot\|.$$

It is clear, since $\|u\| \leq [\![u]\!]$, that W is a larger space than V

$$\text{(2.32)} \qquad V \subset W.$$

Lemma 2.3.

$$\text{(2.33)} \qquad W \subset \{u \in \mathbb{L}^{\alpha}(\Omega);\ D_i u \in \mathbb{L}^2(\Omega),\ 1 \leq i \leq n\}$$

with a continuous injection, where

$$\text{(2.34)} \qquad \begin{cases} \alpha \text{ is any number, } 1 \leq \alpha < +\infty, & \text{if } n = 2 \\ \alpha = \dfrac{2n}{n-2}, & \text{if } n \geq 3. \end{cases}$$

Proof.

This is a consequence of Sobloev inequality (see Sobolev [1], Lions [1], or also Chapter II):

$$\text{(2.35)} \qquad \|\phi\|_{L^{\alpha}(\Omega)} \leq c(\alpha,n)\, \|\text{grad } \phi\|_{\mathbb{L}^2(\Omega)}, \quad \forall \phi \in \mathcal{D}(\Omega),$$

where α is given by (2.34).

If $u \in W$, there exists a sequence of elements $u_m \in \mathcal{V}$ converging to u; by (2.35)

$$\|u_m-u_p\|_{\mathbb{L}^\alpha(\Omega)} \le c'(\alpha,n)\|u_m-u_p\|$$

and

$$(2.36) \qquad \|u_m-u_p\|_Y \le c''\|u_m-u_p\|$$

where Y stands for the space on the right-hand side of inclusion (2.33) and $\|\cdot\|_Y$ is the natural norm of Y

$$\|u\|_Y = \|u\|_{\mathbb{L}^\alpha(\Omega)} + \|u\|.$$

As m and p tend to infinity, the right side of (2.36) converges to 0. Thus u_m is a Cauchy sequence in Y; its limit u belongs to Y. It is clear also that

$$\|u\|_Y \le c''\|u\|$$

with the same c'' as in (2.36) $(c'' = c''(\alpha,n))$.

Theorem 2.3.

Let Ω be any open set in $\mathbb{R}^n$, and let f be given in W', the dual of the space W in (2.31).

Then, there exists a unique $u \in W$ such that

$$(2.37) \qquad \nu((u,v)) = \langle f,v\rangle, \quad \forall v \in W.$$

There exists $p \in L^2_{loc}(\Omega)$, such that (2.7) is satisfied, and (2.8) is true.

Proof.

We apply Theorem 2.2 with the space W, $a(u,v) = \nu((u,v))$, and ℓ replaced by f; we get a unique u satisfying (2.37).

After that, Proposition 1.3 shows the existence of some $p \in L^2_{loc}(\Omega)$ such that (2.7) is satisfied; (2.8) is of course easily verified. Finally, (2.9) is satisfied in some sense depending on the trace theorem available for W (or for the space Y on the right-hand side of inclusion (2.33)).

If Ω is locally lipschitzian, Proposition 1.2 shows that $p \in L^2_{loc}(\bar{\Omega})$

Remark 2.4.

By Lemma 2.3, for $f \in \mathbb{L}^{\alpha'}(\Omega)\left(\frac{1}{\alpha} + \frac{1}{\alpha'} = 1\right)$,

$$(2.38) \qquad v \longmapsto \int_\Omega f.v\,dx$$

is a linear continuous form on W. Thus one can take in Theorem 2.3, any $f \in \mathbb{L}^{\alpha'}(\Omega)$.

2.4 Regularity Results

A classical result is that the solution $u \in H_0^1(\Omega)$ of the Dirichlet problem $-\Delta u + u = f$ belongs to $H^{m+2}(\Omega)$ whenever $f \in H^m(\Omega)$ (and Ω is sufficiently smooth). One naturally wonders whether similar regularity results exist for the Stokes problem.

Theorem 2.4.

If Ω is an open bounded set of class $\mathscr{C}^{m+2}$, m integer, $m \geq 0$ and if f belongs to $\mathbb{H}^m(\Omega)$, then the solution u of (2.6) satisfies (2.7) - (2.9) and

$$u \in \mathbb{H}^{m+2}(\Omega), \quad p \in H^{m+1}(\Omega) \tag{2.39}$$

$$\|u\|_{\mathbb{H}^{m+2}(\Omega)} + \|p\|_{H^{m+1}(\Omega)/\mathbb{R}} \leq c(\nu,m,\Omega)\|f\|_{\mathbb{H}^m(\Omega)} \tag{2.40}$$

The theorem follows from the next proposition.

Proposition 2.2.

Let Ω be an open bounded set of class $\mathscr{C}^r$, $r = \max(m+2,2)$, m integer ≥ 0. Let us suppose that

$$u \in \mathbb{W}^{1,\alpha}(\Omega), \quad p \in L^{\alpha}(\Omega), \quad 1 < \alpha < +\infty, \tag{2.41}$$

are solutions of the generalized Stokes problem.

$$-\nu\Delta u + \text{grad } p = f \quad \text{in } \Omega, \tag{2.42}$$

$$\text{div } u = g \quad \text{in } \Omega, \tag{2.43}$$

$$\gamma_0 u = \phi; \text{ i.e., } u = \phi \text{ on } \Gamma. \tag{2.44}$$

If $f \in \mathbb{W}^{m,\alpha}(\Omega)$, $g \in W^{m+1,\alpha}(\Omega)$ and $\phi \in \mathbb{W}^{m+2-\frac{1}{\alpha},\alpha}(\Gamma)$, [1] then

$$u \in \mathbb{W}^{m+2,\alpha}(\Omega), \quad p \in W^{m+1,\alpha}(\Omega) \tag{2.45}$$

---

[1] $W^{m+2-\frac{1}{\alpha},\alpha}(\Gamma) = \gamma_0 W^{m+2,\alpha}(\Omega)$ and is equipped with the image norm

$$\|\psi\|_{W^{m+2-\frac{1}{\alpha},\alpha}(\Gamma)} = \operatorname*{Inf}_{\gamma_0 u = \psi} \|u\|_{W^{m+2,\alpha}(\Omega)}$$

and there exists a constant $c_0(\alpha,\nu,m,\Omega)$ *such that*

(2.46) $$\|u\|_{W^{m+2,\alpha}(\Omega)} + \|p\|_{W^{m+1,\alpha}(\Omega)/\mathbb{R}}$$

$$\leq c_0\{\|f\|_{W^{m,\alpha}(\Omega)} + \|g\|_{W^{m+1,\alpha}(\Omega)} + \|\phi\|_{W^{m+2-\frac{1}{\alpha},\alpha}(\Gamma)} + d_\alpha\|u\|_{\mathbb{L}^\alpha(\Omega)}\}$$

$d_\alpha = 0$ *for* $\alpha \geq 2$, $d_\alpha = 1$ *for* $1 < \alpha < 2$.

Proof.

This proposition results from the paper of Agmon-Douglis-Nirenberg [2], hereinafter referred to as A.D.N., giving *a priori* estimates of solutions of general elliptic systems.

Let $u_{n+1} = \frac{1}{\nu} p$, $u = (u_1,\cdots,u_{n+1})$, $f = (+\frac{f_1}{\nu},\cdots,\frac{f_n}{\nu},g)$. Then equations (2.42) and (2.43) become

(2.47) $$\sum_{j=1}^{n+1} \ell_{ij}(D)u_j = f_j, \quad 1 \leq i \leq n+1,$$

where $\ell_{ij}(\xi)$, $\xi = (\xi_1,\cdots,\xi_n)\in\mathbb{R}^n$, is the matrix

(2.48) $$\begin{cases} \ell_{ij}(\xi) = |\xi|^2\delta_{ij}, \quad 1 \leq i,j \leq n, \\ \ell_{n+1,j}(\xi) = -\ell_{j,n+1}(\xi) = \xi_j, \quad 1 \leq j \leq n \\ \ell_{n+1,n+1}(\xi) = 0, \end{cases}$$

$$|\xi|^2 = \xi_1^2+\cdots+\xi_n^2.$$

We take (see p. 38), $s_i = 0$, $t_i = 2$, $1 \leq i \leq n$, $s_{n+1} = -1$, $t_{n+1} = 1$. As requested, degree $\ell_{ij}(\xi) \leq s_i+t_j$, $1 \leq i,j \leq n+1$, and we have $\ell'_{ij}(\xi) = \ell_{ij}(\xi)$. We easily compute $L(\xi) = \det \ell'_{ij}(\xi) = |\xi|^{2n}$, so that $L(\xi) \neq 0$ for real $\xi \neq 0$, and this ensures the ellipticity of the system (Condition (1.5), p. 39). It is clear that (1.7) on p. 39 holds with $m = n$. The Supplementary Condition on L is satisfied: $L(\xi+\tau\xi') = 0$ has exactly n roots with positive imaginary part and these roots are all equal to

$$\tau^+(\xi,\xi') = -\xi\cdot\xi' + i\sqrt{|\xi|^2|\xi'|^2-|\xi\cdot\xi'|^2}.$$

Concerning the boundary conditions (see p. 42), there are m boundary conditions and

$$B_{hj} = \delta_{hj} \text{ (the Kronecker symbol) for } 1 \leq h \leq n, \quad 1 \leq j \leq n+1.$$

We take $r_h = -2$ for $h = 1,\cdots,n$. Then, as requested, degree $B_{hj} \leq r_h+t_j$ and we have $B'_{hj} = B_{hj}$.

It remains to check the Complementing Boundary Condition. It is easy to verify that

$$M^+(\xi) = (\tau - \tau^+(\xi))^n$$

where $\tau^+(\xi) = \tau^+(\xi,\nu)$. The matrix with elements $\sum_{j=1}^{N} B'_{hj}(\xi)L^{jk}(\xi)$ is simply the matrix with elements $\ell_{hk}(\xi)$, $1 \leq h,k \leq n$, $-\ell_{h,n+1}(\xi)$, $1 \leq h \leq n$. A combination $\sum_{h=1}^{n} C_h \sum_{j=1}^{N} B'_{hj}L^{jk}$ is then equal to

$$(C_1(\xi+\tau\nu)^2,\cdots,C_n(\xi+\tau\nu)^2, \quad \sum_{i=1}^{n} C_i(\xi_i+\tau\nu_i))$$

and this is zero modulo M^+ only if $C_1 = \cdots = C_n = 0$, and the Complementing Condition holds.

We then apply Theorem 2.5, page 78 of A.D.N. in order to get (2.45) and (2.46) with $d_\alpha = 1$ for all α. According to the remark after Theorem 2.5, one can take $d_\alpha = 0$ for $\alpha \geq 2$ since the solutions u and p of (2.41) - (2.44) are necessarily unique (p is unique up to an additive constant): if (u_*,p_*), (u_{**},p_{**}) are two solutions, then $u = u_*-u_{**}$, $p = p_*-p_{**}$ are solutions of (2.7) - (2.9) with $f = 0$ and hence $u = 0$ and p = constant.

<u>Remark 2.5.</u>

For $\alpha = 2$ and $m\in\mathbb{R}$, $m \geq -1$, one has results similar to those in Theorems 2.4 and 2.5 by using the interpolation techniques of Lions-Magenes [1].

Proposition 2.2 does not assert the existence of u,p satisfying (2.42) - (2.46) (for given f,g,ϕ) but gives only a result on the regularity of an eventual solution. The existence is ensured by the variational method if $\alpha = 2$, $g = 0$, and $\phi = 0$. The following proposition gives a general existence result for $n = 2$ or 3.

<u>Proposition 2.3.</u>

<u>Let Ω be an open set of class $\mathscr{C}^r$, $r = \max(m+2,2)$, m integer ≥ -1, and let $f\in W^{m,\alpha}(\Omega)$, $g\in W^{m+1,\alpha}(\Omega)$, $\phi\in W^{m+2-\hat{\alpha},\alpha}(\Gamma)$, be given satisfying the compatibility condition</u>

(2.49) $$\int_\Omega g dx = \int_\Gamma \phi\cdot\nu d\Gamma.$$

<u>Then there exist unique functions u and p (p is unique up to a constant) which are solutions of (2.42) - (2.44) and satisfy (2.45) and (2.46) with $d_\alpha = 0$ for any α.</u>

<u>Proof.</u>

This is precisely the result proved in Cattabriga [1] when $n=3$ (and even for $m=-1$).

For $n = 2$ one can reduce the problem to a classical biharmonic problem. There exists $v \in W^{m+2,\alpha}(\Omega)$, such that

(2.50) $$\operatorname{div} v = g$$

(2.51) $$\gamma_0 v = \phi .$$

Such v can be defined by

(2.52) $$v = \operatorname{grad} \theta + \{\frac{\partial \sigma}{\partial x_2} , - \frac{\partial \sigma}{\partial x_1}\}$$

where $\theta \in W^{m+3,\alpha}(\Omega)$ is solution of the Neumann problem

(2.53) $$\Delta\theta = g \text{ in } \Omega$$

(2.54) $$\frac{\partial \theta}{\partial \nu} = \phi \cdot \nu \text{ on } \Gamma$$

and $\sigma \in W^{m+3,\alpha}(\Omega)$ will be chosen later.

The Neumann problem (2.53) - (2.54) has a solution θ because of (2.49), and $\theta \in W^{m+3,\alpha}(\Omega)$ by the usual regularity results for the Neumann problem.

The conditions on σ are only boundary conditions on Γ and these are

$\frac{\partial \sigma}{\partial \tau}$ = the tangential derivative of $\sigma = 0$,

$\frac{\partial \sigma}{\partial \nu}$ = the normal derivative of $\sigma = \phi \cdot \tau - \frac{\partial \theta}{\partial \tau}$.

Since $\phi . \tau - \frac{\partial \theta}{\partial \tau} \in W^{m+2-\frac{1}{\alpha},\alpha}(\Gamma)$, there exists a $\sigma \in W^{m+3,\alpha}(\Gamma)$ with $\gamma_0 \sigma = 0$, $\gamma_1 \sigma = \phi . \tau - \frac{\partial \theta}{\partial \tau}$. With these definitions of σ and θ, the vector v in (2.52) belongs to $W^{m+2,\alpha}(\Omega)$ and satsifies (2.50) - (2.51). Moreover, the mapping $\{g,\phi\} \longmapsto v$ is linear and continuous.

Setting $w = u-v$, the problem (2.42) - (2.44) reduces to the problem of finding $w \in W^{m+2,\alpha}(\Omega)$, $p \in W^{m+1,\alpha}(\Omega)$ such that

(2.55) $$-\nu\Delta w + \operatorname{grad} p = f', \quad f' = f + \nu\Delta v,$$

(2.56) $$\operatorname{div} w = 0,$$

(2.57) $$\gamma_0 w = 0.$$

If Ω is simply connected then, because of (2.56), there exists a function ρ such that

(2.58) $$w = (D_2\rho, -D_1\rho).$$

Condition (2.57) amounts to saying that $\rho = \frac{\partial\rho}{\partial\nu} = 0$ on Γ and (2.55) gives after differentiation

$$-\nu\Delta(D_2 w_1 - D_1 w_2) = D_2 f'_1 - D_1 f'_2.$$

Thus we obtain

$$\nu\Delta^2\rho = D_1 f'_2 - D_2 f'_1 \in \mathbb{W}^{m-1,\alpha}(\Omega) \tag{2.59}$$

$$\rho = 0, \quad \frac{\partial\rho}{\partial\nu} = 0 \quad \text{on } \Gamma. \tag{2.60}$$

The biharmonic problem (2.59)(2.60) has a unique solution $\rho \in \mathbb{W}^{m+3,\alpha}(\Omega)$, and the function w defined by (2.58) is a solution of (2.55) - (2.57) and belongs to $\mathbb{W}^{m+2,\alpha}(\Omega)$.

If Ω is not simply connected we first obtain an existence and uniqueness result for (2.55) - (2.57) by the variational method (Theorem 2.1). We notice that $f' \in \mathbb{W}^{m,\alpha}(\Omega)$, i.e., $f' \in \mathbb{L}^{\alpha}(\Omega)$ at least, and $\mathbb{L}^{\alpha}(\Omega) \subset \mathbb{H}^{-1}(\Omega)$ since by the Sobolev inclusion theorem $H_0^1(\Omega) \subset L^{\beta}(\Omega)$, $\forall\beta$, $1 < \beta < +\infty$, in the two dimensional case. The solution w belongs to $\mathbb{H}_0^1(\Omega)$ and $p \in L^2(\Omega)$. If $1 < \alpha \leq 2$, the proof is achieved using Proposition 2.2. If $\alpha > 2$, we obtain more regularity on w by introducing locally the function ρ and recalling that the regularity property for an elliptic equation is a local property (if ρ satisfies (2.60) and $\Delta^2\rho \in W^{m-1,\alpha}(\mathcal{O})$, $\mathcal{O} \subset \Omega$, then $\rho \in W^{m+3,\alpha}_{loc}(\mathcal{O}')$ for $\mathcal{O}' = \mathcal{O} \cup (\mathcal{O} \cap \Gamma)$; for this see, for instance, the synthesis presentation of Agmon-Douglis-Nirenberg [1], Lions-Magenes [1]).

§3. DISCRETIZATION OF THE STOKES EQUATIONS

Section 3.1 deals with the general concept of approximation of a normed space and Section 3.2 contains a general convergence theorem for the approximation of a general variational problem. In the last section and throughout all of Section 4 we then describe some particular approximations of the basic space V of the Navier-Stokes equations. We give the corresponding numerical scheme for the Stokes equations and then apply the general convergence theorem to this case.

In Section 3.3 we consider the finite difference method. The finite element methods will be treated in the next section (Section 4; 4.1 to 4.5).

The approximations of the space V introduced here will be used throughout subsequent chapters, and they will be referred to by (APX1), (APX2),•••.

3.1 Approximation of a Normed Space.

When computational methods are involved, a normed space W must be a approximated by a family $(W_h)_{h\in\mathcal{H}}$ of normed spaces W_h. The set $\mathcal{H}$ of indexes depends on the type of approximation considered: we will consider in the following the main situations for $\mathcal{H}$, i.e., $\mathcal{H} = \mathbb{N}$ (= positive integers) for the Galerkin method, $\mathcal{H} = \prod_{i=1}^{n} (0,h_i^0]$ for finite differences, and $\mathcal{H}$ = a set of triangulations of the domain Ω for finite element methods. The precise form of $\mathcal{H}$ need not be known; we need only to know the existence of a filter on $\mathcal{H}$, and we are concerned with passing to the limit through this filter. For the sake of simplicity we will always speak about passage to the limit as "$h\to 0$", which is strictly speaking the correct terminology for finite differences; definitions and results can be easily adapted to the other cases.

Definition 3.1.

An internal approximation of a normed vector space W is a set consisting of a family of triples $\{W_h,p_h,r_h\}$, $h\in\mathcal{H}$, where

i) W_h is a normed vector space;

ii) p_h is a linear continuous operator from W_h into W;

iii) r_h is a (perhaps nonlinear) operator from W into W_h.

The natural way to compare an element $u\in W$ and an element $u_h\in W_h$ is either to compare p_hu_h and u in W or to compare u_h and r_hu in W_h. The first point of view is certainly more interesting as we make comparisons in a fixed space. Nevertheless comparisons in W_h can also be useful.

Another way to compare an element $u \in W$ and an element $u_h \in W_h$ is to compare a certain image $\bar{\omega}u$ of u in some other space F, with a certain image $p_h u_h$ of u_h in F. This leads to the concept of external approximation of a space W, which contains the concept of internal approximation as a particular case.

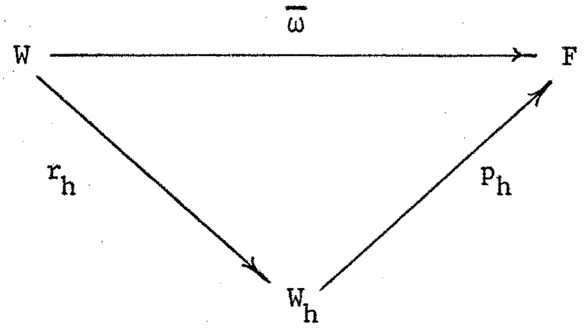


Fig. 4

Definition 3.2.

An external approximation of a normed space W is a set consisting of

i) a normed space F and an isomorphism $\bar{\omega}$ of W into F.

ii) a family of triples $\{W_h, p_h, r_h\}_{h \in \mathcal{H}}$, in which, for each h,

- W_h is a normed space
- p_h a linear continuous mapping of W_h into F
- r_h a (perhaps nonlinear) mapping of W into W_h

When $F = W$ and $\bar{\omega}$ = identity, we get of course an internal approximation of W. It is easy to specialize what follows to internal approximations.

In most cases, W_h are finite dimensional spaces; rather often the operators p_h are injective. In some cases the operators r_h are linear, or only linear on some subspace of W, but there is no need to impose this condition in the general case; also, no continuity property of the r_h is required.

The operators p_h and r_h are called prolongation and restriction operators, respectively. When the spaces W and F are Hilbert spaces, and when the spaces W_h are likewise Hilbert spaces, the approximation is said to be a Hilbert approximation.

Definition 3.3.

For given h, $u \in W$, $u_h \in W_h$, we say that

i) $\|\bar{\omega}u - p_h u_h\|_F$ is the error between u and u_h,

ii) $\|u_h - r_h u\|_{W_h}$ is the discrete error between u and u_h,

iii) $\|\bar{\omega}u - p_h r_h u\|_F$ is the truncation error of u.

We define now stable and convergent approximations.

Definition 3.4. Stable Approximation.

The prolongation operators p_h are said to be stable if their norms

$$\|p_h\| = \sup_{\substack{u_h \in W_h \\ \|u_h\|_{W_h} = 1}} \|p_h u_h\|_F$$

can be majorized independently of h.

The approximation of the space W is said to be stable if the prolongation operators are stable.

Let us now consider what happens when "$h \to 0$".

Definition 3.5.

We will say that a family u_h converges strongly (or weakly) to u if $p_h u_h$ converges to $\overline{\omega} u$ when $h \to 0$ in the strong (or weak) topology of F.

We will say that the family u_h converges discretely to u if

$$\lim_{h \to 0} \|u_h - r_h u\|_h = 0.$$

Definition 3.6.

We will say that an external approximation of a normed space W is convergent if the two following conditions hold:

(C1) for all $u \in W$,

$$\lim_{h \to 0} p_h r_h u = \overline{\omega} u$$

in the strong topology of F.

(C2) for each sequence $u_{h'}$ of elements of $W_{h'}$, $(h' \to 0)$, such that $p_{h'} u_{h'}$ converges to some element ϕ in the weak topology of F, we have $\phi \in \overline{\omega} W$; i.e., $\phi = \overline{\omega} u$ for some $u \in W$.

Remark 3.1.

Condition (C2) disappears in the case of internal approximations.

The following proposition shows that condition (C1) can be in some sense weakened for internal and external approximations.

Proposition 3.1.

Let there be given a stable external approximation of a space W which is convergent in the following restrictive sense: the operators r_h are only defined on a dense subset $\mathcal{W}$ of W and condition (C1) in Definition 3.6 holds only for the u belonging to $\mathcal{W}$ (condition (C2) remains unchanged).

Then it is possible to extend the definition of the restriction operators r_h to the whole space W so that condition (C1) is valid for each $u \in W$ and hence the approximation of W is stable and convergent without any restriction.

Proof.

Let $u \in W$, $u \notin \mathcal{W}$; we must define in some way $r_h u_h \in W_h, \forall h$, so that $p_h r_h u \longrightarrow \bar{\omega} u$ as $h \longrightarrow 0$. This element u can be approximated by elements in $\mathcal{W}$, and these elements in turn can be approximated by elements in the $p_h W_h$; we only have to combine suitably these two approximations.

For each integer $n \geq 1$, there exists $u_n \in \mathcal{W}$ such that $\|u_n - u\|_W \leq \frac{1}{n}$ and hence

$$(3.1) \qquad \|\bar{\omega} u_n - \bar{\omega} u\|_F \leq \frac{c_0}{n} ;$$

where c_0 is the norm of the isomorphism $\bar{\omega}$.

For each fixed integer n, $p_h r_h u_n$ converges to $\bar{\omega} u_n$ in F as $h \longrightarrow 0$. Thus there exists some $\eta_n > 0$, such that $|h| \leq \eta_n$ implies

$$\|p_h r_h u_n - \bar{\omega} u_n\|_F \leq \frac{1}{n}.$$

We can suppose that η_n is less than both η_{n-1} and $\frac{1}{n}$ so that the η_n form a strictly decreasing sequence converging to 0:

$$0 < \eta_{n+1} < \cdots < \eta_1 ; \quad \eta_n \longrightarrow 0.$$

Let us define $r_h u$ by

$$r_h u = r_h u_n \quad \text{for} \quad \eta_{n+1} < |h| \leq \eta_n .$$

It is clear that for $\eta_{n+1} < |h| \leq \eta_n$,

$$\|\bar{\omega} u - p_h r_h u\|_F \leq \|\bar{\omega} u - \bar{\omega} u_n\|_F + \|\bar{\omega} u_n - p_h r_h u_n\|_F + \|p_h r_h u_n - p_h r_h u\|_F \leq \frac{(1+c_0)}{n}$$

and consequently

$$\|\overline{\omega}u-p_h r_h u\|_F \leq \frac{1+c_0}{n}$$

for $|h| \leq \eta_n$. This implies the convergence of $p_h r_h u$ to $\overline{\omega}u$ as $h \to 0$ and completes the proof.

Remark 3.2.

If the mappings r_h are defined on the whole space W and condition (C1) holds for all $u \in \mathcal{W}$, Proposition 3.1 shows us that we can modify the value of $r_h u$ on the complement of $\mathcal{W}$ so that condition (C1) is satisfied for all $u \in W$.

Galerkin Approximation of a Normed Space.

As a very easy example we can define a Galerkin approximation of any separable normed space W.

Let W_h, $h \in \mathbb{N} = \mathcal{H}$, be an increasing sequence of finite dimensional subspaces of W whose union is dense in W. For each h, let p_h be the canonical injection of W_h into W, and for any $u \in W_{h_0}$, let $r_h u = 0$ if $h \leq h_0$, $r_h u = u$ if $h \geq h_0$. It is clear that $p_h r_h u \longrightarrow u$ as $h \longrightarrow \infty$, for any $u \in \bigcup_{h \in \mathbb{N}} W_h$. The operator r_h is only defined on $\mathcal{W} = \bigcup_{h \in \mathbb{N}} W_h$ which is dense in W. Since the prolongation operators have norm one they are stable and according to Proposition 3.1 the definition of the operators r_h can be extended in some way (which does not matter) to the whole space W so that we get a stable convergent internal approximation of W; this is a Galerkin approximation of W.

3.2 A General Convergence Theorem.

Let us discuss now the approximation of the general variational problem (2.12): W is a Hilbert space, $a(u,v)$ is a coercive bilinear continuous form on $W \times W$,

(3.2) $$a(u,u) \geq \alpha\|u\|_W^2, \quad \forall u \in W, \ (\alpha > 0),$$

and ℓ is a linear continuous form on W.

Let u denote the unique solution in W of

(3.3) $$a(u,v) = \langle \ell, v \rangle, \quad \forall v \in W.$$

With respect to the approximation of this element u, let there be given an external stable and convergent Hilbert approximation of the space W, say $\{W_h, p_h, r_h\}_{h \in \mathcal{H}}$. Likewise, for each $h \in \mathcal{H}$, let there be given

i) a continuous bilinear form $a_h(u_h,v_h)$ on $W_h \times W_h$ which is coercive and which, more precisely, satisfies

$$(3.4) \quad \begin{cases} \exists \alpha_0 > 0, \text{ independent of } h, \text{ such that} \\ a_h(u_h,u_h) \geq \alpha_0 \|u_h\|_h, \quad \forall u_h \in W_h \end{cases}$$

where $\|\cdot\|_h$ stands for the norm in W_h,

ii) a continuous linear form on $W_h, \ell_h \in W'_h$, such that:

$$(3.5) \qquad \|\ell_h\|_{*h} \leq \beta,$$

in which $\|\cdot\|_{*h}$ stands for the norm in W'_h, and in which β is independent of h.

We associate now with equation (3.3) the following family of approximate equations:

For fixed $h \in \mathcal{H}$, find $u_h \in W_h$ such that

$$(3.6) \qquad a_h(u_h,v_h) = \langle \ell_h, v_h \rangle, \quad \forall v_h \in W_h.$$

By the preceding hypotheses, Theorem 2.2 (in which W, W', a, ℓ, are replaced by W_h, W'_h, a_h, ℓ_h) asserts now that equation (3.6) has a unique solution; we will say that u_h is the approximate solution of equation (3.3).

A general theorem on the convergence of the approximate solutions u_h to the exact solution will be given after making precise the manner in which the forms a_h and ℓ_h are consistent with the forms a and ℓ. We make the following consistency hypotheses:

(3.7) If the family v_h converges weakly to v as $h \to 0$, and if the family w_h converges strongly to w as $h \to 0$,

$$\lim_{h\to 0} a_h(v_h,w_h) = a(v,w)$$

$$\lim_{h\to 0} a_h(w_h,v_h) = a(w,v).$$

(3.8) If the family v_h converges weakly to v as $h \to 0$, then

$$\lim_{h\to 0} \langle \ell_h, v_h \rangle = \langle \ell, v \rangle.$$

The general convergence theorem is then

Theorem 3.1.

Under the hypotheses (3.2), (3.4), (3.5), (3.7) and (3.8), the solution u_h of (3.6) converges strongly to the solution u of (3.3), as $h \longrightarrow 0$.

Proof.

Putting $v_h = u_h$ in (3.6) and using (3.4) and (3.5), we find

$$a_h(u_h,u_h) = \langle \ell_h,u_h \rangle, \tag{3.9}$$

$$\alpha_0 \| u_h \|_h^2 \leq \| \ell_h \|_{*h} \| u_h \|_h \leq \beta \| u_h \|_h ;$$

hence

$$\| u_h \|_h \leq \beta / \alpha_0 . \tag{3.10}$$

As the operators p_h are stable, there exists a constant c_0 which majorizes the norm of these operators

$$\| p_h \| = \| p_h \|_{\mathcal{L}(W_h,F)} \leq c_0 ; \tag{3.11}$$

and hence

$$\| p_h u_h \|_F \leq \frac{c_0 \beta}{\alpha_0} \tag{3.12}$$

Under these conditions, there exists some $\phi \in F$ and a sequence h' converging to 0, such that

$$\lim_{h' \longrightarrow 0} p_{h'} u_{h'} = \phi$$

in the weak topology of F; according to hypothesis (C2) in Definition 3.6, $\phi \in \bar{\omega} W$, whence $\phi = \bar{\omega} u_*$ for some $u_* \in W$:

$$\lim_{h' \longrightarrow 0} p_{h'} u_{h'} = \bar{\omega} u_* , \quad \text{(weak topology of } F). \tag{3.13}$$

Let us show that $u_* = u$. For a fixed $v \in W$, we write (3.6) with $v_h = r_h v$ and then take the limit with the sequence h' which gives, by using (3.7), (3.8), and (3.13):

$$a_h(u_h, r_h v) = \langle \ell_h, r_h v \rangle$$

$$\lim_{h' \to 0} a_{h'}(u_{h'}, r_{h'} v) = a(u_*, v)$$

$$\lim_{h' \to 0} \langle \ell_{h'}, r_{h'} v \rangle = \langle \ell, v \rangle.$$

Finally

$$a(u_*, v) = \langle \ell, v \rangle,$$

and because $v \in W$ is arbitrary, u_* is a solution of (3.3) and thus $u_* = u$.

One may show in exactly the same way, that from every subsequence of $p_h u_h$ one can extract a subsequence which converges in the weak topology of F to $\bar{\omega} u$. This proves that the family $p_h u_h$ as a whole converges to $\bar{\omega} u$, in the weak topology, as $h \to 0$.

Proof of the Strong Convergence.

Let us consider the expression

$$X_h = a_h(u_h - r_h u, u_h - r_h u),$$

or

$$X_h = a_h(u_h, u_h) - a_h(u_h, r_h u) - a_h(r_h u, u_h) + a_h(r_h u, r_h u).$$

By (3.7), (3.8), and (3.9),

$$\lim_{h \to 0} a_h(u_h, u_h) = \langle \ell, u \rangle$$

$$\lim_{h \to 0} a_h(u_h, r_h u) = \lim_{h \to 0} a_h(r_h u, u_h) = \lim_{h \to 0} a_h(r_h u, r_h u) = a(u,u).$$

Finally

$$\lim_{h \to 0} X_h = -a(u,u) + \langle \ell, u \rangle = 0, \tag{3.14}$$

according to (3.3) (when $v = u$).

With (3.4) and (3.11) we get now

$$0 \leq \alpha_0 \| u_h - r_h u \|_h^2 \leq X_h, \quad \text{whence}$$

$$0 \leq \| p_h u_h - p_h r_h u \|_F^2 \leq \frac{c_0^2}{\alpha_0} X_h \to 0.$$

Using now condition (C1) of Definition 3.6 and

$$\| p_h u_h - \bar{\omega} u \|_F \leq \| p_h u_h - p_h r_h u \|_F + \| p_h r_h u - \bar{\omega} u \|_F,$$

we see that this converges to 0 as $h \to 0$.

The theorem is proved.

Remark 3.3.

As announced in Remark 2.2, point (ii), Theorem 3.1 is applicable to the Galerkin approximation of (3.3) used in the proof of Theorem 2.2. One takes $W_h = W_m, \forall h = m \in \mathcal{H}$, $\mathcal{H} = \mathbb{N}$, and one gets as in the example at the end of Section 3.2 a Galerkin approximation of W. With

$$a_h(v,w) = a(v,w), \langle \ell_h, v\rangle = \langle \ell, v\rangle, \forall v,w \in W,$$

Theorem 3.1 is applicable and shows that u_m converges to u in the strong topology of W as $m \to \infty$.

Remark 3.4.

If W_h is a finite dimensional space and $\{w_{ih}\}_{1\leq i\leq N(h)}$ is a basis of W_h, then the approximate problem (3.6) is equivalent to a regular linear system for the components of u_h in this basis; i.e., if

$$u_h = \sum_{i=1}^{N(h)} \xi_{ih} w_{ih}$$

$$\sum_{i=1}^{N(h)} \xi_{ih} a_h(w_{ih}, w_{jh}) = \langle \ell_h, w_{jh}\rangle, \quad 1 \leq j \leq N(h). \tag{3.15}$$

The solution of (3.15) is effected by the usual methods for algebraic linear systems.

When no basis of W_h can be easily constructed (and this happens sometimes for the Stokes problem), some special method must be found to actually solve (3.6).

3.3 Approximation by Finite Differences.

We study the approximation by finite differences of the space $H_0^1(\Omega)$, then the same for the space V, and finally the approximation of Stokes problem by the corresponding scheme. The approximation of V considered here will be denoted by (APX1).

Notation.

When working with finite differences, h denotes the vector-mesh, $h = (h_1,\cdots,h_n)$ where h_i is the mesh in the x_i direction and thus

$$0 < h_i \leq h_i^0,$$

for some strictly positive numbers h_i^0; hence

$$\mathcal{H} = \prod_{i=1}^{n} (0, h_i^0]. \tag{3.16}$$

We are interested in passing to the limit $h \longrightarrow 0$.

For all $h \in \mathcal{H}$ we define:

i) $\vec{h}_i$ is the vector $h_i e_i$, where e_i has for j^h coordinate δ_{ij} = the symbol of Kronecker.

ii) $\mathcal{R}_h$ is the set of points of $\mathbb{R}^n$ of the form $j_1 h_1 + \cdots + j_n h_n$, in which the j_i are integers of arbitrary sign ($j_i \in \mathbb{Z}$).

iii) $\sigma_h(M)$, $M = (\mu_1,\cdots,\mu_n)$, is the set

$$\prod_{i=1}^{n} [\mu_i - \frac{h_i}{2}, \mu_i + \frac{h_i}{2})$$

and is called a block.

iv) $\sigma_h(M,r)$ is the set $\bigcup\limits_{\substack{1\leq i\leq n \\ -r\leq\alpha\leq+r}} \sigma_h(M + \frac{\alpha}{2}\vec{h}_i)$'

of course $\sigma_h(M,0) = \sigma_h(M)$.

v) w_{hM} is the characteristic function of the block $\sigma_h(M)$.

vi) δ_{ih} (or δ_i if no confusion can arise) is the finite difference operator

$$(\delta_i\phi)(x) = \frac{\phi(x + \frac{1}{2}\vec{h}_i) - \phi(x - \frac{1}{2}\vec{h}_i)}{h_i} \tag{3.17}$$

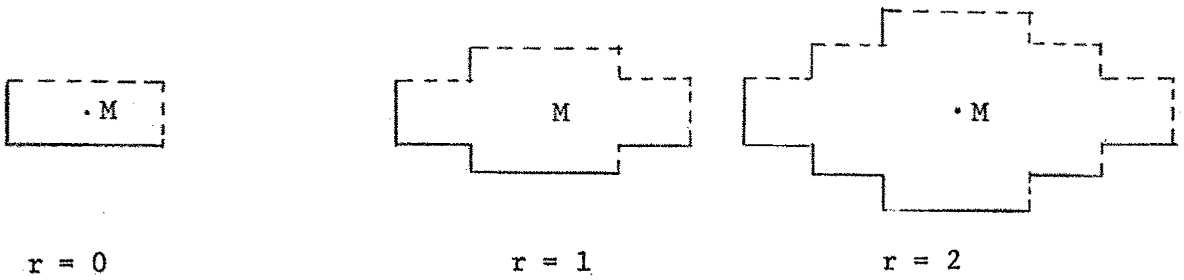
If $j = (j_1,\cdots,j_n) \in \mathbb{N}^n$ is a multi-index, then δ_h^j (or simply δ^j) will denote the operator

$$\delta^j = \delta_1^{j_1} \cdots \delta_n^{j_n}. \tag{3.18}$$

vii) With each open set Ω of $\mathbb{R}^n$ and each non-negative integer r we associate the following point sets

(3.19) $$\overset{\circ}{\Omega}{}_h^r = \{M \in \mathcal{R}_h,\ \sigma_h(M,r) \subset \Omega\}$$

(3.20) $$\Omega_h^r = \{M \in \mathcal{R}_h,\ \sigma_h(M,r) \cap \Omega \neq \phi\}.$$



Examples of Sets $\sigma_h(M,r)$ in the Plane.

viii) Sometimes we will use other finite difference operators such as ∇_{ih} and $\bar{\nabla}_{ih}$ (also denoted ∇_i and $\bar{\nabla}_i$);

(3.21) $$\nabla_{ih}\phi(x) = \frac{\phi(x+\vec{h}_i) - \phi(x)}{h_i}$$

(3.22) $$\bar{\nabla}_{ih}\phi(x) = \frac{\phi(x) - \phi(x-\vec{h}_i)}{h_i}$$

External Approximation of $H_0^1(\Omega)$.

Let Ω be a lipschitzian open bounded set in R^n. Let $W = H_0^1(\Omega)$, $F = L^2(\Omega)^{n+1}$ equipped with the natural hilbertian scalar product, and let $\bar{\omega}$ be the mapping

(3.23) $$u \longmapsto \bar{\omega}u = (u, D_1 u, \cdots, D_n u)$$

from W into F. It is clear that

$$\|\bar{\omega}u\|_F = [\![u]\!]_{H_0^1(\Omega)}$$

so that $\bar{\omega}$ is an isomorphism from W into F.

Space W_h: With the preceding notation, W_h will be the space of step functions

(3.24) $$u_h(x) = \sum_{M \in \overset{\circ}{\Omega}{}_h^1} u_h(M) w_{hM}(x),\ u_h(M) \in \mathbb{R}$$

The functions w_{hM} for $M \in \overset{\circ}{\Omega}{}_h^1$ are linearly independent and span the whole space W_h; they form a basis of W_h. The dimension of W_h is n times the number $N(h)$ of points $M \in \overset{\circ}{\Omega}{}_h^1$; W_h is finite dimensional. This space is provided with the scalar product

(3.25) $$[\![u_h, v_h]\!]_h = \int_\Omega u_h(x)v_h(x)\,dx + \sum_{i=1}^{n} \int_\Omega \delta_i u_h(x)\delta_i v_h(x)\,dx.$$

which makes it a Hilbert space.

The functions u_h and $\delta_i u_h$, $1 \leq i \leq n$, have compact supports in Ω, by the definition of W_h and the set $\mathring{\Omega}_h^1$. Hence they will be considered as vector functions defined on Ω or on $\mathbb{R}^n$.

Operators p_h The prolongation operators p_h are the discrete analog of $\bar{\omega}$:

(3.26) $$p_h u_h = (u_h, \delta_1 u_h, \cdots, \delta_h u_h), \quad \forall u_h \in W_h.$$

The norm of p_h is exactly one,

$$\|p_h u_h\|_F = [\![u_h]\!]_h$$

and they are stable.

Operators r_h: As a consequence of Proposition 3.1 we need only define the operator r_h on $\mathcal{W} = \mathcal{D}(\Omega)$ which is a dense subspace of $H_0^1(\Omega)$; we put

(3.27) $$(r_h u)(M) = u(M), \quad \forall M \in \mathring{\Omega}_h^1, \quad \forall u \in \mathcal{D}(\Omega)$$

which completely defines $r_h u \in W_h$.

Proposition 3.2.

The preceding external approximation of $H_0^1(\Omega)$ is stable and convergent.

Proof.

The approximation is stable since the prolongation operators are stable. We must check now conditions (C1) and (C2) of Definition 3.6.

Lemma 3.1.

Condition (C1) is satisfied: $\forall u \in \mathcal{D}(\Omega)$,

(3.28) $$r_h u \longrightarrow u \quad \text{in} \quad L^2(\Omega),$$

(3.29) $$\delta_i r_h u \longrightarrow D_i u \quad \text{in} \quad L^2(\Omega),$$

as $h \rightarrow 0$.

Proof.

Let $u \in \mathscr{D}(\Omega)$ and let h be sufficiently small so that the support of u is included in the set

$$\Omega(h) = \bigcup_{M \in \mathring{\Omega}_h^1} \sigma_h(M) \tag{3.30}$$

For any $M \in \mathring{\Omega}_h^1$, for any $x \in \sigma_h(M)$, the Taylor formula gives (with $u_h = r_h u$)

$$|u_h(x)-u(x)| = |u(M)-u(x)| \leq c_1(u)|M-x| \leq \frac{c_1(u)}{2}|h|$$

where

$$c_1(u) = \sup_{x \in \Omega} |\text{grad } u(x)| \tag{3.31}$$

$$|h| = \left(\sum_{i=1}^{n} h_i^2\right)^{\frac{1}{2}}. \tag{3.32}$$

Then

$$\sup_{x \in \Omega(h)} |u_h(x)-u(x)| \leq \frac{c_1(u)}{2} |h|. \tag{3.33}$$

On the set $\Omega-\Omega(h)$,

$$|u(x)| \leq c_1(u) d(x,\Gamma) \quad ,$$

$$|u_h(x)-u(x)| \leq c_1(u) d(\Omega(h),\Gamma) . \tag{3.34}$$

Hence

$$\sup_{x \in \Omega} |u_h(x)-u(x)| \leq c_1(u)\left\{\frac{|h|}{2}+d(\Omega(h),\Gamma)\right\} \tag{3.35}$$

which converges to 0 as $h \to 0$; u_h converges to u in $\mathbb{L}^\infty(\Omega)$ and then in $L^2(\Omega)$ since Ω is bounded.

To prove (3.29), we use again Taylor's formula: $\forall M \in \mathring{\Omega}_h^1$, $\forall x \in \sigma_h(M)$,

$$|\delta_i u_h(x)-D_i u(x)| \tag{3.36}$$

$$= \left|\frac{1}{h_i}\left[u_h(M+\tfrac{1}{2}\vec{h}_i)-u_h(M-\tfrac{1}{2}\vec{h}_i)\right]-D_i u(x)\right| \leq c_2(u)|h|,$$

where $c_2(u)$ depends only on the maximum norm of the second derivatives of u.

On the set $\Omega-\Omega(h)$,

$$|D_i u(x)| \leq c'_2(u) d(\Omega(h),\Gamma)$$

and then on the whole set Ω

(3.37) $$|\delta_i u_h(x) - D_i u(x)| \leq c_2(u)|h| + c'_2(u)d(\Gamma).$$

This converges to 0 as $h \to 0$, and shows the convergence of $\delta_i u_h$ to $D_i u$ in the uniform and L^2 norms.

<u>Lemma 3.2.</u>

<u>Condition</u> (C2) <u>is satisfied</u>.

<u>Proof</u>.

Let there be given a sequence $u_{h'} \in W_{h'}$, $h' \to 0$, such that $p_{h'} u_{h'}$ converges to ϕ in the weak topology of F, as $h' \to 0$. This means

(3.38) $$\begin{cases} \lim\limits_{h' \to 0} u_{h'} = \phi_0 \\ \lim\limits_{h' \to 0} \delta_{ih'} u_{h'} = \phi_i, \quad 1 \leq i \leq n \end{cases}$$

in the weak topology of $\mathbb{L}^2(\Omega)$; $\phi = (\phi_0, \cdots, \phi_n)$. As the functions $u_{h'}$, $\delta_i u_{h'}$ have compact support in Ω, we also have

(3.39) $$\begin{cases} \lim\limits_{h' \to 0} u_{h'} = \tilde{\phi}_0 \\ \lim\limits_{h' \to 0} \delta_i u_{h'} = \tilde{\phi}_i, \quad 1 \leq i \leq n, \end{cases}$$

in the weak topology of $\mathbb{L}^2(\mathbb{R}^n)$; here $\tilde{g}$ means the function equal to g in Ω and equal to 0 in the complement of Ω.

A discrete integration by parts formula gives

(3.40) $$\int_{\mathbb{R}^n} \delta_{ih} u_{h'}(x)\sigma(x)dx = -\int_{\mathbb{R}^n} u_{h'}(x)\delta_{ih}\sigma(x)dx,$$

for each $\sigma \in \mathcal{D}(\mathbb{R}^n)$

As $h' \to 0$, the left-hand side of (3.40) converges to

$$\int_{\mathbb{R}^n} \tilde{\phi}_i(x)\sigma(x)dx;$$

the right-hand side converges to

$$-\int_{\mathbb{R}^n} \tilde{\phi}_0(x) D_i\sigma(x)dx,$$

since $\delta_{ih}\sigma$ converges to $D_i\sigma$ in $L^2(\mathbb{R}^n)$ as shown in Lemma 3.1. Then

$$\int_{\mathbb{R}^n} \tilde{\phi}_i(x)\sigma(x)dx = -\int_{\mathbb{R}^n} \tilde{\phi}_0(x)D_i\sigma(x)dx, \quad \forall\sigma\in\mathcal{D}(\mathbb{R}^n)$$

which amounts saying that

$$\tilde{\phi}_i = D_i\tilde{\phi}_0, \quad 1 \leq i \leq n, \tag{3.41}$$

in the distribution sense.

It is clear now that $\tilde{\phi}_0\in\mathbb{H}^1(R^n)$, and since $\tilde{\phi}_0$ vanishes in the complement of Ω, ϕ_0 belongs to $\mathbb{H}_0^1(\Omega)$. Thus $\phi\in\bar{\omega}W$;

$$\phi = \bar{\omega}\phi_0, \quad \phi_0\in\mathbb{H}_0^1(\Omega). \tag{3.42}$$

Discrete Poincaré Inequality.

The following discrete Poincaré inequality (see (1.9)) will allow us to equip the space W_h in (3.24) with another scalar product $((\cdot,\cdot))_h$, the discrete analog of scalar product $((\cdot,\cdot))$ (see (1.11)).

Proposition 3.3.

Let Ω be a set bounded in the x_i direction, and let u_h be a scalar step function of type (3.24) (with $u_h(M)\in\mathbb{R}$). Then

$$|u_h| \leq 2\ell|\delta_{ih}u_h| \tag{3.43}$$

where ℓ is the width of Ω in this direction.

Proof.

For the sake of simplicity we take $i = 1$. Since u_h has a compact support, for any $M\in\mathcal{R}_h$,

$$u_h(M)^2 = \sum_{j=0}^{+\infty} \{[u_h(M-j\vec{h}_1)]^2 - [u_h(M-(j+1)\vec{h}_1)]^2\}$$

$$= h_1 \sum_{j=0}^{+\infty} [\delta_{1h}u_h(M-(j+\tfrac{1}{2})\vec{h}_1)][u_h(M-j\vec{h}_1) + u_h(M-(j+1)\vec{h}_1)] ,$$

$$u_h(M)^2 \leq I = h_1 \sum_{j=-\infty}^{+\infty} |\delta_{1h}u_h(M-(j+\tfrac{1}{2})\vec{h}_1)|[|u_h(M-j\vec{h}_1)|+|u_h(M-(j+1)\vec{h}_1)|]. \tag{3.44}$$

The sums are actually finite. Now for any $i \in \mathbb{Z}$, $u_h(M+i\vec{h}_1)^2$ is majorized by exactly the same expression I. There are less than $\frac{\ell}{h_1}$ values of i such that $u_h(M+i\vec{h}_1) \neq 0$ since the x_1-width of Ω is less than ℓ. Hence

$$\sum_{i=-\infty}^{+\infty} u_h(M+ih_1)^2 \leq \frac{\ell}{h_1} I. \tag{3.45}$$

Let $\mathcal{F}_h(M)$ denote the tube $\bigcup_{i=-\infty}^{+\infty} \sigma_h(M+i\vec{h}_1)$. We can interpret (3.45) as follows:

$$\int_{\mathcal{F}_h(M)} u_h(x)^2 dx = (h_1 \cdots h_n) \sum_{i=-\infty}^{+\infty} u_h(M+i\vec{h}_1)^2$$

$$\leq \ell(h_1 \cdots h_n) \frac{I}{h_1}$$

$$= \ell \int_{\mathcal{F}_h(M)} |\delta_{1h} u_h(x)| \{|u_h(x+\tfrac{1}{2}\vec{h}_1)| + |u_h(x-\tfrac{1}{2}\vec{h}_1)|\} dx.$$

We take summations of the last inequality for all tubes $\mathcal{F}_h(M)$ and obtain

$$\int_{\mathbb{R}^n} u_h(x)^2 dx \leq \ell \int_{\mathbb{R}^n} |\delta_{1h} u_h(x)| \{|u_h(x+\tfrac{1}{2}\vec{h}_1)| + |u_h(x-\tfrac{1}{2}\vec{h}_1)|\} dx.$$

Applying Schwarz's inequality, we obtain

$$|u_h|^2 = \int_{\mathbb{R}^n} u_h(x)^2 dx \leq \ell |\delta_{1h} u_h| \cdot \{2 \int_{\mathbb{R}^n} [|u_h(x+\tfrac{1}{2}\vec{h}_1)|^2 + |u_h(x-\tfrac{1}{2}\vec{h}_1)|^2] dx\}^{\frac{1}{2}}$$

$$\leq 2\ell |\delta_{1h} u_h| \cdot |u_h|$$

and (3.43) follows.

<u>Proposition 3.4.</u>

<u>Let Ω be a bounded lipschitzian set. If we equip the space W_h with the scalar product</u>

$$((u_h, v_h))_h = \sum_{i=1}^{n} \int_\Omega \delta_{ih} u_h \delta_{ih} v_h dx, \tag{3.46}$$

<u>we get again a stable convergent approximation of $H_0^1(\Omega)$.</u>

Proof.

The prolongation operators are stable as a consequence of Proposition 3.3.

Remark 3.5.

Using the difference operators ∇_{ih}, or $\overline{\nabla}_{ih}$, or even any "reasonable" approximation of the differentiation operator $\frac{\partial}{\partial x_i}$, one can define many other similar approximations of the space $H_0^1(\Omega)$. The modifications arise then in (3.25) where δ_{ih} is replaced by $\nabla_{ih},\cdots$; in the set of points $\mathring{\Omega}_h^1$ which must be suitably defined, and in some points of the proof of Lemmas 3.1 and 3.2.

The same Poincaré inequality is valid for the operators ∇_{ih} and $\overline{\nabla}_{ih}$ but not for more general operators.

Remark 3.6.

When Ω is unbounded, one can define an external approximation of $H_0^1(\Omega)$ with a space W_h consisting of either:

- step functions $\sum_{M\in\mathring{\Omega}_h^1} \lambda_M w_{hM}$, which have compact support (we restrict the sum sum to a finite number of points $M\in\mathring{\Omega}_h^1$),
- or step functions $\sum \lambda_M w_{hM}$ for the M in the intersection of $\mathring{\Omega}_h^1$ with some "large" ball: $|x| \leq \rho(h)$, where $\rho(h)\longrightarrow+\infty$ as $h\longrightarrow 0$.

In the second case W_h is finite dimensional but not in the first case.

Without any modification for the other elements of the approximation, one can see that we get a stable convergent approximation of $H_0^1(\Omega)$ for an unbounded locally lipschitzian set Ω.

The discrete Poincaré inequality is available if Ω is bounded in one of the directions $x_1,\cdots,x_n$.

Approximation of the Space V (APX1).

Let Ω be a lipschitzian bounded set in $\mathbb{R}^n$ and let $\mathcal{V}$ be the usual space (1.12) and V its closure in $\mathbb{H}_0^1(\Omega)$.

We define now an approximation of V using finite differences (which will be denoted by (APX1)).

Let $F = \mathbb{L}^2(\Omega)^{n+1}$ equipped with the natural hilbertian scalar product and let us define the mapping $\overline{\omega}\in\mathcal{L}(V,F)$:

$$u\longrightarrow\overline{\omega}u = (u,D_1u,\cdots,D_nu).$$

It is clear that

$$\|\bar{\omega}u\|_F = [\![u]\!]$$

so that $\bar{\omega}$ is an isomorphism from V into F.

Space V_h. We take the following space V_h as the space W_h: V_h is the space of step functions

$$u_h(x) = \sum_{M\in\mathring{\Omega}_h^1} u_h(M)w_{hM}(x), \quad u_h(M)\in\mathbb{R}^n, \tag{3.47}$$

which are discretely divergence free in the following sense:

$$\sum_{i=1}^{n} \nabla_{ih}u_h(M) = 0, \quad \forall M\in\mathring{\Omega}_h^1 \tag{3.48}$$

which amounts to saying

$$\sum_{i=1}^{n} \nabla_{ih}u_h(x) = 0, \quad \forall x\in\Omega(h). \tag{3.49}$$

No basis of V_h is available; it is clear that V_h is a finite dimensional space with dimension less than or equal to $nN(h)-N(h) = (n-1)N(h)$ since all the functions in (3.47) form a space of dimension $nN(h)$ and there are at most $N(h)$ independent linear constraints in (3.48); it is not clear whether the constraints (3.48) are always linearly independent so that V_h is not necessarily of dimension $(n-1)N(h)$.

The space V_h is equipped with one of the scalar products

$$((u_h,v_h))_h = \sum_{i=1}^{n} \int_\Omega \delta_i u_h(x)\cdot\delta_i u_h(x)\,dx \tag{3.50}$$

$$[\![u_h,v_h]\!]_h = \int_\Omega u_h(x)v_h(x)\,dx + ((u_h,v_h))_h. \tag{3.51}$$

Because of Proposition 3.3 (discrete Poincaré inequality), V_h equipped with either one of these scalar products is a Hilbert space.

Operators p_h. These are the discrete analog of $\bar{\omega}$:

$$p_h u_h = \{u_h, D_1 u_h, \cdots, D_n u_h\} \tag{3.52}$$

These operators are stable since by (3.43)

(3.53) $$\|p_h u_h\|_F = [\![u_h]\!]_h \leq c\|u_h\|_h, \quad \forall u_h \in V_h.$$

Operator r_h We only define $r_h u$ for $u \in \mathcal{V}$, a dense subspace of V; $\forall M = (m_1 h_1, \cdots, m_n h_n) \in \mathring{\Omega}_h^1$, $(r_h u)(M) = u_h(M)$ is defined by

(3.54) $$\begin{aligned} u_{ih}(M) &= i^{th} \text{ component of } u_h \\ &= \text{the average value of } u_i \text{ on the face} \\ &\quad x_i = (m_i - \tfrac{1}{2})h_i \text{ of } \sigma_h(M). \end{aligned}$$

This complicated definition of $r_h u$ is necessary if we want u_h to belong to V_h; actually

Lemma 3.3.

For each $u \in \mathcal{V}$, $r_h u \in V_h$

Proof.

One has

$$\nabla_{ih} u_{ih}(M) = \frac{1}{h_1 \cdots h_n} \left\{ \int_{\Sigma_i} u_i(x)\,dx - \int_{\Sigma'_i} u_i(x)\,dx \right\}$$

where Σ_i and Σ'_i respectively are the faces $x_i = (m_i + \frac{1}{2})h_i$ and $x_i = (m_i - \frac{1}{2})h_i$ of $\sigma_h(M)$.

This gives also

$$\nabla_{ih} u_{ih}(M) = \frac{1}{h_1 \cdots h_n} \int_{\Sigma_i \cup \Sigma'_i} u.\nu d\Gamma,$$

where ν stands for the unit vector normal to the boundary of $\sigma_h(M)$ and pointing in the outward direction. Then, for each $M \in \mathring{\Omega}_h^1$,

$$\begin{aligned} \sum_{i=1} \nabla_{ih} u_{ih}(M) &= \frac{1}{h_1 \cdots h_n} \int_{\partial\sigma_h(M)} u.\nu d\Gamma \\ &= \text{(by Stokes formula)} \\ &= \frac{1}{h_1 \cdots h_n} \int_{\sigma_h(M)} \operatorname{div} u \, dx = 0, \quad \text{since } \operatorname{div} u = 0. \end{aligned}$$

Conditions (3.48) are met.

Proposition 3.5.

The preceding external approximation of V is stable and convergent.

Proof.

Stability has been shown already.

The proof of condition (C1) is very similar to the proof of Lemma 3.1 and we do not repeat all the details; for example, for $x \in \sigma_h(M)$, $M \in \mathring{\Omega}_h^1$,

$$|u_{ih}(x)-u_i(x)| = |u_{ih}(\xi)-u_i(x)|,$$

where ξ is some point of the face $x_i = (m_i - \frac{1}{2})h_i$ of $\sigma_h(M)$, and hence

$$|u_{ih}(x)-u_i(x)| \leq c_1(u_i)|x-\xi| \leq c_1(u_i)|h|$$

and (3.35) is replaced by

$$\text{(3.55)} \qquad \sup_{x\in\Omega} |u_h(x)-u(x)| \leq c_1'(u)\{|h| + d(\Omega(h),\Gamma)\}.$$

The proof for condition (C2) is similar to the proof of Lemma 3.2; more precisely, the same proof as for Lemma 3.2 shows that if

$$p_{h'}u_{h'} \longrightarrow \phi \quad \text{in the weak topology of } F,$$

as $h' \longrightarrow 0$, then $\phi = \bar{\omega}u = (u, D_1u, \cdots, D_nu)$, where $u \in H_0^1(\Omega)$. Because of Theorem 1.6, proving that $u \in V$ amounts now to proving that div $u = 0$. This follows from (3.49) as we now show. Let σ be any test function in $\mathscr{D}(\Omega)$, and let us suppose that h is small enough that the support of σ is included in $\Omega(h)$; then (3.49) shows that

$$\int_\Omega (\sum_{i=1}^n (\nabla_{ih}u_{ih})(x))\sigma(x)dx = 0$$

or

$$\text{(3.56)} \qquad \int_{\mathbb{R}^n} (\sum_{i=1}^n (\nabla_{ih}u_{ih})(x))\sigma(x)dx = 0.$$

It is easy to check the discrete integration by parts formula

$$\text{(3.57)} \qquad \int_{\mathbb{R}^n} (\nabla_{ih}\theta)(x).\sigma(x)dx = -\int_{\mathbb{R}^n}\theta(x)(\bar{\nabla}_{ih}\sigma)(x)dx;$$

then (3.56) becomes

$$\text{(3.58)} \qquad \int_{\mathbb{R}^n} \sum_{i=1}^n [u_{ih}(x).(\bar{\nabla}_{ih}\sigma)(x)]dx = 0.$$

With a proof similar to that of Lemma 3.1 (based on Taylor's formula) we see that

$$\bar{\nabla}_{ih}\sigma \longrightarrow -D_i\sigma, \quad \text{as} \quad h \longrightarrow 0, \tag{3.59}$$

in the (uniform and) L^2 norm. Since u_{ih} converges to u_i for the weak topology of $L^2(\mathbb{R}^n)$, letting $h \longrightarrow 0$ in (3.58) gives the result

$$\int_{\mathbb{R}^n} [\sum_{i=1}^{n} u_i(x).D_i\sigma(x)]dx = 0. \tag{3.60}$$

The equality (3.60), true for any $\sigma \in \mathcal{D}(\Omega)$, implies that $\operatorname{div} u = 0$ and then $\phi = \bar{\omega}u$, with $u \in V$.

Remark 3.7.

The Remark 3.5 can be extended to the present case with, however, one restriction: condition (3.48) - (3.49) in the definition of the spaces V_h cannot be replaced by similar relations involving other finite difference operators; for example, it seems impossible to replace (3.49) by

$$\sum_{i=1}^{n} \delta_{ih}u_{ih}(x) = 0, \tag{3.61}$$

since (3.61) requires many more algebraic relations than (3.49) and probably too many relations.

Remark 3.8.

When Ω is unbounded one can define, by using the methods mentioned in Remark 3.5, a stable and convergent external approximation of the space W introduced in (2.31).

Approximation of Stokes Problem.

Using the preceding approximation of $\underset{\sim}{V}$ and the results of Section 3.2, we can propose a finite difference scheme for the approximation of Stokes problem. Let us take, for (3.6),

$$a_h(u_h,v_h) = \nu((u_h,v_h))_h \tag{3.62}$$

$$<\ell_h,v_h> = (f,v_h), \tag{3.63}$$

where V_h and $((\cdot,\cdot))_h$ are the space and scalar product just defined, and ν and f are given as in Section 2.1.

The approximate problem corresponding to (2.6) is then:

Proof.

Stability has been shown already.

The proof of condition (C1) is very similar to the proof of Lemma 3.1 and we do not repeat all the details; for example, for $x \in \sigma_h(M)$, $M \in \mathring{\Omega}_h^1$,

$$|u_{ih}(x)-u_i(x)| = |u_{ih}(\xi)-u_i(x)|,$$

where ξ is some point of the face $x_i = (m_i - \frac{1}{2})h_i$ of $\sigma_h(M)$, and hence

$$|u_{ih}(x)-u_i(x)| \leq c_1(u_i)|x-\xi| \leq c_1(u_i)|h|$$

and (3.35) is replaced by

(3.55) $$\operatorname*{Sup}_{x\in\Omega} |u_h(x)-u(x)| \leq c_1'(u)\{|h| + d(\Omega(h),\Gamma)\}.$$

The proof for condition (C2) is similar to the proof of Lemma 3.2; more precisely, the same proof as for Lemma 3.2 shows that if

$$p_{h'}u_{h'} \longrightarrow \phi \quad \text{in the weak topology of } F,$$

as $h' \longrightarrow 0$, then $\phi = \overline{\omega}u = (u, D_1u, \cdots, D_nu)$, where $u \in H_0^1(\Omega)$. Because of Theorem 1.6, proving that $u \in V$ amounts now to proving that div $u = 0$. This follows from (3.49) as we now show. Let σ be any test function in $\mathcal{D}(\Omega)$, and let us suppose that h is small enough that the support of σ is included in $\Omega(h)$; then (3.49) shows that

$$\int_\Omega (\sum_{i=1}^n (\nabla_{ih}u_{ih})(x))\sigma(x)dx = 0$$

or

(3.56) $$\int_{\mathbb{R}^n} (\sum_{i=1}^n (\nabla_{ih}u_{ih})(x))\sigma(x)dx = 0.$$

It is easy to check the discrete integration by parts formula

(3.57) $$\int_{\mathbb{R}^n} (\nabla_{ih}\theta)(x).\sigma(x)dx = -\int_{\mathbb{R}^n}\theta(x)(\overline{\nabla}_{ih}\sigma)(x)dx;$$

then (3.56) becomes

(3.58) $$\int_{\mathbb{R}^n} \sum_{i=1}^n [u_{ih}(x).(\overline{\nabla}_{ih}\sigma)(x)]dx = 0.$$

With a proof similar to that of Lemma 3.1 (based on Taylor's formula) we see that

(3.59) $$\bar{\nabla}_{ih}\sigma \longrightarrow -D_i\sigma, \quad \text{as} \quad h \longrightarrow 0,$$

in the (uniform and) L^2 norm. Since u_{ih} converges to u_i for the weak topology of $L^2(\mathbb{R}^n)$, letting $h \longrightarrow 0$ in (3.58) gives the result

(3.60) $$\int_{\mathbb{R}^n} [\sum_{i=1}^{n} u_i(x).D_i\sigma(x)]dx = 0.$$

The equality (3.60), true for any $\sigma \in \mathscr{D}(\Omega)$, implies that $\operatorname{div} u = 0$ and then $\phi = \bar{\omega}u$, with $u \in V$.

Remark 3.7.

The Remark 3.5 can be extended to the present case with, however, one restriction: condition (3.48) - (3.49) in the definition of the spaces V_h cannot be replaced by similar relations involving other finite difference operators; for example, it seems impossible to replace (3.49) by

(3.61) $$\sum_{i=1}^{n} \delta_{ih}u_{ih}(x) = 0,$$

since (3.61) requires many more algebraic relations than (3.49) and probably too many relations.

Remark 3.8.

When Ω is unbounded one can define, by using the methods mentioned in Remark 3.5, a stable and convergent external approximation of the space W introduced in (2.31).

Approximation of Stokes Problem.

Using the preceding approximation of V and the results of Section 3.2, we can propose a finite difference scheme for the approximation of Stokes problem. Let us take, for (3.6),

(3.62) $$a_h(u_h,v_h) = \nu((u_h,v_h))_h$$

(3.63) $$<\ell_h,v_h> = (f,v_h),$$

where V_h and $((\cdot,\cdot))_h$ are the space and scalar product just defined, and ν and f are given as in Section 2.1.

The approximate problem corresponding to (2.6) is then:

$$(3.64)\qquad \begin{cases} \underline{\text{To find}}\ \ u_h \in V_h\ \ \underline{\text{such that}} \\ \nu((u_h,v_h))_h = (f,v_h),\ \ \forall v_h \in V_h. \end{cases}$$

Proposition 3.6.

For all $h \in \mathcal{H}$, the solution u_h of (3.64) exists and is unique; moreover as $h \to 0$ the solution u_h of (3.64) converges to the solution u of (2.6) in the following sense:

$$(3.65)\qquad u_h \to u \ \text{ in } \ \mathbb{L}^2(\Omega),$$

$$(3.66)\qquad \delta_{ih}u_h \to D_i u \ \text{ in } \ \mathbb{L}^2(\Omega).$$

Proof.

We only have to check that Theorem 3.1 is applicable. Condition (3.4) is obvious ($\alpha_0 = 1$); for (3.5) we notice that

$$\begin{aligned} |\langle \ell_h, v_h \rangle| &= |(f,v_h)| \leq |f|.|v_h| \\ &\leq \text{(by the discrete Poincaré inequality)} \\ &\leq c(\Omega)|f| \|v_h\|_h, \ \ \forall v_h \in V_h. \end{aligned}$$

Hence

$$(3.67)\qquad \|\ell_h\|_{*h} \leq c(\Omega)|f|,$$

and (3.5) is satisfied.

For (3.7) - (3.8) we notice that

$$p_h v_h \to \bar{\omega} v \ \text{ weakly (resp. } \ p_h w_h \to \bar{\omega} w \ \text{ strongly)}$$

means

$$v_h \to v, \ \text{ and } \ \delta_{ih}v_h \to D_i v \ \text{ in } \ \mathbb{L}^2(\Omega) \ \text{ weakly,}$$

(resp.

$$w_h \to w, \ \text{ and } \ \delta_{ih}w_h \to D_i w \ \text{ in } \ \mathbb{L}^2(\Omega) \ \text{ strongly),}$$

and it is clear that this implies,

$$(\delta_{ih}v_h, \delta_{ih}w_h) \to (D_i v, D_i w)$$

$$((v_h,w_h))_h \to ((v,w)),$$

$$(f,v_h) \to (f,v).$$

Approximation of the Pressure.

We want to present the "approximate" pressure which is implicitly contained in (3.64) as the exact pressure p is implicitly contained in (2.6).

The space V_h in (3.47) is a subspace of the space W_h in (3.24); namely, the space of $v_h \in W_h$ satisfying the linear constraint (3.68).

The form $v_h \mapsto \nu((u_h,v_h))_h - (f,v_h)$ appears as a linear form defined on W_h which vanishes on V_h. Hence introducing the Lagrange multipliers corresponding to the linear constraints (3.48) we find, with the aid of a classical theorem of linear algebra, that there exist numbers $\lambda_M \in \mathbb{R}$, $M \in \mathring{\Omega}_h^1$, such that the equation

$$(3.68) \qquad \nu((u_h,v_h))_h - (f,v_h) = \sum_{M \in \mathring{\Omega}_h^1} \lambda_M \sum_{i=1}^{n} (\nabla_{ih} v_{ih}(M)),$$

holds for each $v_h \in W_h$.

Let us now introduce the operator $D_h \in \mathcal{L}(W_h, L^2(\Omega))$:

$$(3.69) \qquad D_h v_h(x) = \sum_{i=1}^{n} \nabla_{ih} v_{ih}(x), \quad \forall v_h \in W_h,$$

its adjoint $D_h^* \in \mathcal{L}(L^2(\Omega), W_h)$ is defined by

$$(3.70) \qquad (D_h^* \theta, v_h) = (\theta, D_h v_h) \quad \forall v_h \in W_h, \forall \theta \in L^2(\Omega).$$

Let π_h be the step function which vanishes outside $\Omega(h) = \bigcup_{M \in \mathring{\Omega}_h^1} \sigma_h(M)$, and which satisfies

$$(3.71) \qquad \pi_h(x) = \pi_h(M) = \frac{\lambda_M}{h_1 \cdots h_n}, \quad \forall x \in \sigma_h(M),\ M \in \mathring{\Omega}_h^1.$$

Then (3.68) can be written as

$$\nu((u_h,v_h))_h - (f,v_h) = (\pi_h, D_h v_h),$$

or equivalently,

$$(3.72) \qquad \nu((u_h,v_h))_h - (D_h^* \pi_h, v_h) = (f,v_h), \quad \forall v_h \in W_h.$$

Taking successively $v_h = w_{hM} e_j$ for $M \in \mathring{\Omega}_h^1$, $j = 1,\cdots,n$, we can interpret (3.72) as

$$(3.73) \qquad -\nu \sum_{i=1}^{n} \delta_{ih}^2 u_h(M) + (\bar{\nabla}_h \pi_h)(M) = f_h(M), \quad M \in \mathring{\Omega}_h^1$$

where $\overline{\nabla}_h(\pi_h(M)$ is the vector $(\overline{\nabla}_{1h}\pi_h(M),\cdots,\overline{\nabla}_{nh}\pi_h(M))$ and

$$(3.74) \qquad f_h(M) = \frac{1}{h_1\cdots h_n}\int_{\sigma_h(M)} f(x)\,dx.$$

The equations (3.73), and

$$(3.75) \qquad \sum_{i=1}^{n} (\nabla_{ih}u_{ih})(M) = 0, \quad M\in\mathring{\Omega}_h^1,$$

are the discrete form of the equations (2.7) - (2.8); $-D_h^*$ is the "approximation" of p, $-D_h^*$ is a discrete gradient operator.

Remark 3.9.

As indicated in Remark 3.4, the solution of (3.64) is not easy since we do not know any simple basis of V_h. One possiblility for solving (3.64) would be to solve the system (3.73),(3.75), which is a linear system with unknowns

$$u_{1h}(M),\cdots,u_{nh}(M);\quad \pi_h(M),\ M\in\mathring{\Omega}_h^1.$$

This system has a unique solution up to an additive constant for the $\pi_h(M)$; this non-uniqueness makes the resolution of this linear system difficult; moreover, the matrix of the system is ill-conditionned.

More efficient ways for actually computing the approximate solution will be given in Section 5.

The Error.

Let us suppose that the exact solution satisfies $u\in\mathcal{C}^3(\overline{\Omega})$ and $p\in\mathcal{C}^2(\overline{\Omega})$. Then by using Taylor's formula, we have

$$(3.76) \qquad -\nu\sum_{i=1}^{n} (\delta_{ih}^2 r_h u)(M) - (\overline{\nabla}_h p)(M) = f(M) + \varepsilon_h(M)$$

where $r_h u$ is the function of V_h defined by (3.54) and where $\varepsilon_h(M)$ is a "small" vector:

$$(3.77) \qquad |\varepsilon_h(M)| \leq c(u,p)|h|,$$

c(u,p) depending only on the maximum norms of third derivatives of u and second derivatives of p. Let us denote by π'_h the function $\sum_{M\in\Omega_h^1} p(M)w_{hM}$. Then,

the equality (3.76) is equivalent to

(3.78) $$\nu((r_h u, v_h))_h + (\pi'_h, D_h v_h) = (f+\varepsilon_h, v_h)$$

for each $v_h \in W_h$ (space (3.24)) and implies

(3.79) $$\nu((r_h u, v_h))_h = (f+\varepsilon_h, v_h),$$

for each $v_h \in V_h$.

Subtracting this equality from (3.64) we get

(3.80) $$\nu((u_h - r_h u, v_h))_h = (\varepsilon_h, v_h),$$

and then taking $v_h = u_h - r_h u$, we see that

$$\begin{aligned}\nu\|u_h - r_h u\|_h^2 &= (\varepsilon_h, u_h - r_h u)\\ &\leq c(\Omega,u,p)|h|\|u_h - r_h u\|_h.\end{aligned}$$

Hence we find the following estimates for the discrete error:

(3.81) $$\|u_h - r_h u\|_h \leq \frac{1}{\nu}\, c(\Omega,u,p)|h|$$

(3.82) $$|u_h - r_h u| \leq \frac{1}{\nu}\, c'(\Omega,u,p)|h|.$$

§4. DISCRETIZATION OF STOKES EQUATIONS (II)

We study here the discretization of Stokes equations by means of finite element methods. The results are less general here than in the previous section and vary according to the dimension. We successively consider conforming finite elements which are first piecewise polynomials of degree two in the two-dimensional case (Section 4.2), next piecewise polynomials of degree three in the three-dimensional case (Section 4.3), and then piecewise polynomials of degree four in the two-dimensional case (Section 4.4). Finally we consider an external approximation by non-conforming finite elements (any dimension) in Section 4.5.

4.1 Preliminary Results.

We will have to work with peicewise polynomial functions defined on n-simplices. For that purpose, we recall here some definitions and introduce some notations adapted to the situation.

Barycentric Coordinates.

Let there be given in $\mathbb{R}^n$, $(n+1)$ points $A_1,\cdots,A_{n+1}$,(1) with coordinates $a_{1,i},\cdots,a_{n,i}$, $1 \le i \le n+1$, which do not lie in the same hyperplane; this amounts saying that the n vectors $A_1A_2,\cdots,A_1A_{n+1}$ are independent, or that the matrix

$$(4.1) \qquad \mathcal{A} = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n+1} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & & a_{n,n+1} \\ 1 & 1 & & 1 \end{vmatrix}$$

is non-singular. Given any point $P \in \mathbb{R}^n$, with coordinates $x_1,\cdots,x_n$, there exists $(n+1)$ real numbers

$$\lambda_i = \lambda_i(P), \quad 1 \le i \le n+1$$

such that

$$(4.2) \qquad OP = \sum_{i=1}^{n+1} \lambda_i OA_i,$$

$$(4.3) \qquad \sum_{i=1}^{n+1} \lambda_i = 1,$$

(1) In this section dealing with finite elements, the capital letters $A,B,M,P,\cdots$, (sometimes with subscripts) will denote points of the affine space $\mathbb{R}^n$. Couples of such letters, like $AB,\cdots$, denote the vector of $\mathbb{R}^n$ with origin A and terminal point B.

where 0 is the origin of $\mathbb{R}^n$.

To see this it suffices to remark that (4.2)(4.3) are equivalent to the linear system

$$(4.4) \qquad \mathcal{A}\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

which has a unique solution since the matrix $\mathcal{A}$ is non-singular, by hypothesis. The quantities λ_i are called the _barycentric coordinates of_ P, _with respect to the_ $(n+1)$ _points_ $A_1,\cdots,A_{n+1}$. As a consequence of (4.4), the numbers λ_i appear as linear, generally nonhomogeneous functions of the coordinates $x_1,\cdots,x_n$ of P:

$$(4.5) \qquad \lambda_i = \sum_{j=1}^{n} b_{i,j}x_j + b_{i,n+1}, \quad 1 \le i \le n+1,$$

where the matrix $\mathcal{B} = (b_{i,j})$ is the inverse of the matrix $\mathcal{A}$. It is easy to see that the point 0 in (4.2) can be replaced by another point of $\mathbb{R}^n$ without changing the value of the barycentric coordinates; hence

$$(4.6) \qquad \sum_{i=1}^{n+1} \lambda_i PA_i = 0.$$

The barycentric coordinates are also clearly independent of the choice of basis in $\mathbb{R}^n$.

The convex hull of the $(n+1)$ points A_i is exactly the set of points of R^n with barycentric coordinates satisfying the conditions:

$$(4.7) \qquad 0 \le \lambda_i \le 1, \quad 1 \le i \le n+1.$$

This convex hull $\mathcal{S}$ is the _n-simplex generated by the points_ A_i, _which are called the vertices of the n-simplex_. The _barycenter_ G of $\mathcal{S}$ is the point of $\mathcal{S}$ whose barycentric coordinates are all equal and hence equal $\frac{1}{n+1}$. An _m-dimensional face_ of $\mathcal{S}$ is any m-simplex $(1 \le m \le n-1)$ generated by $m+1$ of the vertices of $\mathcal{S}$ (of course these vertices do not lie in a $(m-1)$-dimensional subspace of $\mathbb{R}^n$). A _1-dimensional face is an edge_.

In the two-dimensional case $(n=2)$ the 2-simplices are triangles; the vertices and edges of the simplex are simply the vertices and edges of the triangle. In the three dimensional case, the 3-simplices are tetrahedrons, the two-faces are the four triangles which form its boundary.

An Interpolation Result.

Proposition 4.1.

Let $A_1,\cdots,A_{n+1}$ *be* $(n+1)$ *points of* $\mathbb{R}^n$ *which are not included in a hyperplane. Given* $(n+1)$ *real numbers* $\alpha_1,\cdots,\alpha_{n+1}$, *there exists one and only linear function* u *such that* $u(A_i) = \alpha_i$, $1 \leq i \leq n+1$, *and*

$$u(P) = \sum_{i=1}^{n+1} \alpha_i \lambda_i(P), \quad \forall P \in \mathbb{R}^n, \tag{4.8}$$

where the $\lambda_i(P)$ *are the barycentric coordinates of* P *with respect to* $A_1,\cdots,A_{n+1}$.

Proof. Let

$$u(x) = \sum_{j=1}^{n} \beta_j x_j + \beta_{n+1},$$

be this function. The unknowns are $\beta_1,\cdots,\beta_{n+1}$ which satisfy the following equations asserting that $u(A_i) = \alpha_i$:

$$\sum_{j=1}^{n} \beta_j a_{j,i} + \beta_{n+1} = \alpha_i, \quad 1 \leq i \leq n+1.$$

The matrix of the system is the transposed matrix ${}^t\mathcal{A}$ of $\mathcal{A}$ and thus the function u exists and is unique.

It remains to see that (4.8) is the required function; actually

$$u(A_j) = \alpha_j, \quad 1 \leq j \leq n+1,$$

since $\lambda_i(A_j) = \delta_{ij}$ = the Kronecker symbol, for each i and j.

Remark 4.1.

Higher order interpolation formulas using the barycentric coordinates will be given later (see Sections 4.2, 4.3, 4.4).

Differential Properties.

We give some differential properties of the λ_i considered as functions of the cartesian coordinates $x_1,\cdots,x_n$, of P; here we denote by D the gradient operator $D = (D_1,\cdots,D_n)$.

Lemma 4.1.

(4.9)
$$\sum_{i=1}^{n+1} D\lambda_i = 0$$

(4.10)
$$D\lambda_i(P) \cdot PA_j = \delta_{ij} - \lambda_i(P),\ 1 \leq i,j \leq n+1.$$

Proof.

The identity (4.3) immediately implies (4.9). According to (4.5)

$$\frac{\partial \lambda_i}{\partial x_k} = b_{i,k},\ 1 \leq i \leq n+1,\ 1 \leq k \leq n,$$

and then

$$\begin{aligned} D\lambda_i(P) \cdot PA_j &= D\lambda_i(P) \cdot OA_j - D\lambda_i(P) \cdot OP \\ &= \sum_{k=1}^{n} b_{i,k}a_{k,j} - \sum_{k=1}^{n} b_{i,k}x_k \\ &= \sum_{k=1}^{n} b_{i,k}a_{k,j} + b_{i,n+1} - \lambda_i \\ &= \delta_{ij} - \lambda_i; \end{aligned}$$

for the last equality we note that $\mathcal{B} = \mathcal{A}^{-1}$.

Lemma 4.2.

Let $\mathcal{S}$ be an n-simplex with vertices $A_1, \cdots, A_{n+1}$ and let ρ' be the least upper bound of the diameters of all balls included in $\mathcal{S}$. Then

(4.11)
$$|D\lambda_i| \leq \frac{1}{\rho'},\quad 1 \leq i \leq n+1,$$

where $|D\lambda_i|$ is the euclidian norm of the constant vector $D\lambda_i$.

Proof.

We have

(4.12)
$$|D\lambda_i| = \sup_{\substack{x \in \mathbb{R}^n \\ |x|=1}} D\lambda_i \cdot x.$$

Any vector x, with norm 1, can be written as

$$x = \frac{1}{\rho'} PQ,$$

where P and Q belong to $\mathcal{S}$; denoting by $\mu_1, \cdots, \mu_{n+1}$, the barycentric coordinates of Q with respect to $A_1,\cdots,A_{n+1}$, we have because of (4.2)-(4.3),

$$PQ = \sum_{j=1}^{n+1} \mu_j PA_j,$$

$$\sum_{j=1}^{n+1} \mu_j = 1.$$

Then

$$D\lambda_i \cdot x = \frac{1}{\rho'} (D\lambda_i)\cdot(\sum_{j=1}^{n+1} \mu_j\, PA_j)$$

$$= \frac{1}{\rho'} \sum_{j=1}^{n+1} \mu_j D\lambda_i \cdot PA_j$$

$$= \text{(according to (4.10))}$$

$$= \frac{1}{\rho'} \sum_{j=1}^{n+1} \mu_j(\delta_{ij}-\lambda_i) = \frac{1}{\rho'} (\mu_i-\lambda_i).$$

Since P and Q belong to $\mathcal{S}$, $0 \le \lambda_i \le 1$, $0 \le \mu_i \le 1$ for each i, $1 \le i \le n+1$, and then $-1 \le \mu_i-\lambda_i \le 1$, so that

$$|D\lambda_i \cdot x| \le \frac{1}{\rho'}$$

and (4.11) follows.

Norms of Some Linear Transformations.

Let $\mathcal{S}$ and $\bar{\mathcal{S}}$ be two n-simplices with vertices $A_1,\cdots,A_{n+1}$, and $\bar{A}_1,\cdots,\bar{A}_{n+1}$. We denote by ρ (resp. ρ') the diameter of the smallest ball containing $\mathcal{S}$ (resp. the diameter of largest ball contained in $\mathcal{S}$); with a similar meaning for $\bar{\rho}$, $\bar{\rho'}$.

We can suppose that, up to a translation, $A_1 = \bar{A}_1 = 0$, the origin in $\mathbb{R}^n$, and we denote then by Λ the linear mapping in $\mathbb{R}^n$ such that

$$A_i = \Lambda\, \bar{A}_i, \quad 2 \le i \le n+1. \tag{4.13}$$

The norms of Λ and Λ^{-1} can be majorized as follows in terms of $\rho,\rho',\bar{\rho},\bar{\rho'}$:

Lemma 4.3.

$$\|\Lambda\| \le \frac{\bar{\rho}}{\rho'}, \quad \|\Lambda^{-1}\| \le \frac{\rho}{\bar{\rho'}} \tag{4.14}$$

Proof.

As in the proof of Lemma 4.2, let x be any vector in $\mathbb{R}^n$ with norm 1. Then

$$x = \frac{1}{\rho'} PQ,$$

where P and Q belong to $\mathcal{D}$. It is clear that

$$\Lambda x = \frac{1}{\rho'} \overline{P}\overline{Q},$$

where $\overline{P} = \Lambda P$, $\overline{Q} = \Lambda Q$. But $\overline{P}$ and $\overline{Q}$ belong to $\overline{\mathcal{D}}$ too, since

$$OP = \sum_{i=1}^{n+1} \lambda_i OA_i, \quad 0 \leq \lambda_i \leq 1,$$

implies

$$\Lambda OP = \sum_{i=1}^{n+1} \lambda_i \, O\overline{A}_i,$$

so that the barycentric coordinates of $\overline{P}$ with respect to $\overline{A}_1,\cdots,\overline{A}_{n+1}$, are the same as the barycentric coordinates of P with respect to $A_1,\cdots,A_{n+1}$. Hence $|\overline{P}\overline{Q}| \leq \overline{\rho}$, and

$$|\Lambda x| \leq \frac{\overline{\rho}}{\rho'} .$$

The first inequality (4.14) is proved. The second inequality is obvious when interchanging the role of $\mathcal{D}$ and $\overline{\mathcal{D}}$.

When handling divergence free vector functions, the following lemma will be useful:

Lemma 4.4.

Let $x \mapsto u(x)$ be a divergence free vector function defined on $\mathcal{D}$ (or on $\mathbb{R}^n_x$) and let $\overline{x} \mapsto \overline{u}(\overline{x})$ be defined on $\overline{\mathcal{D}}$ by

$$(4.15) \qquad \overline{u}(\overline{x}) = \Lambda u(\Lambda^{-1}\overline{x}), \ \forall \overline{x} \in \overline{\mathcal{D}} (\text{or } \mathbb{R}^n_{\overline{x}}).$$

Then $\overline{u}$ is a divergence free vector function too.

Proof.

Let (α_{ij}) and $(\beta_{k\ell})$ denote the elements of Λ and Λ^{-1}. Then

$$\frac{\partial \bar{u}_i}{\partial \bar{x}_j}(\bar{x}) = \frac{\partial}{\partial \bar{x}_j} \sum_\ell \alpha_{i\ell} u_\ell (\Lambda^{-1}\bar{x})$$

$$= \sum_{\ell,k} \alpha_{i\ell} \frac{\partial u_\ell}{\partial x_k} \cdot \frac{\partial x_k}{\partial \bar{x}_j}$$

$$= \sum_{\ell,k} \alpha_{i\ell} \beta_{kj} \frac{\partial u_\ell}{\partial x_k} (\Lambda^{-1}\bar{x})$$

and

$$(\text{div } \bar{u})(\bar{x}) = \sum_i \frac{\partial \bar{u}_i}{\partial \bar{x}_i}(\bar{x}) = \sum_{i,k,\ell} \alpha_{i\ell} \beta_{kj} \frac{\partial u_\ell}{\partial x_k} (\Lambda^{-1}\bar{x})$$

$$= \sum_k \frac{\partial u_k}{\partial x_k} (\Lambda^{-1}\bar{x}) = 0.$$

Regular Triangulations of an Open Set Ω.

Let Ω be an open bounded set in $\mathbb{R}^n$.

Let $\mathcal{T}_h$ be a family of n-simplices; such a family will be called an admissible triangulation of Ω if the following conditions are satisfied:

(4.16) $$\Omega(h) = \bigcup_{\mathcal{S} \in \mathcal{T}_h} \mathcal{S} \subset \Omega$$

(4.17) If $\mathcal{S}$ and $\mathcal{S}' \in \mathcal{T}_h$, then $\mathring{\mathcal{S}} \cap \mathring{\mathcal{S}}' = \phi$, (where $\mathring{\mathcal{S}}$ is the interior of $\mathcal{S}$[1]) and, either $\mathcal{S} \cap \mathcal{S}'$ is empty or $\mathcal{S} \cap \mathcal{S}'$ is exactly a whole m-face for both $\mathcal{S}$ and $\mathcal{S}'$ (any m, $0 \leq m \leq n-1$).

We will denote by $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$ the family of all admissible triangulations of Ω; with each admissible triangulation $\mathcal{T}_h$ we associate the following three numbers

(4.18) $$\rho(h) = \sup_{\mathcal{S} \in \mathcal{T}_h} \rho_{\mathcal{S}}$$

(4.19) $$\rho'(h) = \inf_{\mathcal{S} \in \mathcal{T}_h} \rho'_{\mathcal{S}}$$

(4.20) $$\sigma(h) = \sup_{\mathcal{S} \in \mathcal{T}_h} \frac{\rho_{\mathcal{S}}}{\rho'_{\mathcal{S}}}$$

---

(1) i.e., the points of $\mathcal{S}$ with barycentric coordinates, with respect to the vertices of $\mathcal{S}$, satisfying $0 < \lambda_i < 1$, $1 \leq i \leq n+1$.

where, as before, $\rho = \rho_{\mathcal{S}}$ is the diameter of the smallest ball containing $\mathcal{S}$, and $\rho' = \rho'_{\mathcal{S}}$ is the diameter of the greatest ball contained in $\mathcal{S}$.

For finite element methods, we are concerned with passage to the limit $\rho(h) \longrightarrow 0$. It will appear later that some restrictions on $\sigma(h)$ are necessary to get convergent approximations.

A subfamily of the family of admissible triangulations $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$ will be called a <u>regular triangulation</u> of Ω if $\sigma(h)$ remains bounded as $\rho(h) \longrightarrow 0$,

(4.21) $$\sigma(h) \leq \alpha < +\infty, \quad \rho(h) \longrightarrow 0$$

and $\Omega(h)$ converges to Ω in the following sense:

(4.22) For each compact set $K \subset \Omega$, there exists $\delta = \delta(K) > 0$ such that $\rho(h) \leq \delta(K) \Rightarrow \Omega(h) \supset K$.

$\mathcal{H}_\alpha$ will denote the set of admissible triangulations of Ω satsifying (4.21) and (4.22).

<u>Remark 4.2.</u>

In the two dimensional case the 2-simplex is a triangle and it is known that

$$\frac{1}{2 \tan \frac{\theta}{2}} \leq \frac{\rho_{\mathcal{S}}}{\rho'_{\mathcal{S}}} \leq \frac{2}{\sin \theta}$$

where θ is the smallest angle of $\mathcal{S}$.

The condition (4.21) amounts then to saying that the smallest angle of all the triangles $\mathcal{S} \in \mathcal{T}_h$ remains bounded from below

(4.23) $$\theta \geq \theta_0 > 0.$$

Our purpose now will be to associate to a regular family of triangulations $\{\mathcal{T}_h\}_{h \in \mathcal{H}_\alpha}$ of Ω, various types of approximations of the function spaces we are concerned with.

4.2 <u>Finite Elements of Degree</u> 2(n=2).

Let Ω be a lipschitzian open bounded set in $\mathbb{R}^n$. We describe an internal approximation of $\mathbb{H}_0^1(\Omega)$ (any n) and then an external approximation of V (n=2 only). The approximate functions are piecewise polynomials of degree 2.

Approximation of $\mathbb{H}_0^1(\Omega)$.

Let $\mathcal{T}_h$ be any admissible triangulation of Ω.

Space W_h

This is the space of continuous vector functions, which vanish outside $\Omega(h)$

$$(4.24) \qquad \Omega(h) = \bigcup_{\mathcal{S}\in\mathcal{T}_h} \mathcal{S}$$

and whose components are polynomials of degree two [1] on each simplex $\mathcal{S}\in\mathcal{T}_h$.

This space W_h is a finite dimensional subspace of $\mathbb{H}_0^1(\Omega)$. We equip it with the scalar product induced by $\mathbb{H}_0^1(\Omega)$:

$$(4.25) \qquad ((u_h,v_h))_h = ((u_h,v_h)), \quad \forall u_h,v_h \in W_h.$$

A Basis of W_h.

If $\mathcal{S}$ is an n-simplex we denote as before by $A_1,\cdots,A_{n+1}$, the vertices of $\mathcal{S}$; we denote also by A_{ij} the mid-point of A_iA_j.

We first have

Lemma 4.5.

A polynomial of degree less than or equal to two is uniquely defined by its values at the points A_i, A_{ij}, $1 \le i,j \le n+1$ (the vertices and the mid-points of the edges of an n-simplex $\mathcal{S}$).

Moreover, this polynomial is given in terms of the barycentric coordinates with respect to $A_1,\cdots,A_{n+1}$, by the formula:

$$(4.26) \qquad \phi(x) = \sum_{i=1}^{n+1} (2(\lambda_i(x))^2-\lambda_i(x))\phi(A_i) + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \lambda_i(x)\lambda_j(x)\phi(A_{ij}).$$

Proof.

Let us show first that (4.26) satisfies the requirements. The function on the right-hand side of (4.26) is a polynomial of degree two since the $\lambda_i(x)$ are linear nonhomogeneous functions of $x_1,\cdots,x_n$ (see (4.5)). Besides, if $\psi(x)$ denotes this function,

[1] Roughly speaking, a polynomial of degree two means a polynomial of degree less than or equal to two.

$$\psi(A_k) = \phi(A_k) \quad \text{since} \quad \lambda_i(A_k) = \delta_{ik}$$

$$\psi(A_{k\ell}) = \phi(A_{k\ell}) \quad \text{since} \quad \lambda_i(A_{k\ell}) = \frac{\delta_{ik}+\delta_{i\ell}}{2}$$

Thus ψ is as required.

Now, a polynomial of degree two has the form

$$\phi(x) = \alpha_0 + \sum_{i=1}^{n} (\alpha_i x_i + \beta_i x_i^2) + \sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_{ij} x_i x_j . \tag{4.27}$$

and ϕ is defined by $\frac{(n+1)(n+2)}{2}$ unknown coefficients $\alpha_0, \alpha_i, \beta_i, \alpha_{ij}$. There are $(n+1)$ points A_i, $\frac{n(n+1)}{2}$ points A_{ij}, and hence the conditions on ϕ:

$$\phi(A_i) = \text{given}, \quad \phi(A_{ij}) = \text{given}, \tag{4.28}$$

are $\frac{(n+1)(n+2)}{2}$ linear equations for the unknown coefficients. According to (4.26) this system has a solution for any set of data in (4.28); then the linear system is a regular system (1), and the solution found in (4.26) is unique.

Now let us denote by $\mathcal{V}_h$ the set of vertices and mid-edges of the n-simplices $\mathcal{S} \in \mathcal{T}_h$. We denote also by $\mathring{\mathcal{V}}_h$ those points of $\mathcal{V}_h$ which belong to the interior of $\Omega(h)$. According to the preceding lemma there is at most one function u_h in W_h which takes given values at the points $A \in \mathring{\mathcal{V}}_h$. Actually we have more.

<u>Lemma 4.6.</u>

<u>There exists one and only one function u_h in W_h which takes given values at the points $M \in \mathring{\mathcal{V}}_h$.</u>

<u>Proof.</u>

We saw that such a function is necessarily unique. Now, by Lemma 4.5, there exists a function u_h whose components are piecewise polynomials of degree two, which takes given values at the points $M \in \mathring{\mathcal{V}}_h$, and vanish at the points $M \in \mathcal{V}_h - \mathring{\mathcal{V}}_h$ and outside $\Omega(h)$. We just have to check that this function is continuous. On each $(n-1)$-face $\mathcal{S}'$ of a simplex $\mathcal{S} \in \mathcal{T}_h$, each component u_{ih}

---

(1) We use the well-known property that, in finite dimensional spaces, the linear operators which are onto, are one-to-one and onto.

of u_h is a polynomial of degree two which has two (perhaps different) values u_{ih}^+ and u_{ih}^-. But u_{ih}^+ and u_{ih}^- are polynomials of degree less than or equal to two in $(n-1)$ variables, which are equal at the vertices and the mid-points of the edges of $\mathcal{S}'$; then Lemma 4.5 applied to a $(n-1)$ dimensional simplex shows that $u_{ih}^+ = u_{ih}^-$ on $\mathcal{S}'$. Therefore u_h is continuous, and u_h belongs to W_h.

Repeating the argument of the preceding proof, we see that there exists a unique scalar continuous function, which is a polynomial of degree two on each simplex $\mathcal{S}\in\mathcal{T}_h$, and which takes on given values at the points $M\in\mathring{\mathcal{V}}_h$, and vanishes outside $\Omega(h)$. Let us denote by w_{hM} the function of this type defined by

$$(4.29)\qquad w_{hM}(M) = 1,\ w_{hM}(P) = 0,\ \forall P\in\mathring{\mathcal{V}}_h,\ P \neq M,\ (M\in\mathring{\mathcal{V}}_h).$$

Finally, we have

Lemma 4.7.

The functions $w_{hM}e_i$, $M\in\mathring{\mathcal{V}}_h$, $i = 1,\cdots,n$, form a basis of W_h, and the dimension of W_h is $nN(h)$ where $N(h)$ is the number of points in $\mathring{\mathcal{V}}_h$.

Proof.

These functions are linearly independent, and, clearly, each function $u_h\in W_h$ can be written

$$u_h(x) = \sum_{M\in\mathring{\mathcal{V}}_h} \sum_{i=1}^{n} u_{ih}(M)e_i w_{hM}(x)$$

or

$$(4.30)\qquad u_h = \sum_{M\in\mathring{\mathcal{V}}_h} u_h(M)w_{hM}.$$

Operator p_h. The prolongation operator p_h is the identity,

$$(4.31)\qquad p_h u_h = u_h,\ \forall u_h\in W_h.$$

The p_h's have norm one and thus are stable.

Operator r_h. We define $r_h u$ for $u\in\mathcal{D}(\Omega)$; we set:

$$(4.32)\qquad (r_h u)(M) = u(M),\ \forall M\in\mathring{\mathcal{V}}_h.$$

Proposition 4.2.

The preceding internal approximation of $\mathbb{H}_0^1(\Omega)$ is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.

Proof.

We only have to prove that for each $u \in \mathcal{D}(\Omega)$,

$$p_h r_h u \to u \quad \text{in} \quad \mathbb{H}_0^1(\Omega),$$

as $\rho(h) \longrightarrow 0$, $h \in \mathcal{H}_\alpha$.

If h is sufficiently small, $\Omega(h)$ contains the support of u, and then the next lemma will show that

$$\|p_h r_h u - u\| \leq c(u)\rho^2(h)\cdot\sigma(h) \leq c(u)\alpha\rho^2(h), \tag{4.33}$$

and the result follows.

Lemma 4.8.

Let $\mathcal{S}$ be an n-simplex, ϕ a scalar function in $\mathcal{C}^3(\mathcal{S})$, and let $\tilde{\phi}$ be the interpolating polynomial of degree two such that,

$$\tilde{\phi}(A_i) = \phi(A_i), \quad \tilde{\phi}(A_{ij}) = \phi(A_{ij})$$

for $1 \leq i,j \leq n+1$.

Then, we have

$$\sup_{x\in\mathcal{S}} |\phi(x) - \tilde{\phi}(x)| \leq c(\phi)\rho_{\mathcal{S}}^3 \tag{4.34}$$

$$\sup_{x\in\mathcal{S}} \left| \frac{\partial\phi}{\partial x_i}(x) - \frac{\partial\tilde{\phi}}{\partial x_i}(x)\right| \leq c(\phi)\frac{\rho_{\mathcal{S}}^3}{\rho'_{\mathcal{S}}} \tag{4.35}$$

where $c(\phi)$ depends on the maximum norm of the third derivatives of ϕ.

This lemma is a particular case of general theorems concerning polynomial interpolation on a simplex in connection with finite elements.

Polynomial Interpolation on a Simplex.

Let $\mathcal{S}$ be an n-simplex and let $\mathcal{E}$ be a finite set of points having the following property: for any family of given numbers $\gamma_M \in \mathbb{R}$, $M \in \mathcal{E}$, there exists a unique polynomial p of degree less than or equal to k such that

$$p(M) = \gamma_M, \quad \forall M \in \mathcal{E}. \tag{4.36}$$

Such a set $\mathcal{E}$ is called k-unisolvent by Ciarlet-Raviart [1]; for example, according to Proposition 4.1 and Lemma 4.5, the vertices $A_1,\cdots,A_{n+1}$ of $\mathcal{S}$ are 1-unisolvent, the points A_i, A_{ij}, $1 \le i,j \le n+1$, are 2-unisolvent.

Let us denote by p_i the polynomial of degree k such that

$$(4.37) \qquad p_i(M_i) = 1, \quad p_i(M_j) = 0, \quad M_j \ne M_i, \; M_j \in \mathcal{E}.$$

Then the polynomial p in (4.36) can be written as

$$(4.38) \qquad p = \sum_{M_i \in \mathcal{E}} \gamma_{M_i} p_i$$

Now let us suppose that a function ϕ is given, $\phi \in \mathcal{C}^{k+1}(\mathcal{S})$, and let $\tilde{\phi}$ be the interpolating polynomial of degree k defined by

$$(4.39) \qquad \tilde{\phi}(M) = \phi(M), \quad \forall M \in \mathcal{E}$$

i.e.,

$$(4.40) \qquad \tilde{\phi} = \sum_{M_i \in \mathcal{E}} \phi(M_i) p_i .$$

Using Taylor's formula it is proved that for any multi-index $j = (j_1,\cdots,j_n)$ with $[j] = j_1+\cdots+j_n \le k$, one has

$$(4.41) \qquad D^j \tilde{\phi}(P) = D^j \phi(P) + \frac{1}{(k+1)!} \sum_{M_i \in \mathcal{E}} \sum_{[\ell]=k+1} \{D^\ell \phi(P_i) \cdot M_i P^\ell\} D^j p_i(P),$$

where P_i is some point of the open interval (M_i,P),

$$D^\ell = D_1^{\ell_1} \cdots D_n^{\ell_n}; \; M_i P^\ell = \varepsilon_{1i}^{\ell_1} \cdots \varepsilon_{ni}^{\ell_n}$$

for $M_i P = (\varepsilon_{1i},\cdots,\varepsilon_{ni})$, $\ell = (\ell_1,\cdots,\ell_n)$.

The error between ϕ and $\tilde{\phi}$ is majorized on $\mathcal{S}$ by

$$(4.42) \qquad \sup_{x \in \mathcal{S}} |D^j \phi(P) - D^j \tilde{\phi}(P)| \le c\eta_{k+1}(\phi) \frac{\rho_{\mathcal{S}}^{k+1}}{\rho'^{m}_{\mathcal{S}}} ,$$

for $[j] = j_1+\cdots j_n = m \le k$, where $\rho_{\mathcal{S}}$ and $\rho'_{\mathcal{S}}$ are defined in Section 4.1(1).

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(1)
$$\eta_k(\phi) = \sup_x \sup_{[j]=k} \{|D^j \phi(x)|\}.$$

The supremum in x is taken on $\mathcal{S}$; elsewhere when using this notation, the supremum is understood on the whole support of ϕ.

This is a consequence of (4.41) and the following estimation of p_i

(4.43) $$\sup_{x\in\mathcal{S}} |D^j p_i(x)| \leq \frac{c}{\rho_{\mathcal{S}}^{m}} \quad \text{for} \quad [j] = m \leq k.$$

For the proofs of (4.41) and (4.43), the reader is referred to Ciarlet-Raviart [1]; for the particular case of Lemma 4.8, see also Ciarlet-Wagshall [1].

Approximation of V (APX 2)

Here Ω is an open bounded set in $\mathbb{R}^2$; we shall define an external approximation of the space V.

Space F, Operator $\bar{\omega}$.

The space F is $H_0^1(\Omega)$ and $\bar{\omega}$ is the identity

(4.44) $$\bar{\omega}u = u, \quad \forall u \in V;$$

$\bar{\omega}$ is an isomorphism from V into F.

Let $\mathcal{T}_h$ be any admissible triangulation of Ω.

Space V_h.

V_h is a subspace of the space W_h previously defined. It is the space of continuous vector functions which vanish outside

(4.45) $$\Omega(h) = \bigcup_{\mathcal{S}\in\mathcal{T}_h} \mathcal{S}$$

and whose components are polynomials of degree two on each simplex $\mathcal{S}\in\mathcal{T}_h$ such that

(4.46) $$\int_{\mathcal{S}} \operatorname{div} u_h dx = 0, \quad \forall \mathcal{S}\in\mathcal{T}_h.$$

The condition (4.46) is a discrete form of the condition $\operatorname{div} u = 0$. The functions $u_h \in V_h$ belong to $H_0^1(\Omega)$, but not to V, $V_h \not\subset V$. We do not have a simple basis of V_h; according to Lemma 4.7, any function $u_h \in V_h$ can be written as

$$u_h = \sum_{M\in \mathring{\mathcal{V}}_h} u_h(M) w_{hM}$$

but the functions w_{hM} do not belong to V_h. Lemma 4.7 and (4.46) shows also that

$$\dim V_h \leq 2N(h) - N'(h),$$

where $N(h)$ is the number of points in $\mathring{\mathcal{V}}_h$ and $N'(h)$ is the number of triangles $\mathcal{A}\in\mathcal{T}_h$.

We provide the space V_h with the scalar product of $\mathbb{H}_0^1(\Omega)$ (as W_h):

$$((u_h,v_h))_h = ((u_h,v_h)). \tag{4.47}$$

Operator p_h

The operator p_h is the identity (recall that $V_h \subset \mathbb{H}_0^1(\Omega)$). The prolongation operators have norm one and are thus stable.

Operator r_h.

The restriction operators are more difficult to define because of condition (4.46) which must be satisfied by $r_h u$.

Let u be an element of $\mathcal{V}$; we set

$$r_h u = u_h = u_h^1 + u_h^2, \tag{4.48}$$

where u_h^1 and u_h^2 belong separately to W_h; u_h^1 is defined like in (4.32) by

$$u_h^1(M) = u(M), \quad \forall M \in \mathring{\mathcal{V}}_h. \tag{4.49}$$

There is no reason for u_h^1 to belong to V_h, and actually u_h^2 will be a "small corrector" so that $u_h^1+u_h^2 \in V_h$. We define u_h^2 by its values at the points $M\in\mathring{\mathcal{V}}_h$; if $M = A_i$ is the vertex of a triangle then $u_h^2(A_i) = 0$; if $M = A_{ij}$ is the mid-point of an edge, then, letting ν_{ij} denote one of the two unit vectors orthogonal to A_iA_j, we set:

$$\begin{cases} u_h^2(A_{ij})\cdot A_iA_j = 0 \\ u_h^2(A_{ij})\cdot \nu_{ij} \\ \quad = -\{u_h^1(A_{ij}) + \frac{1}{4}\, u_h^1(A_i) + \frac{1}{4}\, u_h^1(A_j)\}\cdot\nu_{ij} + \frac{3}{2}\int_0^1 u(tA_i+(1-t)A_j)\cdot\nu_{ij}\,dt \end{cases} \tag{4.50}$$

<u>Lemma 4.9.</u>

u_h <u>defined by</u> (4.48)(4.49)(4.50) <u>belongs to</u> V_h.

<u>Proof.</u>

The main idea in (4.50) was to choose u_h^2 so that

$$\int_{A_i}^{A_j} u_h\cdot\nu_{ij}\,d\ell = \int_{A_i}^{A_j} u\cdot\nu_{ij}\,d\ell. \tag{4.51}$$

If we show that (4.51) is satisfied, we will then have for any triangle $\mathcal{T}$:

$$\int_{\mathcal{T}} \operatorname{div} u_h \, dx = \int_{\partial\mathcal{T}} u_h \cdot \nu d\ell = \int_{\partial\mathcal{T}} u \cdot \nu d\ell = \int_{\mathcal{T}} \operatorname{div} u \, dx = 0,$$

since $u \in \mathcal{V}$ (ν = unit vector normal to $\partial\mathcal{T}$ pointing outward with respect to $\mathcal{T}$).

Let us prove (4.51). The function u_h^2 is equal to

$$u_h^2 = \sum_{\substack{M \in \mathring{\mathcal{V}}_h \\ M = A_{k\ell}}} u_h^2(M) w_{hM} \tag{4.52}$$

On the segment $\overline{A_i A_j}$, the function $w_{hA_{ij}}$ is the only function w_{hM} in the preceding sum which is not identically equal to 0. By the definition of $w_{hA_{ij}}$ one easily checks that

$$w_{hA_{ij}}(tA_i + (1-t)A_j) = 4t(1-t), \quad 0 < t < 1. \tag{4.53}$$

Likewise

$$u_h^1 = \sum_{M \in \mathring{\mathcal{V}}_h} u_h^1 \quad (M) w_{hM}$$

where the only functions w_{hM} which do not vanish on $A_i A_j$ are $w_{hA_i}, w_{hA_j}, w_{hA_{ij}}$.

It is easily shown that

$$\begin{cases} w_{hA_i}(tA_i + (1-t)A_j) = (t-1)(2t-1) \\ w_{hA_j}(tA_i + (1-t)A_j) = t(2t-1). \end{cases} \tag{4.54}$$

Then

$$\frac{1}{|A_i A_j|} \int_{A_i}^{A_j} u_h(x) \cdot \nu_{ij} d\ell = \int_0^1 u_h(tA_i + (1-t)A_j) \cdot \nu_{ij} dt$$

$$= \frac{2}{3} u_h^2(A_{ij}) \cdot \nu_{ij} + \frac{2}{3} u_h^1(A_{ij}) \cdot \nu_{ij} + \frac{1}{6} \{u_h^1(A_i) + u_h^1(A_j)\} \cdot \nu_{ij}$$

$$= \text{(by (4.50))}$$

$$= \int_0^1 u(tA_i + (1-t)A_j) \cdot \nu_{ij} dt = \frac{1}{|A_i A_j|} \int_{A_i}^{A_j} u(x) \cdot \nu_{ij} d\ell.$$

The lemma is proved.

Proposition 4.3.

The preceding external approximation of V is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.

Proof.

Let us check first the condition (c2) of Definition 3.6.

We have to show that if a sequence $p_{h'}u_{h'}, u_{h'} \in V_{h'}$, converges weakly to ϕ in F, then $\phi = \bar{\omega}u \in V$. According to Theorem 1.6, we need only show that

$$\text{div } u = 0. \tag{4.55}$$

Let θ be any function of $\mathcal{D}(\Omega)$; by (4.46), we have

$$\int_\Omega (\text{div } u_h)\theta_h dx = 0, \tag{4.56}$$

where θ_h is the step function which is equal on each $\mathcal{A} \in \mathcal{T}_h$ to the average value of θ on $\mathcal{A}$, and which vanishes outside $\Omega(h)$. It is easy to see that when support $\theta \subset \Omega(h)$,

$$\sup_{x\in\Omega} |\theta_h(x) - \theta(x)| \leq c(\theta)\rho_h,$$

so that θ_h converges to θ in the L^∞ and L^2 norms; thus we can pass to the limit in (4.56) and obtain

$$\int_\Omega \text{div } u\cdot\theta dx = 0, \quad \forall\theta\in\mathcal{D}(\Omega)$$

This proves (4.55).

The condition (c1) of Definition 3.6 is

$$\lim_{h\to 0} p_h r_h u = \bar{\omega}u, \quad \forall u \in \mathcal{V}. \tag{4.57}$$

This is equivalent to

$$\lim_{h\to 0} \|u - r_h u\| = 0, \quad \forall u \in \mathcal{V}. \tag{4.58}$$

Let us suppose that ρ_h is sufficiently small so that $\Omega(h)$ contains the support of u. Because of Lemma 4.8 and (4.42), on each triangle $\mathcal{A} \in \mathcal{T}_h$:

$$(4.59)\quad \begin{cases} \sup_{x\in \mathcal{S}} |u(x)-u_h^1(x)| \leq cn_3(u)\rho_{\mathcal{S}}^3 \\ \sup_{x\in \mathcal{S}} |D_i u(x)-D_i u_h^1(x)| \leq cn_3(u)\, \dfrac{\rho_{\mathcal{S}}^3}{\rho'_{\mathcal{S}}} \end{cases}$$

By the proof of Lemma 4.9,

$$(4.60)\quad \frac{2}{3}\, u_h^2(A_{ij})\cdot \nu_{ij} = \frac{1}{|A_iA_j|}\int_{A_i}^{A_j} u_h^2(x)\cdot \nu_{ij}\,d\ell$$

$$= \frac{1}{|A_iA_j|}\int_{A_i}^{A_j} [u(x)-u_h^1\ (x)]\cdot \nu_{ij}\,d\ell.$$

$$= \int_0^1 (u-u_h^1)(tA_i+(1-t)A_j)\cdot \nu_{ij}\,dt$$

Hence, with (4.60) estimated by (4.59),

$$(4.61)\quad |u_h^2(A_{ij})| = |u_h^2(A_{ij})\cdot \nu_{ij}| \leq cn_3(u)\rho_{\mathcal{S}}^3.$$

Now by (4.43), we obtain

$$(4.62)\quad \begin{cases} \sup_{x\in \mathcal{S}} |w_{hM}(x)| \leq c \\ \sup_{x\in \mathcal{S}} |D_i w_{hM}(x)| \leq \dfrac{c}{\rho'_{\mathcal{S}}}, \quad i = 1,\cdots,n \end{cases}$$

Next, combining (4.61) - (4.62) with (4.52), we get

$$\begin{cases} \sup_{x\in \mathcal{S}} |u_h^2(x)| \leq cn_3(u)\rho_{\mathcal{S}}^3 \\ \sup_{x\in \mathcal{S}} |D_i u_h^2(x)| \leq cn_3(u)\, \dfrac{\rho_{\mathcal{S}}^3}{\rho'_{\mathcal{S}}}. \end{cases}$$

Finally, combining (4.59) and last inequalities, it follows that

$$(4.63)\quad \begin{cases} \sup_{x\in \Omega} |u(x)-u_h(x)| \leq cn_3(u)\rho(h)^3 \\ \sup_{x\in \Omega} |D_i u(x)-D_i u_h(x)| \leq cn_3(u)\rho(h)^2\sigma(h) \leq cn_3(u)\alpha\rho(h)^2. \end{cases}$$

The proof is achieved.

Remark 4.3.

If Ω is a polygon, it is possible to choose the triangulation $\mathcal{T}_h$ such that $\Omega(h) = \Omega$, and this is usually done in practical computations. In this case we can extend the preceding computation to any $u \in \mathbb{H}_0^1(\Omega) \cap \mathbb{C}^3(\overline{\Omega})$ and we find

(4.64) $$\|u - r_h u\| < c\eta_3(u)\sigma(h)\rho(h)^2.$$

Approximation of Stokes Problem.

Using the preceding approximation of V and Section 3.2 we can propose a finite element scheme for the approximation of a two-dimensional Stokes problem.

Let us take in (3.6)

(4.65) $$a_h(u_h, v_h) = \nu((u_h, v_h)),$$

(4.66) $$< \ell_h, v_h > = (f, v_h)$$

where ν and f are given as in Section 2.1 (see Theorem 2.1).

The approximate problem (3.6) is then

(4.67) $$\begin{cases} \underline{\text{To find}}\ u_h \in V_h\ \underline{\text{such that}} \\ \nu((u_h, v_h)) = (f, v_h),\ \forall v_h \in V_h. \end{cases}$$

The solution u_h of (4.67) exists and is unique; moreover, we have

Proposition 4.4.

If $\rho(h) \longrightarrow 0$, with $\sigma(h) \le \alpha$ (i.e., $h \in \mathcal{H}_\alpha$), the solution u_h of (4.67) converges to the solution u of (2.6) in the $\mathbb{H}_0^1(\Omega)$ norm.

Proof.

It is easy to see that Theorem 3.1 is applicable, and the conclusion gives exactly the convergence result announced.

Approximation of the Pressure.

We introduce the approximation of the pressure, as in Section 3.3.

The form

$$v_h \longmapsto \nu((u_h, v_h)) - (f, v_h)$$

is a linear form on W_h, which vanishes on V_h. Since V_h is characterized by the set of linear constraints (4.46), we know that there exists a family of numbers $\lambda_{\mathcal{S}}$, $\mathcal{S} \in \mathcal{T}_h$, which are the Lagrange multipliers associated with the constraints (4.46), such that

$$\nu((u_h, v_h)) - (f, v_h) = \sum_{\mathcal{S} \in \mathcal{T}_h} \lambda_{\mathcal{S}} \left(\int_{\mathcal{S}} \operatorname{div} v_h dx \right), \quad \forall v_h \in W_h.$$

Let $\chi_{h\mathcal{S}}$ denote the characteristic function of $\mathcal{S}$ and let π_h denote the step function

$$\text{(4.69)} \quad \begin{cases} \pi_h = \sum_{\mathcal{S} \in \mathcal{T}_h} \pi_h(\mathcal{S}) \chi_{h\mathcal{S}} \\ \pi_h(\mathcal{S}) = \dfrac{\lambda_{\mathcal{S}}}{(\text{meas } \mathcal{S})} \end{cases}$$

We then have

$$\text{(4.69)} \qquad \nu((u_h, v_h)) - (\pi_h, \operatorname{div} v_h) = (f, v_h), \quad \forall v_h \in W_h,$$

which is the discrete analog of equation

$$\text{(4.70)} \qquad \nu((u,v)) - (p, \operatorname{div} v) = (f,v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

Remark 4.4.

Since no basis of V_h is available, the solution of (4.67) is not easy. The computation can be effected by the algorithms studied in Section 5.

The Error between u and u_h.

Let us suppose that Ω has a polygonal boundary $(\Omega \subset \mathbb{R}^2)$ and that $u \in \mathcal{C}^3(\bar{\Omega})$ and $p \in \mathcal{C}^1(\bar{\Omega})$. Then according to Remark 4.3,

$$\text{(4.71)} \qquad \|u - r_h u\| \leq c(u, \alpha) \rho(h)^2.$$

We can take $v = v_h = u_h - r_h u$ in (4.69) and (4.70); subtracting then (4.70) from (4.69) there remains

$$\text{(4.72)} \qquad \nu((u_h - u, u_h - r_h u)) = (\pi_h - p, \operatorname{div}(u_h - r_h u)).$$

Let π'_h denote the step function

$$\pi'_h = \sum_{\mathcal{S} \in \mathcal{T}_h} \frac{1}{(\text{meas } \mathcal{S})} \left(\int_{\mathcal{S}} p(x) dx \right) \chi_{h\mathcal{S}}.$$

Then the right-hand side of (4.72) is equal to

$$(\pi'_h - p,\ \operatorname{div}(u_h - r_h u))$$

and is majorized by

$$|\pi'_h - p| \cdot |\operatorname{div}(u_h - r_h u)| \le \sqrt{2}|\pi'_h - p| \|u_h - r_h u\|.$$

Hence

$$\nu((u_h - u, u_h - r_h u)) \le \sqrt{2}|\pi'_h - p| \|u_h - r_h u\|,$$

$$\nu\|u_h - r_h u\|^2 \le \{\sqrt{2}|\pi'_h - p| + \nu\|u - r_h u\|\}\|u_h - r_h u\|$$

(4.73) $$\|u_h - r_h u\| \le \frac{\sqrt{2}}{\nu}|\pi'_h - p| + \|u - r_h u\|$$

It is easy to see that $|\pi'_h - p|$ is majorized by $c\eta_1(p)\rho(h)$ and then we have at least

(4.74) $$\|u_h - r_h u\| \le c\eta_1(u,p)\rho(h).$$

This estimation is perhaps not optimal.

When the boundary of Ω is not a polygon, an additional error of order $\rho(h)$ appears in the right-hand side of (4.74).

4.3 <u>Finite Elements of Degree</u> $3(n = 3)$.

Let Ω be a lipschitzian open bounded set in $\mathbb{R}^3$. We describe first an internal approximation of $\mathbb{H}_0^1(\Omega)$ and then an external approximation of V. The approximate functions are piecewise polynomials of degree 3.

<u>Approximation of</u> $\mathbb{H}_0^1(\Omega)$.

Let $\mathcal{T}_h$ be an admissible triangulation of Ω and let

(4.75) $$\Omega(h) = \bigcup_{\mathcal{S} \in \mathcal{T}_h} \mathcal{S}.$$

If $\mathcal{S}$ is a 3-simplex (i.e., a tetrahedron) we denote by $A_1, \cdots, A_4$, the vertices of $\mathcal{S}$ and by $B_1, \cdots, B_4$, the barycenter of the 2-faces $\mathcal{S}'_1, \cdots, \mathcal{S}'_4$. We denote by $\mathcal{E}_h^1$ the set of vertices of the simplices $\mathcal{S} \in \mathcal{T}_h$ and by $\mathcal{E}_h^2$ the set of barycenters of the 2-faces of the simplices $\mathcal{S}$ belonging to $\mathcal{T}_h$; $\mathcal{E}_h = \mathcal{E}_h^1 \cup \mathcal{E}_h^2$.

We first prove the following result.

Lemma 4.10.

A polynomial of degree three in $\mathbb{R}^3$ is uniquely defined by its values at the points A_i, B_i, $1 \leq i \leq 4$, and the values of its first derivatives at the points A_i. Moreover, the polynomial is given in terms of the barycentric coordinates with respect to $A_1, \cdots, A_4$, by the formula

$$(4.76) \quad \phi(x) = \sum_{i=1}^{4} [-2(\lambda_i(x))^3 + 3(\lambda_i(x))^2]\phi(A_i)$$
$$+ \frac{1}{6} \sum_{i=1}^{4} \frac{\lambda_1(x)\cdots\lambda_4(x)}{\lambda_i(x)} [27\phi(B_i) - 7 \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{4} \phi(A_\alpha)]$$
$$+ \sum_{\substack{i,j=1 \\ i\neq j}}^{4} (\lambda_i(x))^2 \lambda_j(x) [D\phi(A_i)\cdot A_i A_j]$$
$$- \sum_{i=1}^{4} \frac{\lambda_1(x)\cdots\lambda_4(x)}{\lambda_i(x)} [D\phi(A_i)\cdot A_i A_j].$$

Proof.

The proof is exactly the same as the proof of Lemma 4.5. The coefficients of ϕ are the solutions of a linear system with as many equations as unknowns; we just have to check that the polynomial on the right-hand side of (4.76) fulfills all the required conditions for any set of given data $\phi(A_i)$, $\phi(B_i)$, $D\phi(A_i)$.

It follows from this lemma that a scalar function ϕ_h which is defined on $\Omega(h)$ and is a polynomial of degree three on each simplex $\mathcal{S} \in \mathcal{T}_h$ is completely known if the values of ϕ_h are given at the points $A_i \in \mathcal{E}_h^1$ and $B_i \in \mathcal{E}_h^2$ and also the values of $D\phi_h$ are given at the points $A_i \in \mathcal{E}_h^1$. Such a function ϕ_h is differentiable on each $\mathcal{S}$, $\mathcal{S} \in \mathcal{T}_h$, but there is no reason for such a function to be differentiable or even continuous in all of $\Omega(h)$. Actually, this function ϕ_h is at least continuous: on a two face $\mathcal{S}'$ of a tetrahedron $\mathcal{S} \in \mathcal{T}_h$, ϕ_h has two, perhaps different, values ϕ_h^+ and ϕ_h^-; but ϕ_h^+ and ϕ_h^- are polynomials of degree three which take the same values at the vertices and at the barycenter of $\mathcal{S}'$; the first derivatives of ϕ_h^+ and ϕ_h^- at the vertices of $\mathcal{S}'$ are also equal (these are the derivatives of ϕ_h which are tangential with respect to $\mathcal{S}'$). Then it can be proved exactly as in Lemmas 4.5 and 4.6 that $\phi_h^+ = \phi_h^-$.

We denote then by $w_{hM}, M \in \mathcal{E}_h$, the scalar function which is a piecewise polynomial of degree three on $\Omega(h)$ with

$$(4.77)\qquad \begin{cases} w_{hM}(M) = 1, \quad w_{hM}(P) = 0 \quad \forall P \in \mathcal{E}_h \quad P \neq M \\ Dw_{hM}(P) = 0, \quad \forall P \in \mathcal{E}_h^1. \end{cases}$$

For $M \in \mathcal{E}_h$, $i = 1,2,3$, $w_{hM}^{(i)}$ is the scalar function which is a piecewise polynomial of degree three on $\Omega(h)$ such that

$$(4.78)\qquad \begin{cases} w_{hM}^{(i)}(P) = 0, \quad \forall P \in \mathcal{E}_h \\ Dw_{hM}^{(i)}(P) = 0 \quad \forall P \in \mathcal{E}_h^1, \quad P \neq M \\ Dw_{hM}^{(i)}(M) = e_i, \quad i = 1,2,3. \end{cases}$$

All of the functions w_{hM}, $w_{hM}^{(i)}$ are continuous on $\Omega(h)$.

Space W_h

The space W_h is the space of continuous vector functions u_h from Ω into $\mathbb{R}^3$, of type

$$(4.79)\qquad u_h = \sum_{M \in \mathcal{E}_h} u_h(M) w_{hM} + \sum_{M \in \mathcal{E}_h^1} \sum_{i=1}^{3} D_i u_h(M) w_{hM}^{(i)},$$

which vanish outside $\Omega(h)$.

It is clear that $u_h(M) = 0$ for any $M \in \mathcal{E}_h \cup \partial\Omega(h)$; but since the tangential derivatives of u_h vanish on the faces of the tetrahedrons  which are included in $\Omega(h)$, the derivatives $D_i u_h(M)$, $M \in \mathcal{E}_h^1 \cup \partial\Omega(h)$ are not independent.

The space W_h is a finite dimensional subspace of $\mathbb{H}_0^1(\Omega)$; we provide it with the scalar product induced by $\mathbb{H}_0^1(\Omega)$

$$(4.80)\qquad ((u_h, v_h))_h = ((u_h, v_h)), \quad \forall u_h, v_h \in W_h.$$

Operator p_h.

The prolongation operator p_h is the identity; the p_h are stable.

Operator r_h.

For $u \in \mathcal{D}(\Omega)$, we define $u_h = r_h u$ by $u_h = 0$ if the support of u is not included in $\Omega(h)$, and if the support is included in $\Omega(h)$,

$$(4.81)\qquad \begin{cases} u_h(M) = u(M), & \forall M \in \mathcal{E}_h, \\ Du_h(M) = Du(M), & \forall M \in \mathcal{E}_h^1. \end{cases}$$

<u>Proposition 4.5.</u>

<u>The preceding internal approximation of</u> $\mathbb{H}_0^1(\Omega)$ <u>is stable and convergent, provided</u> h <u>belongs to a regular triangulation</u> $\mathcal{H}_\alpha$ <u>of</u> Ω.

<u>Proof.</u>

We only have to prove that for each $u \in \mathcal{D}(\Omega)$,

$$u_h = r_h u \longrightarrow u \quad \text{in} \quad \mathbb{H}_0^1(\Omega)$$

as $\rho(h) \longrightarrow 0$, $h \in \mathcal{H}_\alpha$.

This is proved like Proposition 4.2. The analog of (4.42) for Hermite type interpolation polynomials (see Ciarlet-Raviart [1]) shows that

$$(4.82)\qquad \begin{cases} \sup\limits_{x \in \mathcal{S}} |u(x)-u_h(x)| \le c\eta_4(u)\rho^4_{\mathcal{S}} \\ \sup\limits_{x \in \mathcal{S}} |Du(x)-Du_h(x)| \le c\eta_4(u)\, \dfrac{\rho^4_{\mathcal{S}}}{\rho'_{\mathcal{S}}} \end{cases}$$

Hence

$$(4.83)\qquad \begin{cases} \sup\limits_{x \in \Omega} |u(x)-u_h(x)| \le c(u)\rho^4(h) \\ \sup\limits_{x \in \Omega} |Du(x)-Du_h(x)| \le c(u)\rho^3(h)\sigma(h) \end{cases}$$

and in particular

$$(4.84)\qquad \|u-u_h\| \le \alpha c(u)\rho^3(h)$$

provided supp $u \subset \Omega(h)$.

<u>Approximation of</u> <u>V</u> (APX 3).

We recall that Ω is a lipschitzian bounded set in $\mathbb{R}^3$.

<u>Space F, Operator</u> $\overline{\omega}$.

The space F is $\mathbb{H}_0^1(\Omega)$ and $\overline{\omega}$ is the identity.

Let $\mathcal{T}_h$ be an admissible triangulation of Ω.

<u>Space</u> V_h.

V_h is a subspace of the previous space W_h; it is the space of $u_h \in W_h$

such that

$$\int_{\mathcal{S}} \operatorname{div} u_h dx = 0, \ \forall \mathcal{S} \in \mathcal{T}_h. \tag{4.85}$$

We provide the space V_h with the scalar product (4.80) induced by $\mathbb{H}_0^1(\Omega)$ and W_h.

Operator p_h.

This operator is the identity (recall that $V_h \subset \mathbb{H}_0^1(\Omega)$).

Operator r_h.

The construction of r_h is based on the same principle as in Section 4.2.

Let u be an element of $\mathcal{V}$; we set

$$r_h u = u_h^1 + u_h^2 \tag{4.86}$$

where u_h^1 and u_h^2 separately belong to W_h; u_h^1 is defined exactly as in (4.81)

$$\begin{cases} u_h^1(M) = u(M), \ \forall M \in \mathcal{E}_h \\ Du_h^1(M) = Du(M), \ \forall M \in \mathcal{E}_h^1 \end{cases} \tag{4.87}$$

The corrector u_h^2 is defined by

$$u_h^2(M) = 0, \ Du_h^2(M) = 0, \ \forall M \in \mathcal{E}_h^1 \tag{4.88}$$

and at the points $M \in \mathcal{E}_h^2$, the component of $u_h^2(M)$ which is tangent to the face $\mathcal{S}'$ whose M is the barycenter, is zero; the normal components $u^2{}_h(M) \cdot \nu$ is characterized by the condition that

$$\int_{\mathcal{S}'} u_h(x) \cdot \nu d\Gamma = \int_{\mathcal{S}'} u(x) \cdot \nu d\Gamma \tag{4.89}$$

One can prove [(1)] that there exists some constant d such that

$$\int_{\mathcal{S}'} u_h^2(x) \cdot \nu d\Gamma = d \cdot (\text{area } \mathcal{S}') u_h^2(M) \cdot \nu$$

and (4.89) means that

$$u_h^2(M) \cdot \nu = \frac{1}{d \cdot (\text{area } \mathcal{S}')} \int_{\mathcal{S}'} (u - u_h^1)(x) \cdot \nu d\Gamma. \tag{4.90}$$

---

(1) The principle of the proof is similar to that of Lemma 4.9.

It is clear then that u_h belongs to V_h since, for each $\mathcal{S} \in \mathcal{T}_h$,

$$\int_{\mathcal{S}} \operatorname{div} u_h dx = \int_{\partial \mathcal{S}} u_h \cdot \nu d\Gamma = \int_{\partial \mathcal{S}} u \cdot \nu d\Gamma = \int_{\mathcal{S}} \operatorname{div} u \, dx = 0 \quad .$$

Proposition 4.6.

The preceding external approximation of V is stable and convergent, provided h belongs to a set of regular triangulations $\mathcal{H}_\alpha$ of Ω.

The proof of this proposition follows the same lines as the proof of Proposition 4.3.

The approximation of Stokes problem can then be studied exactly as in Section 4.2.

4.4 An Internal Approximation of V.

We suppose here that Ω is a bounded and simply-connected open subset of $\mathbb{R}^2$ with a lipschitzian boundary.

In the two dimensional case, the condition $\operatorname{div} u = 0$ is

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{4.91}$$

and implies that there exists a function ψ (the stream function), such that

$$u_1 = \frac{\partial \psi}{\partial x_2}, \quad u_2 = -\frac{\partial \psi}{\partial x_1}; \tag{4.92}$$

the function ψ exists locally for any set Ω, and globally for a simply connected set Ω.

In the present case we can associate to each function u in V the corresponding stream function ψ. The condition $u = 0$ on $\partial\Omega$ amounts to saying that the tangential and normal derivatives of ψ on $\partial\Omega$ vanish. Then ψ is constant on $\partial\Omega$ and since ψ is only defined up to an additive constant, we can suppose that $\psi = 0$ on Γ and hence $\psi \in H_0^2(\Omega)$.

Therefore the mapping

$$\psi \longrightarrow u = \{ \frac{\partial \psi}{\partial x_2} , - \frac{\partial \psi}{\partial x_1} \} \tag{4.93}$$

is an isomorphism from $H_0^2(\Omega)$ onto V.

Our purpose is now to construct an approximation of $H_0^2(\Omega)$ by piecewise polynomial functions of degree 5 and then to obtain with the isomorphism (4.93) an internal approximation of V.

Internal Approximation of $H_0^2(\Omega)$.

Let $\mathcal{T}_h$ be an admissible triangulation of Ω, and let

(4.94) $$\Omega(h) = \bigcup_{\Delta \in \mathcal{T}_h} \Delta.$$

A 2-simplex is a triangle. Let Δ be some triangle with vertices A_1, A_2, A_3; we denote by B_1, B_2, B_3, or by A_{23}, A_{13}, A_{12}, the mid-points of the edges A_2A_3, A_1A_3, and A_1A_2; ν_{ij} denotes one of the unit vectors normal to the edge A_iA_j, $1 \leq i,j \leq 3$.

We first notice the following result:

Lemma 4.11.

A polynomial ϕ of degree 5 in $\mathbb{R}^2$ is uniquely defined by the following values of ϕ and its derivatives

(4.95) $$D^\alpha \phi(A_i), \quad 1 \leq i \leq 3, \quad [\alpha] \leq 2,$$

(4.96) $$\frac{\partial \phi}{\partial \nu_{ij}}(A_{ij}), \quad 1 \leq i,j \leq 3, \quad i \neq j,$$

where the A_i are the vertices of a triangle Δ and the A_{ij} are the mid-points of the edges.

Principle of the Proof.

We see that there are as many unknowns (21 coefficients for ϕ) as linear equations for these unknowns (the 21 conditions corresponding to (4.95)-(4.96)).

As in Lemmas 4.5 and 4.10 it is then sufficient to show that a solution does in fact exist for any set of data in (4.95)-(4.96) and this can be proved by an explicit construction of ϕ leading to a formula similar to (4.26) or (4.76). We omit the very technical proof of this point which can be found in A. Ženíšek [1] or M. Zlámal [1] (1)

It follows from this lemma that a scalar function ψ_h which is defined on $\Omega(h)$ and is a polynomial of degree five on each triangle $\Delta \in \mathcal{T}_h$, is completely known if the values of ψ_h are given at the points $A_i \in \mathcal{E}_h^1$, $B_i \in \mathcal{E}_h^2$,

(1) The principle of the construction is the following: let $\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates with respect to A_1, A_2, A_3. The affine mapping $x \mapsto y = (\lambda_1(x), \lambda_2(x))$, maps the triangle Δ, on the triangle $\hat{\Delta}$:

$$y_1 = \lambda_1 \geq 0, \; y_2 = \lambda_2 \geq 0, \; 0 \leq y_1 + y_2 = \lambda_1 + \lambda_2 \leq 1.$$

The construction of $\phi(x(y))$ on $\hat{\Delta}$ is elementary; then using the inverse mapping $y \mapsto x$, we obtain the function $\phi(x)$.

and if also the values of the first and second derivatives are given at the points $A_i \in \mathcal{E}_h^1$, where

$$(4.97)\quad \begin{cases} \mathcal{E}_h^1 = \text{set of vertices of the triangles } \Delta \in \mathcal{T}_h. \\ \mathcal{E}_h^2 = \text{set of mid-points of the edges of the triangles } \Delta \in \mathcal{T}_h. \\ \mathcal{E}_h = \mathcal{E}_h^1 \cup \mathcal{E}_h^2. \end{cases}$$

Such a function ϕ_h is infinitely differentiable on each Δ, $\Delta \in \mathcal{T}_h$, but there is no reason for such a function to be so smooth in all of $\Omega(h)$. Actually, the function ϕ_h is continuously differentiable in $\Omega(h)$. Let ϕ_h^+ and ϕ_h^- denote the values of ϕ_h on two sides of the edge A_1A_2 of a triangle $\Delta \in \mathcal{T}_h$; ϕ_h^+, ϕ_h^- are polynomials of degree less than or equal to 5 on A_1A_2 and they are equal together with their first and second derivatives at the points A_1 and A_2 (six independent conditions) and hence $\phi_h^+ = \phi_h^-$. The tangential derivatives $\frac{\partial \phi_h^+}{\partial \tau}$ and $\frac{\partial \phi_h^-}{\partial \tau}$, $\tau = \frac{A_1A_2}{|A_1A_2|}$ are also necessarily equal. Let us show then that the normal derivatives $\frac{\partial \phi_h^+}{\partial \nu_{12}}$ and $\frac{\partial \phi_h^-}{\partial \nu_{12}}$ are equal on A_1A_2. These derivatives are polynomials of degree less than or equal to 4 on A_1A_2; they are equal at A_1 and A_2 together with their first derivatives, and they are equal at A_{12}. Therefore they are equal on A_1A_2. This shows that ϕ_h is continuously differentiable on $\Omega(h)$.

To each point $M \in \mathcal{E}_h^2$ we associate the function ψ_{hM}^0 which is a piecewise polynomial of degree 5 on $\Omega(h)$ and such that,

$$(4.98)\quad \begin{cases} \frac{\partial}{\partial \nu} \psi_{hM}^0(M) = 1 \text{ and all the other nodal values of } \psi_{hM}^0 \\ \text{are zero, i.e.,} \\ \frac{\partial \psi_{hM}^0}{\partial \nu}(P) = 0, \ \forall P \in \mathcal{E}_h^2, \ P \neq M, \\ D^\alpha \psi_{hM}^0(P) = 0, \ \forall P \in \mathcal{E}_h^1, \ [\alpha] \leq 2. \end{cases}$$

To each point $M \in \mathcal{E}_h^1$, we associate the six functions $\psi_{hM}^1, \cdots, \psi_{hM}^6$ defined as follows: they are piecewise polynomials of degree 5 on $\Omega(h)$ and

$$(4.99)\quad \begin{cases} \psi_{hM}^1(M) = 1, \text{ all the other nodal values of } \psi_{hM}^1 \text{ are zero} \end{cases}$$

$$(4.100)\quad \begin{cases} \text{for } i = 1 \text{ or } 2, \ D_j \psi_{hM}^{i+1}(M) = \delta_{ij}, \text{ and all the other nodal values} \\ \text{of } \psi_{hM}^{i+1} \text{ are zero} \end{cases}$$

$$(4.101)\quad \begin{cases} D_1^2\psi_{hM}^4(M) = 1, \quad D_1D_2\psi_{hM}^5(M) = 1, \quad D_2^2\psi_{hM}^6(M) = 1, \\ \text{and all the other nodal values of } \psi_{hM}^4 \\ (\text{resp. } \psi_{hM}^5, \text{ resp. } \psi_{hM}^6) \text{ are zero.} \end{cases}$$

All these functions are continuously differentiable on $\Omega(h)$.

Space X_h.

The space X_h is the space of continuously differentiable scalar functions on Ω (or $\mathbb{R}^2$) of type:

$$(4.102)\qquad \psi_h = \sum_{M\in\mathcal{E}_h^2} \xi_M^0\psi_{hM}^0 + \sum_{i=1}^{6} \sum_{M\in\mathcal{E}_h^1} \xi_M^i\psi_{hM}^i, \quad \xi_M^j \in \mathbb{R}.$$

These functions vanish outside $\Omega(h)$, and since they are continuously differentiable in Ω,

$$(4.103)\qquad \begin{cases} D^\alpha\psi_h(M) = 0, \quad \forall M\in\mathcal{E}_h^1 \cap \partial\Omega(h), \quad [\alpha] \le 1, \\ \dfrac{\partial\psi_h}{\partial\nu}(M) = 0, \quad \forall M\in\mathcal{E}_h^2 \cap \partial\Omega(h). \end{cases}$$

The space X_h is a finite dimensional subspace of $H_0^2(\Omega)$; we provide it with the scalar product induced by $H_0^2(\Omega)$

$$(4.104)\qquad ((\psi_h,\phi_h))_h = ((\psi_h,\phi_h))_{H_0^2(\Omega)}, \quad \forall\psi_h,\phi_h \in X_h.$$

Operator p_h.

p_h = the identity, as $X_h \subset H_0^2(\Omega)$.

Operator r_h.

For $\psi\in\mathcal{D}(\Omega)$ (a dense subspace of $H_0^2(\Omega)$), we define $r_h\psi = \psi_h$ by its nodal values

$$(4.105)\qquad \begin{cases} D^\alpha\psi_h(M) = D^\alpha\psi(M), \quad \forall M\in\mathcal{E}_h^1, \quad [\alpha] \le 2 \\ \dfrac{\partial\psi_h}{\partial\nu_{ij}}(A_{ij}) = \dfrac{\partial\psi}{\partial\nu_{ij}}(A_{ij}), \quad \forall A_{ij}\in\mathcal{E}_h^2. \end{cases}$$

Proposition 4.7.

(X_h,p_h,r_h) defines a stable and convergent internal approximation of $H_0^2(\Omega)$, provided h belongs to a regular triangualtion $\mathcal{H}_\alpha$ of Ω.

Proof.

We only have to prove that, for each $\psi \in \mathcal{D}(\Omega)$,

(4.106) $$r_h\psi \longrightarrow \psi \quad \text{in} \quad H_0^2(\Omega), \quad \text{as} \quad \rho(h)\longrightarrow 0.$$

This follows from an analog of (4.42) and (4.82) for Hermite type polynomial interpolation (see Ciarlet-Raviart [1], G. Strang and G. Fix [1], A. Ženíšek [1], M. Zlámal [1]):

(4.107) $$\sup_{x\in\mathcal{A}} |\psi_h(x)-\psi(x)| \leq c\eta_6(\psi)\rho^5_{\mathcal{A}}$$

(4.108) $$\sup_{x\in\mathcal{A}} |D_i\psi_h(x)-D_i\psi(x)| \leq c\eta_6(\psi)\frac{\rho^5_{\mathcal{A}}}{\rho'_{\mathcal{A}}}, \quad i = 1,2$$

(4.109) $$\sup_{x\in\mathcal{A}} |D^\alpha\psi_h(x)-D^\alpha\psi(x)| \leq c\eta_6(\psi)\frac{\rho^5_{\mathcal{A}}}{\rho'^2_{\mathcal{A}}}, \quad [\alpha] = 2.$$

Therefore

(4.110) $$\|\psi_h-\psi\|_{H_0^2(\Omega(h))} \leq c(\psi)\alpha^2\rho(h)^3$$

and it is clear by (4.22) that

(4.111) $$\|\psi\|_{H_0^2(\Omega-\Omega(h))} \longrightarrow 0 \quad \text{as} \quad \rho(h)\longrightarrow 0.$$

Internal Approximation of V (APX 4)

We recall that Ω is a bounded simply connected set of $\mathbb{R}^2$. We define an internal approximation of V, using the preceding approximation of $H_0^2(\Omega)$ and the isomorphism (4.93).

Let there be given an admissible triangulation $\mathcal{T}_h$ of Ω. We associate with $\mathcal{T}_h$, the space V_h, and the operators p_h, r_h, as follows.

Space V_h.

It is the space of continuous vector functions u_h defined on Ω (or $\mathbb{R}^2$), of type

(4.112) $$u_h = \{ \frac{\partial\psi_h}{\partial x_2}, -\frac{\partial\psi_h}{\partial x_1} \},$$

ψ_h belonging to the previous space X_h.

It is clear that u_h vanishes outside $\Omega(h)$ and is continuous since ψ_h is continuously differentiable, and that $\operatorname{div} u_h = 0$. Therefore $u_h \in V$, and V_h is a finite dimensional subspace of V. We provide it with the scalar product induced by V

(4.113) $$((u_h,v_h))_h = ((u_h,v_h)).$$

In particular,

(4.114) $$\|u_h\|_h = \left(\sum_{[\alpha]=2} |D^\alpha \psi_h|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \leq \|\psi_h\|_{H_0^2(\Omega)}$$

Operator p_h: the identity.

Operator r_h:

Let u belong to $\mathcal{V}$, and let ψ denote the corresponding stream function (see (4.93)); clearly, $\psi \in \mathcal{D}(\Omega)$ and we can define $\psi_h \in X_h$ by (4.105). Then we set

(4.115) $$u_h = r_h u = \{ \frac{\partial \psi_h}{\partial x_2} , - \frac{\partial \psi_h}{\partial x_1} \} \in V_h.$$

Proposition 4.8.

The preceding internal approximation of V is stable and convergent if $\rho(h) \to 0$, with h belonging to a regular triangulation $\mathcal{T}_\alpha$ of Ω.

Proof.

We just have to show that

(4.116) $$u_h = r_h u \to u \quad \text{in} \quad V, \quad \forall u \in \mathcal{V}.$$

According to (4.93),(4.114),(4.115), we have

$$\|u_h - u\| \leq \|\psi_h - \psi\|_{H_0^2(\Omega)}$$

The convergence (4.116) follows then from (4.110) and (4.111).

Approximation of Stokes Problem.

We take for (3.6),

(4.117) $$a_h(u_h,v_h) = \nu((u_h,v_h))_h = \nu((u_h,v_h))$$

(4.118) $$\langle \ell_h, v_h \rangle = \langle f, v_h \rangle,$$

where ν and f are given as in Section 2.1 (see Theorem 2.1).

The approximate problem associated with (2.6) is

$$(4.119)\quad \begin{cases} \underline{\text{To find}}\ \ u_h \in V_h\ \ \underline{\text{such that}} \\ \nu((u_h,v_h)) = \langle f,v_h\rangle,\ \ \forall v_h \in V_h. \end{cases}$$

The solution u_h of (4.119) exists and is unique and, according to Theorem 3.1 and Proposition 4.8,

$$(4.120)\qquad u_h \longrightarrow u \ \text{ in } V \text{ strongly if } \rho(h)\longrightarrow 0,\ \ h\in\mathcal{H}_\alpha.$$

<u>The Error between u and u_h.</u>

Let us suppose that Ω has a polygonal boundary, so that we can choose triangualtions $\mathcal{T}_h$ such that $\Omega(h) = \Omega$. Let us suppose that the solution u of Stokes problem is so smooth that $u\in\mathcal{C}^5(\Omega)$; then, by (4.110) and (4.114), (1)

$$(4.121)\qquad \|u-r_hu\| \le c(u,\alpha)\rho(h)^3.$$

The equations

$$\nu((u,v)) = \langle f,v\rangle,\ \ \forall v\in V,$$

$$\nu((u_h,v_h)) = \langle f,v_h\rangle,\ \ \forall v_h\in V,$$

give

$$\nu((u-u_h,r_hu-u_h)) = 0.$$

Therefore,

$$\|u-u_h\|^2 = ((u-u_h,u-r_hu)) \le \|u-u_h\|\ \|u-r_hu\|$$

$$(4.122)\qquad \|u-u_h\| \le \|u-r_hu\|$$

and by (4.121), we obtain

$$(4.123)\qquad \|u-u_h\| \le c(u,\alpha)\rho(h)^3.$$

<u>Remark 4.5.</u>

(i) An internal approximation of V with piecewise polynomials of degree 6 is constructed in F. Thomasset [1].

(ii) Internal approximations of V are not available if $n = 3$ or if Ω is not simply connected.

---

(1) For the sake of simplicity (4.110) was proved for $\psi\in\mathcal{D}(\Omega)$; the proof is valid for any $\psi\in\mathcal{C}^6(\bar\Omega)\cap H_0^2(\Omega)$.

4.5 Non-Conforming Finite Elements.

Because of the condition $\operatorname{div} u = 0$, it is not possible to approximate V by the most simple finite elements, the piecewise linear continuous functions. This was shown by M. Fortin [2]. Our purpose here is to describe an approximation of V by linear, non-conforming finite elements, which means in this case, piecewise linear but discontinuous functions. This leads to the approximation of V denoted by (APX 5). Then we associate with this approximation of V a new approximation scheme for Stokes problem.

Approximation of $\mathbb{H}_0^1(\Omega)$.

We suppose that Ω is a bounded lipschitzian open set in $\mathbb{R}^n$ and in this section we will approximate $\mathbb{H}_0^1(\Omega)$ by non-conforming piecewise linear finite elements.

Let $\mathcal{T}_h$ denote an admissible triangulation of Ω. If $\mathcal{S}\in\mathcal{T}_h$, we denote by $A_1,\cdots,A_{n+1}$ its vertices, by $\mathcal{S}_i$ the $(n-1)$-face which does not contain A_i, and by B_i the barycenter of the face $\mathcal{S}_i$. If G denotes the barycenter of $\mathcal{S}$, then since the barycentric coordinates of B_i with respect to the A_j, $j \neq i$, are equal to $1/n$, we have

$$(4.124)\qquad GB_i = \sum_{j\neq i} \frac{GA_j}{n} = \sum_{j=1}^{n+1} \frac{GA_j}{n} - \frac{GA_i}{n}$$

or

$$(4.125)\qquad GB_i = -\frac{1}{n} GA_i,$$

since $\sum_{j=1}^{n+1} GA_j = 0$ (the barycentric coordinates of G with respect to $A_1,\cdots,A_{n+1}$, are equal to $\frac{1}{n+1}$). We deduce from this, that

$$(4.126)\qquad nB_iB_j = n(GB_j-GB_i) = GA_i-GA_j = -A_iA_j,$$

and therefore the vectors B_1B_j, $j = 2,\cdots,n+1$, are linearly independent like the vectors A_1A_j, $j = 2,\cdots,n+1$. Because of this, the barycentric coordinates of a point P, with respect to $B_1,\cdots,B_{n+1}$, can be defined, and we denote by $\mu_1,\cdots,\mu_{n+1}$, these coordinates. We remark also that for each given set of $(n+1)$ numbers $\beta_1,\cdots,\beta_{n+1}$, there exists one and only one linear function taking on at the points $B_1,\cdots,B_{n+1}$, the values $\beta_1,\cdots,\beta_{n+1}$, and this function u is

$$(4.127)\qquad u(P) = \sum_{i=1}^{n+1} \beta_i\mu_i(P);$$

(see Proposition 4.1).

Space W_h.

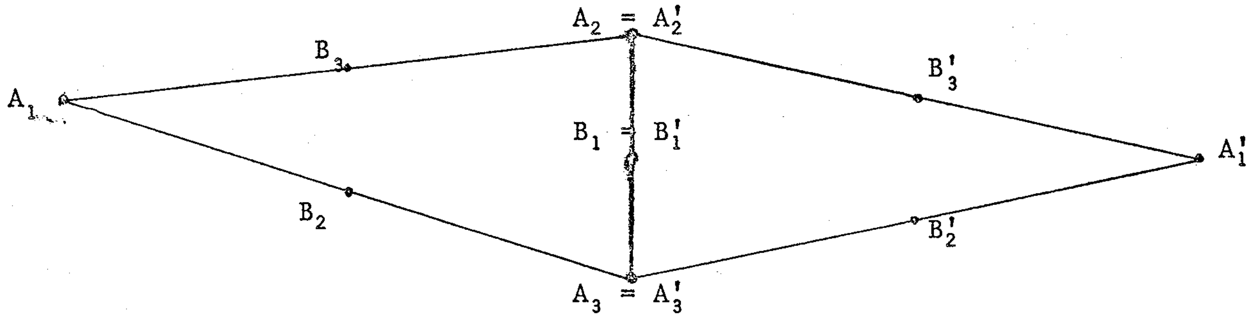
W_h is the space of vector-functions u_h which are linear on each $\mathcal{S} \in \mathcal{T}_h$, vanish outside $\Omega(h)$ (1) and such that the value of u_h at the barycenter B_i of some (n-1) dimensional face $\mathcal{S}_i$ of a simplex $\mathcal{S} \in \mathcal{T}_h$ is zero, if this face belongs to the boundary of $\Omega(h)$; if this face intersects the interior of $\Omega(h)$ then the values of u_h at B_i are the same when B_i is considered as a point of the two different adjacent simplices.

Let $\mathcal{V}_h$ denote the set of points B_i which are barycenters of an (n-1) dimensional face of a simplex $\mathcal{S} \in \mathcal{T}_h$ and which belong to the interior of $\Omega(h)$. A function $u_h \in W_h$ is completely characterized by its values at the points $B_i \in \mathcal{V}_h$.

We denote by w_{hB}, B a point of $\mathcal{V}_h$, the scalar function which is linear on each simplex $\mathcal{S} \in \mathcal{T}_h$, satisfies the same boundary and matching condition that the functions of W_h satisfy and, moreover,

(4.128) $$w_{hB}(B) = 1, \quad w_{hB}(M) = 0 \quad \forall M \in \mathcal{V}_h, \quad M \neq B.$$

Such a function w_{hB} has a support equal to the two simplices which are adjacent to B.



Two adjacent triangles (n=2).

Lemma 4.12.

The functions $w_{hB}e_i$ of W_h, where $B \in \mathcal{V}_h$ and $1 \leq i \leq n$, form a basis of W_h. Hence the dimension of W_h is $nN(h)$, $N(h)$ being the number of points in $\mathcal{V}_h$.

Proof.

It is clear that these functions are linearly independent and that they span the whole space W_h: by Proposition 4.1, any $u_h \in W_h$ can be written as

(4.129) $$u_h = \sum_{B \in \mathcal{V}_h} u_h(B) w_{hB}.$$

---

(1) As before, $\Omega(h) = \bigcup_{\mathcal{S} \in \mathcal{T}_h} \mathcal{S}$.

The space W_h is not included in $\mathbb{H}_0^1(\Omega)$; actually the derivative $D_i u_h$ of some function $u_h \in W_h$ is the sum of Dirac distributions located on the faces of the simplices and of a step function $D_{ih} u_h$ defined almost everywhere by

(4.130) $$D_{ih} u_h(x) = D_i u_h(x), \quad \forall x \in \mathcal{S}, \quad \forall \mathcal{S} \in \mathcal{T}_h .$$

Since u_h is linear on $\mathcal{S}$, $D_{ih} u_h$ is constant on each simplex.

We equip W_h with the following scalar product :

(4.131) $$[\![u_h, v_h]\!]_h = (u_h, v_h) + \sum_{i=1}^{n} (D_{ih} u_h, D_{ih} v_h)$$

which is the discrete analog of the scalar product of $\mathbb{H}_0^1(\Omega)$:

(4.132) $$[\![u, v]\!] = (u,v) + \sum_{i=1}^{n} (D_i u, D_i v).$$

Space F, Operators $\bar{\omega}$, p_h.

We take, as in Section 3.3, $F = \mathbb{L}^2(\Omega)^{n+1}$, and for $\bar{\omega}$ the isomorphism

(4.133) $$u \in \mathbb{H}_0^1(\Omega) \mapsto \bar{\omega} u = (u, D_1 u, \cdots, D_n u) \in F.$$

Similarly, the operator p_h is defined by

(4.134) $$u_h \in W_h \longmapsto p_h u_h = (u_h, \tilde{D}_{1h} u_h, \cdots, \tilde{D}_{nh} u_h) \in F.$$

The operators p_h each have norm equal to 1 and are stable.

Operator r_h.

We define $r_h u = u_h$, for $u \in \mathcal{D}(\Omega)$, by

(4.135) $$u_h(B) = u(B), \quad \forall B \in \mathcal{V}_h .$$

Proposition 4.9.

If h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω, the preceding approximation of $\mathbb{H}_0^1(\Omega)$ is stable and convergent.

Proof.

We have to check the conditions (c1) and (c2) of the Definition 3.6.

For condition (c2) we have to prove that, for each $u \in \mathcal{D}(\Omega)$, (4.136) $$u_h \longrightarrow u \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \rho(h) \longrightarrow 0,$$

(4.137) $$D_{ih}u_h \longrightarrow D_i u \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \rho(h) \longrightarrow 0.$$

On each simplex $\mathcal{S}$ we can apply the result (4.42) to each component of u; this gives:

(4.138) $$\begin{cases} \displaystyle\sup_{x\in\mathcal{S}} |u(x)-u_h(x)| \leq c\eta_2(u)\rho_{\mathcal{S}}^2, \\ \displaystyle\sup_{x\in\mathcal{S}} |D_i u(x) - \dot{D}_i u_h(x)| \leq c\eta_2(u) \frac{\rho_{\mathcal{S}}^2}{\rho'_{\mathcal{S}}} . \end{cases}$$

Therefore

(4.139) $$\|p_h u_h - \bar{\omega}u\|_F \leq c(u)\alpha\rho(h) + [\![u]\!]_{\mathbb{H}_0^1(\Omega-\Omega(h))}$$

and this goes to 0 as $\rho(h) \longrightarrow 0$.

To prove the condition (cl) let us suppose that $p_{h'}u_{h'}$ converges weakly in F to $\phi = (\phi_0,\cdots,\phi_n)$; this means that

(4.140) $$u_{h'} \longrightarrow \phi_0 \text{ in } \mathbb{L}^2(\Omega) \quad \text{weakly},$$

(4.141) $$D_{ih}u_{h'} \longrightarrow \phi_i \quad \text{in} \quad \mathbb{L}^2(\Omega) \quad \text{weakly}, \quad 1 \leq i \leq n.$$

Since the functions have compact supports included in Ω, (4.140) and (4.141) amount to saying that:

(4.142) $$\tilde{u}_{h'} \longrightarrow \tilde{\phi}_0 \quad \text{in} \quad \mathbb{L}^2(\mathbb{R}^n) \quad \text{weakly},$$

(4.143) $$D_{ih}\tilde{u}_{h'} \longrightarrow \tilde{\phi}_i \quad \text{in} \quad \mathbb{L}^2(\mathbb{R}^n) \quad \text{weakly}, \quad 1 \leq i \leq n,$$

($\tilde{g}$ is the function equal to g in Ω and to 0 in $\complement\Omega$).

If we show that

(4.144) $$\tilde{\phi}_i = D_i\tilde{\phi}_0, \quad 1 \leq i \leq n,$$

it will follow that $\tilde{\phi}_0 \in \mathbb{H}^1(\mathbb{R}^n)$ and hence $\phi_0 \in \mathbb{H}_0^1(\Omega)$ with $\phi_i = D_i\phi_0$, which amounts to saying that $\phi = \bar{\omega}u$, $u = \phi_0$.

Let θ be any test function in $\mathcal{D}(\mathbb{R}^n)$; then:

$$\int_{\mathbb{R}^n} \tilde{u}_{h'}(D_i\theta)dx \longrightarrow \int_{\mathbb{R}^n} \Phi_0(D_i\theta)dx,$$

$$\int_{\mathbb{R}^n} (D_{ih'}\tilde{u}_{h'})\theta dx \longrightarrow \int_{\mathbb{R}^n} \Phi_i\theta dx.$$

The equality (4.144) is proved if we show that

$$\int_{\mathbb{R}^n} \Phi_i\theta dx = -\int_{\mathbb{R}^n} \Phi_0(D_i\theta)dx, \quad \forall\theta\in\mathcal{D}(\mathbb{R}^n),$$

or that

(4.145) $\mathcal{Y}_{h'} = \int_{\mathbb{R}^n} \tilde{u}_{h'}(D_i\theta)dx + \int_{\mathbb{R}^n} (D_{ih'}\tilde{u}_{h'})\theta dx \longrightarrow 0, \quad \text{as} \quad \rho(h')\longrightarrow 0,$

for each $\theta\in\mathcal{D}(\mathbb{R}^n)$.

The proof of (4.145) is the goal of the next three lemmas.

<u>Lemma 4.13.</u>

(4.146) $$\mathcal{Y}_h = \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} u_h\theta\nu_{i,\mathcal{S}'}d\Gamma,$$

<u>where $\partial^+\mathcal{S}$ is the set of the (n-1)-dimensional faces of $\mathcal{S}$[(1)], and $\nu_{i,\mathcal{S}'}$ is the i^{th} component of the unit vector $\nu_{\mathcal{S}'}$ which is normal to $\mathcal{S}'$ and is pointing outward with respect to $\mathcal{S}$.</u>

<u>Proof.</u>

Since the functions vanish outside $\Omega(h)$, we have

$$\mathcal{Y}_h = \int_{\Omega(h)} [u_h(D_i\theta) + (D_{ih}u_h)\theta]dx = \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\mathcal{S}} [u_h(D_i\theta) + (D_iu_h)\theta]dx$$

$$= \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\mathcal{S}} D_i(u_h\theta)dx.$$

The Green-Stokes formula gives

$$\int_{\mathcal{S}} D_i(u_h\theta)dx = \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} u_h\theta\nu_{i,\mathcal{S}'}d\Gamma,$$

(1) There are (n+1) such faces.

and (4.146) follows.

Lemma 4.14.

$$\mathcal{J}_h = \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (u_h(x)-u_h(B))(\theta(x)-\theta(B))\nu_{i,\mathcal{S}'}d\Gamma, \tag{4.147}$$

where $B = B_{\mathcal{S}'}$ is the barycenter of $\mathcal{S}'$.

Proof.

We show first that

$$\mathcal{J}_h = \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (u_h(x)-u_h(B))\theta(x)\nu_{i,\mathcal{S}'}d\Gamma. \tag{4.148}$$

To prove this equality we just have to show that

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} u_h(B)\theta(x)\nu_{i,\mathcal{S}'}d\Gamma = 0. \tag{4.149}$$

But for a face $\mathcal{S}'$ belonging to the boundary of $\Omega(h)$, $u_h(B) = 0$ and the contribution of this face in the sum is zero. If $\mathcal{S}'$ belongs to the boundary of two adjacent simplices, this face contributes to the sum two opposite terms: the $u_h(B)$ and $\theta(x)$ are the same and the $\nu_{i,\mathcal{S}'}$ are equal but with opposite signs when $\mathcal{S}'$ is considered as part of the boundary of the two simplices. Hence the sum (4.149) is zero.

The equality (4.147) is then easily deduced from (4.148) if we prove that

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (u_h(x)-u_h(B))\theta(B)\nu_{i,\mathcal{S}'}d\Gamma = 0. \tag{4.150}$$

But to prove (4.150) we simply note that

$$\int_{\mathcal{S}'} [u_h(x)-u_h(B)]\theta(B)\nu_{i,\mathcal{S}'}d\Gamma = 0,$$

since $\theta(B)$ and $\nu_{i,\mathcal{S}'}$ are constant on $\mathcal{S}'$ and since

$$\int_{\mathcal{S}'} u_h(x)d\Gamma = u_h(B)\int_{\mathcal{S}'} d\Gamma, \tag{4.151}$$

because u_h is linear on $\mathcal{S}'$, and B is the barycenter of $\mathcal{S}'$.

Lemma 4.15.

(4.152) $\quad \gamma_{h'} \longrightarrow 0$ as $\rho(h') \longrightarrow 0$, with $\sigma(h') \leq \alpha$.

Proof.

Since $|\nu_{i,\Delta'}| \leq 1$ and

$$|\theta(x)-\theta(B)| \leq c(\theta)\rho_{\Delta}, \ \forall x \in \Delta',$$

we get the estimate,

$$(4.153) \qquad \left| \int_{\Delta'} (u_h(x)-u_h(B))(\theta(x)-\theta(B))\nu_{i,\Delta'} d\Gamma \right|$$

$$\leq c(\theta)\rho_{\Delta} \int_{\Delta'} |u_h(x)-u_h(B)| d\Gamma.$$

Since $u_h(x)-u_h(B)$ is a linear function on Δ which vanishes at $x = B$, we can write it on Δ as,

$$u_h(x)-u_h(B) = \sum_{i=1}^{n} \frac{\partial u_h}{\partial x_i} \cdot (x_i-\beta_i),$$

where $\beta_1,\cdots,\beta_n$, are the coordinates of B. Therefore

$$|u_h(x)-u_h(B)| \leq \rho_{\Delta} \sum_{i=1}^{n} \left| \frac{\partial u_h}{\partial x_i} \right|, \ \forall x \in \Delta,$$

and

$$c(\theta)\rho_{\Delta} \int_{\Delta'} |u_h(x)-u_h(B)| d\Gamma \leq c(\theta)\rho_{\Delta}^2 \int_{\Delta'} \left(\sum_{i=1}^{n} \left| \frac{\partial u_h}{\partial x_i} \right| \right) d\Gamma$$

$$= c(\theta)\rho_{\Delta}^2 (\text{meas}_{n-1} \Delta') \left\{ \sum_{i=1}^{n} \left| \frac{\partial u_h}{\partial x_i} \right| \right\}$$

Let us denote by $A_1,\cdots,A_{n+1}$, the vertices of Δ, and let us suppose that Δ' is the face containing $A_2,\cdots,A_{n+1}$; let ξ = the distance between A_1 and Δ'. The following elementary formula is well known,

$$(4.154) \qquad \text{meas}_n(\Delta) = \frac{1}{n} \xi \ \text{meas}_{n-1}(\Delta')$$

Hence

$$(4.155) \qquad \text{meas}_{n-1}(\Delta') = \frac{n}{\xi} \text{meas}_n(\Delta) \leq \frac{n}{2\rho'_{\Delta}} \text{meas}_n(\Delta),$$

since

$$(4.156) \qquad 2\rho'_{\Delta} \leq \xi$$

The reason (4.156) holds is that the largest ball included in Δ has a diameter equal to $2\rho'_{\mathcal{S}}$, and this ball is included in the set bounded by the hyperplane containing $A_2,\cdots,A_{n+1}$, and the parallel hyperplane issued from A_1.

We then have

$$c(\theta)\rho_{\mathcal{S}} \int_{\mathcal{S}} |u_h(x)-u_h(B)| \; d\Gamma \leq \frac{c(\theta)n}{2} \frac{\rho^2_{\mathcal{S}}}{\rho'_{\mathcal{S}}} (\text{meas}_n \mathcal{S}) \left\{ \sum_{i=1}^{n} \left| \frac{\partial u_h}{\partial x_i} \right| \right\}$$

$$\leq \frac{c(\theta)n\alpha}{2} \rho(h) \int_{\mathcal{S}} \sum_{i=1}^{n} \frac{\partial u_h}{\partial x_i} \; dx.$$

Combining this with (4.153) and (4.147) we obtain the following estimates for γ_h:

$$|\gamma_h| \leq \frac{c(\theta)n(n+1)}{2} \alpha\rho(h) \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{i=1}^{n} \int \frac{\partial u_h}{\partial x_i} \; dx \quad ,$$

$$|\gamma_h| \leq c(\theta;n,\alpha)\rho(h) \int_{\Omega} |\text{grad } u_h| dx$$

$\leq$ (by the Schwarz inequality)

$$(4.157) \qquad |\gamma_h| \leq c(\theta,n,\alpha,\Omega)\rho(h)[\![u_h]\!]_h.$$

According to (4.136)(4.137), the sequence $[\![u_{h'}]\!]_{h'}$ is bounded and therefore $\gamma_{h'} \longrightarrow 0$ as $\rho(h') \longrightarrow 0$.

Discrete Poincaré Inequality.

The following discrete Poincaré inequality will allow us to endow the space W_h described above, with another scalar product $((\cdot,\cdot))_h$, the discrete analog of the scalar product $((\cdot,\cdot))$ of $H_0^1(\Omega)$ (see (1.11) and Proposition 3.3).

Proposition 4.10.

Let us suppose that Ω *is a bounded set in* $\mathbb{R}^n$. *Then there exists a constant* $c(\Omega,\alpha)$ *depending only on* Ω *and the constant* α *in* (4.21) *such that the inequality*

$$(4.158) \qquad |u_h|_{L^2(\Omega)} \leq c(\Omega) \sum_{i=1}^{n} |D_{ih}u_h|^2_{L^2(\Omega)},$$

holds for any scalar function of type (4.129)

$$(4.159) \qquad u_h = \sum_{B\in\mathcal{V}_h} u_h(B) w_{hB}.$$

A similar inequality holds for the vector functions of type (4.129):

$$|u_h|_{\mathbb{L}^2(\Omega)} \leq c'(\Omega) \ \|u_h\|_h, \quad \forall u_h \in W_h \tag{4.160}$$

where

$$\|u_h\|_h = \{ \sum_{i=1}^{n} |D_{ih} u_h|^2_{\mathbb{L}^2(\Omega)} \}^{\frac{1}{2}} \tag{4.161}$$

Proof.

The inequality (4.160) follows immediately from (4.158). In order to prove (4.158), we will show that

$$\left|\int_\Omega u_h \ \theta \ dx\right| \leq c(\Omega)|\theta|_{L^2(\Omega)} \|u_h\|_h, \tag{4.162}$$

for each u_h of type (4.159) and for each θ in $\mathcal{D}(\Omega)$; (4.162) implies (4.158) since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$.

Let us denote by χ the solution of the Dirichlet problem

$$\Delta\chi = \theta \ \text{ in } \ \Omega, \quad \chi \in H_0^1(\Omega).$$

The function χ is $\mathcal{C}^\infty$ on Ω and

$$\|\chi\|_{H^2(\Omega)} \leq c_0(\Omega)|\theta|_{L^2(\Omega)} \ ^{(1)}. \tag{4.163}$$

We then have

$$\int_\Omega u_h \ \theta \ dx = \sum_{\mathcal{S} \in \mathcal{T}_h} \int u_h \ \Delta\chi \ dx$$

and the Green formula implies

$$\int_{\mathcal{S}} u_h \ \Delta\chi \ dx = \int_{\partial\mathcal{S}} u_h \frac{\partial\chi}{\partial\nu} \ d\Gamma - \int_{\mathcal{S}} \text{grad } u_h \cdot \text{grad } \chi \ dx.$$

Hence

$$\left|\int_\Omega u_h \ \theta \ dx\right| \leq |((u_h,\chi))_h| + |\mathcal{X}_h|$$

(1) Strictly speaking this inequality is true only if Ω is smooth enough; in the general case (4.163) is valid if we define χ by $\Delta\chi = \theta$, $\chi \in H_0^1(\Omega')$ where Ω' is smooth and $\Omega' \supset \bar{\Omega}$. This makes no change in the following.

$$(4.164) \qquad \left|\int_\Omega u_h\, \theta\, dx\right| \le \|u_h\|_h \|\chi\|_{H^1(\Omega)} + |\mathcal{J}_h|$$

$$\le c(\Omega)|\theta|_{L^2(\Omega)} \|u_h\|_h + |\mathcal{J}_h|$$

where

$$\mathcal{J}_h = \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\partial\mathcal{S}} u_h \frac{\partial\chi}{\partial\nu}\, d\Gamma.$$

The next lemmas will give an estimate for $\mathcal{J}_h$ which together with (4.164) will give (4.162).

Lemma 4.16.

Using the same notations as in Lemma 4.14, we have

$$\mathcal{J}_h = \sum_{i=1}^{n} \mathcal{J}_h^i,$$

with

$$(4.165) \qquad \mathcal{J}_h^i = \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (u_h(x)-u_h(B))(\chi_i(x)-\chi_i(B))\nu_{i,\mathcal{S}'}\, d\Gamma,$$

where

$$\chi_i = \frac{\partial\chi}{\partial x_i} \quad .$$

Proof.

We write

$$\frac{\partial\chi}{\partial\nu} = \sum_{i=1}^{n} \chi_i\, \nu_{i,\mathcal{S}'}$$

on the face $\mathcal{S}'$, and we then proceed exactly as in Lemma 4.14.

Lemma 4.17.

$$(4.166) \quad |\mathcal{J}_h^i| \le c(n)\sqrt{\alpha\rho(h)}\, \|u_h\|_h \left(\sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (\chi_i(x)-\chi_i(B))^2 d\Gamma\right)^{\frac{1}{2}}$$

Proof.

The proof is similar to that of Lemma 4.15. We write on the face $\mathcal{S}'$,

$$(4.167) \qquad u_h(x)-u_h(B) = \sum_{i=1}^{n} \frac{\partial u_h}{\partial x_i}(x_i-\beta_i),$$

where $\beta_1,\cdots,\beta_n$, are the coordinates of B. Hence

$$(4.168) \qquad |u_h(x)-u_h(B)| \le \rho_{\mathcal{S}}|\text{grad } u_h|, \quad \forall x\in \mathcal{S},$$

and

$$\left|\int_{\mathcal{S}'} (u_h(x)-u_h(B))(\chi_i(x)-\chi_i(B))\nu_{i,\mathcal{S}'}d\Gamma\right|$$

$$\le \rho_{\mathcal{S}}\left(\int_{\mathcal{S}'}|\text{grad } u_h|^2 d\Gamma\right)^{\frac{1}{2}} \left(\int_{\mathcal{S}'} (\chi_i(x)-\chi_i(B))^2 d\Gamma\right)^{\frac{1}{2}}.$$

But, since grad u_h is contant on $\mathcal{S}$,

$$\int_{\mathcal{S}'} (\text{grad } u_h)^2 d\Gamma = \text{meas}_{n-1}(\mathcal{S}')\cdot|\text{grad } u_h|^2$$

$$\le \text{(because of (4.155))}$$

$$\le \frac{n}{2\rho'_{\mathcal{S}}}\text{meas}_n(\mathcal{S}) \cdot |\text{grad } u_h|^2$$

$$\le \frac{n}{2\rho'_{\mathcal{S}}}\int_{\mathcal{S}}|\text{grad } u_h|^2 \, dx$$

and

$$(4.169) \qquad |\mathcal{J}_h^i| \le c(n) \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \frac{\rho_{\mathcal{S}}}{\sqrt{\rho'_{\mathcal{S}}}} \cdot$$

$$\cdot \left(\int_{\mathcal{S}}|\text{grad } u_h|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathcal{S}'} |\chi_i(x)-\chi_i(B)|^2 d\Gamma\right)^{\frac{1}{2}}.$$

We then obtain (4.166) by using (4.20),(4.21) and applying the Schwarz inequality to (4.169).

Lemma 4.18.

$$(4.170) \qquad \int_{\mathcal{S}'} (\chi_i(x)-\chi_i(B))^2 d\Gamma \le c(n)\alpha^2 \int_{\mathcal{S}}(\text{grad } \chi_i)^2 dx \quad ,$$

$$(4.171) \qquad \sum_{\mathcal{S}\in\mathcal{T}_h} \sum_{\mathcal{S}'\in\partial^+\mathcal{S}} \int_{\mathcal{S}'} (\chi_i(x)-\chi_i(B))^2 d\Gamma$$

$$\le c(n)\alpha^2 \int_{\Omega} (\text{grad } \chi_i)^2 \, dx \le c(n,\Omega)\alpha^2|\theta|^2_{L^2(\Omega)}.$$

Proof.

The inequalities (4.171) follow directly from (4.170) and (4.164).

The inequality (4.170) is an obvious consequence of the trace theorems in $H^1(\mathcal{K})$ if we replace the constant $c(n)\alpha^2$ in the right-hand side of (4.170) by some constant $c(\mathcal{K})$ depending on the particular simplex $\mathcal{K}$; the interest of (4.170) is that this inequality is uniformly valid with respect to the simplices $\mathcal{K}$ in $\mathcal{T}_h$.

To prove (4.170) we make some transformation in the coordinates which maps $\mathcal{K}$ on a fixed simplex $\overline{\mathcal{K}}$ and then we apply the trace theorem inequality in $\overline{\mathcal{K}}$ and come back to $\mathcal{K}$.

For simplicity we suppose that $A_1 = 0$, that $\mathcal{K}'$ is contained in the hyperplane $x = 0$, and that the vertices of $\mathcal{K}'$ are $A_1,\cdots,A_n$; the referential simplex is the simplex $\overline{\mathcal{K}}$ with vertices $\overline{A}_1,\cdots,\overline{A}_{n+1}$, $\overline{A}_1 = 0$, and $\overline{A}_1\overline{A}_{i+1} = e_i$, $i = 1,\cdots,n$. The face corresponding to $\mathcal{K}'$ is the face $\overline{\mathcal{K}}'$ with vertices $\overline{A}_1,\cdots,\overline{A}_n$. Let Λ denote the linear operator in $\mathbb{R}^n$ which is defined by

$$A_i = \Lambda\overline{A}_i, \quad i = 2,\cdots,n+1$$

and let Λ' be the linear mapping in $\mathbb{R}^{n-1}$, which is defined by

$$A_i = \Lambda'\overline{A}_i, \quad i = 2,\cdots,n.$$

A change of coordinates for the integral in the left hand side of (4.170) gives

$$\int_{\mathcal{K}'} \sigma(x)^2 d\Gamma_{\mathcal{K}'} = \frac{1}{|\det \Lambda'|}\int_{\overline{\mathcal{K}}'} \overline{\sigma}^2(\overline{x}) \quad d\Gamma_{\overline{\mathcal{K}}'}$$

where

$$(4.172) \qquad \sigma(x) = \chi_i(x)-\chi_i(B)$$

and

$$(4.173) \qquad \overline{\sigma}(\overline{x}) = \sigma(x), \quad \overline{x} = \Lambda^{-1}x.$$

For the simplex $\overline{\mathcal{K}}$, the trace theorem inequality and the Poincaré inequality give (recall that $\sigma(B) = 0$)

$$\int_{\overline{\mathcal{K}}'} \overline{\sigma}^2(\overline{x})d\Gamma \leq c(\overline{\mathcal{K}}) \sum_{j=1}^{n} \int \left(\frac{\partial\overline{\sigma}}{\partial\overline{x}_j}(\overline{x})\right)^2 d\overline{x}.$$

We come back to $\mathcal{D}$ and the coordinates x_i. We write

$$\frac{\partial\bar{\sigma}}{\partial\bar{x}_j}(\bar{x}) = \sum_{k=1}^{n} \Lambda^{-1}_{kj}\left(\frac{\partial\sigma}{\partial x_k}\right)(\Lambda\bar{x}),$$

$$\sum_{j=1}^{n}\left|\frac{\partial\bar{\sigma}}{\partial\bar{x}_j}(\bar{x})\right|^2 \leq \|\Lambda^{-1}\|^2 \sum_{k=1}^{n}\left|\frac{\partial\sigma}{\partial x_k}(\Lambda\bar{x})\right|^2,$$

and hence

$$\int_{\bar{\mathcal{D}}} \bar{\sigma}^2(\bar{x})\,d\Gamma \leq c(\bar{\mathcal{D}})\|\Lambda^{-1}\|^2 \int_{\bar{\mathcal{D}}}\left(\sum_{k=1}^{n}\left|\frac{\partial\sigma}{\partial x_k}(\Lambda\bar{x})\right|^2\right)d\bar{x}$$

$$\leq c(\bar{\mathcal{D}})\,|\det\Lambda|\,\|\Lambda^{-1}\|^2 \int_{\mathcal{D}} (\operatorname{grad}\sigma)^2 dx.$$

We arrive at

$$\int_{\mathcal{D}'} (\chi_i(x) - \chi_i(B))^2 d\Gamma \leq c\,\frac{|\det\Lambda|}{|\det\Lambda'|}\,\|\Lambda^{-1}\|^2 \int_{\mathcal{D}} (\operatorname{grad}\chi_i)^2 dx.$$

In order to prove (4.170), it remains to show that

$$\frac{|\det\Lambda|}{|\det\Lambda'|}\,\|\Lambda^{-1}\|^2 \leq c\alpha^2. \tag{4.174}$$

Since

$$\frac{\det\Lambda}{\det\Lambda'} = \frac{\det(\Lambda')^{-1}}{\det\Lambda^{-1}} = \Lambda_{nn} \quad (1)$$

and

$$|\Lambda_{nn}| \leq \|\Lambda\|,$$

the left-hand side of (4.174) is majorized by

$$\|\Lambda\|^2\,\|\Lambda^{-1}\|^2$$

Because of Lemma 4.3, this term is majorized by

---

(1) Λ_{nn} is the (n,n) element of Λ; note that $\Lambda_{in} = 0$, $1 \leq i \leq n-1$, due to our choice of the coordinate axes.

$$\left(\frac{\rho_{\Delta}\rho_{\bar{\Delta}}}{\rho'_{\Delta}\rho'_{\bar{\Delta}}}\right)^2 \le c(\bar{\Delta})\alpha^2$$

and (4.174) follows.

The proof of Lemma 4.18 is complete.

Lemma 4.19.

(4.175) $$|\delta_h| \le c(n)\sqrt{\rho(h)}\ \alpha^{\frac{3}{2}} \|u_h\|_h |\theta|_{L^2(\Omega)}$$

The proof of this lemma is now obvious. As mentioned above, this finishes the proof of (4.158) and, more precisely, shows that

(4.176) $$|u|_{L^2(\Omega)} \le (c(\Omega) + c(n)\alpha^{\frac{3}{2}}\ \rho(h)^{\frac{1}{2}})\ \|u_h\|_h;$$

$\rho(h)$ is bounded by the diameter of Ω and actually goes to 0 as $h \longrightarrow 0$.

Proposition 4.11.

Let Ω be a bounded lipschitzian set. Let us suppose that we equip the space W_h with the scalar product

(4.177) $$((u_h, v_h))_h = \sum_{i=1}^{n} (D_{ih}u_h, D_{ih}v_h),$$

and leave the other unchanged in the statement of Proposition 4.9. Then this approximation of $\mathbb{H}_0^1(\Omega)$ is again stable and convergent.

Proof.

The only difference between this and Proposition 4.9 comes from the stability of the operators p_h and this difficulty is completely overcome by Proposition 4.10 and (4.160), which give

(4.178) $$\|u_h\|_h \le [\![u_h]\!]_h \le c(\Omega)\|u_h\|_h, \quad \forall u_h \in W_h.$$

Approximation of V (APX 5)

Let Ω be a lipschitzian bounded set in $\mathbb{R}^n$ and let $\mathcal{V}$ be the usual space (1.12) and V its closure in $\mathbb{H}_0^1(\Omega)$.

We now define an approximation of V similar to the preceding approximation of $\mathbb{H}_0^1(\Omega)$.

As previously we take $F = \mathbb{L}^2(\Omega)^{n+1}$, and $\bar{\omega} \in \mathcal{L}(V,F)$ as the linear operator

(4.179) $$u \in V \longmapsto \bar{\omega} u = \{u, D_1 u, \cdots, D_n u\}.$$

Space V_h.

We take as space V_h a subspace of the preceding space W_h

(4.180) $$V_h = \{u_h \in W_h \mid \sum_{i=1}^{n} D_{ih} u_{ih} = 0\}$$

The condition in (4.180) concerning the divergence of u_h is equivalent to

(4.181) $$\text{div } u_h = 0 \text{ in } \mathcal{S}, \ \forall \mathcal{S} \in \mathcal{T}_h.$$

We equip the space V_h with the scalar product $((u_h, v_h))_h$ induced by W_h.

Operator p_h.

As before,

(4.181) $$p_h u_h = \{u_h, D_{1h} u_h, \cdots, D_{nh} u_h\}.$$

The operators p_h are stable because of the inequality (4.160) (or (4.178)).

Operator r_h.

We have to define $r_h u = u_h \in V_h$, for $u \in \mathcal{V}$. Since u_h must satisfy the condition (4.181), the operator r_h used for the approximation of $\mathbb{H}_0^1(\Omega)$ does not satisfy all the requirements. We choose instead the following operator r_h: $u_h = r_h u$ is characterized by the values of $u_h(B)$, $B \in \mathcal{U}_h$; if $B \in \mathcal{U}_h$, B is the barycenter of some $(n-1)$-face $\mathcal{S}'$ of some n-simplex $\mathcal{S} \in \mathcal{T}_h$; we set

(4.182) $$u_h(B) = \frac{1}{\text{meas}_{n-1}(\mathcal{S}')} \int_{\mathcal{S}'} u \, d\Gamma.$$

Let us show that $u_h \in V_h$; since div u_h is constant on each simplex $\mathcal{S}$, the condition (4.181) is equivalent to

$$\int_{\mathcal{S}} \text{div } u_h \, dx = 0, \ \forall \mathcal{S} \in \mathcal{T}_h.$$

Applying the Green formula, we get

$$\int_{\Delta} \operatorname{div} u_h \, dx = \sum_{\Delta' \in \partial^+\Delta} \int_{\Delta'} u_h \cdot \nu_{\Delta'} d\Gamma$$

$$= \text{(by (4.182))}$$

$$= \sum_{\Delta' \in \partial^+\Delta} \int_{\Delta'} u \cdot \nu_{\Delta'} d\Gamma$$

$$= \int_{\partial\Delta} u \cdot \nu d\Gamma = \int_{\Delta} \operatorname{div} u \, dx,$$

and this last integral is zero since $\operatorname{div} u = 0$.

Proposition 4.12.

The previous external approximation of V is stable and convergent, provided h belongs to a regular triangulation $\mathcal{H}_\alpha$ of Ω.

Proof.

We noticed already that the p_h are stable. Let us check the condition (C2) of Definition 3.6. For that, let us suppose that

(4.183) $$p_{h'} u_{h'} \longrightarrow \phi \ \text{ in } \ F, \ \text{ weakly.}$$

Exactly as in Proposition 4.9 we see that

(4.184) $$\phi = \overline{\omega} u, \quad u \in \mathbb{H}_0^1(\Omega)$$

Here, we must prove moreover that $u \in V$, i.e., $\operatorname{div} u = 0$. But (4.183) means in particular

$$\sum_{i=1}^{n} D_{ih'} u_{ih'} \longrightarrow \operatorname{div} u \ \text{ in } \ L^2(\Omega), \ \text{ weakly,}$$

and since $\sum_{i=1}^{n} D_{ih'} u_{ih'}$ is identically zero, $\operatorname{div} u$ is zero.

Let us check the condition (C1); if $u \in \mathcal{V}$ we denote by u_h the function $r_h u$ and by v_h the function of W_h defined by

$$v_h(B) = u(B), \quad \forall B \in \mathcal{U}_h.$$

It was proved in Proposition 4.9 that

(4.185) $$\| p_h v_h - \overline{\omega} u \|_F \leq c(u) \alpha \rho(h) + [\![u]\!]_{\mathbb{H}^1(\Omega - \Omega(h))}$$

It suffices now to show that

(4.186) $$\|p_h u_h - p_h v_h\|_F = [\![u_h - v_h]\!]_h \longrightarrow 0, \quad \text{as} \quad \rho(h) \longrightarrow 0$$

Because of the inequality (4.178), it suffices to prove that

$$\|u_h - v_h\|_h \longrightarrow 0, \quad \text{as} \quad \rho(h) \longrightarrow 0.$$

Each B of $\mathcal{U}_h$ is the barycenter of some face $\mathcal{A}'$ of some simplex $\mathcal{A}$; we can write

(4.187) $$u(x) = u(B) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(B) \cdot (x_i - \beta_i) + \sigma(x),$$

where $(\beta_1, \cdots, \beta_n)$ are the coordinates of B and

(4.188) $$|\sigma(x)| \leq c(u) \rho_{\mathcal{A}}^2, \quad \forall x \in \mathcal{A}.$$

Integrating (4.187) on $\mathcal{A}'$, we find

$$u_h(B) = v_h(B) + \left(\int_{\mathcal{A}'} \sigma(x)dx\right) \left(\int_{\mathcal{A}'} dx\right)^{-1}$$

since $\int_{\mathcal{A}} (x_i - \beta_i)\, dx = 0$. Because of (4.188),

(4.189) $$u_h(B) - v_h(B) = \varepsilon_h(B),$$

with

(4.190) $$|\varepsilon_h(B)| \leq c(u) \rho_{\mathcal{A}}^2.$$

Inside the simplex $\mathcal{A}$ with faces $\mathcal{A}_1, \cdots, \mathcal{A}_{n+1}$,

$$u_h(x) - v_h(x) = \sum_{i=1}^{n+1} \varepsilon_h(B_i) \mu_i(x)$$

where $\mu_1, \cdots, \mu_n$, are the barycentric coordinates of x with respect to $B_1, \cdots, B_{n+1}$. Therefore, in $\mathcal{A}$,

$$|\text{grad}(u_h - v_h)| \leq c(u) \rho_{\mathcal{A}}^2 \sum_{i=1}^{n+1} |\text{grad } \mu_i|$$

and by Lemma 4.2 and (4.21),

$$|\mathrm{grad}(u_h - v_h)| \leq c(u)\,\frac{\rho^2}{\rho'}.$$

Therefore in all Ω,

$$(4.191) \qquad |D_{ih}(u_h - v_h)(x)| \leq c(u)\alpha\rho(h)$$

and this implies

$$(4.192) \qquad \|u_h - v_h\|_h \leq c(u,\alpha,\Omega)\rho(h)$$

so that

$$(4.193) \qquad \|p_h u_h - \overline{\omega} u\|_F \leq c(u)\alpha\rho(h) + [\![u]\!]_{H^1(\Omega - \Omega(h))}.$$

<u>Approximation of the Stokes Problem</u>

Using the preceding approximation of V and the general results of Section 3.2, we can propose another approximation scheme of the Stokes problem.

Let f belong to $\mathbb{L}^2(\Omega)$, and $\nu > 0$. We set, with the preceding notations,

$$(4.194) \qquad a_h(u_h,v_h) = \nu((u_h,v_h))_h, \quad \forall u_h, v_h \in V_h$$

$$(4.195) \qquad \langle \ell_h, v_n \rangle = (f,v_h), \ \forall v_h \in V_h.$$

The approximation problem is

$$(4.196) \qquad \begin{cases} \text{To find } u_h \in V_h, \text{ such that} \\ \nu((u_h,v_h))_h = (f,v_h), \quad \forall v_h \in V_h. \end{cases}$$

The solution u_h of (4.196) exists and is unique. If $\rho(h) \longrightarrow 0$, with h belonging to a regular triangulation $\mathcal{H}_\alpha$, then the following convergence results hold

$$(4.197) \qquad \begin{cases} u_h \longrightarrow u \text{ in } \mathbb{L}^2(\Omega) \text{ strongly,} \\ D_{ih} u_h \longrightarrow D_i u \text{ in } \mathbb{L}^2(\Omega) \text{ strongly, } 1 \leq i \leq n. \end{cases}$$

This follows of course from Theorem 3.1.

We can, as in Section 3.3 and as for the other approximations, introduce the discrete pressure. It is a step function π_h of the type

(4.198) $$\pi_h = \sum_{\mathcal{S} \in \mathcal{T}_h} \pi_h(\mathcal{S}) \chi_{h\mathcal{S}},$$

where $\pi_h(\mathcal{S})$ is the value of π_h on $\mathcal{S}$, $\pi_h(\mathcal{S}) \in \mathbb{R}$, and $\chi_{h\mathcal{S}}$ is the characteristic function of $\mathcal{S}$. This function π_h is such that

(4.199) $$\nu((u_h, v_h))_h - (\pi_h, \operatorname{div}_h v_h) = (f, v_h), \quad \forall v_h \in W_h,$$

where

(4.200) $$\operatorname{div}_h v_h = \sum_{i=1}^{n} D_{ih} v_{ih}.$$

The error between u and u_h, the solutions of (2.6) and (4.199) respectively, can be estimated as in Section 3.3. Let us suppose that Ω has a polygonal boundary, that $\Omega(h) = \Omega$, and that $u \in \mathcal{C}^3(\overline{\Omega})$, $p \in \mathcal{C}^1(\overline{\Omega})$. We can define an approximation $r_h u$ by a formula similar to (4.182), and it is not difficult to see that the estimation (4.193) still holds:

(4.201) $$\| p_h r_h u - \overline{\omega} u \|_F \leq c(u, \alpha) \rho(h).$$

We will prove later the following lemma.

<u>Lemma 4.20</u>.

<u>Let</u> u, p <u>denote the exact solution of</u> (2.6)-(2.8) <u>and let us suppose that</u> $u \in \mathcal{C}^3(\overline{\Omega})$, $p \in \mathcal{C}^1(\overline{\Omega})$. <u>Then</u>,

(4.202) $$a_h(u, v_h) = (f, v_h) + \ell_h(v_h), \quad \forall v_h \in V_h,$$

<u>where</u>

(4.203) $$|\ell_h(v_h)| \leq c(u,p) \rho(h) \, \|v_h\|_h.$$

If we admit this lemma temporarily, we see that

$$a_h(u_h - u, v_h) = -\ell_h(v_h), \quad \forall v_h \in V_h,$$

$$a_h(u_h - r_h u, v_h) = a_h(u - r_h u, v_h) - \ell_h(v_h).$$

Taking $v_h = u_h - r_h u$ and using (4.195) we obtain

$$\nu\|u_h - r_h u\|_h^2 \leq \nu\|u - r_h u\|_h \|u_h - r_h u\|_h + |\ell_h(u_h - r_h u)|.$$

The estimates (4.201) and (4.203) then give

$$\text{(4.204)} \qquad \nu\|u_h - r_h u\|_h \leq c(u,p,\alpha,\Omega)\rho(h).$$

More precisely, the constant c in (4.204) depends only on the norms of u in $\mathcal{C}^3(\overline{\Omega})$ and of p in $\mathcal{C}^1(\overline{\Omega})$.

Proof of Lemma 4.20.

We take the scalar product, in $L^2(\Omega)$, of the equation

$$\text{(4.205)} \qquad -\nu\Delta u + \operatorname{grad} p = f,$$

with v_h; since $\Omega = \Omega(h)$, we find

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \{-\nu(\Delta u, v_h)_{\mathcal{S}} + (\operatorname{grad} p, v_h)_{\mathcal{S}} - (f, v_h)_{\mathcal{S}}\} = 0.$$

The Green formula applied in each simplex $\mathcal{S}$ gives

$$\sum_{\mathcal{S}\in\mathcal{T}_h} \{-\nu(\Delta u, v_h)_{\mathcal{S}} + (\operatorname{grad} p, v_h) - (f, v_h)_{\mathcal{S}}\}$$

$$= a_h(u, v_h) - (f, v_h) - \ell_h(v_h) = 0,$$

where

$$\ell_h(v_h) = \sum_{\mathcal{S}\in\mathcal{T}_h} \int_{\partial\mathcal{S}} (\nu \frac{\partial u}{\partial\vec{\nu}} v_h - p v_h \vec{\nu})\, d\Gamma \text{ (1)}$$

The estimate (4.203) of ℓ_h is then proved exactly as in Lemmas 4.13, 4.14 and 4.15.

Remark 4.6.

A simple basis for V_h is available in the two-dimensional case. See the work of Crouzeix [1].

Non-conforming finite elements which are piecewise polynomials of degree

(1) The unit vector normal to $\partial\mathcal{S}$ is denoted $\vec{\nu}$ in this formula, to avoid any confusion with the constant $\nu > 0$.

$k > 1$ have also been studied for the approximation of, either $H_0^1(\Omega)$, or the space V.

M. Fortin [2] has pointed out that the space V cannot be approximated by conforming finite elements of degree one (i.e., piecewise linear functions). For this reason the approximation studied in this section is certainly very useful.

§5. NUMERICAL ALGORITHMS

We saw that discretization of the Stokes equations does not solve completely the problem of numerical approximation of these equations; for the actual computation of the solution, we must have a basis of the space V_h such that the analog of (3.6) leads to an algebraic linear system for the components of u_h, with a sufficiently sparse matrix. This occurs only with the schemes corresponding to (APX 4) and (APX 5); for the discrete problem associated with (APX 1) -- (APX 3) we do not even have a basis of V_h.

In Sections 5.1 to 5.3 we will study two algorithms which are very useful for the practical solution of the discretized equations. In Sections 5.1 and 5.2 we consider the continuous case and in Section 5.3 we show rapidly how they can be adapted to the discrete problems.

The results proved in Section 5.4 are related to this problem but they also show how incompressible fluids can be considered as the limit of "slightly" compressible fluids.

5.1 Uzawa Algorithm

We interpreted in Proposition 2.1 the Stokes problem as a variational problem, an optimization problem with linear constraints. The algorithms described in this section and the next are classical algorithms of optimization. We will present these algorithms and study their convergence without any direct reference to optimization theory, although the idea of the algorithm and the proof of the convergence result from optimization theory.

Let us consider the functions u and p defined by Theorem 2.1; we will obtain u,p as limits of sequences u^m,p^m which are much easier to compute than u and p.

We start the algorithm with an arbitrary element p^0,

$$p^0 \in L^2(\Omega). \tag{5.1}$$

When p^m is known, we define u^{m+1} and p^{m+1} $(m \geq 0)$, by the conditions

$$\begin{cases} u^{m+1} \in H_0^1(\Omega) \quad \text{and} \\ \nu((u^{m+1},v)) - (p^m, \operatorname{div} v) = (f,v), \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{5.2}$$

$$(5.3)\qquad \begin{cases} p^{m+1} \in L^2(\Omega) \quad \text{and} \\ (p^{m+1}-p^m,q) + \rho(\text{div } u^{m+1},q) = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

We suppose that $\rho > 0$ is a fixed number; a condition that ρ must satisfy will be given later.

The existence and uniqueness of the solution u^{m+1} of (5.2) is very easy, because of the projection theorem (Theorem 2.2). Actually u^{m+1} is simply a solution of the Dirichlet problem

$$(5.4)\qquad \begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega) \\ -\nu\Delta u^{m+1} = \text{grad } p^m + f \in H^{-1}(\Omega). \end{cases}$$

When u^{m+1} is known, p^{m+1} is explicitly given by (5.3) which is equivalent to

$$(5.5)\qquad p^{m+1} = p^m - \rho \text{ div } u^{m+1} \in L^2(\Omega).$$

<u>Convergence of the Algorithm</u>

<u>Theorem 5.1</u>.

<u>If the number</u> ρ <u>satisfies</u>

$$(5.6)\qquad 0 < \rho < \frac{2\nu}{n},$$

<u>then as</u> $m \to \infty$, u^{m+1} <u>converges to</u> u <u>in</u> $\mathbb{H}_0^1(\Omega)$ <u>and</u> p^{m+1} <u>converges to</u> p <u>weakly in</u> $L^2(\Omega)/\mathbb{R}$.

<u>Proof</u>.

The equation (2.7) which is satisfied by u and p is equivalent to

$$(5.7)\qquad \nu((u,v)) - (p, \text{div } v) = (f,v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

Let us take $v = u^{m+1}-u$ in equations (5.2) and (5.7) and then let us subtract the resulting equations; this gives:

$$\nu\|u^{m+1} - u\|^2 = (p^m - p, \text{ div } u^{m+1})$$

or

$$(5.8)\qquad \nu\|v^{m+1}\|^2 = (q^m, \text{ div } v^{m+1}),$$

where we have set

$$(5.9)\qquad v^{m+1} = u^{m+1} - u.$$

(5.10) $$q^m = p^m - p.$$

Taking $q = p^{m+1} - p$ in (5.3), we get:

$$(q^{m+1}-q^m, q^{m+1}) + \rho(\text{div } v^{m+1}, q^{m+1}) = 0,$$

or equivalently

(5.11) $$|q^{m+1}|^2 - |q^m|^2 + |q^{m+1}-q^m|^2 = -2\rho(\text{div } v^{m+1}, q^{m+1}).$$

We next multiply equation (5.8) by 2ρ, and then add equation (5.11), obtaining

(5.12) $$|q^{m+1}|^2 - |q^m|^2 + |q^{m+1}-q^m|^2 + 2\rho\nu\|v^{m+1}\|^2$$

$$= -2\rho(\text{div } v^{m+1}, q^{m+1}-q^m).$$

We majorize the right-hand side of (5.12) by

$$2\rho|\text{div } v^{m+1}|\ |q^{m+1}-q^m|$$

which is less than or equal to

$$2\rho\sqrt{n}\|v^{m+1}\|\ |q^{m+1}-q^m|$$

since

(5.13) $$|\text{div } v| \leq \sqrt{n}\|v\|, \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

We can then majorize the last expression by

$$\delta|q^{m+1}-q^m|^2 + \frac{\rho^2 n}{\delta}\|v^{m+1}\|^2,$$

where $0 < \delta < 1$ is arbitrary at the present time. Hence

(5.14) $$|q^{m+1}|^2 - |q^m|^2 + (1-\delta)|q^{m+1}-q^m|^2 + \rho(2\nu - \frac{\rho n}{\delta})\|v^{m+1}\|^2 \leq 0.$$

If we add the inequalities (5.14) for $m = 0,\cdots,N$, we find

(5.15) $$|q^{N+1}|^2 + (1-\delta)\sum_{m=0}^{N}|q^{m+1}-q^m|^2 + (2\nu - \frac{\rho n}{\delta})\rho\sum_{m=0}^{N}\|v^{m+1}\|^2 \leq |q^0|^2.$$

Because of condition (5.6), there exists some δ such that

$$0 < \frac{\rho n}{2\nu} < \delta < 1,$$

and hence

$$(2\nu - \frac{\rho n}{\delta}) > 0.$$

With such a δ fixed, the inequality (5.15) shows that

$$(5.16) \quad \begin{cases} |q^{m+1}-q^m|^2 = |p^{m+1}-p^m|^2 \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty, \\ \|v^{m+1}\|^2 = \|u^{m+1}-u\|^2 \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \end{cases}$$

The convergence of u^{m+1} to u is thereby proved. Now by (5.14) we see also that the sequence p^m is bounded in $L^2(\Omega)$. We can then extract from p^m a subsequence $p^{m'}$ converging weakly in $L^2(\Omega)$ to some element p_*. The equation (5.2) gives in the limit

$$\nu((u,v)) - (p_*, \text{div } v) = (f,v), \quad \forall v \in \mathbb{H}_0^1(\Omega),$$

and by comparison with (5.7), we get

$$(p-p_*, \text{div } v) = 0, \quad \forall v \in \mathbb{H}_0^1(\Omega),$$

whence

$$\text{grad}(p-p_*) = 0, \quad p_* = p + \text{const.}$$

From any subsequence of p^m, we can extract a subsequence converging weakly in $L^2(\Omega)$ to $p + c$; hence the sequence p^m converges as a whole to p for the weak topology of $L^2(\Omega)/\mathbb{R}$.

Remark 5.1.

Let us choose p by demanding the condition

$$\int_\Omega p(x)dx = 0.$$

Let us suppose that p^0 in $L^2(\Omega)$ is chosen so that

$$\int_\Omega p^0(x)dx = 0.$$

Then clearly we have

$$\int_\Omega p^m(x)dx = 0, \quad \forall m \geq 1$$

and the whole sequence p^m converges to p, weakly, in the space $L^2(\Omega)$.

5.2 Arrow-Hurwicz Algorithm.

In this case too, the functions u and p are the limits of two sequences u^m, p^m which are recursively defined.

We start the algorithm with arbitrary elements u^0, p^0,

$$u^0 \in \mathbb{H}_0^1(\Omega), \quad p^0 \in L^2(\Omega). \tag{5.17}$$

When p^m and u^m are known, we define p^{m+1} and u^{m+1} as the solutions of

$$\begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega) \text{ and} \\ ((u^{m+1}-u^m, v)) + \rho\nu((u^m, v)) - \rho(p^m, \operatorname{div} v) \\ \qquad = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \end{cases} \tag{5.18}$$

$$\begin{cases} p^{m+1} \in L^2(\Omega) \text{ and} \\ \alpha(p^{m+1}-p^m, q) + \rho(\operatorname{div} u^{m+1}, q) = 0, \quad \forall q \in L^2(\Omega). \end{cases} \tag{5.19}$$

We suppose that ρ and α are two stricly positive numbers; conditions on ρ and α will appear later.

The existence and uniqueness of $u^{m+1} \in \mathbb{H}_0^1(\Omega)$ satisfying (5.18) is easily established with the projection theorem; u^{m+1} is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u^{m+1} = -\Delta u^m + \rho\nu\Delta u^m - \rho \operatorname{grad} p^m + f \\ u^{m+1} \in \mathbb{H}_0^1(\Omega). \end{cases} \tag{5.20}$$

Then p^{m+1} is explicitly given by (5.19) which is equivalent to

$$p^{m+1} = p^m - \frac{\rho}{\alpha} \operatorname{div} u^{m+1} \in L^2(\Omega). \tag{5.21}$$

Convergence of the Algorithm.

Theorem 5.2.

If the numbers α and ρ satisfy

$$0 < \rho < \frac{2\alpha\nu}{\alpha\nu^2+n}, \tag{5.22}$$

then, as $m \to \infty$, u^m converges to u in $\mathbb{H}_0^1(\Omega)$ and p^m converges to p weakly in $L^2(\Omega)/R$.

Proof.

Let

$$v^m = u^m - u, \tag{5.23}$$

$$q^m = p^m - p. \tag{5.24}$$

Equations (5.18) and (5.7) give

$$((v^{m+1}-v^m,v)) + \rho\nu((v^m,v)) = \rho(q^m, \text{div } v), \quad \forall v\in H_0^1(\Omega),$$

and taking $v = v^{m+1}$ we obtain

$$\begin{aligned}(5.25)\qquad \|v^{m+1}\|^2 - \|v^m\|^2 + \|v^{m+1}-v^m\|^2 + 2\rho\nu\|v^{m+1}\|^2 &\\ = 2\rho\nu((v^{m+1},v^{m+1}-v^m)) + 2\rho(q^m, \text{div } v^{m+1}) &\\ \leq \delta\|v^{m+1}-v^m\|^2 + \frac{\rho^2\nu^2}{\delta}\|v^{m+1}\|^2 + 2\rho(q^m, \text{div } v^{m+1}), &\end{aligned}$$

where $\delta > 0$ is arbitrary at the present time.

Equation (5.19), with $q = q^{m+1}$ can be written as

$$\begin{aligned}\alpha|q^{m+1}|^2 - \alpha|q^m|^2 + \alpha|q^{m+1}-q^m|^2 &= -2\rho(q^{m+1}, \text{div } u^{m+1})\\ &= -2\rho(q^m, \text{div } v^{m+1}) - 2\rho(q^{m+1}-q^m, \text{div } v^{m+1})\\ &\leq -2\rho(q^m, \text{div } v^{m+1}) + 2\rho|q^{m+1}-q^m|\ |\text{div } v^{m+1}|\\ &\leq \text{(by (5.13))}\\ &\leq -2\rho(q^m, \text{div } v^{m+1}) + 2\rho\sqrt{n}|q^{m+1}-q^m|\ \|v^{m+1}\|.\end{aligned}$$

Finally, with the same δ as before,

$$\begin{aligned}(5.26)\qquad \alpha|q^{m+1}|^2 - \alpha|q^m|^2 + \alpha|q^{m+1}-q^m|^2 &\leq -2\rho(q^m, \text{div } v^{m+1})\\ &+ \alpha\delta|q^{m+1}-q^m|^2 + \frac{\rho^2 n}{\alpha\delta}\|v^{m+1}\|^2.\end{aligned}$$

Adding inequalities (5.25) and (5.26), we get

$$\begin{aligned}(5.27)\qquad \alpha|q^{m+1}|^2 + \|v^{m+1}\|^2 - \alpha|q^m|^2 - \|v^m\|^2 + \alpha(1-\delta)|q^{m+1}-q^m|^2 &\\ + (1-\delta)\|v^{m+1}-v^m\|^2 + \rho\left(2\nu - \frac{\rho\nu^2}{\delta} - \frac{\rho n}{\alpha\delta}\right)\|v^{m+1}\|^2 \leq 0. &\end{aligned}$$

If condition (5.22) holds, then

$$2\nu > \rho\nu^2 + \frac{\rho n}{\alpha},$$

and for some $0 < \delta < 1$ sufficiently close to 1, we have again

$$2\nu > \frac{1}{\delta}(\rho\nu^2 + \frac{\rho n}{\alpha})$$

so that

$$\rho(2\nu - \frac{\rho\nu^2}{\delta} - \frac{\rho n}{\alpha\delta}) > 0. \tag{5.28}$$

By adding inequalities (5.27) for $m = 0,\cdots,N$, we obtain an inequality of the same type as (5.15), and the proof is completed as for Theorem 5.1.

Remark 5.2.

It is easy to extend the Remark 5.1 to this algorithm.

5.3 Discrete Form of these Algorithms

We describe the discrete form of these algorithms in the case of finite differences (approximation APX 1).

In order to actually compute the step functions u_h and π_h which are solutions of (3.64), (3.71), (3.73), we define two sequences of step functions u_h^m, π_h^m, of the type

$$u_h^m = \sum_{M\in\mathring{\Omega}_h^1} \xi_M w_{hM}, \quad \xi_M \in \mathbb{R}^n \quad (\text{i.e., } u_h^m \in W_h) \tag{5.29}$$

$$\pi_h^m = \sum_{M\in\mathring{\Omega}_h^1} \eta_M w_{hM}, \quad \eta_M \in \mathbb{R}, \tag{5.30}$$

which are recursively defined by the analog of one of the preceding algorithms.

Uzawa Algorithm

We start with an arbitrary π_h^0 of type (5.30). When π_h^m is known, we define u_h^{m+1} and π_h^{m+1} by

$$\begin{cases} u_h^{m+1} \in W_h \quad \text{and} \\ \nu((u_h^{m+1}, v_h))_h - (\pi_h^m, D_h v_h) = (f, v_h), \quad \forall v_h \in W_h \end{cases} \tag{5.31}$$

(5.32) $$\pi_h^{m+1}(M) = \pi_h^m(M) - \rho(D_h u_h^{m+1})(M)$$

where D_h is the discrete divergence operator defined by (3.69).

If ρ satisfies the same condition (5.6), a repetition of the proof of Theorem 5.1 shows that, as $m \longrightarrow \infty$

(5.33) $$u_h^m \longrightarrow u_h \quad \text{in } W_h,$$

(5.34) $$\pi_h^m \longrightarrow \pi_h \quad \text{up to a constant;}$$

the convergence holds for any norm on the finite dimensional spaces considered.

Arrow-Hurwicz Algorithm

We start with arbitrary u_h^0, π_h^0 of type (5.29) and (5.30) respectively. When u_h^m, π_h^m are known, we define u_h^{m+1}, π_h^{m+1} by

(5.35) $$u_h^{m+1} \in W_h \quad \text{and}$$

$$((u_h^{m+1} - u_h^m, v_h))_h + \rho\nu((u_h^m, v_h))_h$$

$$-\rho(\pi_h^m, D_h v_h) = (f, v_h), \quad \forall v_h \in W_h.$$

(5.36) $$\pi_h^{m+1}(M) = \pi_h^m(M) - \frac{\rho}{\alpha} D_h u_h^{m+1}(M), \quad \forall M \in \mathring{\Omega}_h^1.$$

If ρ satisfies the condition (5.22), an extension of the proof of Theorem 5.2 gives the convergence (5.33) - (5.34).

Discrete Arrow-Hurwicz Algorithm

The problems (5.31) and (5.35) are discrete Dirichlet problems and their solution is easy and quite standard. Nevertheless, it is interesting to notice that in the finite dimensional case, we can use another form of Arrow-Hurwicz algorithm, for which we do not have any boundary value problem to solve during the iteration process.

When u_h^m, π_h^m are known, we define u_h^{m+1} by

(5.37) $$u_h^{m+1} \in W_h \quad \text{and,}$$

$$(u_h^{m+1} - u_h^m, v_h) + \rho\nu((u_h^m, v_h))_h - \rho(\pi_h^m, D_h v_h) = (f, v_h), \quad \forall v_h \in W_h,$$

and then π_h^{m+1} is again defined by (5.36). The variational equation (5.37) is equivalent to the following equations

$$(5.38)\qquad u_h^{m+1}(M) = u_h^m(M) + \rho\nu \sum_{i=1}^{n} (\delta_{ih}^2 u_h^m)(M)$$
$$-\rho(\overline{\nabla}_h \pi_h^m)(M) + f_h(M)., \quad \forall M \in \mathring{\Omega}_h^1,$$

where $\overline{\nabla}_h$ and f_h were defined in (3.74)

The proof of Theorem 5.2 can be extended to this situation as follows. Since W_h is a finite dimensional space, all the norms defined on W_h are equivalent, and hence there exists some constant $S(h)$ depending on h, such that

$$(5.39)\qquad \|u_h\|_h \leq S(h)|u_h|, \quad \forall u_h \in W_h \quad .$$

We will compute $S(h)$, and use this remark extensively in Chapter III ($S(h) = 2(\sum_{i=1}^{n} \frac{1}{h_i^2})^{\frac{1}{2}}$).

Now we have: if ρ satisfies

$$(5.40)\qquad 0 < \rho < \frac{2\alpha\nu}{\alpha\nu^2 S^2(h)+n} ,$$

then the convergences (5.33) - (5.34) are also true for the algorithm (5.36) - (5.37).

The proof is the same as for Theorem 5.2. The inequality (5.25) is just replaced by

$$(5.41)\qquad (v_h^m = u_h^m - u_h, \quad \kappa_h^m = \pi_h^m - \pi_h):$$
$$|v_h^{m+1}|^2 - |v_h^m|^2 + |v_h^{m+1} - v_h^m| + 2\rho\nu\|v_h^{m+1}\|_h^2$$
$$= 2\rho\nu((v_h^{m+1}, v_h^{m+1} - v_h^m))_h + 2\rho(\kappa_h^m, D_h v_h^{m+1})$$
$$\leq 2\rho\nu\|v_h^{m+1}\|_h \|v_h^{m+1} - v_h^m\|_h + 2\rho(\kappa_h^m, D_h v_h^{m+1})$$
$$\leq 2\rho\nu S(h) \|v_h^{m+1}\|_h |v_h^{m+1} - v_h^m| + 2\rho(\kappa_h^m, D_h v_h^{m+1})$$
$$\leq \delta|v_h^{m+1} - v_h^m|^2 + \frac{\rho^2\nu^2 S^2(h)}{\delta}\|v_h^{m+1}\|_h^2$$
$$+ 2\rho(\kappa_h^m, D_h v_h^{m+1}).$$

The inequality (5.27) is accordingly changed to

$$\text{(5.42)}\qquad \begin{aligned} &\alpha|\kappa_h^{m+1}|^2 + |v_h^{m+1}|^2 - \alpha|\kappa_h^m|^2 - |v_h^m|^2 \\ &+ \alpha(1-\delta)|\kappa_h^{m+1}-\kappa_h^m|^2 + (1-\delta)|v_h^{m+1}-v_h^m|^2 \\ &+ \rho\left(2\nu - \frac{\rho\nu^2 S^2(\bar{h})}{\delta} - \frac{\rho n}{\alpha\delta}\right)\|v_h^{m+1}\|_h^2 \leq 0; \end{aligned}$$

and because of (5.40), inequality (5.42) leads to the same conclusion as (5.27).

§6. SLIGHTLY COMPRESSIBLE FLUIDS

The stationary linearized equations of slightly compressible fluids are

(6.1) $$-\nu\Delta u_\varepsilon - \frac{1}{\varepsilon}\,\text{grad div } u_\varepsilon = f \quad \text{in } \Omega,$$

(6.2) $$u_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

where $\varepsilon > 0$ is "small". Equations (6.1) - (6.2) are also the stationary Lamé equations of elasticity.

We will show that equations (6.1) - (6.2) have a unique solution u_ε for $\varepsilon > 0$ fixed, and that u_ε converges to the solution u of the Stokes equations as $\varepsilon \longrightarrow 0$.

At first, equations (6.1) - (6.2) were used as "approximate" equations for Stokes equations -- one way to overcome the difficulty "div u = 0", was to solve equations (6.1) - (6.2) with ε sufficiently small in place of solving the Stokes equations. Nowadays, since many efficient algorithms are known for solving Stokes equations, and since the discretization of (6.1)-(6.2) leads to a very ill-conditioned matrix for very small ε, one can try to do the converse: compute u_ε for small ε by using Stokes equations.

In Section 6.1 we show the relation between u_ε and u and in Section 6.2 we give an asymptotic development of u_ε as $\varepsilon \longrightarrow 0$. Then in Section 6.3 we show how one can proceed to compute u_ε, for small ε, using this asymptotic development.

6.1 Convergence of u_ε to u.

Theorem 6.1.

Let Ω be a bounded lipschitzian domain in $\mathbb{R}^n$.
For $\varepsilon > 0$ fixed, there exists a unique $u_\varepsilon \in \mathbb{H}^1_0(\Omega)$ which satisfies (6.1).
When $\varepsilon \longrightarrow 0$,

(6.3) $$u_\varepsilon \longrightarrow u \quad \text{in the norm of } \mathbb{H}^1_0(\Omega),$$

(6.4) $$-\frac{\text{div } u_\varepsilon}{\varepsilon} \longrightarrow p \quad \text{in the norm of } L^2(\Omega),$$

where u and p are defined by (2.6) - (2.9) *and moreover*

(6.5) $$\int_\Omega p(x)\,dx = 0.$$

<u>Proof</u>.

It is easy to show that the problem (6.1) - (6.2) is equivalent to the following variational problem:

<u>To find</u> $u_\varepsilon \in \mathbb{H}_0^1(\Omega)$ <u>such that</u>

$$(6.6) \qquad \nu((u_\varepsilon,v)) + \frac{1}{\varepsilon}(\operatorname{div} u_\varepsilon, \operatorname{div} v) = (f,v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

Actually, if u_ε satisfies (6.1) - (6.2) then $u_\varepsilon \in \mathbb{H}_0^1(\Omega)$ and satisfies (6.6) for each $v \in \mathcal{D}(\Omega)$. By a continuity argument, it satisfies (6.6) too, for each $v \in \mathbb{H}_0^1(\Omega)$. Conversely, if $u_\varepsilon \in \mathbb{H}_0^1(\Omega)$ is a solution of (6.6) then u_ε satisfies (6.1) in the distribution sense and (6.2) in the sense of trace theorems.

The existence and uniqueness of u_ε satisfying (6.6) results from the projection theorem: we apply Theorem 2.2 with

$$W = \mathbb{H}_0^1(\Omega), \; a(u,v) = \nu((u,v)) + \frac{1}{\varepsilon}(\operatorname{div} u, \operatorname{div} v),$$

$$\langle \ell, v \rangle = (f,v).$$

The coercivity of a and the continuity of a and ℓ are obvious.

To prove (6.3) let us substract (2.7) from (6.1); this gives

$$(6.7) \qquad -\nu\Delta(u_\varepsilon - u) - \frac{1}{\varepsilon}\operatorname{grad}\operatorname{div} u_\varepsilon = + \operatorname{grad} p$$

and thus

$$(6.8) \quad \nu((u_\varepsilon - u, v)) + \frac{1}{\varepsilon}(\operatorname{div} u_\varepsilon, \operatorname{div} v) = -(p, \operatorname{div} v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

Equation (6.8) follows easily from (6.7) for $v \in \mathcal{D}(\Omega)$; by a continuity argument, (6.8) is satisfied for each $v \in \mathbb{H}_0^1(\Omega)$.

Let us put $v = u_\varepsilon - u$ in (6.8); we obtain

$$\begin{aligned} &\nu\|u_\varepsilon - u\|^2 + \frac{1}{\varepsilon}|\operatorname{div} u_\varepsilon|^2 \\ &= -(p, \operatorname{div} u_\varepsilon) \le |p|\,|\operatorname{div} u_\varepsilon| \\ &\le \frac{1}{2\varepsilon}|\operatorname{div} u_\varepsilon|^2 + \frac{\varepsilon}{2}|p|^2 \end{aligned}$$

so that

$$(6.9) \qquad \nu\|u_\varepsilon - u\|^2 + \frac{1}{2\varepsilon}|\operatorname{div} u_\varepsilon|^2 \le \frac{\varepsilon}{2}|p|^2.$$

This proves (6.3). Consequently, (6.7) shows that

(6.10) $$\frac{\partial}{\partial x_i}\left(\frac{\operatorname{div} u_\varepsilon}{\varepsilon}\right) \longrightarrow -\frac{\partial p}{\partial x_i},$$

in the norm of $H^{-1}(\Omega)$, for $i = 1,\cdots,n$ ($\Delta(u_\varepsilon - u)$ converges to 0 in $H^{-1}(\Omega)$, because of (6.3)).

According to the following lemma,

(6.11) $$\left|p + \frac{\operatorname{div} u_\varepsilon}{\varepsilon}\right| \leq \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i}\left(p + \frac{\operatorname{div} u_\varepsilon}{\varepsilon}\right)\right\|_{H^{-1}(\Omega)},$$

since, because u_ε vanishes on $\partial\Omega$ and (6.5) holds,

$$\int_\Omega \left(p + \frac{\operatorname{div} u_\varepsilon}{\varepsilon}\right) dx = 0 \quad .$$

The convergence (6.4) is proved.

Lemma 6.1.

Let Ω be a bounded lipschitzian domain in $\mathbb{R}^n$. Then there exists a constant $c = c(\Omega)$ depending only on Ω, such that

(6.12) $$|\sigma|_{L^2(\Omega)} \leq c(\Omega) \left\{\left|\int_\Omega \sigma\, dx\right| + \sum_{i=1}^{n} \left\|\frac{\partial\sigma}{\partial x_i}\right\|_{H^{-1}(\Omega)}\right\},$$

for every σ in $L^2(\Omega)$.

Proof.

Let us denote by $[\sigma]$ the expression between the brackets on the right-hand side of (6.12); $[\sigma]$ is a norm on $L^2(\Omega)$: it is obviously a semi-norm and, if $[\sigma] = 0$, then σ is a constant since $\frac{\partial\sigma}{\partial x_i} = 0$, $i = 1,\cdots,n$, and this constant is zero since $\int_\Omega \sigma\, dx = 0$.

It is clear that there exists a constant $c' = c'(\Omega)$ such that

(6.13) $$[\sigma] \leq c'(\Omega)\,|\sigma|_{L^2(\Omega)}, \quad \forall \sigma \in L^2(\Omega).$$

If we show that $L^2(\Omega)$ is complete for the norm $[\sigma]$ then, by the closed graph theorem, $[\sigma]$ and $|\sigma|$ will be two equivalent norms on $L^2(\Omega)$ and (6.12) will be proved.

In order to show that $L^2(\Omega)$ is complete for the norm $[\sigma]$, let us consider a sequence σ_m, which is a Cauchy sequence for this norm. Then the integrals $\int_\Omega \sigma_m dx$ form a Cauchy sequence in R and the derivatives $\frac{\partial \sigma_m}{\partial x_i}$ are Cauchy sequences in $H^{-1}(\Omega)$:

$$(6.14) \qquad \int_\Omega \sigma_m dx \longrightarrow \lambda \quad \text{as} \quad m \longrightarrow \infty,$$

$$(6.15) \qquad \frac{\partial \sigma_m}{\partial x_i} \longrightarrow \chi_i \quad \text{as} \quad m \longrightarrow \infty, \quad \text{in} \quad H^{-1}(\Omega), \quad 1 \leq i \leq n.$$

Because of (6.15) and the de Rham [1] theorem, there exists some distribution σ such that

$$\chi_i = \frac{\partial \sigma}{\partial x_i}, \quad 1 \leq i \leq n.$$

Proposition 1.2 shows that $\sigma \in L^2(\Omega)$. We can then choose σ so that

$$\int_\Omega \sigma \, dx = \lambda$$

and it is easy to see that the sequence σ_m converges to this element σ of $L^2(\Omega)$ in the norm $[\sigma]$.

Remark 6.1.

If Ω is not connected, (6.12) is true if we replace $\left|\int_\Omega \sigma \, dx\right|$ by

$$\sum_j \left|\int_{\Omega_j} \sigma \, dx\right|$$

where the Ω_j are the connected components of Ω. For extending Theorem 6.1 to this case we just have to define p by

$$(6.16) \qquad \int_{\Omega_j} p dx = 0, \quad \forall \Omega_j.$$

6.2 Asymptotic Development of u_ε.

From now on we denote by u^0 and p^0 the solution of Stokes problem (2.6) - (2.9) which satisfies (6.5) (in place of u and p).

We will show that u_ε has an asymptotic development

$$(6.17) \qquad u_\varepsilon = u^0 + \varepsilon u + \varepsilon u^2 + \cdots + \varepsilon^N u^N + \cdots$$

where all the u^i belong to the space $H_0^1(\Omega)$.

The functions u^i and some auxiliary functions p^i are recursively defined as follows:

(6.18) u^0, p^0, are already known;

when u^{m-1}, p^{m-1} are known $(m \geq 1)$, we define u^m and p^m as the solutions of the nonhomogeneous Stokes problem

$$u^m \in \mathbb{H}_0^1(\Omega)\ ,\ p^m \in L^2(\Omega), \tag{6.19}$$

$$-\nu\Delta u^m + \text{grad } p^m = 0 \tag{6.20}$$

$$\text{div } u^m = -p^{m-1} \tag{6.21}$$

$$\int_\Omega p^m(x)\ dx = 0. \tag{6.22}$$

The existence and uniqueness of u^m and p^m follow Proposition 2.3 in the two and three-dimensional cases (see also Remark 2.6). The condition (6.22) is useful in two ways: it ensures the complete uniqueness of p^m which is otherwise only unique up to an additive constant; it also ensures the compatibility condition necessary for the level $m + 1$:

$$\int_\Omega \text{div } u^{m+1}\ dx = \int_\Gamma u^{m+1}\cdot\nu\ d\Gamma = 0 = -\int_\Omega p^m\ dx. \tag{6.23}$$

We denote by u_ε^N, p_ε^N, $N \geq 1$, the quantities

$$u_\varepsilon^N = \sum_{m=0}^{N} \varepsilon^m u^m, \tag{6.24}$$

$$p_\varepsilon^N = \sum_{m=0}^{N} \varepsilon^m p^m. \tag{6.25}$$

<u>Theorem 6.2.</u>

<u>Let Ω be a bounded domain of class $\mathscr{C}^2$ in $\mathbb{R}^2$ or $\mathbb{R}^3$.</u>

<u>Then for each $m \geq 1$, there exist functions u^m, p^m, uniquely defined by</u> (6.19) - (6.22).

<u>For each</u> $N \geq 0$, <u>as</u> $\varepsilon \longrightarrow 0$,

$$\frac{u_\varepsilon - u_\varepsilon^N}{\varepsilon^N} \longrightarrow 0 \quad \text{in the } \mathbb{H}_0^1(\Omega) \text{ norm}, \tag{6.26}$$

$$\frac{1}{\varepsilon^N}\left(-\frac{\text{div } u_\varepsilon}{\varepsilon} - p_\varepsilon^N\right) \longrightarrow 0 \quad \text{in the } L^2(\Omega) \text{ norm}. \tag{6.27}$$

Proof.

The existence and uniqueness results have been established as previously remarked.

We multiply (6.20) by ε^m and add these equations for $m = 1,\cdots,N$. We then add the resulting equation to the equation satisfied by u^0 and p^0 (formerly denoted by u and p)

$$-\nu\Delta u^0 + \text{grad } p^0 = f.$$

After expanding we obtain

$$-\nu\Delta u_\varepsilon^N - \frac{1}{\varepsilon} \text{grad div } u_\varepsilon^N = f - \varepsilon^N \text{ grad } p^N.$$

By comparison with (6.3) we find

(6.28) $$u_\varepsilon - u_\varepsilon^N \in \mathbb{H}_0^1(\Omega), \quad p^N \in L^2(\Omega),$$

(6.29) $$-\nu\Delta(u_\varepsilon - u_\varepsilon^N) - \frac{1}{\varepsilon} \text{grad div}(u_\varepsilon - u_\varepsilon^N) = +\varepsilon^N \text{ grad } p^N.$$

As done for (6.8), we show that (6.29) is equivalent to

(6.30) $$\nu((u_\varepsilon - u_\varepsilon^N, v)) + \frac{1}{\varepsilon}(\text{div}(u_\varepsilon - u_\varepsilon^N), \text{div } v) = -\varepsilon^N(p^N, \text{div } v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

Putting $v = u_\varepsilon - u_\varepsilon^N$ in (6.30) we get

$$\begin{aligned}\nu\|u_\varepsilon - u_\varepsilon^N\|^2 &+ \frac{1}{\varepsilon}|\text{div}(u_\varepsilon - u_\varepsilon^N)|^2 \\ &= -\varepsilon^N(p^N, \text{div}(u_\varepsilon - u_\varepsilon^N)) \\ &\leq \varepsilon^N|p^N|\,|\text{div}(u_\varepsilon - u_\varepsilon^N)| \\ &\leq \frac{1}{2\varepsilon}|\text{div}(u_\varepsilon - u_\varepsilon^N)|^2 + \frac{\varepsilon^{2N+1}}{2}|p^N|^2\end{aligned}$$

so that

(6.31) $$\nu\|u_\varepsilon - u_\varepsilon^N\|^2 + \frac{1}{2\varepsilon}|\text{div}(u_\varepsilon - u_\varepsilon^N)|^2 \leq \frac{\varepsilon^{2N+1}}{2}|p^N|^2.$$

The inequality (6.31) clearly implies (6.26). This, in turn, implies that $\frac{1}{\varepsilon^N}\Delta(u_\varepsilon - u_\varepsilon^N) \longrightarrow 0$ in $\mathbb{H}^{-1}(\Omega)$ and hence (6.29) shows that

(6.32) $$\frac{1}{\varepsilon^{N+1}} \frac{\partial}{\partial x_i} \text{div}(u_\varepsilon - u_\varepsilon^N) \longrightarrow \frac{\partial p^N}{\partial x_i} \quad \text{in } H^{-1}(\Omega), \quad 1 \leq i \leq n.$$

But

$$- \frac{1}{\varepsilon} \text{grad div } u_\varepsilon^N = - \frac{1}{\varepsilon} \sum_{m=1}^{N} \varepsilon^m \text{ grad div } u^m$$

$$= \frac{1}{\varepsilon} \sum_{m=1}^{N} \varepsilon^m \text{ grad } p^{m-1} = \text{grad}(p_\varepsilon^N - \varepsilon^N p^N),$$

which along with (6.32) implies that

$$\frac{1}{\varepsilon^N} \frac{\partial}{\partial x_i} \left(- \frac{\text{div } u_\varepsilon}{\varepsilon} - p_\varepsilon^N\right) \longrightarrow 0 \quad \text{in} \quad H^{-1}(\Omega)$$

as $\varepsilon \longrightarrow 0$. Finally (6.27) results from (6.22), (6.25), and Lemma 6.1.

Remark 6.2.

Remark 6.1 can be easily adapted to Theorem 6.2: this theorem holds for non-connected sets Ω, provided we replace condition (6.22) by the conditions

$$\int_{\Omega_j} p^m \, dx = 0 \tag{6.33}$$

on each connected component Ω_j of Ω.

Remark 6.3.

Theorem 6.2 can be extended to higher dimensions, provided one proves the existence of u^m, p^m, satisfying (6.19) - (6.22).

It suffices to show that the divergence operator maps $\mathbb{H}_0^1(\Omega)$ onto $L^2(\Omega)/\mathbb{R}$ and this is most probably true although unproved (except for $n = 2,3$). See also Remark 2.6.

6.3 Numerical Algorithms.

Let us show rapidly how one can extend the algorithms described in Section 5 to algorithms for solving the nonhomogeneous Stokes problems (6.19) - (6.22) which is, at this point, the only difficulty for practical computation of the asymptotic development (6.17) of u_ε.

We only describe the adaptation of the Uzawa algorithm.

Changing our notation, we write problem (6.19) - (6.22) as the problem: to find v,p such that

$$v \in \mathbb{H}_0^1(\Omega), \quad p \in L^2(\Omega) \tag{6.34}$$

$$-\nu\Delta v + \text{grad } p = 0, \tag{6.35}$$

(6.36) $$\operatorname{div} v = \phi,$$

(6.37) $$\int_\Omega p(x)\, dx = 0,$$

where ϕ is given with

(6.38) $$\int_\Omega \phi(x)\, dx = 0.$$

The existence and uniqueness of v and p are known.

We start the algorithm with any

(6.39) $$p^0 \in L^2(\Omega), \quad \text{such that} \quad \int_\Omega p^0(x)\, dx = 0.$$

When p^m is known, we define v^{m+1} and p^{m+1} $(m \geq 0)$ by

(6.40) $$\begin{cases} v^{m+1} \in \mathbb{H}_0^1(\Omega) \quad \text{and} \\ \nu((v^{m+1}, w)) - (p^m, \operatorname{div} w) = 0, \quad \forall w \in \mathbb{H}_0^1(\Omega) \end{cases}$$

(6.41) $$\begin{cases} p^{m+1} \in L^2(\Omega) \quad \text{and} \\ (p^{m+1} - p^m, \theta) + \rho(\operatorname{div} v^{m+1} - \phi, \theta) = 0, \quad \forall \theta \in L^2(\Omega). \end{cases}$$

The equation (6.40) is a Dirichlet problem for v^{m+1}:

(6.42) $$\begin{cases} v^{m+1} \in \mathbb{H}_0^1(\Omega) \\ -\nu\Delta v^{m+1} = -\operatorname{grad} p^m \in \mathbb{H}^{-1}(\Omega) \end{cases}$$

and (6.41) gives p^{m+1} directly as

(6.43) $$p^{m+1} = p^m - \rho(\operatorname{div} v^{m+1} - \phi) \in L^2(\Omega).$$

We notice that

(6.44) $$\int_\Omega p^m\, dx = 0, \quad \forall m \geq 0.$$

Exactly as for Theorem 5.1 (see also Remark 5.1), one can prove the following result.

Theorem 6.3.

If the number ρ *satisfies*

$$0 < \rho < \frac{2\nu}{n} \tag{6.45}$$

then, as $m \to \infty$, v^{m+1} *converges to* v *in the norm of* $H_0^1(\Omega)$ *and* p^{m+1} *converges to* p *in* $L^2(\Omega)$ *weakly.*

CHAPTER II

Stationary Navier-Stokes Equations

INTRODUCTION

In this chapter we will be concerned with the stationary Navier-Stokes equations from the same point of view as in the previous chapter, i.e., existence, uniqueness, and numerical approximation of the solution. However, there are three important differences from the linear case; these are:

- The introduction of the compactness methods. For passing to the limit in the nonlinear term we need strong convergence results; these are obtained by compactness arguments.

- Some technical difficulties related to the nonlinear term and connected with the Sobolev inequalities. Their consequence is a treatment of the equation which varies slightly according to the dimension of the space.

- The non-uniqueness of solutions, in general. Uniqueness occurs only when "the data are small enough, or the viscosity is large enough."

In Section 1 we describe some existence, uniqueness, and regularity results in various situations (Ω bounded or not, homogeneous or inhomogeneous equations;$\cdots$). In Section 2 we prove a discrete Sobolev inequality and a discrete compactness theorem for step function spaces considered in the approximation (APX 1) of V (approximation of V by finite differences). The similar results for the approximations (APX 2),$\cdots$,(APX 4), are already available as consequences of the theorem in the continuous case.[1] Section 3 deals with the approximation of the stationary problem: discretization and resolution of the discretized problems.

[1] For the approximation (APX 5), such results are not proved yet.

§1. EXISTENCE AND UNIQUENESS THEOREMS.

In this section we study some existence and uniqueness results for the stationary (nonlinear) Navier-Stokes equations. The existence results are obtained by constructing approximate solutions to the equation by the Galerkin Method, and then passing to the limit, as in the linear case. As we already said, for passing to the limit, we need, in the nonlinear case, some strong convergence properties of the sequence and these are obtained by compactness methods.

In Section 1.1 we recall the Sobolev inequalities and a compactness theorem for the Sobolev spaces; this theorem is of course the basic tool for the compactness method. In Section 1.2 we give a variational formulation of the homogeneous Navier-Stokes equations (i.e., the Navier-Stokes equations with homogeneous boundary conditions); we study some properties of a nonlinear (trilinear) form which occurs in the variational formulation. We then give a general existence theorem and a rather restricted uniqueness result. In Section 1.3 we consider the case where the set Ω is unbounded and we give regularity results for solutions. Section 1.4 deals with inhomogeneous Navier-Stokes equations.

1.1 Sobolev Inequalities and Compactness Theorems.

Imbedding Theorems.

We recall the Sobolev imbedding theorems which will be used frequently from now on. Let m be an integer and p any finite number greater than or equal to one, $p \geq 1$; then, if $\frac{1}{p} - \frac{m}{n} = \frac{1}{q} > 0$ the space $W^{m,p}(\mathbb{R}^n)$ is included in $L^q(\mathbb{R}^n)$ and the injection is continuous. If $u \in W^{m,p}(\mathbb{R}^n)$ and $\frac{1}{p} - \frac{m}{n} = 0$ then u belongs to $L^q(\mathcal{O})$ for any bounded set $\mathcal{O}$ and any q, $1 \leq q < \infty$. If $\frac{1}{p} - \frac{m}{n} < 0$ then a function in $W^{m,p}(R^n)$ is almost everywhere equal to a continuous function; such a function has also some Hölder or Lipschitz continuity properties but such properties will not be used here; if a function belongs to $W^{m,p}(R^n)$ with $\frac{1}{p} - \frac{m}{n} < 0$ then the derivatives of order α belong to $W^{m-\alpha,p}(R^n)$ and some imbedding results of preceding type hold for these derivatives if $\frac{1}{p} - \frac{m-\alpha}{n} > 0$.

For $u \in W^{m,p}(R^n)$, $m \geq 1$, $1 \leq p < \infty$

$$(1.1)\quad \begin{cases} \text{if } \frac{1}{p}-\frac{m}{n}=\frac{1}{q}>0, & |u|_{L^q(\mathbb{R}^n)} \le c(m,p,n)\|u\|_{W^{m,p}(\mathbb{R}^n)}, \\ \text{if } \frac{1}{p}-\frac{m}{n}=0, & |u|_{L^q(\mathcal{O})} \le c(m,p,n,q,\mathcal{O})\,\|u\|_{W^{m,p}(\mathbb{R}^n)} \\ & \forall \text{ bounded set } \mathcal{O}\subset\mathbb{R}^n, \forall q,\ 1\le q<\infty, \\ \text{if } \frac{1}{p}-\frac{m}{n}<0, & |u|_{\mathcal{C}^0(\mathcal{O})} \le c(m,n,p,\mathcal{O})\,\|u\|_{W^{m,p}(\mathbb{R}^n)}, \\ & \forall \text{ bounded set } \mathcal{O},\ \mathcal{O}\subset\mathbb{R}^n. \end{cases}$$

If Ω is any open set of $\mathbb{R}^n$, results similar to (1.1) can usually be obtained if Ω is sufficiently smooth so that

(1.2) <u>There exists a continuous linear prolongation operator</u> $\Pi\in\mathcal{L}(W^{m,p}(\Omega),\ W^{m,p}(\mathbb{R}^n))$.

Property (1.2) is satisfied by a locally lipschitzian set Ω. When (1.2) is satisfied, the properties (1.1) applied to Πu, $u\in W^{m,p}(\Omega)$ give in particular, assuming that $u\in W^{m,p}(\Omega)$, $m\ge 1$, $1<p<\infty$, and (1.2) holds:

$$(1.3)\quad \begin{cases} \text{if } \frac{1}{p}-\frac{m}{n}=\frac{1}{q}>0, & |u|_{L^q(\Omega)} \le c(m,p,n,\Omega)\,\|u\|_{W^{m,p}(\Omega)}, \\ \text{if } \frac{1}{p}-\frac{m}{n}=0, & |u|_{L^q(\mathcal{O})} \le c(m,p,n,q,\mathcal{O},\Omega)\,\|u\|_{W^{m,p}(\Omega)}, \\ & \text{any } q,\ 1\le q<\infty, \text{ any bounded set } \mathcal{O}\subset\overline{\Omega}, \\ \text{if } \frac{1}{p}-\frac{m}{n}<0, & |u|_{\mathcal{C}^0(\mathcal{O})} \le c(m,p,n,q,\Omega,\mathcal{O})\,\|u\|_{W^{m,p}(\Omega)}, \\ & \text{any bounded set } \mathcal{O},\ \mathcal{O}\subset\overline{\Omega}. \end{cases}$$

When $u\in \overset{\circ}{W}{}^{m,p}(\Omega)$, the function $\tilde{u}$ which is equal to u in Ω and to 0 in $\complement\Omega$, belongs to $W^{m,p}(\mathbb{R}^n)$ and hence the properties (1.3) are valid without any hypothesis on Ω.

The case of particular interest for us is the case $p = 2$, $m = 1$, i.e., the case $H_0^1(\Omega)$. Without any regularity property required for Ω we have for

$u \in H_0^1(\Omega)$

$$(1.4)\quad \begin{cases} n = 2, & |u|_{L^q(\Omega)} \le c(q,\mathcal{O},\Omega)\,\|u\|_{H_0^1(\Omega)} \\ & \forall \text{ bounded set } \mathcal{O}\subset\Omega,\ \forall q,\ 1 \le q < \infty \\ n = 3, & |u|_{L^6(\Omega)} \le c(\Omega)\,\|u\|_{H_0^1(\Omega)} \\ n = 4, & |u|_{L^4(\Omega)} \le c(\Omega)\,\|u\|_{H_0^1(\Omega)} \\ n \ge 3, & |u|_{L^{\frac{2n}{n-2}}(\Omega)} \le c(\Omega)\,\|u\|_{H_0^1(\Omega)}. \end{cases}$$

Compactness Theorems.

Theorem 1.1.

Let Ω be any bounded open set of R^n satisfying (1.2). Then the imbedding

$$(1.5)\qquad W^{1,p}(\Omega) \subset L^{q_1}(\Omega)$$

is compact for any q_1, $1 \le q_1 < \infty$, if $p \ge n$; and for any q_1, $1 \le q_1 < q$ (q given by $\frac{1}{p} - \frac{1}{n} = \frac{1}{q}$) if $1 \le p < n$.

With the same values of p and q_1, the imbedding

$$(1.6)\qquad \overset{\circ}{W}{}^{1,p}(\Omega) \subset L^{q_1}(\Omega)$$

is compact for any bounded open set Ω.

As a particular case of Theorem 1.1 we notice that for any unbounded set Ω, if $u \in W^{1,p}(\Omega)$, then the restriction of u to $\mathcal{O}$, $\mathcal{O}\subset\bar{\mathcal{O}}\subset\Omega$, $\mathcal{O}$ bounded, belongs to $L^{q_1}(\mathcal{O})$ and this restriction mapping is compact

$$(1.7)\qquad W^{1,p}(\Omega) \longrightarrow L^{q_1}(\Omega)$$

(same values of p and q_1).

For all the preceding properties of Sobolev spaces, the reader is referred to the references mentioned in the first section of Chapter I (see also at the end the comments on Chapter I).

1.2 The Homogeneous Navier-Stokes Equations.

Let Ω be a lipschitzian, bounded open set in $\mathbb{R}^n$ with boundary Γ, let $f \in \mathbb{L}^2(\Omega)$ be a given vector function. We are looking for a vector function

$u = (u_1, \cdots, u_n)$ and a scalar function p, representing the velocity and the pressure of the fluid, which are defined in Ω and satisfy the following equations and boundary conditions:

$$-\nu\Delta u + \sum_{i=1}^{n} u_i D_i u + \text{grad } p = f \quad \text{in} \quad \Omega \tag{1.8}$$

$$\text{div } u = 0 \quad \text{in} \quad \Omega \tag{1.9}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{1.10}$$

Exactly as in Section 2.1 of Chapter I, if f,u,p are smooth functions satisfying (1.8) - (1.10) then $u \in V$ and, for each $v \in \mathcal{V}$,

$$\nu((u,v)) + b(u,u,v) = (f,v) \tag{1.11}$$

where

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i (D_i v_j) w_j \, dx. \tag{1.12}$$

A continuity argument shows moreover that equation (1.11) is satisfied by any $v \in V$. Conversely, if u is a smooth function in V such that (1.11) holds for each $v \in \mathcal{V}$ then because of Proposition 1.3, Chapter I, there exists a distribution p such that (1.8) is satisfied, and (1.9)-(1.10) are satisfied since $u \in V$.

For u and v in V, the expression $b(u,u,v)$ does not necessarily make sense and then the variational formulation of (1.8) - (1.10) is not exactly: "to find $u \in V$ such that (1.11) holds for each $v \in V$." The variational formulation will be slightly different, and this will be stated after studying some properties of the form $b(u,v,w)$.

Let us introduce first the following space:

$$\tilde{V} = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^n(\Omega); \tag{1.13}$$

of course $\mathbb{H}_0^1(\Omega) \cap \mathbb{L}^n(\Omega)$ and $\tilde{V}$ are equipped with the norm

$$\|u\|_{\mathbb{H}_0^1(\Omega)} + |u|_{\mathbb{L}^n(\Omega)}. \tag{1.14}$$

In general $\tilde{V}$ is a subspace of V, different from V but, because of (1.4), $\tilde{V} = V$ for $n = 2,3,$ or 4 (and Ω bounded).

(1.15) V_s = the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega)\cap\mathbb{H}^s(\Omega)$, $(s \geq 1)$;

it is understood again that $\mathbb{H}_0^1(\Omega)\cap\mathbb{H}^s(\Omega)$ and V_s are equipped with the hilbertian norm

$$\{\|u\|^2_{\mathbb{H}_0^1(\Omega)} + \|u\|^2_{\mathbb{H}^s(\Omega)}\}^{\frac{1}{2}}; \tag{1.16}$$

V_s is included in V.

<u>The Trilinear Form</u> b.

The form b is trilinear and continuous on various spaces among the spaces V, $\tilde{V}$, V_s. The most convenient result concerning b is the following result which is independent of any property of Ω.

<u>Lemma 1.1.</u>

The form b is defined and trilinear continuous on $\mathbb{H}_0^1(\Omega)\times\mathbb{H}_0^1(\Omega)\times(\mathbb{H}_0^1(\Omega)\cap\mathbb{L}^n(\Omega))$, Ω *bounded or unbounded, any dimension of space* $\mathbb{R}^n$.

<u>Proof.</u>

If $u,v\in V$ and $w\in\tilde{V}$, then because of (1.4) $(n \geq 3)$:

$$u_i\in L^{\frac{2n}{n-2}}(\Omega),\ D_iv_j\in L^2(\Omega),\ w_j\in L^n(\Omega),\ 1 \leq i,j \leq n.$$

By the Hölder inequality, $u_i(D_iv_j)w_j$ belongs to $L^1(\Omega)$ and

$$\left|\int_\Omega u_iD_iv_jw_j\,dx\right| \leq |u_i|_{L^{\frac{2n}{n-2}}(\Omega)}\,|D_iv_j|_{L^2(\Omega)}\,|w_j|_{L^n(\Omega)}. \tag{1.17}$$

Then $b(u,v,w)$ is well defined and

$$|b(u,v,w)| \leq c(n)\,\|u\|_{\mathbb{H}_0^1(\Omega)}\,\|v\|_{\mathbb{H}_0^1(\Omega)}\,\|w\|_{\mathbb{H}_0^1(\Omega)\cap\mathbb{L}^n(\Omega)}. \tag{1.18}$$

The form b is obviously trilinear and (1.18) ensures the continuity of b.

When $n = 2$, we have the same result, $(\mathbb{H}_0^1(\Omega)\cap\mathbb{L}^2(\Omega) = \mathbb{H}_0^1(\Omega))$, but (1.17) must be replaced by

$$\left|\int_\Omega u_iD_iv_jw_j\,dx\right| \leq |u_i|_{L^4(\Omega)}\,|D_iv_j|_{L^2(\Omega)}\,|w_j|_{L^4(\Omega)}. \tag{1.19}$$

In particular one has

Lemma 1.2.

For any open set Ω, b *is a trilinear continuous form on* $V \times V \times \tilde{V}$. *If* Ω *is bounded and* $n \leq 4$, b *is trilinear continuous on* $V \times V \times V$.

We will prove, when needed, other properties of b similar to those given before; the proof will be always the same as in Lemma 1.1 (use of Hölder's inequality and the imbedding theorem (1.4)).

We denote by $B(u,v)$, $u,v \in \mathbb{H}_0^1(\Omega)$, the linear continuous form on $\tilde{V}$ defined by

$$(1.20) \qquad \langle B(u,v),w\rangle = b(u,v,w), \quad u,v \in \mathbb{H}_0^1(\Omega), \ \forall w \in \tilde{V}.$$

For $u = v$, we write

$$(1.21) \qquad B(u) = B(u,u), \quad u \in \mathbb{H}_0^1(\Omega).$$

Another fundamental property of b is the following

Lemma 1.3.

For any open set Ω,

$$(1.22) \qquad b(u,v,v) = 0, \quad \forall u \in V, \quad v \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^n(\Omega)$$

$$(1.23) \qquad b(u,v,w) = -b(u,w,v), \quad \forall u \in V, \quad v,w, \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^n(\Omega).$$

Proof.

Property (1.23) is a consequence of (1.22) when we replace v by v + w, and we use the multilinear properties of b.

In order to prove (1.22), it suffices to show this equality for $u \in \mathcal{V}$ and $v \in \mathcal{D}(\Omega)$. But for such u and v

$$\int_\Omega u_i \, D_i v_j \, v_j \, dx = \int_\Omega u_i \, D_i \frac{(v_j)^2}{2} dx$$

$$= -\frac{1}{2} \int_\Omega D_i u_i (v_j)^2 \, dx \ ,$$

$$(1.24) \qquad b(u,v,v) = -\frac{1}{2} \sum_{j=1}^{n} \int_\Omega \operatorname{div} u (v_j)^2 \, dx = 0.$$

Variational Formulation.

For Ω bounded, and n arbitrary, we associate with (1.8) - (1.10) the problem

(1.25) To find $u \in V$ such that

$$\nu((u,v)) + b(u,u,v) = (f,v), \quad \forall v \in \tilde{V},$$

(f given in $\mathbb{L}^2(\Omega)$). It is clear from (1.11) and (1.13) that if u and p are smooth functions satisfying (1.8) - (1.10), then u satisfies (1.25). Conversely if $u \in V$ satisfies (1.25), then

(1.26) $$\langle -\nu\Delta u + \sum_i u_i D_i u - f,\ v\rangle = 0, \quad \forall v \in \mathcal{V};$$

$\Delta u \in \mathbb{H}^{-1}(\Omega)$, $f \in \mathbb{L}^2(\Omega)$, and $u_i D_i u \in \mathbb{L}^{n'}(\Omega)$ $(\frac{1}{n'} = 1 - \frac{1}{n})$, since $u_i \in \mathbb{L}^{\frac{2n}{n-2}}(\Omega)$ by (1.4) and $D_i u \in \mathbb{L}^2(\Omega)$. According then to Proposition 1.3 and Remark 1.5 of Chapter I, there exists a distribution $p \in L^1_{loc}(\Omega)$, such that (1.8) is satisfied in the distribution sense; then (1.9) and (1.10) are satisfied respectively in the distribution and the trace theorem senses.

Theorem 1.2.

Let Ω be a bounded set in $\mathbb{R}^n$ and let f be given in $\mathbb{H}^{-1}(\Omega)$.

Then Problem (1.25) has at least one solution $u \in V$ and there exists a distribution $p \in L^1_{loc}(\Omega)$ such that (1.8)-(1.9) are satisfied.

Proof.

We have only to prove the existence of u; the existence of p and the interpretation of (1.8)-(1.9) have already been shown.

The existence of u is proved by the Galerkin method: we construct an approximate solution of (1.25) and then pass to the limit.

The space $\tilde{V}$ is separable as a subspace of $\mathbb{H}^1_0(\Omega)$. Because of (1.13) there exists a sequence $w_1, \cdots, w_m, \cdots$, of elements of $\mathcal{V}$ which is free and total in $\tilde{V}$. This sequence is also free and total in V.

For each fixed integer $m \geq 1$, we would like to define an approximate solution u_m of (1.25) by

(1.27) $$u_m = \sum_{i=1}^{m} \xi_{i,m} w_i, \quad \xi_{i,m} \in \mathbb{R}$$

(1.28) $$\nu((u_m, w_k)) + b(u_m, u_m, w_k) = \langle f, w_k\rangle, \quad k = 1,\cdots,m.$$

The equations (1.27)–(1.28) are a system of nonlinear equations for $\xi_{1,m},\cdots,\xi_{m,m}$, and the existence of a solution of this system is not obvious, but follows from the next lemma.

Lemma 1.4.

Let X *be a finite dimensional Hilbert space with scalar product* $[\cdot,\cdot]$ *and norm* $[\cdot]$ *and let* P *be a continuous map from* X *into itself such that*

$$[P(\xi),\xi] > 0 \quad \text{for} \quad [\xi] = k > 0. \tag{1.29}$$

Then there exists $\xi \in X$, $[\xi] \leq k$, *such that*

$$P(\xi) = 0. \tag{1.30}$$

The proof of Lemma 1.4 follows the proof of Theorem 1.2. We apply this lemma for proving the existence of u_m, as follows:

X = the space spanned by $w_1,\cdots,w_m$; the scalar product on X is the scalar product $((\cdot,\cdot))$ induced by V, and $P = P_m$ is defined by

$$[P_m(u),v] = ((P_m(u),v)) = \nu((u,v)) + b(u,u,v) - (f,v), \quad \forall u,v \in X.$$

The continuity of the mapping P_m is obvious; let us show (1.29).

$$\begin{aligned}[P_m(u),u] &= \nu\|u\|^2 + b(u,u,u) - \langle f,u\rangle \\ &= (\text{by } (1.22)) \\ &= \nu\|u\|^2 - \langle f,u\rangle \\ &\geq \nu\|u\|^2 - \|f\|_{V'}\|u\| \quad ,\end{aligned}$$

$$[P_m(u),u] \geq \|u\|\,(\nu\|u\| - \|f\|_{V'}). \tag{1.31}$$

It follows that $[P_m u,u] > 0$ for $\|u\| = k$, and k large enough; more precisely, $k > \frac{1}{\nu}\|f\|_{V'}$. The hypotheses of Lemma 1.4 are satisfied and there exists a solution u_m of (1.27)–(1.28).

Passage to the Limit.

We multiply (1.28) by $\xi_{k,m}$ and add the corresponding equalities for $k = 1,\cdots,m$; this gives

$$\nu\|u_m\|^2 + b(u_m,u_m,u_m) = \langle f,u_m\rangle$$

or, because of (1.24),

$$\nu\|u_m\|^2 = \langle f,u_m\rangle \leq \|f\|_{V'} \|u_m\| \quad .$$

We obtain then the *a priori* estimate:

(1.32) $$\|u_m\| \leq \frac{1}{\nu} \|f\|_{V'} \quad .$$

Since the sequence u_m remains bounded in. V, there exists some u in V and a subsequence $m' \longrightarrow \infty$ such that

(1.33) $u_{m'} \rightharpoonup u$ for the weak topology of V.

The compactness theorem 1.1 shows in particular that the injection of V into $\mathbb{L}^2(\Omega)$ is compact, so we have also

(1.34) $u_{m'} \longrightarrow u$ in the norm of $\mathbb{L}^2(\Omega)$.

Let us admit for a short time the following lemma .

Lemma 1.5.

If u_μ converges to u in V weakly and in $\mathbb{L}^2(\Omega)$ strongly, then

(1.35) $b(u_\mu,u_\mu,v) \longrightarrow b(u,u,v), \quad \forall v \in \mathcal{V}.$

Then we can pass to the limit in (1.28) with the subsequence $m' \longrightarrow \infty$. From (1.33), (1.34), (1.35) we find that

(1.36) $\nu((u,v)) + b(u,u,v) = \langle f,v\rangle$

for any $v = w_1,\cdots,w_m\cdots$. Equation (1.36) is also true for any v which is a linear combination of $w_1,\cdots,w_m,\cdots$. Since these combinations are dense in $\tilde{V}$, a continuity argument shows finally that (1.36) holds for each $v \in \tilde{V}$ and that u is a solution of (1.25).

Proof of Lemma 1.4.

This is an easy consequence of the Brouwer fixed point theorem.

Suppose that P has no zero in the ball D of X centered at 0 and with radius k. Then the following application

$$\xi \longmapsto S(\xi) = -k \frac{P(\xi)}{[P(\xi)]}$$

maps D into itself and is continuous. The Brouwer theorem implies then that S has a fixed point in D: there exists $\xi_0 \in D$, such that

$$- k \frac{P(\xi_0)}{[P(\xi_0)]} = \xi_0 .$$

If we take the norm of both sides of this equation we see that $[\xi_0] = k$, and if we take the scalar product of each side with ξ_0, we find

$$[\xi_0]^2 = k^2 = - k \frac{[P(\xi_0),\xi_0]}{[P(\xi_0)]} .$$

This equality contradicts (1.29) and thus $P(\xi)$ must vanish at some point of D.

Proof of Lemma 1.5.

It is easy to show, as for (1.22)-(1.23), that

$$\begin{aligned} b(u_\mu,u_\mu,v) &= - b(u_\mu,v,u_\mu) \\ &= - \sum_{i,j=1}^{n} \int_\Omega u_{\mu i} u_{\mu j} \, D_i v_j \, dx. \end{aligned}$$

But $u_{\mu i}$ converges to u_i in $L^2(\Omega)$ strongly; since $D_i v_j \in L^\infty(\Omega)$, it is easy to check that

$$\int_\Omega u_{\mu i} u_{\mu j} \, D_i v_j \, dx \longrightarrow \int_\Omega u_i u_j \, D_i v_j \, dx.$$

Hence $b(u_\mu,v,u_\mu)$ converges to $b(u,v,u) = - b(u,u,v)$.

Uniqueness.

For uniqueness we only have the following result:

Theorem 1.3.

If $n \leq 4$ and if ν is sufficiently large or f "sufficiently small" so that

(1.37) $$\nu^2 > c(n) \, \|f\|_{V'},$$

then there exists a unique solution u of (1.25).

The constant $c(n)$ in (1.37) is the constant $c(n)$ in (1.18); its estimation is connected with the estimation of the constants in (1.4) and this is given for instance in Lions [1].

Proof of Theorem 1.3.

We can take $v = u$ in (1.25) since $\tilde{V} = V$ for $n \leq 4$; we obtain with (1.22)

(1.38) $$\nu \, \|u\|^2 = \langle f,u \rangle \leq \|f\|_{V'} \, \|u\|$$

so that any solution u of (1.25) satisfies

$$\|u\| \leq \frac{1}{\nu} \|f\|_{V'}. \tag{1.39}$$

Now let u_* and u_{**} be two different solutions of (1.25) and let $u = u_* - u_{**}$. We substract the equations (1.25) corresponding to u_* and u_{**} and we obtain

$$\nu((u,v)) + b(u_*,u,v) + b(u,u_*,v) = 0, \quad \forall v \in V. \tag{1.40}$$

We take $v = u$ in (1.40) and use again (1.22); hence:

$$\nu\|u\|^2 = - b(u,u_*,u).$$

With (1.18) and (1.39) (for $u = u_*$) this gives

$$\nu\|u\|^2 \leq c(n) \|u\|^2 \|u_*\|$$

$$\leq \frac{c(n)}{\nu} \|f\|_{V'} \|u\|^2 ,$$

$$(\nu - \frac{c(n)}{\nu} \|f\|_{V'}) \|u\|^2 \leq 0 .$$

Because of (1.37) this inequality implies $\|u\| = 0$, which means $u_* = u_{**}$.

Remark 1.1.

The solution of (1.25) is probably not unique if (1.37) is not satisfied or at least for ν small enough (f fixed).

Using the bifurcation theory, it has been proven for problems very similar to (1.25) that for ν greater than some critical value ν_*, the solution is unique and, for ν smaller than this value there exist several solutions. This was proved by P. Rabinowitz for the stationary convection problem and by Velte for stationary inhomogeneous Navier-Stokes equations. See Remark 1.7 and the references cited in the "Comments and Bibliography".

Remark 1.2.

For $n > 4$, Theorem 1.2 shows the existence of solutions u of (1.25) satisfying (1.39): the majoration (1.32) and (1.33) give indeed:

$$\|u\| \leq \varliminf_{m' \to \infty} \|u_{m'}\| \leq \frac{1}{\nu} \|f\|_{V'}. \tag{1.41}$$

Nevertheless, the proof of Theorem 1.3 cannot be extended to this case; (1.40) holds for each $v \in \tilde{V}$ and it is not possible to take $v = u$.

1.3 The Homogeneous Navier-Stokes Equations (continued).

The Unbounded Case.

We can study the case for Ω unbounded by introducing the same spaces as in Section 2.3, Chapter I. Let us recall that

(1.42) $\quad Y =$ the completed space of $\mathcal{V}$ for the norm $\|\cdot\|$.

Let us consider also the space $\tilde{Y}$

(1.43) $\quad \tilde{Y} =$ the closure of $\mathcal{V}$ in the space $Y \cap \mathbb{L}^n(\Omega)$ equipped with the norm

(1.44) $$\|u\| + \|u\|_{\mathbb{L}^n(\Omega)}\ ^{(1)}.$$

We recall that because of Lemma 2.3, Chapter I, we have the continuous injection

(1.45) $$Y \subset \{u \in \mathbb{L}^\alpha(\Omega),\ D_i u \in \mathbb{L}^2(\Omega),\ \ 1 \leq i \leq n\}$$

for $n \geq 3$, where

(1.46) $$\alpha = \frac{2n}{n-2}.$$

Since Ω is unbounded the spaces $L^\gamma(\Omega)$ are not decreasing with increasing γ as in the bounded case and $\tilde{Y} \neq Y$ even for $n \leq 4$. Lemma 1.2 cannot be extended to the unbounded case; however, we have:

Lemma 1.6.

For $n \geq 3$, the form b is defined and trilinear continuous on $Y \times Y \times \tilde{Y}$ and

(1.47) $$b(u,v,v) = 0, \quad \forall u \in Y, \quad v \in \tilde{Y},$$

(1.48) $$b(u,v,w) = -\,b(u,w,v), \quad \forall u \in Y, \quad v,w \in \tilde{Y}.$$

Proof.

The inequality (1.17) ($n \geq 3$) is valid; for $u,v \in Y$, $w \in \tilde{Y}$ we then have

$$\left|\int_\Omega u_i D_i v_j w_j \, dx\right| \leq c\|u\|_Y \|v\|_Y \|w\|_{\tilde{Y}}$$

so that

(1.49) $$|b(u,v,w)| \leq c(n)\ \|u\|_Y \|v\|_Y \|w\|_{\tilde{Y}}.$$

---

(1) For a smooth open set Ω, $\tilde{Y}$ is probably equal to $Y \cap \mathbb{L}^n(\Omega)$, but this result is not proved.

The relations (1.47) and (1.48) are proved exactly as (1.22) and (1.23): we prove them for $u,v,w \subset \mathcal{V}$ and then pass to the limit.

The variational formulation of Problem (1.8) - (1.10) for Ω unbounded and $n \geq 3$ is set as follows:

(1.50) <u>To find</u> $u \in Y$ <u>such that</u>

$$\nu((u,v)) + b(u,u,v) = \langle f,v \rangle, \quad \forall v \in \tilde{Y}.$$

<u>Theorem 1.4</u>.

<u>Let</u> Ω <u>be any open set in</u> $\mathbb{R}^n$, $n \geq 3$, <u>and let</u> f <u>be given in</u> Y', <u>the dual space of</u> Y.

<u>Then there exists at least one</u> u <u>in</u> Y <u>which satisfies</u> (1.50).

<u>Proof</u>.

The proof is very similar to the proof of Theorem 1.2 (the bounded case). There exists a sequence $w_1,\cdots,w_m,\cdots$, of elements of $\mathcal{V}$ which is free and total in $\tilde{Y}$ and hence in Y; this sequence is not perhaps the same sequence as before.

We define an approximate solution u_m by

$$u_m = \sum_{i=1}^{m} \xi_{i,m} w_i, \quad \xi_{i,m} \in \mathbb{R}, \tag{1.51}$$

$$\nu((u_m,w_k)) + b(u_m,u_m,w_k) = \langle f,w_k \rangle, \quad k = 1,\cdots,m. \tag{1.52}$$

The existence of u_m satisfying (1.51)-(1.52) is proved exactly as before, using Lemma 1.4. We have then an <u>a priori</u> estimate analogous to (1.32):

$$\|u_m\| \leq \frac{1}{\nu}\|f\|_{Y'}, \quad (\|\cdot\| = \text{the norm in } Y). \tag{1.53}$$

There exists therefore a subsequence $m' \to \infty$ and an element $u \in Y$ such that

$$u_{m'} \to u \text{ weakly in } Y. \tag{1.54}$$

The proof finishes as in the bounded case, except for the passage to the limit in the nonlinear term $b(u_{m'},u_{m'},v)$; it is not true that $u_{m'}$ converges to u in $\mathbb{L}^2(\Omega)$ strongly since u does not even belong to $\mathbb{L}^2(\Omega)$, in general $(Y \not\subset \mathbb{L}^2(\Omega))$. Nevertheless, we have

Lemma 1.7.

If u_μ *converges to* u *in* Y *weakly, then,*

(1.55) $$b(u_\mu,u_\mu,v) \longrightarrow b(u,u,v), \quad \forall v \in \mathcal{V}.$$

Proof.

We can show that

(1.56) $$u_\mu \longrightarrow u \quad \text{in} \quad L^2_{loc}(\overline{\Omega}) \quad \text{strongly},$$

which means that

(1.57) $$u_\mu \longrightarrow u \quad \text{in} \quad L^2(\mathcal{O}),$$

for each bounded set $\mathcal{O} \subset \Omega$.

Actually, let $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\psi = 1$ on $\mathcal{O}$, and let Ω' be a bounded subset of Ω containing the support of ψ. Then the functions ψu_μ belong to $H^1_0(\Omega')$ and since u_μ converges to u weakly in Y,

$$\psi u_\mu \longrightarrow \psi u, \quad \text{weakly in} \quad H^1_0(\Omega').$$

Hence $\psi u_\mu \longrightarrow \psi u$ strongly in $L^2(\Omega')$; in particular

$$\int_{\mathcal{O}} |u_\mu - u|^2 dx \leq \int_{\Omega'} \psi^2 |u_\mu - u|^2 dx \longrightarrow 0,$$

and (1.57) follows.

Since u_μ converges to u for the L^2 norm, on the support of v, the convergence (1.55) is now proved as in the bounded case:

$$b(u_\mu,u_\mu,v) = -b(u_\mu,v,u_\mu) \longrightarrow -b(u,v,u) = b(u,u,v).$$

Remark 1.3.

For $n = 2$, an element u of Y does not belong in general to any space $\mathbb{L}^\beta(\Omega)$. For this reason the proof of Lemma 1.6 fails and b is not defined on $Y \times Y \times \tilde{Y}$.

We can replace (1.50) by the problem: to find $u \in Y$ such that

$$\nu((u,v)) + b(u,u,v) = \langle f,v \rangle, \quad \forall v \in \mathcal{V}.$$

The same proof as for Theorem 1.4 shows that such a u always exists provided f is given in Y'.

Remark 1.4.

Since $Y \neq \tilde{Y}$ (for any n), we cannot put $v = u$ in (1.50). Therefore the proof of Theorem 1.2 cannot be extended to the unbounded case even for $n \leq 4$.

Regularity of the Solution.

If the dimension n of the space is less than or equal to three, we can obtain some information about any solution u of (1.25) or (1.50) by reiterating the following simple procedure: the information we have on u gives us some regularity property of the nonlinear term

$$\sum_{i=1}^{n} u_i D_i u.$$

We then write (1.8) - (1.10) as

$$(1.58) \qquad -\nu\Delta u + \text{grad}\, p = f - \sum_{i=1}^{n} u_i D_i u, \quad \text{in } \Omega,$$

$$(1.59) \qquad \text{div}\, u = 0, \quad \text{in } \Omega,$$

$$(1.60) \qquad u = 0 \quad \text{on } \Gamma;$$

using the available regularity properties of f and Proposition 2.2 and Theorem 2.4 of Chapter I we obtain new information on the regularity of u. If the properties of u thus obtained are better than before, we can reiterate the procedure.

Let us show, for example, the following result:

Proposition 1.1.

Let Ω be an open set of class $\mathscr{C}^\infty$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let f be given in $\mathbb{C}^\infty(\bar{\Omega})$.

Then any solution $\{u,p\}$ of (1.8), (1.25) belongs to $\mathbb{C}^\infty(\bar{\Omega}) \times \mathscr{C}^\infty(\bar{\Omega})$.

Proof.

Let us start first with the case where Ω is bounded.

The nonlinear term $\sum_{i=1}^{n} u_i D_i u$ is also equal to $\sum_{i=1}^{n} D_i(u_i u)$, because of (1.59). If $n = 2$, u_i belongs to $L^\alpha(\Omega)$ for any α, $1 \leq \alpha < +\infty$ (by (1.4)), and then $u_i u_j$ belongs to $L^\alpha(\Omega)$ for any such α, and $D_i(u_i u_j)$ belongs to $W^{-1,\alpha}(\Omega)$ for any such α. Proposition 2.2, Chapter I, shows us that u belongs then to $W^{1,\alpha}(\Omega)$, and p belongs to $L^\alpha(\Omega)$, for any α. For $\alpha > 2$, $W^{1,\alpha}(\Omega) \subset L^\infty(\Omega)$ because of (1.3); hence $u_i D_i u \in \mathbb{L}^\alpha(\Omega)$ for any α. Then, Proposition I.2.2 shows us that $u \in \mathbb{W}^{2,\alpha}(\Omega)$, $p \in W^{1,\alpha}(\Omega)$ for any α. It is easy to check that $u_i D_i u \in \mathbb{W}^{1,\alpha}(\Omega)$, so that $u \in \mathbb{W}^{3,\alpha}(\Omega)$. Repeating this procedure we find in particular that

$$(1.61) \qquad u \in \mathbb{H}^m(\Omega), \quad p \in H^m(\Omega), \quad \text{for any } m \geq 1.$$

The same properties hold for any derivative of u or p; (1.3) implies therefore that any derivative of u or p belongs to $\mathscr{C}(\overline{\Omega})$, and this is the property announced.

For $n = 3$, we notice that $u_i \in L^6(\Omega)$ (by (1.4)) and then

$$u_i D_i u_j \in L^{\frac{3}{2}}(\Omega).$$

Proposition I.2.2 implies that $u \in \mathbb{W}^{2,\frac{3}{2}}(\Omega)$; but (1.3) shows us that $u \in \mathbb{L}^{\alpha}(\Omega)$ for any α, $1 \leq \alpha < +\infty$ ($p = \frac{3}{2}$, $m = 2$, $n = 3$). Therefore

$$D_i(u_i u_j) \in W^{-1,\alpha}(\Omega),$$

for any α, and at this point we only need to repeat the proof given for $n = 2$.

If Ω is unbounded, we obtain the same regularity on any compact subset of $\overline{\Omega}$ by applying the preceding technique to ψu where ψ is a cut-off function, $\psi \in \mathscr{D}(\mathbb{R}^n)$, $\psi = 1$ on the compact subset of Ω.

Remark 1.4.

(i) It is clear that we can assume less regularity for f and obtain less regularity for u and p.

(ii) The same technique gives nothing for $n \geq 4$. For instance, for Ω bounded and $n = 4$, if we write the nonlinear term as $D_i(u_i u)$, we just have $u_i u_j \in L^2(\Omega)$, $D_i(u_i u) \in \mathbb{H}^{-1}(\Omega)$, so that $u \in \mathbb{H}_0^1(\Omega)$; if we write the nonlinear terms as $u_i D_i u$, we have $u_i D_i u \in \mathbb{L}^{\frac{4}{3}}(\Omega)$, so that $u \in \mathbb{W}^{2,\frac{4}{3}}(\Omega)$; but this gives nothing more than $u_i \in L^4(\Omega)$, $D_i u_j \in L^2(\Omega)$, which was known before.

1.4 The Inhomogeneous Navier-Stokes Equations.

Let Ω be an open bounded set in $\mathbb{R}^n$. We consider here the following inhomogeneous Navier-Stokes problem: let there be given two vector functions f and ϕ defined respectively on Ω and Γ and satisfying some conditions which will be specified later; to find u and p such that

$$(1.62) \qquad -\nu\Delta u + \sum_{i=1}^{n} u_i D_i u + \text{grad } p = f, \quad \text{in } \Omega,$$

(1.63) $$\operatorname{div} u = 0, \quad \text{in} \quad \Omega,$$

(1.64) $$u = \phi, \quad \text{on} \quad \Gamma.$$

We will suppose that Ω is of class $\mathscr{C}^2$, that f is given in $\mathbb{H}^{-1}(\Omega)$ and that ϕ is given in the following slightly restrictive way:

(1.65) $$\phi = \operatorname{rot} \zeta$$

where

(1.66) $$\zeta \in \mathbb{H}^2(\Omega), \quad D_i\zeta \in \mathbb{L}^n(\Omega), \quad \zeta \in \mathbb{L}^\infty(\Omega),$$

and rot denotes the usual rotational operator for $n = 2,3$; and for $n \geq 4$, rot denotes a linear differential operator with constant coefficients, such that $\operatorname{div}(\operatorname{rot} \zeta) \equiv 0$ (1).

<u>Theorem 1.5</u>.

<u>Under the above hypotheses, there exists at least one</u> $u \in \mathbb{H}^1(\Omega)$, <u>and a distribution</u> p <u>on</u> Ω, <u>such that</u> (1.62) - (1.64) <u>hold</u>.

<u>Proof</u>.

Let ψ be any vector function belonging to $\mathbb{H}^1(\Omega) \cap \mathbb{L}^n(\Omega)$ such that

(1.67) $$\begin{cases} \psi \in \mathbb{H}^1(\Omega) \cap \mathbb{L}^n(\Omega), \quad \operatorname{div} \psi = 0 \\ \psi = \phi \quad \text{on} \quad \Gamma \end{cases}.$$

Let us set

$$\hat{u} = u - \psi.$$

Then u belongs to $\mathbb{H}^1(\Omega)$ and satisfies (1.63); (1.64) amounts to saying that

(1.68) $$\hat{u} \in V.$$

Equation (1.62) is equivalent to

(1.69) $$-\nu\Delta\hat{u} + \sum_{i=1}^{n} \hat{u}_i D_i \hat{u} + \sum_{i=1}^{n} \hat{u}_i D_i \psi + \sum_{i=1}^{n} \psi_i D_i \hat{u} + \operatorname{grad} p = \hat{f}$$

---

(1) $\operatorname{rot} \zeta = (R_1\zeta, \cdots, R_n\zeta)$, $R_i\zeta = \sum_{j,k} \alpha_{ijk} D_j \zeta_k$; it suffices that $\sum_{i=1}^{n} \alpha_{ijk} = 0$, $\forall j,k$, $1 \leq j,k \leq n$.

where

$$\hat{f} = f + \nu\Delta\psi - \sum_{i=1}^{n} \psi_i D_i \psi.$$

We remark that

$$\hat{f} \in \mathbb{H}^{-1}(\Omega),$$

which we show as follows. It is clear that $f + \nu\Delta\psi \in \mathbb{H}^{-1}(\Omega)$; we notice moreover that $\psi_i D_i \in \mathbb{L}^{\alpha'}(\Omega)$, $1/\alpha' + 1/\alpha = 1$, $\alpha = \frac{2n}{n-2}$ if $n \geq 3$, any $\alpha > 2$ if $n = 2$; since $\mathbb{H}_0^1(\Omega) \subset \mathbb{L}^{\alpha}(\Omega)$, $\psi_i D_i \psi$ belongs to $\mathbb{H}^{-1}(\Omega)$ too.

As in Section 1.2 we can show that Problem (1.68)–(1.69) is solved if we find a $\hat{u}$ in V such that

$$(1.70) \quad \nu((\hat{u},v)) + b(\hat{u},\hat{u},v) + b(\hat{u},\psi,v) + b(\psi,\hat{u},v) = \langle \hat{f},v \rangle, \quad \forall v \in \tilde{V}.$$

The existence of a $\hat{u} \in V$ satisfying (1.70) can be proved exactly as in Theorem 1.2, provided there exists some $\beta > 0$, such that

$$\nu\|v\|^2 + b(v,v,v) + b(v,\psi,v) + b(\psi,v,v) \geq \beta\|v\|^2 \quad \forall v \in \tilde{V},$$

or because of (1.22)

$$(1.71) \quad \nu\|v\|^2 + b(v,\psi,v) \geq \beta\|v\|^2, \quad \forall v \in \tilde{V}.$$

Now (1.71) will certainly be satisfied if we can find ψ which satisfies (1.67) and

$$(1.72) \quad |b(v,\psi,v)| \leq \frac{\nu}{2}\|v\|^2, \quad \forall v \in \tilde{V}.$$

In order to show this, we will prove the following lemma:

Lemma 1.8.

For any $\gamma > 0$, there exists some $\psi = \psi(\gamma)$ satisfying (1.67) and

$$(1.73) \quad |b(v,\psi,v)| \leq \gamma\|v\|^2, \quad \forall v \in \tilde{V}.$$

Before this we prove two other lemmas.

Lemma 1.9.

Let $\rho(x) = d(x,\Gamma)$ = the distance from x to Γ. For any $\varepsilon > 0$, there exists a function $\theta_\varepsilon \in \mathcal{C}^2(\overline{\Omega})$ such that

(1.74) $\theta_\varepsilon = 1$ <u>in some neighborhood of</u> Γ <u>(which depends on</u> ε).

(1.75) $\theta_\varepsilon = 0$ <u>if</u> $\rho(x) \geq 2\delta(\varepsilon)$, $\delta(\varepsilon) = \exp(-\frac{1}{\varepsilon})$

(1.76) $|D_k\theta_\varepsilon(x)| \leq \frac{\varepsilon}{\rho(x)}$ <u>if</u> $\rho(x) \leq 2\delta(\varepsilon)$, $k = 1,\cdots,n$.

<u>Proof</u>.

Let us consider with E. Hopf [2], the function $\lambda \longrightarrow \xi_\varepsilon(\lambda)$ defined for $\lambda \geq 0$ by

$$\xi_\varepsilon(\lambda) = \begin{cases} 1 & \text{if } \lambda < \delta(\varepsilon)^2 \\ \varepsilon \log(\frac{\delta(\varepsilon)}{\lambda}) & \text{if } \delta(\varepsilon)^2 < \lambda < \delta(\varepsilon) \\ 0 & \text{if } \lambda > \delta(\varepsilon) \end{cases} \tag{1.77}$$

and let us denote by χ_ε the function

$$\chi_\varepsilon(x) = \xi_\varepsilon(\rho(x)). \tag{1.78}$$

Since the function ρ belongs to $\mathscr{C}^2(\bar{\Omega})$, the function χ_ε satisfies (1.74) - (1.76) and θ_ε is obtained by regularization of χ_ε.

<u>Lemma 1.10</u>.

<u>There exists a positive constant</u> c_1 <u>depending only on</u> Ω <u>such that</u>

$$\left|\frac{1}{\rho} v\right|_{L^2(\Omega)} \leq c_1 \|v\|_{H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega). \tag{1.79}$$

<u>Proof</u>.

By using a partition of unity subordinated to a covering of Γ, and local coordinates near the boundary, we reduce the problem to the same problem with Ω = a half-space = $\{x = (x_n, x'),\ x_n > 0,\ x' = (x_1,\cdots,x_{n-1}) \in \mathbb{R}^{n-1}\}$. In this case $\rho(x) = x_n$, and it is sufficient to check that

$$\int_\Omega \frac{|v(x)|^2}{x_n^2}\, dx \leq c_1 \int_\Omega |D_n v(x)|^2\, dx, \quad \forall v \in \mathscr{D}(\Omega). \tag{1.80}$$

This inequality is obvious if one proves the following one dimensional inequality:

$$\int_0^{+\infty} \left|\frac{v(s)}{s}\right|^2 ds \leq 2 \int_0^{+\infty} |v'(s)|^2\, ds, \quad \forall v \in \mathscr{D}(0, +\infty). \tag{1.81}$$

This is a classical Hardy inequality. In order to prove it, we write $s = e^{\sigma}$, $t = e^{\tau}$ and

$$\frac{v(s)}{s} = \frac{1}{s}\int_0^s w(t)dt, \qquad v' = w,$$

$$\int_0^{+\infty} \frac{|v(s)|^2}{|s|^2}\, ds = \int_{-\infty}^{+\infty} e^{-\sigma} \left(\int_0^{e^{\sigma}} w(t)dt\right)^2 d\sigma$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \mathcal{Y}(\sigma-\tau)\, e^{-\frac{\sigma-\tau}{2}}\, w(e^{\tau}) e^{\frac{\tau}{2}}\, d\tau\right)^2 d\sigma,$$

where $\mathcal{Y}$ stands for the Heaviside function, $\mathcal{Y}(\sigma) = 1$, for $\sigma > 0$ and $\mathcal{Y}(\sigma) = 0$ for $\sigma < 0$. By the usual convolution inequality we majorize the last quantity by

$$\left(\int_{-\infty}^{+\infty} \mathcal{Y}(\sigma)\, e^{-\frac{\sigma}{2}}\, d\sigma\right)^2 \cdot \int_{-\infty}^{+\infty} |w(e^{\tau})|^2\, e^{\tau} d\tau = 4 \int_0^{+\infty} |w(t)|^2\, dt,$$

and (1.81) follows.

Proof of Lemma 1.8.

Let us now show that

$$\psi = \operatorname{rot}(\theta_{\varepsilon}\zeta)$$

satisfies (1.67) and (1.73); (1.67) is obvious because of (1.65) and (1.74),

$$\psi_j(x) = 0 \quad \text{if} \quad \rho(x) > 2\delta(\varepsilon)$$

and

$$(1.82) \qquad |\psi_j(x)| \leq c_2\left(\frac{\varepsilon}{\rho(x)}\, |\zeta(x)| + |D\zeta(x)|\right) \quad \text{if} \quad \rho(x) \leq 2\delta(\varepsilon)$$

where

$$|D\zeta(x)| = \left\{\sum_{i,j=1}^{n} |D_i\zeta_j(x)|^2\right\}^{\frac{1}{2}}.$$

As we supposed that $\zeta_i \in L^{\infty}(\Omega)$, we deduce from (1.82) that

$$|\psi_j(x)| \leq c_3\left(\frac{\varepsilon}{\rho(x)} + |D\rho(x)|\right), \quad \forall j, \rho(x) \leq 2\delta(\varepsilon).$$

We have therefore

$$(1.83)\quad |v_i\psi_j|_{L^2(\Omega)} \le c_4 \{\varepsilon \left|\frac{v_i}{\rho}\right|_{L^2} + (\int_{\rho\le 2\delta(\varepsilon)} v_i^2 |D\zeta|^2 dx)^{\frac{1}{2}}\}.$$

But, using Hölder's inequality, we see that

$$(\int_{\rho\le 2\delta(\varepsilon)} v_i^2 |D\zeta|^2 dx)^{\frac{1}{2}} \le \mu(\varepsilon)|v_i|_{L^\alpha(\Omega)}$$

where $\frac{1}{\alpha} = \frac{1}{2} - \frac{1}{n}$ and

$$\mu(\varepsilon) = \{\int_{\rho(x)\le 2\delta(\varepsilon)} |D\zeta(x)|^n dx\}^{\frac{1}{n}};$$

$\mu(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ since $D_i\zeta_j \in L^n(\Omega)$, $1 \le i,j \le n$.

With this last majorization, (1.4), and Lemma 1.10, (1.83) gives

$$(1.84)\quad |v_i\psi_j|_{L^2(\Omega)} \le c_5(\varepsilon\|v\| + \mu(\varepsilon)|v|_{\mathbb{L}^\alpha(\Omega)}) \le c_6(\varepsilon+\mu(\varepsilon))\|v\|, \quad 1 \le i,j \le n.$$

Now it is easy to check (1.73); for each $v \in \mathcal{V}$,

$$b(v,\psi,v) = - b(v,v,\psi)$$

$$(1.85)\quad |b(v,v,\psi)| \le \|v\| \{\sum_{i,j=1}^{n} |v_i\psi_j|\}$$

$$\le \text{(by (1.84)}$$

$$\le c_7(\varepsilon + \mu(\varepsilon))\|v\|^2.$$

If ε is sufficiently small so that

$$c_7(\varepsilon + \mu(\varepsilon)) \le \gamma,$$

we obtain (1.73) for each $v \in \mathcal{V}$ and by continuity, for each $v \in \tilde{V}$.

Remark 1.5.

(i) For $n \le 3$, the conditions (1.66) reduce to

$$\zeta \in \mathbb{H}^2(\Omega),$$

because of the Sobolev imbedding theorems (see (1.3)).

(ii) It is easy to write the boundary condition in the form (1.65) for the classical problems of hydrodynamics, such as the cavitation problem, the Taylor problem .....

Remark 1.6.

It is easy to extend Proposition 1.1 to inhomogeneous problems. With the hypotheses of this proposition and moreover assuming that $\phi \in \mathcal{C}^\infty(\Gamma)$ the solution $\{u,p\}$ of (1.62) - (1.64) belongs to $\mathcal{C}^\infty(\overline{\Omega}) \times \mathcal{C}^\infty(\overline{\Omega})$.

To prove this we proceed as in Proposition 1.1, directly on the equations (1.62) - (1.64) (i.e., without introducing $\hat{u}$).

A uniqueness result similar to Theorem 1.3 holds: for $n \leq 4$, ν "large", and f "small", there is uniqueness:

Theorem 1.6.

We suppose that $n \leq 4$, that the norm of ϕ in $\mathbb{L}^n(\Omega)$ is sufficiently small so that

$$|b(v,\phi,v)| \leq \frac{\nu}{2} \|v\|^2, \quad \forall v \in V, \tag{1.86}$$ [1]

and ν is sufficiently large so that

$$\nu^2 > 4c(n) \|\hat{f}\|_{V'}, \tag{1.87}$$

where $c(n)$ is the constant in (1.18) and

$$\hat{f} = f + \nu\Delta\phi - \sum_{i=1}^{n} \phi_i D_i \phi. \tag{1.88}$$

Then, there exists a unique solution u,p of (1.62) - (1.64) [2].

Proof.

It was proved in Lemmas 1.1, 1.2, 1.3 that

$$b(v,\phi,v) = -b(v,v,\phi),$$

and

$$|b(v,v,\phi)| \leq c \|v\|^2 |\phi|_{\mathbb{L}^n(\Omega)}.$$

Therefore condition (1.86) is satisfied if $|\phi|_{\mathbb{L}^n(\Omega)}$ is small enough: this

(1) For $n = 2$ replace $\mathbb{L}^n(\Omega)$ by $L^\alpha(\Omega)$ for some $\alpha > 2$.

(2) As always, p is unique up to a constant.

means that (1.72) is satisfied with $\psi = \phi$ and *we do not need in this case the previous construction of* ψ. Nevertheless, the proof of existence goes along the same lines, with $\psi = \phi$.

If u_1 is a solution of (1.62) - (1.64), then $\hat{u}_1 = u_1 - \phi$ is a solution of (1.70) with $\psi = \phi$. Taking $v = \hat{u}_1$ in (1.70) we get

$$\nu \| \hat{u}_1 \|^2 = -b(\hat{u}_1, \phi, \hat{u}_1) + <\hat{f}, \hat{u}_1> \leq \frac{\nu}{2} \| \hat{u}_1 \|^2 + \| \hat{f} \|_{V'} \| \hat{u}_1 \|$$

by using (1.86), and therefore

$$\| \hat{u}_1 \| \leq \frac{2}{\nu} \| \hat{f} \|_{V'}. \tag{1.89}$$

Let us suppose that u_0, u_1 are two solutions of (1.62) - (1.64); let $\hat{u}_0 = u_0 - \phi$, $\hat{u}_1 = u_1 - \phi$, $\hat{u} = \hat{u}_0 - \hat{u}_1$; $\hat{u}_0$ and $\hat{u}_1$ satisfy (1.70) with $\psi = \phi$:

$$\nu((\hat{u}_0, v)) + b(\hat{u}_0, \hat{u}_0, v) + b(\hat{u}_0, \phi, v) + b(\phi, \hat{u}_0, v) = <\hat{f}, v>, \qquad v \quad V,$$

$$\nu((\hat{u}_1, v)) + b(\hat{u}_1, \hat{u}_1, v) + b(\hat{u}_1, \phi, v) + b(\phi, \hat{u}_1, v) = <\hat{f}, v>, \qquad v \quad V.$$

We take $v = \hat{u}$ in these equations, and take the difference between them; after expanding and using (1.22) we find:

$$\nu \| \hat{u} \|^2 = -b(\hat{u}, \hat{u}_1, \hat{u}) - b(\hat{u}, \phi, \hat{u}). \tag{1.90}$$

Because of (1.86),

$$-b(\hat{u}, \phi, \hat{u}) \leq \frac{\nu}{2} \| \hat{u} \|^2.$$

By (1.18),

$$-b(\hat{u}, \hat{u}_1, \hat{u}) \leq c(n) \| \hat{u}_1 \| \, \| \hat{u} \|^2,$$

and because of (1.89), this is majorized by

$$\frac{2}{\nu} c(n) \| \hat{f} \|_{V'} \| \hat{u} \|^2.$$

We finally arrive at the inequality

$$\left(\frac{\nu}{2} - \frac{2}{\nu} c(n) \| \hat{f} \|_{V'}\right) \| \hat{u} \|^2 \leq 0,$$

and because of (1.87), this implies $\hat{u} = 0$.

If $\hat{u}_0 = \hat{u}_1$, it is clear that grad p_0 = grad p_1, so the difference between p_0 and p_1 is constant.

<u>Remark</u> <u>1.7</u>.

Nonuniqueness results for Problem (1.62) - (1.64) have been proved, in the two-dimensional case, for certain configurations (cf., Velte [1], [2]).

§2. DISCRETE INEQUALITIES AND COMPACTNESS THEOREMS.

Before going through the numerical approximation of the stationary Navier-Stokes equations, we must introduce new tools: the discrete analog, for step functions, of the Sobolev inequalities and of the compactness theorem, Theorem 1.1. These and some further Sobolev-type inequalities are the goals set for this section.

2.1 Discrete Sobolev Inequalities.

The notations are those used for finite differences; see Section 3.3, Chapter I. We recall in particular that $\mathcal{R}_h$ is the set of points with coordinates $m_1h_1,\cdots,m_nh_n$, $m_i \in \mathbb{Z}$, $h = (h_1,\cdots,h_n)$ $h_i > 0$; w_{hM} is the characteristic function of the block

$$(2.1) \qquad \sigma_h(M) = \prod_{i=1}^{n} [\mu_i - \frac{h_i}{2}, \mu_i + \frac{h_i}{2}), \qquad M = (\mu_1,\cdots,\mu_n),$$

and δ_{ih} is the difference operator

$$(2.2) \qquad \delta_{ih}\phi(x) = \frac{1}{h_i}[\phi(x + \frac{1}{2}\vec{h}_i) - \phi(x - \frac{1}{2}\vec{h}_i)]$$

where $\vec{h}_i$ is the vector with i^{th} component h_i, and all other components 0.

Theorem 2.1.

Let p denote some number such that $1 \leq p < n$, and let q be defined by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

There exists a constant $c = c(n,p)$ depending only on n and p such that

$$(2.3) \qquad |u_h|_{L^q(\mathbb{R}^n)} \leq c(n,p) \sum_{i=1}^{n} |\delta_{ih}u_h|_{L^p(\mathbb{R}^n)},$$

for each step function u_h

$$(2.4) \qquad u_h = \sum_{M\in\mathcal{R}_h} u_h(M)w_{hM},$$

with compact support.

Proof.

i) Let us consider the scalar function

$$s \longmapsto g(s) = |s|^{\frac{(n-1)p}{n-p}}$$

Since $\frac{(n-1)p}{n-p} \geq 1$, this function is differentiable with derivative

$$g'(s) = \frac{(n-1)p}{n-p} |s|^{\frac{(n-1)p}{n-p} - 2} s.$$

The Taylor formula can be written

$$g(s_1)-g(s_2) = (s_1-s_2)g'(\lambda s_1+(1-\lambda)s_2), \quad \lambda \in (0,1)$$

and gives

$$|g(s_1)-g(s_2)| \leq |s_1-s_2| \frac{(n-1)p}{n-p} |\lambda s_1+(1-\lambda)s_2|^{\frac{n(p-1)}{n-p}}$$

$$\leq \frac{(n-1)p}{n-p} |s_1-s_2| \{|s_1|+|s_2|\}^{\frac{n(p-1)}{n-p}},$$

$$(2.5) \qquad |g(s_1)-g(s_2)| \leq c_1(n,p) |s_1-s_2| \{|s_1|^{\frac{n(p-1)}{n-p}} + |s_2|^{\frac{n(p-1)}{n-p}}\}.$$

ii) Let M belong to $\mathcal{R}_h$; we apply (2.5) with

$$s_1 = u_h(M-r\vec{h}_i), \quad s_2 = u_h(M-(r+1)\vec{h}_i);$$

$$|u_h(M-r\vec{h}_i)|^{\frac{(n-1)p}{n-p}} - |u_h(M-(r+1)\vec{h}_i)|^{\frac{(n-1)p}{n-p}}$$

$$\leq c_1(n,p) |u_h(M-r\vec{h}_i)-u_h(M-(r+1)\vec{h}_i)| \cdot \{|u_h(M-r\vec{h}_i)|^{\frac{n(p-1)}{n-p}} + |u_h(M-(r+1)\vec{h}_i)|^{\frac{n(p-1)}{n-p}}\}.$$

Summing these inequalities for $r \geq 0$, we find (the sum is actually finite):

$$(2.6) \qquad |u_h(M)|^{\frac{(n-1)p}{n-p}} \leq c_1(n,p)h_i \sum_{r=0}^{+\infty} |\delta_{ih}u_h(M-(r+\tfrac{1}{2})\vec{h}_i)| \cdot \{|u_h(M-r\vec{h}_i)|^{\frac{n(p-1)}{n-p}} + |u_h(M-(r+1)\vec{h}_i)|^{\frac{n(p-1)}{n-p}}.$$

We strengthen inequality (2.6) by replacing the sum on the right-hand side by the sum for $r \in Z$; we can then interpret the sum as an integral and majorize it by

$$c_1(n,p)\int_{-\infty}^{+\infty}|\delta_{ih}u_h(\hat{\mu}_i,\xi_i)|\cdot\{\sum_{\alpha=-1}^{1}|u_h(\hat{\mu}_i,\xi_i+\frac{\alpha}{2}h_i)|^{\frac{n(p-1)}{n-p}}\}d\xi_i,$$

where $(\mu_1,\cdots,\mu_n)$ are the coordinates of M and $\hat{\mu}_i=(\mu_1,\cdots,\mu_{i-1},\mu_{i+1},\cdots,\mu_n)$. In a similar way we denote by $\hat{x}_i$ the vector $(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)$ and then write $x=(\hat{x}_i,x_i)$.

For any $x\in\sigma_h(M)$, inequality (2.6) gives now

$$(2.7)\qquad |u_h(x)|^{\frac{(n-1)p}{n-p}}=|u_h(M)|^{\frac{(n-1)p}{n-p}}$$

$$\leq c_1(n,p)\int_{-\infty}^{+\infty}|\delta_{ih}u_h(\hat{x}_i,\xi_i)|$$

$$\cdot\{\sum_{\alpha=-1}^{1}|u_h(\hat{x}_i,\xi_i+\frac{\alpha h_i}{2})|^{\frac{n(p-1)}{n-p}}\}\,d\xi_i.$$

Let us now set

$$(2.8)\qquad w_i(x)=w_i(\hat{x}_i)=\sup_{x_i\in\mathbb{R}}|u_h(x)|^{\frac{p}{n-p}}$$

Then, $|w_i(\hat{x}_i)|^{n-1}$ is majorized by the right-hand side of (2.7), hence

$$\int_{\mathbb{R}^{n-1}}w_i(\hat{x}_i)^{n-1}\,d\hat{x}_i$$

$$\leq c_1(n,p)\int_{\mathbb{R}^n}|\delta_{ih}u_h(\hat{x}_i,\xi_i)|$$

$$\cdot\{\sum_{\alpha=-1}^{1}|u_h(\hat{x}_i,\xi_i+\frac{\alpha h_i}{2})|^{\frac{n(p-1)}{n-p}}\}\;d\hat{x}_i d\xi_i.$$

$\leq$ (by Hölder's inequality)

$$\leq c_1(n,p)|\delta_{ih}u_h|_{L^p(\mathbb{R}^n)}\cdot\left(\sum_{\alpha=-1}^{+1}\int_{\mathbb{R}^n}|u_h(\hat{x}_i,\xi_i+\alpha h_i)|^q\,d\hat{x}_i d\xi_i\right)^{\frac{p-1}{p}};$$

therefore

$$(2.9)\qquad \int_{\mathbb{R}^{n-1}}w_i(\hat{x}_i)^{n-1}\,d\hat{x}_i\leq c_2(n,p)|\delta_{ih}u_h|_{L^p(\mathbb{R}^n)}|u_h|_{L^q(\mathbb{R}^n)}^{\frac{(p-1)q}{p}}.$$

Now we have

$$\int_{\mathbb{R}^n} |u_h|^q \, dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} \operatorname{Sup}_{x_i} |u_h(\hat{x}_i, x_i)|^{\frac{p}{n-p}} \, dx$$

$$\leq \int_{\mathbb{R}^n} \prod_{i=1}^{n} w_i(\hat{x}_i) \, dx.$$

According to the inequality given in the next lemma, this is majorized by

$$\prod_{i=1}^{n} \{\int_{\mathbb{R}^{n-1}} |w_i(\hat{x}_i)|^{n-1} \, d\hat{x}_i\}^{\frac{1}{n-1}}$$

and, because of (2.9),

$$|u_h|^q_{L^q(\mathbb{R}^n)} \leq c_3(n,p) \cdot \{\prod_{i=1}^{n} |\delta_{ih} u_h|^{\frac{1}{n-1}}_{L^p(\mathbb{R}^n)}\} \, |u_h|^{\frac{n}{n-1}\frac{(p-1)q}{p}}_{L^q(\mathbb{R}^n)}$$

$$|u_h|^{\frac{n}{n-1}}_{L^q(\mathbb{R}^n)} \leq c_3(n,p) \{\prod_{i=1}^{n} |\delta_{ih} u_h|^{\frac{1}{n-1}}_{L^p(\mathbb{R}^n)}\}$$

$$|u_h|_{L^q(\mathbb{R}^n)} \leq c_4(n,p) \, \{\prod_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}\}^{\frac{1}{n}}$$

$$|u_h|_{L^q(\mathbb{R}^n)} \leq c_5(n,p) \sum_{i=1}^{n} |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)} .$$

<u>Lemma 2.1</u>.

<u>Let</u> $w_1, \cdots, w_n$, <u>be</u> n <u>measurable bounded functions on</u> $\mathbb{R}^n$, <u>with compact supports, and</u> w_i <u>independent of</u> x_i.

<u>Then</u>

$$\text{(2.10)} \qquad \int_{\mathbb{R}^n} (\prod_{i=1}^{n} w_i(\hat{x}_i)) \, dx \leq \prod_{i=1}^{n} \{\int_{\mathbb{R}^{n-1}} |w_i(\hat{x}_i)|^{n-1} \, d\hat{x}_i\}^{\frac{1}{n-1}} .$$

This is a particular case of an inequality of E. Gagliardo [1]; see also Lions [1], page 31.

<u>Remark 2.1</u>.

For $p = n$, if the support of u_h is included in a bounded set Ω, then

$$(2.11)\qquad |u_h|_{L^q(\mathbb{R}^n)} \le c(n,q,\Omega) \sum_{i=1}^{n} |\delta_{ih}u_h|_{L^p(\mathbb{R}^n)},$$

for each u_h of type (2.4), and for any q, $1 \le q < +\infty$. Actually any such q greater than p can be written as $\frac{p_1 n}{n-p_1}$ with $1 \le p_1 < n$. The inequality (2.3) is then applicable, with $c = c(n,p_1) = c'(n,q)$:

$$|u_h|_{L^q(\mathbb{R}^n)} \le c'(n,q) \sum_{i=1}^{n} |\delta_{ih}u_h|_{L^{p_1}(\mathbb{R}^n)}.$$

The Hölder inequality shows us that

$$|\delta_{ih}u_h|_{L^{p_1}(\mathbb{R}^n)} \le (\text{meas } \Omega')^{\frac{1}{p_1}-\frac{1}{p}} \quad |\delta_{ih}u_h|_{L^p(\mathbb{R}^n)}$$

where Ω' contains the support of $\delta_{ih}u_h$. If we suppose that $|h|$ is bounded by 1 (or by some constant d), (meas Ω') is bounded by (meas Ω)×Const.; then combining the last two inequalities, we obtain (2.11).

In the two and three dimensional cases, we prove another related inequality which will be useful.

<u>Proposition 2.1.</u>
<u>Let us suppose that the dimension of the space is two or three.</u>
<u>For any step function u_h of type (2.4) with compact support, we have:</u>

$$(2.12)\qquad |u_h|_{L^4(\mathbb{R}^2)} \le 2^{\frac{1}{4}}\cdot 3^{\frac{1}{4}}\cdot |u_h|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} \cdot \{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^2)}\}^{\frac{1}{4}}, \quad \underline{\text{if}} \quad n = 2,$$

$$(2.13)\qquad |u_h|_{L^4(\mathbb{R}^3)} \le 2^{\frac{1}{2}}\cdot 3^{\frac{3}{4}}\cdot |u_h|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)} \cdot \{\sum_{i=1}^{3} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\}^{\frac{3}{8}} \quad \underline{\text{if}} \quad n = 3.$$

<u>Proof.</u>
We use the inequality (2.7) with $n = 2$ and $p = \frac{4}{3}$; a more precise analysis of the proof of (2.7) shows that $c_1(n,p) = 2$ in the present case; actually $g(s) = s^2$ and for (2.5) we have clearly

$$|g(s_1)-g(s_2)| \le 2|s_1-s_2|\{|s_1| + |s_2|\}.$$

We then have, for any $x \in \sigma_h(M)$, and $M \in \mathcal{R}_h$,

$$(2.14)\quad |u_h(x)|^2 \le 2\int_{-\infty}^{+\infty} |\delta_{ih}u_h(\hat{x}_i,\xi_i)|\{\sum_{\alpha=-1}^{+1} |u_h(\hat{x}_i,\xi_i + \frac{\alpha h_i}{2})|\}\, d\xi_i.$$

Since the right-hand side of (2.14) is independent of x_i, we obtain

$$(2.15)\quad \operatorname*{Sup}_{x_i} |u_h(x)|^2 \le 2\int_{-\infty}^{+\infty} |\delta_{ih}u_h(\hat{x}_i,\xi_i)|\ \{\sum_{\alpha=-1}^{+1} |u_h(\hat{x}_i,\xi_i + \frac{\alpha h_i}{2})|\}\, d\xi_i.$$

Now we may write, for the two dimensional case,

$$(2.16)\quad \int_{\mathbb{R}^2} |u_h(x)|^4\, dx \le \int_{\mathbb{R}^2} [\operatorname*{Sup}_{x_1} |u_h(x)|^2][\operatorname*{Sup}_{x_2} |u_h(x)|^2]\, dx$$

$$\le \{\int_{-\infty}^{+\infty} [\operatorname*{Sup}_{x_1} |u_h(x)|^2]\, dx_2\}\{\int_{-\infty}^{+\infty} [\operatorname*{Sup}_{x_2} |u_h(x)|^2]\, dx_1\}$$

$\le$ (because of (2.15))

$$\le 4\{\int_{\mathbb{R}^2} |\delta_{1h}u_h(\xi_1,x_2)|[\sum_{\alpha=-1}^{+1} |u_h(\xi_1 + \frac{\alpha h_1}{2}, x_2)|]\, d\xi_1 dx_2\}$$

$$\cdot \{\int_{\mathbb{R}^2} |\delta_{2h}u_h(x_1,\xi_2)|[\sum_{\alpha=-1}^{+1} |u_h(x_1,\xi_2 + \frac{\alpha h_2}{2})|]\, dx_1 d\xi_2\}.$$

By the Schwarz inequality and since

$$(2.17)\quad \int_{\mathbb{R}^2} |u_h(\xi_1 + \frac{\alpha h_1}{2}, x_2)|^2\, d\xi_1 dx_2 = \int_{\mathbb{R}^2} |u_h(x)|^2 dx_1 dx_2 = |u_h|^2_{L^2(\mathbb{R}^2)} ,$$

the last expression is majorized by

$$36\{|\delta_{1h}u_h|_{L^2(\mathbb{R}^2)} |u_h|_{L^2(\mathbb{R}^2)}\}\{|\delta_{2h}u_h|_{L^2(\mathbb{R}^2)} |u_h|_{L^2(\mathbb{R}^2)}\}$$

$$\le 18|u_h|^2_{L^2(\mathbb{R}^2)} \{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^2)}\}.$$

Hence (2.12) is proved.

In the three dimensional case, using (2.12) and (2.15), we write

$$(2.18)\quad \int_{\mathbb{R}^3} |u_h(x)|^4 dx \le 18\int\{[\int |u_h|^2 dx_1 dx_2][\sum_{i=1}^{2} \int |\delta_{ih}u_h|^2 dx_1 dx_2]\} dx_3$$

$$\le 18\{\operatorname*{Sup}_{x_3} \int |u_h|^2 dx_1 dx_2\}\{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\}$$

$\leq$ (because of (2.15))

$$\leq 36\{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\}\{\int_{\mathbb{R}^3} |\delta_{3h}u_h(\hat{x}_3,x_3)| \; [\sum_{\alpha=-1}^{+1} |u_h(\hat{x}_3,x_3 + \frac{\alpha h_3}{2})|]dx\}$$

$\leq$ (by Schwarz's inequality)

$$\leq 3^3 2^2 \; |u_h|_{L^2(\mathbb{R}^3)} |\delta_{3h}u_h|_{L^2(\mathbb{R}^3)} \; \{\sum_{i=1}^{2} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\}$$

$$\leq 3^3 2^2 \; |u_h|_{L^2(\mathbb{R}^3)} \; \{\sum_{i=1}^{3} |\delta_{ih}u_h|^2_{L^2(\mathbb{R}^3)}\}^{\frac{3}{2}},$$

and (2.13) is proved.

Remark 2.2.

The inequalities (2.3), (2.11), and (2.12) can be extended by continuity to classes of step functions with unbounded support.

2.2 A Discrete Compactness Theorem.

We give here a discrete analog of Theorem 1.1, more precisely of the fact that the injection of $\overset{\circ}{W}{}^{1,p}(\Omega)$ into $L^{q_1}(\Omega)$ is compact if

(2.19) $\quad 1 \leq p < n$ *and* q_1 *is any number, such that* $1 \leq q_1 < q$, *where* $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$,

(2.20) $\quad p = n$ *and* q_1 *is any number*, $1 \leq q_1 < +\infty$.

Theorem 2.2.

Let $\mathcal{E}_h$ *be a family, maybe empty, of step functions of type* (2.4) *and let*

(2.21)
$$\mathcal{E} = \bigcup_{|h| \leq c_0} \mathcal{E}_h.$$

Let us suppose that

(2.22) *the functions* u_h *of* $\mathcal{E}$ *have their supports included in some fixed bounded measurable subset of* $\mathbb{R}^n$, *say* Ω,

(2.23)
$$\sup_{u_h \in \mathcal{E}} \{|u_h|_{L^p(\mathbb{R}^n)} + \sum_{i=1}^{n} |\delta_{ih}u_h|_{L^p(\mathbb{R}^n)}\} < +\infty.$$

Then if p *and* q_1 *satisfy conditions* (2.19)-(2.20), *the family* $\mathcal{E}$ *is relatively compact in* $L^{q_1}(\mathbb{R}^n)$ (*or* $L^{q_1}(\Omega)$).

Proof.

According to a theorem of M. Riesz [1], we must prove the following two properties:

i) For each $\varepsilon > 0$, there exists a compact set $K \subset \Omega$, such that

$$\int_{\Omega-K} |u_h|^{q_1}\, dx \leq \varepsilon, \quad \forall u_h \in \mathcal{E}. \tag{2.24}$$

ii) For each $\varepsilon > 0$, there exists $\eta > 0$ such that,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leq \varepsilon, \tag{2.25}$$

for any $u_h \in \mathcal{E}$ and any $\ell = (\ell_1, \cdots, \ell_n)$, with $|\ell| \leq \eta$; τ_ℓ denotes the translation operator

$$(\tau_\ell \phi)(x) = \phi(x + \ell). \tag{2.26}$$

Proof of (i).

Because of the Sobolev inequalities (2.3) and (2.11), the family $\mathcal{E}$ is bounded in $L^q(\Omega)$ where q is given by (2.19) if $p < n$, and q is some fixed number, $q > q_1$, otherwise ($p = n$).

By the Hölder inequality, we then get

$$\int_{\Omega-K} |u_h|^{q_1}\, dx \leq \left(\int_{\Omega-K} dx\right)^{1-\frac{q_1}{q}} \left(\int_{\Omega-K} |u_h|^q\, dx\right)^{\frac{q_1}{q}}$$

$$\int_{\Omega-K} |u_h|^{q_1}\, dx \leq c(\text{meas}(\Omega-K))^{1-\frac{q_1}{q}}, \quad \forall u_h \in \mathcal{E}. \tag{2.27}$$

The right-hand side of (2.27) (and hence the left-hand side) can be made less than ε, by choosing the compact K sufficiently large; (i) is proved.

Proof of (ii).

First, we show that 2.25 may be replaced by a similar condition on $|\tau_\ell u_h - u_h|_{L^p}$ (condition 2.30 below).

Case (a): $q_1 \leq p$. For any $f \in L^q(\Omega)$ we have $f \in L^{q_1}(\Omega)$ as well as $f \in L^p(\Omega)$ since Ω is bounded and $q_1 \leq p < q$. Also, $0 \leq \frac{1}{q_1} - \frac{1}{p} < 1$. By the Hölder inequality,

$$|f|_{L^{q_1}(\Omega)} \leq (\text{meas }\Omega)^{\frac{1}{q_1} - \frac{1}{p}} \cdot |f|_{L^p(\Omega)} = \text{Const.} \cdot |f|_{L^p(\Omega)}.$$

<u>Case (b)</u>: $q_1 > p$. For any function $f \in L^q(\Omega)$ we can write, using the Hölder inequality,

$$\int |f|^{q_1}\, dx = \int |f|^{\theta q_1} |f|^{(1-\theta)q_1}\, dx$$
$$\leq \left(\int |f|^{q_1\theta\rho}\, dx\right)^{\frac{1}{\rho}} \left(\int |f|^{q_1(1-\theta)\rho'}\, dx\right)^{\frac{1}{\rho'}}$$

where $\theta \in (0,1)$, $\rho > 1$, and as usual $\frac{1}{\rho'} + \frac{1}{\rho} = 1$. We can chose θ and ρ so tha

$$q_1\theta\rho = p, \quad q_1(1-\theta)\rho' = q;$$

this defines ρ and θ uniquely, and these numbers belong to the specified intervals, $(\theta q_1(q-p) = p(q-q_1)$, and $\rho(q-q_1) = q-p)$.

Then

$$\int |f|^{q_1}\, dx \leq \left(\int |f|^p\, dx\right)^{\frac{1}{\rho}} \left(\int |f|^q\, dx\right)^{\frac{1}{\rho'}}. \tag{2.28}$$

In particular, for any ℓ and u_h,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leq |\tau_\ell u_h - u_h|^{1-\theta}_{L^q(\mathbb{R}^n)} \cdot |\tau_\ell u_h - u_h|^{\theta}_{L^p(\mathbb{R}^n)}$$

$$\leq \{|\tau_\ell u_h|_{L^q(\mathbb{R}^n)} + |u_h|_{L^q(\mathbb{R}^n)}\}^{1-\theta} |\tau_\ell u_h - u_h|^{\theta}_{L^p(\mathbb{R}^n)}$$

$$= 2^{1-\theta} |u_h|^{1-\theta}_{L^q(\mathbb{R}^n)} \; |\tau_\ell u_h - u_h|^{\theta}_{L^p(\mathbb{R}^n)}.$$

Since the family $\mathcal{E}$ is bounded in $L^q(\mathbb{R}^n)$,

$$|\tau_\ell u_h - u_h|_{L^{q_1}(\mathbb{R}^n)} \leq c |\tau_\ell u_h - u_h|^{\theta}_{L^p(\mathbb{R}^n)}. \tag{2.29}$$

Inequality (2.29) shows us that it suffices to prove condition (ii) with q_1 replaced by p:

$$\begin{cases} \forall \varepsilon > 0, \ \exists \eta, \text{ such that} \\ |\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)} \leq \varepsilon \\ \text{for } |\ell| \leq \eta \text{ and } u_h \in \mathcal{E}. \end{cases} \tag{2.30}$$

The proof of (2.30) follows easily from (2.23) and the next two lemmas.

Lemma 2.2.

(2.31) $$|\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)} \leq \sum_{i=1}^{n} |\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathbb{R}^n)}$$

where $\vec{\ell}_i$ *denotes the vector* $\ell_i \vec{h}_i$.

Proof.

Denoting by I the identity operator, one can check easily the identity

(2.32) $$\tau_\ell - I = \sum_{i=1}^{n} \tau_{\vec{\ell}_1} \cdots \tau_{\vec{\ell}_{i-1}} (\tau_{\vec{\ell}_i} - I).$$

This identity allows us to majorize the norm $|\tau_\ell u_h - u_h|_{L^p(\mathbb{R}^n)}$ by

$$\sum_{i=1}^{n} |\tau_{\vec{\ell}_1} \cdots \tau_{\vec{\ell}_{i-1}} (\tau_{\vec{\ell}_i} u_h - u_h)|_{L^p(\mathbb{R}^n)}$$

We obtain (2.31) recalling that

(2.33) $$|\tau_\alpha f|_{L^p(\mathbb{R}^n)} = |f|_{L^p(\mathbb{R}^n)}$$

for any $\alpha \in \mathbb{R}^n$ and any function $f \in L^p(\mathbb{R}^n)$.

Lemma 2.3.

(2.34) $$|\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathbb{R}^n)} \leq c(|\ell_i| + |\ell_i|^{\frac{1}{p}}) |\delta_{ih} u_h|_{L^p(\mathbb{R}^n)}, \quad 1 \leq i \leq n.$$

Proof.

Since

$$|\tau_{\vec{\ell}_i} u_h - u_h|_{L^p(\mathbb{R}^n)} = |\tau_{-\vec{\ell}_i} u_h - u_h|_{L^p(\mathbb{R}^n)},$$

we can suppose that $\ell_i \geq 0$ and we then set

(2.35) $$\begin{cases} \ell_i = (\alpha_i + \beta_i) h_i, & \text{where } \alpha_i \text{ is an integer } \geq 0, \\ \text{and } 0 \leq \beta_i < 1, & 1 \leq i \leq n. \end{cases}$$

We write

$$(2.36)\qquad \tau_{\vec{\ell}_1} u_h - u_h = \sum_{j=0}^{\alpha_1 - 1} \tau_{j\vec{h}_1}(\tau_{\vec{h}_1} u_h - u_h) + \tau_{\alpha_1 h_1}(\tau_{\beta_1 \vec{h}_1} u_h - u_h).$$

From (2.36) and (2.33), we get the majoration

$$(2.37)\quad |\tau_{\vec{\ell}_1} u_h - u_h|_{L^p(\mathbb{R}^n)} \leq \sum_{j=0}^{\alpha_1 - 1} |\tau_{\vec{h}_1} u_h - u_h|_{L^p(\mathbb{R}^n)} + |\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathbb{R}^n)}.$$

But

$$\tau_{\vec{h}_1} - I = h_1 \tau_{\frac{\vec{h}_1}{2}} \delta_{1h}$$

and the sum on the right-hand side of (2.37) is equal to

$$\alpha_1 h_1 \; |\delta_{1h} u_h|_{L^p(\mathbb{R}^n)};$$

since $\alpha_1 h_1 \leq \ell_1$, we obtain

$$(2.38)\qquad |\tau_{\vec{\ell}_1} u_h - u_h|_{L^p(\mathbb{R}^n)} \leq \ell_1 |\delta_{1h} u_h|_{L^p(\mathbb{R}^n)} + |\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathbb{R}^n)}.$$

Let us now majorize the norm of $\tau_{\beta_1 \vec{h}_1} u_h - u_h$.

For $x \in \sigma_h(M)$, $x = (x_1, \cdots, x_n)$, and $M \in \mathcal{R}_h$, $M = (m_1 h_1, \cdots, m_n h_n)$, we have

$$\tau_{\beta_1 \vec{h}_1} u_h(x) - u_h(x) = \begin{cases} 0 & \text{if} \quad (m_1 - \frac{1}{2})h_1 < x_1 < (m_1 - \beta_1 + \frac{1}{2})h_1 \\ h_1 \delta_{1h} u_h(M + \frac{\vec{h}_1}{2}) & \text{if} \quad (m_1 - \beta_1 + \frac{1}{2})h_1 < x_1 < (m + \frac{1}{2})h_1. \end{cases}$$

Hence

$$\int_{\sigma_h(M)} |\tau_{\beta_1 \vec{h}_1} u_h - u_h|^p \, dx = \beta_1 h_1^p \int_{\sigma_h(M)} |\delta_{1h} u_h(x + \frac{\vec{h}_1}{2})|^p \, dx,$$

and summing these equations for the different points M of $\mathcal{R}_h$, we obtain

$$|\tau_{\beta_1 \vec{h}_1} u_h - u_h|_{L^p(\mathbb{R}^n)} = \beta_1^{\frac{1}{p}} h_1 |\tau_{\frac{\vec{h}_1}{2}} \delta_{1h} u_h|_{L^p(\mathbb{R}^n)}$$

Since h is bounded and $\beta_1 h_1 \leq \ell_1$, this is less than

$$c \ell_1^{\frac{1}{p}} \; |\delta_{1h} u_h|_{L^p(\mathbb{R}^n)}$$

and (2.34) follows for $i = 1$; the proof is the same for $i = 2, \cdots, n$.

Remark 2.3.

In the most common applications of Theorem 2.2, the family $\mathcal{E}$ is a sequence of elements u_{h_m}, where h_m is converging to zero. Hence $\mathcal{E}_{h_m} = \{u_{h_m}\}$ and $\mathcal{E}_h$ is empty if h is not an h_m.

Remark 2.4.

Let us suppose that Ω is bounded and that

$$\mathcal{E} = \bigcup_{|h|<c_0} \mathcal{E}_h \ , \ \mathcal{E}_h = \{u_h \in W_h, \ \|u_h\|_h \leq 1\} \tag{2.39}$$

where W_h is the approximation of $\mathbb{H}_0^1(\Omega)$ by finite differences,(APX 1). We infer from Theorem 2.2, that $\mathcal{E}$ is a relatively compact set in $\mathbb{L}^2(\Omega)$. The following set $\mathcal{E}'$, which is a subset of $\mathcal{E}$, is also relatively compact in $\mathbb{L}^2(\Omega)$:

$$\mathcal{E}' = \bigcup_{|h|\leq c_0} \mathcal{E}'_h, \ \mathcal{E}'_h = \{u_h \in V_h, \ \|u_h\|_h \leq 1\}; \tag{2.40}$$

V_h corresponds to the approximation (APX 1) of V.

§3. APPROXIMATION OF THE STATIONARY NAVIER-STOKES EQUATIONS.

We discuss here the approximation of the stationary Navier-Stokes equations by numerical schemes of the same type as those used for the linear Stokes problem. We give in Section 3.1 a general convergence theorem; we then apply it in Section 3.2 to numerical schemes based on the approximations (APX 1),$\cdots$,(APX 4) of the space V. In Section 3.3 we extend to the nonlinear case the numerical algorithms discussed in Section 5, Chapter I.

All of this section appears to be an extension to the nonlinear case of the results obtained in the linear case, in Sections 3,4 and 5. Nevertheless, the results are not as strong here as in the linear case because, in particular, of the nonuniqueness of solutions of the exact problem. Moreover, the convergence of numerical schemes based on the approximation (APX 5) of V is not known, because of the lack in this case of a discrete compactness theorem of the type contained in Section 2, for finite differences.

3.1 A General Convergence Theorem.

Let Ω be a bounded lipschitzian open set in $\mathbb{R}^n$ and let f be given in $\mathbb{L}^2(\Omega)$. By Theorem 1.2 there exists at least one u in V such that

$$(3.1) \qquad \nu((u,v)) + b(u,u,v) = (f,v), \quad \forall v \in \tilde{V}.$$

Because of Theorem 1.3, this u is unique if $n \leq 4$ and if ν is sufficiently large.

Our purpose here is to discuss the approximation of Problem 3.1 .

Let there be given first an external, stable and convergent Hilbert approximation of the space V, say $(\overline{\omega},F)$, $(V_h,p_h,r_h)_{h\in\mathcal{H}}$, where the V_h are finite dimensional; at this point this approximation could be, in particular, any of the approximations (APX 1),$\cdots$,(APX 5), described in Chapter I.

Let us suppose that we are given some consistent approximation of the bilinear form $\nu((u,v))$, and the linear form (f,v), satisfying the same hypotheses as in Section 3, Chapter I:

(i) for each $h\in\mathcal{H}$, $a_h(u_h,v_h)$ is a bilinear continuous form on $V_h \times V_h$, uniformly coercive in the sense

$$(3.2) \qquad \exists\alpha_0 > 0 \text{ independent of } h, \text{ such that}$$
$$a_h(u_h,u_h) \geq \alpha_0 \|u_h\|_h^2, \quad \forall u_h \in V_h,$$

where $\|\cdot\|_h$ stands for the norm in V_h.

(ii) for each $h \in \mathcal{H}$, ℓ_h is a linear continuous form on V_h, such that

(3.3) $$\|\ell_h\|_{V'_h} \leq \beta.$$

The required consistency hypotheses are :

(3.4) If the family v_h converges weakly to v as $h \to 0$ and if the family w_h converges strongly to w as $h \to 0$ (1), then

$$\lim_{h \to 0} a_h(v_h, w_h) = \nu((v,w)),$$
$$\lim_{h \to 0} a_h(w_h, v_h) = \nu((w,v)).$$

(3.5) If the family v_h converges weakly to v as $h \to 0$, then

$$\lim_{h \to 0} \langle \ell_h, v_h \rangle = (f,v).$$

For the approximation of the form b, we suppose that we are given a trilinear continuous form $b_h(u_h, v_h, w_h)$, on V_h, such that:

(3.6) $$b_h(u_h, v_h, v_h) = 0, \quad \forall u_h, v_h \in V_h.$$

(3.7) if the family v_h converges weakly to v, as $h \to 0$, and if w belongs to $\mathcal{V}$, then

$$\lim_{h \to 0} b_h(v_h, v_h, r_h w) = b(v,v,w).$$

Sometimes it will be useful to be more precise about the continuity of b_h and we will require

(3.8) $$|b_h(u_h, v_h, w_h)| \leq c(n,\Omega) \|u_h\|_h \|v_h\|_h \|w_h\|_h \quad \forall u_h, v_h, w_h \in V_h,$$

where the constant $c = c(n,\Omega)$ depends on n and Ω <u>but not on</u> h: an inequality

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(1) We recall that this means

$$p_h v_h \longrightarrow \bar{\omega} v \text{ in } F \text{ weakly},$$
$$p_h w_h \longrightarrow \bar{\omega} w \text{ in } F \text{ strongly}.$$

such as (3.8) with c depending on h is obvious, since such an inequality is equivalent to the continuity of the trilinear form b_h.

We can now define an approximate problem for (3.1):

(3.9)
$$\begin{cases} \text{To find } u_h \in V_h, \text{ such that} \\ a_h(u_h,v_h) + b_h(u_h,u_h,v_h) = \langle \ell_h,v_h \rangle. \end{cases}$$

We then have

Proposition 3.1.

For each h, there exists at least one u_h in V_h, which is a solution of (3.9).

If (3.8) holds and if

(3.10)
$$\alpha_0^2 > c(n,\Omega)\beta,$$

then, u_h is unique.

Proof.

The existence of u_h follows from Lemma 1.4. We apply this lemma with $X = V_h$ which is a finite dimensional Hilbert space for the scalar product $((\cdot,\cdot))_h$. We define the operator P from V_h into V_h by,

(3.11)
$$((P(u_h),v_h))_h = a_h(u_h,v_h)+b_h(u_h,u_h,v_h) - \langle \ell_h,v_h \rangle, \quad \forall u_h,v_h \in V_h.$$

The operator P is continuous and there remains only to check (1.29); but, with (3.6),

$$\begin{aligned} ((P(u_h),u_h))_h &= a_h(u_h,u_h) - \langle \ell_h,u_h \rangle \\ &\geq \text{(by (3.2)-(3.3))} \\ &\geq \alpha_0 \|u_h\|_h^2 - \|\ell_h\|_{V'_h} \|u_h\|_h \\ &\geq (\alpha_0 \|u_h\|_h - \beta) \|u_h\|_h. \end{aligned}$$

Therefore

$$((P(u_h),u_h))_h > 0,$$

provided

$$\|u_h\|_h = k, \quad \text{and} \quad k > \frac{\beta}{\alpha_0}.$$

Lemma 1.4 gives the existence of at least one u_h such that

$$P(u_h) = 0$$

or

$$((P(u_h),v_h))_h = 0, \quad \forall v_h \in V_h, \tag{3.12}$$

which is exactly equation (3.9).

Let us suppose that (3.8) and (3.10) hold and let us show that u_h is unique. If u^*_h and u^{**}_h are two solutions of (3.9) and if $u_h = u^*_h - u^{**}_h$, then

$$a_h(u_h,v_h) + b_h(u^*_h,u^*_h,v_h) - b_h(u^{**}_h,u^{**}_h,v_h) = 0, \quad \forall v_h \in V_h;$$

taking $v_h = u_h$ and using (3.6) we find

$$a_h(u_h,u_h) = b_h(u_h,u^*_h,u_h).$$

Because of (3.2) and (3.8), we have

$$\alpha_0 \|u_h\|_h^2 \leq c\|u^*_h\|_h \|u_h\|_h^2. \tag{3.13}$$

If we set $v_h = u^*_h$ in equation (3.9) satisfied by u^*_h, we find

$$a_h(u^*_h,u^*_h) = \langle \ell_h,u^*_h \rangle,$$

and with (3.2) and (3.3),

$$\alpha_0 \|u^*_h\|_h^2 \leq \beta\|u^*_h\|_h \ ,$$

$$\|u^*_h\|_h \leq \frac{\beta}{\alpha_0} \ . \tag{3.14}$$

Using this majoration in (3.13) gives,

$$(\alpha_0 - \frac{c\beta}{\alpha_0}) \|u_h\|_h^2 \leq 0; \tag{3.15}$$

if (3.10) holds, this shows that $u_h = 0$.

Theorem 3.1.

We assume that conditions (3.2) to (3.7) are satisfied; u_h is some solution of (3.9).

If $n \leq 4$, the family $\{p_h u_h\}$ contains subsequences which are strongly *convergent in* F. *Any such subsequence converges to* $\bar{\omega}u$, *where* u *is some solution of* (3.1). *If the solution of* (3.1) *is unique, the whole family* $\{p_h u_h\}$ *converges to* $\bar{\omega}u$.

If $n \geq 5$, *we have the same conclusions, with only weak convergences in* F.

Proof.

We suppose that n is arbitrary.

Putting $v_h = u_h$ in (3.9), and using (3.2),(3.3) and (3.6) we find

$$a_h(u_h,u_h) = \langle \ell_h, u_h \rangle \quad ,$$

$$\alpha_0 \| u_h \|_h \leq \beta. \tag{3.16}$$

Since the p_h are stable, the family $p_h u_h$ is bounded in F; therefore there exists some subsequence $h' \longrightarrow 0$, and some $\phi \in F$ such that

$$p_{h'} u_{h'} \longrightarrow \phi \quad \text{in } F \text{ weakly.}$$

The condition (C2) for the approximation of a space shows that, necessarily $\phi \in \bar{\omega} V$, or $\phi = \bar{\omega}u$, $u \in V$:

$$p_{h'} u_{h'} \longrightarrow \bar{\omega}u \quad \text{in } F \text{ weakly}, \quad h' \longrightarrow 0. \tag{3.17}$$

Let v be an element of V and let us write (3.9) with $v_h = r_h v$:

$$a_h(u_h, r_h v) + b_h(u_h, u_h, r_h v) = \langle \ell_h, r_h v \rangle. \tag{3.18}$$

As $h' \longrightarrow 0$, according to (3.4),(3.5),(3.7),

$$a_{h'}(u_{h'}, r_{h'} v) \longrightarrow \nu((u,v)),$$

$$b_{h'}(u_{h'}, u_{h'}, r_{h'} v) \longrightarrow b(u,u,v),$$

$$\langle \ell_{h'}, r_{h'} v \rangle \longrightarrow (f,v).$$

Hence u belongs to V and satisfies

$$\nu((u,v)) + b(u,u,v) = (f,v), \quad \forall v \in \mathcal{V}. \tag{3.19}$$

If $n \leq 4$, equation (3.19) holds, by continuity, for each $v \in V$; if $n \geq 5$, we find, by continuity, that (3.19) is satisfied for each $v \in \tilde{V}$. In both cases, u is a solution of the stationary Navier-Stokes equations.

It can be proved by exactly the same method, that any convergent subsequence of $p_{h'}u_{h'}$, converges to $\bar{\omega}u$, where u is some solution of (3.1). If this solution is unique, the whole family $p_h u_h$ converges to $\bar{\omega}u$ in F weakly.

Let us show the strong convergence when $n \leq 4$.

In order to prove this strong convergence, we consider as in the linear case (see Theorem I.3.1) the expression

$$X_h = a_h(u_h - r_h u, u_h - r_h u).$$

Expanding this expression and using (3.9) with $v_h = u_h$ we find

$$X_h = \langle \ell_h, u_h \rangle - a_h(u_h, r_h u) - a_h(r_h u, u_h) + a(r_h u, r_h u).$$

Because of (2.4), (3.5), and (3.17),

$$X_{h'} \longrightarrow \langle f, u \rangle - a(u,u), \quad \text{as} \quad h' \longrightarrow 0.$$

We take $v = u$ in (3.1) and use (1.22); this gives

$$\langle f, u \rangle = a(u,u),^{(1)}$$

hence

$$X_{h'} \longrightarrow 0 \quad \text{as} \quad h' \longrightarrow 0. \tag{3.20}$$

We then finish the proof as in the linear case; the inequality (3.2) shows that

$$\|u_{h'} - r_{h'}u\|_{h'} \longrightarrow 0.$$

Since the p_h are stable, this implies

$$\|p_{h'}u_{h'} - p_{h'}r_{h'}u\|_F \leq \|p_{h'}\|_{\mathcal{L}(V_{h'},F)} \|u_{h'} - r_{h'}u\|_{h'} \longrightarrow 0.$$

Then we write

$$\|p_{h'}u_{h'} - \bar{\omega}u\|_F \leq \|p_{h'}u_{h'} - p_{h'}r_{h'}u\|_F + \|p_{h'}r_{h'}u - \bar{\omega}u\|_F,$$

---

(1) This equation does not hold for $n \geq 5$, and this is the reason the proof cannot be extended to these cases.

and the two terms on the right-hand side of this inequality converge to 0 as $h' \longrightarrow 0$.[1]

3.2 Applications.

In this section we apply the general convergence theorem to approximation schemes corresponding to the approximations (APX 1),...,(APX 4) of V.

Approximation (APX 1)

We choose a_h and ℓ_h as in the linear case (see I. (3.62) and (3.63)),

$$a_h(u_h,v_h) = \nu((u_h,v_h))_h \tag{3.21}$$

$$\langle \ell_h,v_h\rangle = (f,v_h). \tag{3.22}$$

Before defining b_h, we introduce the trilinear form $\hat{b}(u,v,w)$,

$$\hat{b}(u,v,w) = b'(u,v,w) + b''(u,v,w) \tag{3.23}$$

$$b'(u,v,w) = \frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega u_i(D_i v_j)w_j dx \tag{3.24}$$

$$b''(u,v,w) = -\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega u_i v_j(D_i w_j)dx. \tag{3.25}$$

It is not difficult to see that

$$\hat{b}(u,u,v) = b(u,u,v), \quad \forall u\in V, \ \forall v\in \mathcal{V}, \tag{3.26}$$

but $\hat{b}$ and b are otherwise different.

We now define b_h as,

$$b_h(u_h,v_h,w_h) = b'_h(u_h,v_h,w_h) + b''_h(u_h,v_h,w_h) \tag{3.27}$$

$$b'_h(u_h,v_h,w_h) = \frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega u_{ih}(\delta_{ih}v_{jh})w_{jh}dx \tag{3.28}$$

$$b''_h(u_h,v_h,w_h) = -\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega u_{ih}v_{jh}(\delta_{ih}w_{jh})dx. \tag{3.29}$$

It is clear that b'_h, b''_h, and hence b_h are trilinear forms on V_h; since

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[1] We recall that for each $u\in V$, and each $h\in\mathcal{H}$, there exists $r_h u\in V_h$, such that

$$p_h r_h u \longrightarrow \bar{\omega} u, \text{ in } F \text{ strongly, as } h\longrightarrow 0$$

(see Proposition 3.1).

V_h has a finite dimension, these forms are continuous.

We have to check (3.6) and (3.7); (3.6) is obvious with our choice of the form b_h, and (3.7) is the purpose of the next lemma.

Lemma 3.1.

If $p_h u_h$ converges weakly to $\bar{\omega} u$, then

$$b_h(u_h,u_h,r_h v) \longrightarrow b(u,u,v), \quad \forall v \in \mathcal{V}. \tag{3.30}$$

Proof.

Saying that $p_h u_h$ converges weakly to $\bar{\omega} u$ means that

$$u_h \longrightarrow u \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{weakly} \tag{3.31}$$

and

$$\delta_{ih} u_h \longrightarrow D_i u \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{weakly}, \quad 1 \leq i \leq n. \tag{3.32}$$

The Compactness Theorem 2.2 is applicable and shows that

$$u_h \longrightarrow u \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{strongly}. \tag{3.33}$$

We know that if $v \in \mathcal{V}$, $p_h r_h v$ converges to $\bar{\omega} v$ in F strongly; but the proofs of Lemma I.3.1 and of Proposition I.3.5 show actually that

$$r_h v \longrightarrow v \quad \text{in the norm of } \mathbb{L}^\infty(\Omega), \tag{3.34}$$

$$\delta_{ih} r_h v \longrightarrow D_i v \quad \text{in the norm of } \mathbb{L}^\infty(\Omega). \tag{3.35}$$

If we prove that

$$b_h'(u_h,u_h,r_h v) \longrightarrow b'(u,u,v) \tag{3.36}$$

$$b_h''(u_h,u_h,r_h v) \longrightarrow b''(u,u,v), \tag{3.37}$$

then, according to (3.27), the proof of (3.7) will be complete.

For (3.36) we write

$$|b_h'(u_h,u_h,r_h v) - b'(u,u,v)| \leq c_0 \sum_{i,j=1}^{n} \left| \int_\Omega (u_{ih} v_{jh} - u_i v_j) \delta_{ih} u_{jh} \, dx \right|$$

$$+ c_0 \sum_{i,j=1}^{n} \left| \int_\Omega u_i v_j (\delta_{ih} u_{jh} - D_i u_j) \, dx \right|.$$

All the preceding integrals converge to 0, and (3.36) is proved; the proof of (3.37) is similar.

The convergence result given by Theorem 3.1 is the following one:

(3.38) $$u_{h'} \longrightarrow u \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{weakly},$$

(3.39) $$\delta_{ih'}u_{h'} \longrightarrow D_i u \quad \text{in } \mathbb{L}^2(\Omega) \quad \text{weakly}, \quad 1 \leq i \leq n.$$

Exactly as in the linear case it can be shown that there exists some step function

(3.40) $$\pi_h = \sum_{M \in \mathring{\Omega}^1_h} \pi_h(M) w_{hM}$$

such that

(3.41) $$\nu((u_h, v_h))_h + (\overline{\nabla}_h \pi_h, v_h) = (f, v_h), \quad \forall v_h \in W_h;$$

therefore a solution u_h of (3.9) is a step function

(3.42) $$u_h = \sum_{M \in \mathring{\Omega}^1_h} u_h(M) w_{hM}$$

such that

(3.43) $$\sum_{i=1}^{n} (\nabla_{ih} u_{ih})(M) = 0, \quad \forall M \in \mathring{\Omega}^1_h$$

and

(3.44) $$-\nu \sum_{i=1}^{n} \delta^2_{ih} u_h(M) + \frac{1}{2} \sum_{i=1}^{n} u_{ih}(M) \delta_{ih} u_h(M)$$
$$- \frac{1}{2} \sum_{i=1}^{n} \delta_{ih}(u_{ih} u_h)(M) + \overline{\nabla}_h \pi_h(M) = f_h(M), \quad \forall M \in \mathring{\Omega}^1_h$$

where

(3.45) $$f_h(M) = \frac{1}{h_1 \cdots h_n} \int_{\sigma_h(M)} f(x)\,dx.$$

When condition (3.8) and some condition similar to (3.10) are satisfied, u_h and u are unique and the error between u and u_h can be estimated as in the linear case, if moreover $u \in \mathcal{C}^3(\overline{\Omega})$ and $p \in \mathcal{C}^2(\overline{\Omega})$.

Using the Taylor formula we can write,

(3.46) $$-\nu \sum_{i=1}^{n} (\delta_{ih} r_h u)(M) + \frac{1}{2} \sum_{i=1}^{n} (r_h u)_i (M) \delta_{ih} (r_h u)(M)$$

$$- \frac{1}{2} \sum_{i=1}^{n} \delta_{ih} (r_h u)_i r_h u)(M) + (\overline{\nabla}_h p)(M)$$

$$f(M) + \varepsilon_h(M), \quad \forall M \in \mathring{\Omega}_h^1,$$

with

(3.47) $$|\varepsilon_h(M)| \leq c(u,p)|h|,$$

where $c(u,p)$ depends only on the maximum norms of the second and third derivatives of u, and of the second derivatives of p. Equations (3.46) show that

(3.48) $$\nu((r_h u, v_h))_h + b_h(r_h u, r_h u, v_h) - (\overline{\nabla}_h \pi'_h, v_h) = (f + \varepsilon_h, v_h),$$

for each $v_h \in W_h$ where π'_h is the step function

(3.49) $$\pi'_h = \sum_{M \in \mathring{\Omega}_h^1} p(M) w_{hM}.$$

Subtracting (3.48) from (3.9) gives

$$\nu((u_h - r_h u, v_h))_h = -b_h(u_h, u_h, v_h) + b_h(r_h u, r_h u, v_h) + (\varepsilon_h, v_h), \quad \forall v_h \in W_h.$$

We take $v_h = u_h - r_h u$ and use (3.6) to get:

$$\nu \|u_h - r_h u\|_h^2 = -b_h(u_h - r_h u, u_h, u_h - r_h u) + (\varepsilon_h, u_h - r_h u),$$

and with (3.8) and (3.47),

$$\nu \|u_h - r_h u\|_h^2 \leq c(n,\Omega) \|u_h\|_h \|u_h - r_h u\|_h^2 + c(u,p) \|u_h - r_h u\|_h |h|$$

$$\nu \|u_h - r_h u\|_h \leq c(n,\Omega) \|u_h\|_h \|u_h - r_h u\|_h + c(u,p)|h|$$

$$\leq \text{(by (3.16))}$$

$$\leq \frac{\beta}{\alpha_0} c(n,\Omega) \|u_h - r_h u\|_h + c(u,p)|h|.$$

Finally

(3.50) $$(\nu - \frac{\beta}{\alpha_0} c(n,\Omega)) \|u_h - r_h u\|_h \leq c(u,p)|h|.$$

If

$$\nu\alpha_0 > \beta c(n,\Omega), \tag{3.51}$$

(the constant $c(n,\Omega)$ in (3.8)), this gives the following majoration of the error

$$\|u_h - r_h u\|_h \leq \frac{c(u,p)}{[\nu - \frac{\beta}{\alpha_0} c(n,\Omega)]} |h|. \tag{3.52}$$

Approximation (APX 2)

In the two dimensional case we can associate to the approximation (APX 2) of V described in Section I.4.2, a new discretization scheme for the Navier-Stokes equations.

We recall that V_h is a subspace of $\mathbb{H}_0^1(\Omega)$ for this approximation, and we take as in the linear case

$$a_h(u_h,v_h) = \nu((u_h,v_h)), \tag{3.53}$$

$$\langle \ell_h, v_h \rangle = (f,v_h), \tag{3.54}$$

where $((\cdot,\cdot))$ is the scalar product in V_h and in $\mathbb{H}_0^1(\Omega)$.

We define the form b_h by

$$b_h(u_h,v_h,w_h) = \hat{b}(u_h,v_h,w_h), \quad \forall u_h,v_h,w_h \in V_h, \tag{3.55}$$

$\hat{b}$ defined by (3.23) - (3.25); since V_h is a space of bounded vector functions, the forms $\hat{b}$ are defined on V_h; they are trilinear, hence continuous.

Condition (3.6) is obviously satisfied with our choice of b_h; condition (3.7) is the purpose of the next lemma.

Lemma 3.2.

If $p_h u_h = u_h$ converges weakly to $\bar{\omega} u = u$, then

$$b_h(u_h,u_h,r_h v) \to b(u,u,v), \quad \forall v \in \mathcal{V}. \tag{3.56}$$

Proof.

We recall that $F = \mathbb{H}_0^1(\Omega)$ and $\bar{\omega}$ and p_h are the identity. Saying that $p_h u_h$ converges weakly in F to $\bar{\omega} u$, amounts to saying that

$$u_h \longrightarrow u \text{ in } \mathbb{H}_0^1(\Omega) \text{ weakly.} \tag{3.57}$$

The Compactness Theorem 1.1 shows then that

(3.58) $$u_h \longrightarrow u \quad \text{in } \mathbb{L}^2(\Omega) \text{ strongly.}$$

The proof of Lemma 3.2 will be the same as that of Lemma 3.1, if we observe that

(3.59) $$r_h v \longrightarrow v \quad \text{in the norm of } \mathbb{L}^\infty(\Omega)$$

(3.60) $$D_i r_h v \longrightarrow D_i v \quad \text{in the norm of } \mathbb{L}^\infty(\Omega), \quad 1 \leq i \leq n,$$

which was actually proved in Proposition I.4.3 (see (4.63)).

The weak (or strong) convergence result given by Theorem 3.1 is the following one:

<u>If</u> $\rho(h) \longrightarrow 0$, <u>with</u> $\sigma(h) \leq \alpha$ (i.e. $h \in \mathcal{H}_\alpha$),
<u>then</u>

(3.61) $$u_{h'} \longrightarrow u \quad \underline{\text{in}}\ \mathbb{H}_0^1(\Omega) \quad \underline{\text{weakly}}\ (\underline{\text{or}}\ \underline{\text{strongly}}).$$

Exactly as in the linear case (see Section I.4.2), we can show that there exists a step function π_h,

(3.62) $$\pi_h = \sum_{\mathcal{S} \in \mathcal{T}_h} \pi_h(\mathcal{S}) \chi_{h\mathcal{S}},$$

($\chi_{h\mathcal{S}}$ = the characteristic function of the simplex $\mathcal{S}$), such that

(3.63) $$\nu((u_h,v_h)) + b_h(u_h,u_h,v_h) - (\pi_h, \operatorname{div} v_h) = (f,v_h), \quad \forall v_h \in W_h.$$

This equation is the discrete analog of

(3.64) $$\nu((u,v)) + b(u,u,v) - (p, \operatorname{div} v) = (f,v), \quad \forall v \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^n(\Omega).$$

Since the dimension of the space is $n = 2$, the form $\hat{b}$ is trilinear continuous on $\mathbb{H}_0^1(\Omega)$ and there exists some constant $\hat{c}$ such that

(3.65) $$|\hat{b}(u,v,w)| \leq \hat{c}\|u\| \|v\| \|w\|, \quad \forall u,v,w \in \mathbb{H}_0^1(\Omega).$$

This can be shown by the same method as (1.18); hence (3.8) holds with

$$c(n,\Omega) = \hat{c}.$$

We will get an estimation of the error assuming that $u \in \mathcal{C}^3(\overline{\Omega})$, $p \in \mathcal{C}^1(\overline{\Omega})$, and an hypothesis similar to (3.51). We take $v_h = r_h u - u_h$ in (3.63) and $v = r_h u - u_h$ in (3.64); subtracting these equations we find

$$(3.66)\qquad \nu((u-u_h, r_h u-u_h)) + \hat{b}(u,u,r_h u-u_h) - \hat{b}(u_h,u_h,r_h u-u_h) - (p-\pi_h, \operatorname{div}(r_h u-u_h)) = 0.$$

Since $\operatorname{div}(r_h u-u_h)$ is a step function which is constant on each simplex $\mathcal{S}\in\mathcal{T}_h$, we have

$$(p-\pi_h, \operatorname{div}(r_h u-u_h)) = (p-\pi'_h, \operatorname{div}(r_h u-u_h))$$

where π'_h is defined by

$$(3.67)\qquad \pi'_h = \sum_{\mathcal{S}\in\mathcal{T}_h} \frac{1}{(\text{meas } \mathcal{S})} \left(\int_{\mathcal{S}} p(x)dx\right)\chi_{h\mathcal{S}}.$$

Hence

$$(3.68)\qquad |(p-\pi_h, \operatorname{div}(r_h u-u_h)| \le |\pi'_h-p|\ |\operatorname{div}(r_h u-u_h)| \le \sqrt{2}\ |\pi'_h-p|\ \|u_h-r_h u\|.$$

We then estimate the difference

$$\hat{b}(u_h,u_h,r_h u-u) - \hat{b}(u,u,r_h u-u)$$

$$= \hat{b}(u_h-r_h u,u_h,r_h u-u_h) + \hat{b}(r_h u,u_h,r_h u-u_h) - \hat{b}(u,u,r_h u-u)$$

$$= \hat{b}(u_h-r_h u,u_h,r_h u-u_h) + \hat{b}(r_h u,r_h u-u,r_h u-u_h) + \hat{b}(r_h u-u,u,r_h u-u_h).$$

The absolute value of this sum can be majorized because of (3.65) by

$$(3.69)\qquad \hat{c}\|u_h\|\ \|r_h u-u_h\|^2 + \hat{c}\ \{\|r_h u\| + \|u\|\}\ \|r_h u-u\|\ \|r_h u-u_h\|.$$

We recall that

$$\nu\|u_h\|^2 = \langle f,u_h\rangle$$

and therefore

$$\|u_h\| \le \frac{1}{\nu}|f|.$$

Hence the sum (3.69) is less than or equal to

$$(3.70)\qquad \frac{\hat{c}}{\nu}|f|\ \|u_h-r_h u\|^2 + \hat{c}\{\|r_h u\| + \|u\|\}\ \|r_h u-u\|\ \|r_h u-u_h\|.$$

With the majorations (3.68) and (3.70), we get from (3.66)

$$(\nu - \frac{\hat{c}}{\nu} |f|) \|u_h - r_h u\|^2 \le \nu \|u_h - r_h u\| \, \|u - r_h u\| + \sqrt{2} |\pi'_h - p| \, \|u_h - r_h u\|$$

$$+ \hat{c}\{\|r_h u\| + \|u\|\} \|u_h - r_h u\| \, \|u - r_h u\|,$$

and finally

$$(3.71) \quad (\nu - \frac{\hat{c}}{\nu} |f|) \|u_h - r_h u\| \le \nu \|u - r_h u\| + \sqrt{2} |\pi'_h - p| + \hat{c}\{\|r_h u\| + \|u\|\} \|u - r_h u\| .$$

Under the assumption

$$(3.72) \qquad \nu^2 > \hat{c}|f|,$$

the inequality (3.71) gives a majoration of the error between u_h and $r_h u$ and hence between u and u_h.

<u>Approximation</u> (APX 3).

As in the linear case, the method here is very similar to the method used for the approximation (APX 2).

<u>Approximation</u> (APX 4).

We recall that Ω must be a simply connected open set in $\mathbb{R}^2$.

Since (APX 4) is an internal approximation of V, the simplest scheme (3.9) associated with this approximation is

(3.73) To find $u_h \in V_h$ such that

$$\nu((u_h, v_h)) + b(u_h, u_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in V_h .$$

Theorem 3.1 is applicable and shows that

$$(3.74) \qquad u_h \longrightarrow u \text{ in } V, \text{ as } \rho(h) \longrightarrow 0,$$

provided $\sigma(h) \le \alpha$ (i.e. $h \in \mathcal{H}_\alpha$).

The error between u and u_h can be estimated as follows (if we have uniqueness): we take $v = u - u_h$ in the variational equation satisfied by u:

$$\nu((u,v)) + b(u,u,v) = (f,v), \quad \forall v \in V.$$

We then take $v = r_h u - u_h$ in (3.73); we subtract these equations and find

$$(3.75) \quad \nu \|u_h - u\|^2 = \nu((u - u_h, u - r_h u)) + b(u_h, u_h, r_h u - u_h) - b(u, u, r_h u - u_h).$$

The difference

$$b(u_h, u_h, r_h u - u_h) - b(u, u, r_h u - u_h)$$

is equal to

$$b(u_h-u,u_h,r_hu-u_h) + b(u,u_h-u,r_hu-u_h)$$

$$= b(u_h-u,u_h,r_hu-u) + b(u_h-u,u_h,u-u_h) + b(u,u_h-u,r_hu-u)$$

$$= b(u_h-u,u_h,r_hu-u) + b(u_h-u,u,u-u_h) + b(u,u_h-u,r_hu-u).$$

Because of (1.18) this is majorized by

$$c\{\|u\| + \|u_h\|\} \|u_h-u\| \|r_hu-u\| + c\|u\| \|u-u_h\|^2.$$

We recall that

$$\nu\|u\|^2 = \langle f,u\rangle,$$

$$\|u\| \leq \frac{1}{\nu}\|f\|_{V'},$$

and similarly

$$\nu\|u_h\|^2 = \langle f,u_h\rangle$$

$$\|u_h\| \leq \frac{1}{\nu}\|f\|_{V'} .$$

Therefore the last expression is majorized by

$$\frac{2c}{\nu}|f| \|u_h-u\| \|r_hu-u\| + \frac{c}{\nu}|f| \|u-u_h\|^2.$$

We deduce then from (3.75)

$$(\nu - \frac{c}{\nu}|f|) \|u_h-u\|^2 \leq (\nu + \frac{2c}{\nu}|f|) \|u_h-u\| \|r_hu-u\|,$$

$$(\nu - \frac{c}{\nu}|f|) \|u_h-u\| \leq (\nu + \frac{2c}{\nu}|f|) \|r_hu-u\|. \tag{3.76}$$

If

$$\nu^2 > c\|f\|_{V'} , \tag{3.77}$$

inequality (3.76) shows that the error $\|u_h-u\|$ has the same order as $\|r_hu-u\|$. Since c is the constant $c(n)$, $n = 2$, in (1.18), the inequality (3.77) is exactly inequality (1.37) which ensures the uniqueness of the solution u of the exact problem.

Approximation (APX 5).

If we consider the approximation (APX 5) of V, we can set

$$a_h(u_h,v_h) = \nu((u_h,v_h))_h \tag{3.78}$$

$$\langle \ell_h,v_h\rangle = (f,v_h) \tag{3.79}$$

(3.80) $$b_h(u_h,v_h,w_h) = \frac{1}{2} \sum_{i,j=1}^{n} \int_\Omega u_{ih}[(D_{ih}v_{jh})w_{jh} - v_{jh}(D_{ih}w_{jh})]dx.$$

Proposition 3.1 is applicable and shows the existence of at least one $u_h \in V_h$ such that

(3.81) $$\nu((u_h,v_h))_h + b_h(u_h,u_h,v_h) = (f,v_h), \quad \forall v_h \in V_h.$$

Theorem 3.1 cannot be applied since we do not know if condition (3.7) is satisfied, because of the lack of discrete compactness theorems for this type of approximation.

3.3 Numerical Algorithms.

The following analysis is restricted to the dimensions $n \leq 4$. We wish to extend to the nonlinear case the numerical algorithms described, for the linear case, in Section 5, Chapter I.

We observe that the stationary Navier-Stokes equations are not the Euler equations of an optimization problem like the Stokes equations. The following algorithms are then some extension of the Uzawa and Arrow-Hurwicz algorithms classically related to optimization problems.

In the sequel of this section we will always use the trilinear form $\hat{b}(u,v,w)$ defined by (3.23) - (3.25). This is a trilinear continuous form on $\mathbb{H}_0^1(\Omega) \times \mathbb{H}_0^1(\Omega) \times \mathbb{H}_0^1(\Omega)$ and there exists some constant $\hat{c} = \hat{c}(n)$ such that

(3.82) $$|\hat{b}(u,v,w)| \leq \hat{c}(n) \|u\| \|v\| \|w\|, \quad \forall u,v,w \in \mathbb{H}_0^1(\Omega), \ (n \leq 4).$$

We already noticed that

(3.83) $$\hat{b}(u,v,w) = b(u,v,w) \quad \text{if} \quad u \in V, v,w \in \mathbb{H}_0^1(\Omega)$$

(3.84) $$\hat{b}(u,v,v) = 0, \qquad u,v \in \mathbb{H}_0^1(\Omega).$$

Uzawa Algorithm.

In order to approximate the solutions of (1.8) - (1.11) we will construct as in Section I.5, two sequences of elements

(3.85) $$u^m \in \mathbb{H}_0^1(\Omega), \quad p^m \in \mathbb{L}^2(\Omega).$$

This construction is relatively easy because the condition $\operatorname{div} u = 0$ will disappear in the approximate equations.

We start the algorithm with an arbitrary element p^0:

$$(3.86) \qquad p^0 \in L^2(\Omega).$$

When p^m is known $(m \geq 0)$, we define u^{m+1} as some solution of

$$(3.87) \qquad \begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega), \quad \text{and} \\ \nu((u^{m+1},v)) + \hat{b}(u^{m+1},u^{m+1},v) = (p^m, \operatorname{div} v) + (f,v), \ \forall v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

We then define p^{m+1} by

$$(3.88) \qquad \begin{cases} p^{m+1} \in L^2(\Omega), \quad \text{and} \\ (p^{m+1}-p^m,q) + \rho(\operatorname{div} u^{m+1},q) = 0, \ \forall q \in L^2(\Omega). \end{cases}$$

Later we will give the conditions that the number $\rho > 0$ must satisfy.

The existence of u^{m+1} satisfying (3.87) is not obvious , but can be proved using the Galerkin method, exactly as in Theorem 1.2. Therefore we will skip the proof. It is not difficult to see that u^{m+1} is the solution of the following nonlinear Dirichlet problem:

$$(3.89) \qquad \begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega) \\ -\nu\Delta u^{m+1} + \sum_{i=1}^{n} u_i^{m+1} D_i u^{m+1} + \frac{1}{2}(\operatorname{div} u^{m+1}) u^{m+1} \\ \qquad\qquad = - \operatorname{grad} p^m + f \in \mathbb{H}^{-1}(\Omega). \end{cases}$$

The solution of (3.87)-(3.89) is not unique, in general. When u^{m+1} is known, p^{m+1} is explicitly given by (3.88) which is equivalent to

$$(3.90) \qquad p^{m+1} = p^m - \rho \operatorname{div} u^{m+1} \in L^2(\Omega).$$

To investigate convergence we will assume that

$$(3.91) \qquad \nu - \frac{c(n)}{\nu} \|f\|_{V'} = \bar{\nu} > 0;$$

with (3.82), (3.83) and Theorem 1.3, the condition (3.91) implies the uniqueness of the solution of (1.8) - (1.11); p is unique up to an additive constant; we fix this constant by asking that

$$(3.92) \qquad \int_\Omega p(x)\,dx = 0.$$

Proposition 3.2.

We assume that $n \leq 4$ and that condition (3.91) holds. We suppose also that the number ρ satisfies

(3.93) $$0 < \rho < \frac{2\bar{\nu}}{n} .$$

Then, as $m \longrightarrow \infty$,

(3.94) $$u^m \longrightarrow u \text{ in the norm of } \mathbb{H}_0^1(\Omega) ,$$

(3.95) $$p^m \longrightarrow p \text{ in } \mathbb{L}^2(\Omega), \text{ weakly,}$$

where $\{u,p\}$ is the unique solution of (1.8) - (1.11) which satisfies (3.92).

Proof.

We set

(3.96) $$\begin{cases} v^{m+1} = u^{m+1} - u \\ q^{m+1} = p^{m+1} - p \end{cases}$$

and we proceed as in the proof of Theorem I.5.1.

We subtract equation (3.87) from the equation

(3.97) $$\nu((u,v)) + \hat{b}(u,u,v) - (p,\operatorname{div} v) = \langle f,v \rangle, \quad \forall v \in \mathbb{H}_0^1(\Omega)$$

and we take $v = v^{m+1}$ to obtain

$$\begin{aligned} \nu\|v^{m+1}\|^2 &= (q^m, \operatorname{div} v^{m+1}) + \hat{b}(u,u,v^{m+1}) - \hat{b}(u^{m+1},u^{m+1},v^{m+1}) \\ &= -(q^{m+1}-q^m, \operatorname{div} v^{m+1}) - (q^{m+1}, \operatorname{div} v^{m+1}) \\ &\qquad + \hat{b}(v^{m+1},u,v^{m+1}) \\ &\leq \text{(by (3.82) and (1.39))} \\ &\leq -(q^{m+1}-q^m, \operatorname{div} v^{m+1}) - (q^{m+1}, \operatorname{div} v^{m+1}) \\ &\qquad + \frac{\hat{c}}{\nu} \|f\|_{V'} \|v^{m+1}\|^2. \end{aligned} \tag{3.98}$$

We take $q = q^{m+1}$ in (3.88) and we find,

(3.99) $$|q^{m+1}|^2 - |q^m|^2 + |q^{m+1} - q^m|^2 = -2\rho(\operatorname{div} v^{m+1},q^{m+1}).$$

We multiply the last inequality in (3.98) by 2ρ and add it to (3.99); this gives

$$(3.100)\qquad |q^{m+1}|^2 - |q^m|^2 + |q^{m+1}-q^m|^2 + (\nu - \frac{\hat{c}}{\nu}\|f\|_{V'})\|v^{m+1}\|^2 \leq -2\rho(q^{m+1}-q^m, \operatorname{div} v^{m+1}).$$

This inequality is similar to equation (5.12) in the proof of Theorem I.5.1 with ν replaced by $\overline{\nu}$ (see (3.91)). The proof can be completed exactly as in Theorem I.5.1.

Remark 3.1.

In the general case, when uniqueness is not assumed, we can prove weak convergence results for the average values

$$\frac{1}{N}\sum_{m=1}^{N} u_m, \qquad \frac{1}{N}\sum_{m=1}^{N} p^m.$$

These sequences are bounded in $\mathbb{H}_0^1(\Omega)$ and $L^2(\Omega)$, and every weakly convergent subsequence converges to a couple $\{u,p\}$ which is a solution of (1.8) - (1.11).

Arrow-Hurwicz Algorithm.

We construct a sequence of couples $\{u^m,p^m\}$ defined as follows.

We start the algorithm with arbitrary elements

$$(3.101)\qquad u^0 \in \mathbb{H}_0^1(\Omega), \quad p^0 \in L^2(\Omega).$$

When p^m, u^m are known, we define p^{m+1}, u^{m+1}, as solutions of

$$(3.102)\quad \begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega) \quad \text{and} \\ ((u^{m+1}-u^m,v)) + \rho\nu((u^m,v)) + b(u^m,u^{m+1},v) - \rho(p^m, \operatorname{div} v) = \rho(f,v), \\ \qquad\qquad \forall v \in \mathbb{H}_0^1(\Omega) \end{cases}$$

$$(3.103)\qquad \begin{cases} p^{m+1} \in L^2(\Omega) \quad \text{and} \\ \alpha(p^{m+1}-p^m,q) + \rho(\operatorname{div} u^{m+1},q) = 0, \quad \forall q \in L^2(\Omega). \end{cases}$$

We suppose that ρ and α are two strictly positive numbers; conditions on ρ and α will be given later.

The existence and uniqueness of $u^{m+1} \in \mathbb{H}_0^1(\Omega)$ satisfying (3.102) is easy with the projection theorem; (3.102) is a linear variational equation equivalent to the Dirichlet problem

(3.104) $$\begin{cases} u^{m+1} \in \mathbb{H}_0^1(\Omega) \\ -\Delta u^{m+1} + \rho \sum_{i=1}^{n} u_i^m D_i u^{m+1} + \frac{\rho}{2}(\operatorname{div} u^m) u^{m+1} = -\Delta u^m + \rho\nu\Delta u^m + \rho \operatorname{grad} p^m + f. \end{cases}$$

Then p^{m+1} is explicitly given by (3.103) which is equivalent to

(3.105) $$p^{m+1} = p^m - \frac{\rho}{\alpha} \operatorname{div} u^{m+1} \in L^2(\Omega).$$

Convergence can be proved under stronger conditions than those used in Proposition 3.2.

Proposition 3.3.

We assume that $n \leq 4$, *that*

(3.106) $$\nu - \frac{2\hat{c}}{\nu} \|f\|_{V'} - \frac{4\hat{c}^2}{\nu^2} \|f\|_{V'}^2 = \nu^* > 0$$

and that

(3.107) $$0 < \rho < \frac{\alpha\nu^*}{2(n+\nu^2\alpha)} .$$

Then, as $m \longrightarrow \infty$,

(3.108) $$u^m \longrightarrow u \text{ in the norm of } \mathbb{H}_0^1(\Omega) ,$$

(3.109) $$p^m \longrightarrow p \text{ in } \mathbb{L}^2(\Omega) \text{ weakly},$$

where $\{u,p\}$ *is the unique solution of* (1.8) - (1.11) *which satisfies* (3.92).

Proof.

We use again the notation (3.96). We take $v = 2v^{m+1}$ in (3.97) and (3.102) and subtract these equations; this gives

$$\|v^{m+1}\|^2 - \|v^m\|^2 + \|v^{m+1}-v^m\|^2 + 2\rho\nu\|v^{m+1}\|^2$$

$$= 2\rho\nu((v^{m+1},v^{m+1}-v^m)) + 2\rho(q^m, \operatorname{div} v^{m+1})$$

$$+ 2b(u^m,u^{m+1},v^{m+1}) - 2b(u,u,v^{m+1})$$

$$\leq \frac{1}{4}\|v^{m+1}-v^m\|^2 + 4\rho^2\nu^2\|v^{m+1}\|^2 + 2\rho(q^m, \operatorname{div} v^{m+1})$$

$$+ 2b(v^m-v^{m+1},u,v^{m+1}) + 2b(v^{m+1},u,v^{m+1})$$

(3.110) $\leq$ (because of (3.82) and (1.39))

$$\leq \frac{1}{4}\|v^{m+1}-v^m\|^2 + 4\rho^2\nu^2\|v^{m+1}\|^2 + 2\rho(q^m, \operatorname{div} v^{m+1})$$

$$+ \frac{2\hat{c}}{\nu}\|f\|_{V'}\|v^{m+1}-v^m\|\,\|v^{m+1}\| + \frac{2\hat{c}}{\nu}\|f\|_{V'}\|v^{m+1}\|^2$$

$$\leq \frac{1}{2}\|v^{m+1}-v^m\|^2 + \left(4\rho^2\nu^2 + \frac{4\hat{c}^2}{\nu^2}\|f\|_{V'}^2 + \frac{2\hat{c}}{\nu}\|f\|_{V'}\right)\|v^{m+1}\|^2$$

$$+ 2\rho(q^m, \operatorname{div} v^{m+1}).$$

We take $q = 2q^{m+1}$ in (3.103):

$$\alpha|q^{m+1}|^2 - \alpha|q^m|^2 = -2\rho(q^{m+1}, \operatorname{div} v^{m+1})$$

$$= -2\rho(q^m, \operatorname{div} v^{m+1}) - 2\rho(q^{m+1}-q^m, \operatorname{div} v^{m+1})$$

(3.111)

$$\leq -2\rho(q^m, \operatorname{div} v^{m+1}) + 2\rho\sqrt{n}|q^{m+1}-q^m|\,\|v^{m+1}\|$$

$$\leq \frac{\alpha}{2}|q^{m+1}-q^m|^2 + \frac{2\rho^2 n}{\alpha}\|v^{m+1}\|^2 - 2\rho(q^m, \operatorname{div} v^{m+1})$$

(see (5.13), (5.25), (5.26), Chapter I). We add the last inequality in (3.110) to the last inequality in (3.111) and obtain

$$\alpha|q^{m+1}|^2-\alpha|q^m|^2 + \frac{1}{2}|q^{m+1}-q^m|^2 + \|v^{m+1}\|^2 - \|v^m\|^2 + \frac{1}{2}\|v^{m+1}-v^m\|^2$$

(3.112)

$$+ 2\rho\left(\nu - \frac{2\hat{c}}{\nu}\|f\|_{V'} - \frac{4\hat{c}^2}{\nu^2}\|f\|_{V'}^2 - 2\rho\nu^2 - \frac{2\rho n}{\alpha}\right)\|v^{m+1}\|^2 \leq 0.$$

The conditions (3.106)-(3.107) ensure that the coefficient of $\|v^{m+1}\|^2$ in (3.112) is strictly positive; this inequality is then similar to the inequality (5.27) in the proof of Theorem I.5.2; we finish the proof as in that theorem.

CHAPTER III
The Evolution Navier-Stokes Equations

INTRODUCTION

This final chapter deals with the full Navier-Stokes Equations; i.e., the evolution nonlinear case. First we describe a few basic results concerning existence and uniqueness of solutions, and then study the approximation of these equations by several methods.

In Section 1 we rapidly study the linear evolution equations (evolution Stokes equations). This section contains some technical lemmas appropriate for the study of evolution equations. Section 2 gives compactness theorems which will enable us to obtain strong convergence results in the evolution case, and to pass to the limit in the nonlinear terms. Section 3 contains the variational formulation of the problem (weak or turbulent solutions, according to J. Leray [1], [2], [3]; E. Höpf [2]) and the main results of existence and uniqueness of solution (the dimension of the space is $n = 2$ or 3); the existence is based on the construction of an approximate solution by the Galerkin method. In Section 4 further existence and uniqueness results are presented; there existence is obtained by semi-discretization in time, and is valid for any dimension of the space.

In the final section we study the approximation of the evolution Navier-Stokes equations, in the two and three dimensional cases. Several schemes are considered corresponding to a classical discretization in the time variable (implicit, Cranck-Nicholson, explicit) associated with any of the discretizations in the space variables introduced in Chapter I (finite differences, finite elements). We conclude with a study of the nonlinear stability of these schemes, establishing sufficient conditions for stability and proving the convergence of all these schemes when they are stable.

§1. THE LINEAR CASE.

In this section we develop some results of existence, uniqueness, and regularity of the solutions of the linearized Navier-Stokes equations. After introducing some notation useful in the linear as well as in the nonlinear case (Section 1.1), we give the classical and variational formulations of the problem and the statement of the main existence and uniqueness result (Section 1.2); the proofs of the existence and of the uniqueness are then given in Sections 1.3 and 1.4.

1.1 Notation.

Let Ω be an open lipschitzian set in $\mathbb{R}^n$; for simplicity we suppose Ω bounded, and we refer to the remarks in Section 1.5 for the unbounded case. We recall the definition of the spaces $\mathcal{V}$, V, H, used in the previous chapters and which will be the basic spaces in this chapter too:

$$(1.1)\qquad \mathcal{V} = \{u \in \mathcal{D}(\Omega),\ \ \text{div}\ u = 0\}$$

$$(1.2)\qquad V = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega),$$

$$(1.3)\qquad H = \text{the closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\Omega).$$

The space H is equipped with the scalar product $(\cdot,\cdot)$ induced by $\mathbb{L}^2(\Omega)$; the space V is a Hilbert space with the scalar product

$$(1.4)\qquad ((u,v)) = \sum_{i=1}^{n} (D_i u, D_i v),$$

since Ω is bounded.

The space V is contained in H, is dense in H and the injection is continuous. Let H' and V' denote the dual spaces of H and V, and let i denote the injection mapping from V into H. The adjoint operator i' is linear continuous from H' into V', is one to one since $i(V) = V$ is dense in H and $i'(H')$ is dense in V' since i is one to one; therefore H' can be identified with a dense subspace of V'. Moreover, by the Riesz representation theorem, we can identify H and H', and we arrive at the inclusions

$$(1.5)\qquad V \subset H \equiv H' \subset V',$$

where each space is dense in the following one and the injections are continuous.

As a consequence of the previous identifications, the scalar product in H of $f \in H$ and $u \in V$ is the same as the scalar product of f and u in the duality between V' and V:

$$(1.6)\qquad \langle f,u\rangle = (f,u),\quad \forall f \in H,\ \forall u \in V.$$

For each u in V, the form

(1.7) $$v \in V \longmapsto ((u,v)) \in \mathbb{R}$$

is linear and continuous on V; therefore, there exists an element of V' which we denote by Au such that

(1.8) $$\langle Au,v\rangle = ((u,v)), \quad \forall v \in V.$$

It is easy to see that the mapping $u \longmapsto Au$ is linear and continuous, and by Theorem I.2.2 and Remark I.2.2, is an isomorphism from V onto V'.

If Ω is unbounded, the space V is equipped with the scalar product

(1.9) $$[\![u,v]\!] = ((u,v)) + (u,v);$$

the inclusions (1.5) hold. The operator A is linear continuous from V into V' but is not in general an isomorphism; for every $\varepsilon > 0$, $A + \varepsilon I$ is an isomorphism from V onto V'.

Let a,b be two extended real numbers, $-\infty \leq a < b \leq \infty$, and let X be a Banach space. For given α, $1 \leq \alpha < +\infty$, $L^{\alpha}(a,b;X)$ denotes the space of L^{α}-integrable functions from $[a,b]$ into X, which is a Banach space with the norm

(1.10) $$\left\{\int_a^b \|f(t)\|_X^{\alpha}\, dt\right\}^{\frac{1}{\alpha}}.$$

The space $L^{\infty}(a,b;X)$ is the space of essentially bounded functions from $[a,b]$ into X, and is equipped with the Banach norm

(1.11) $$\operatorname*{Ess\,Sup}_{[a,b]} \|f(t)\|_X.$$

The space $\mathscr{C}([a,b];X)$ is the space of continuous functions from $[a,b]$ into X and if $-\infty < a < b < \infty$ is equipped with the Banach norm

(1.12) $$\sup_{t \in [a,b]} \|f(t)\|_X.$$

Most often the interval $[a,b]$ will be the interval $[0,T]$, $T > 0$ fixed; when no confusion can arise, we will use the following more condensed notations,

(1.13) $$^{\alpha}(X) = L^{\alpha}(0,T;X), \quad 1 \leq \alpha \leq +\infty$$

(1.14) $$\mathscr{C}(X) = \mathscr{C}([0,T];X).$$

The remainder of this Section 1.1 is devoted to the proof of the following technical lemma concerning the derivatives of functions with values in a Banach space.

Lemma 1.1.

Let X be a given Banach space with dual X' and let u and g be two functions belonging to $L^1(a,b;X)$. Then, the following three conditions are equivalent

(i) u *is absolutely continuous and* $u' = g$,

$$u(t_1) - u(t_0) = \int_{t_0}^{t_1} g(s)ds, \quad \forall t_0,t_1 \in [a,b]; \tag{1.15}$$

(ii) *For each test function* $\phi \in \mathcal{D}((a,b))$,

$$\int_a^b u(t)\phi'(t)dt = - \int_a^b g(t)\phi(t)dt \quad (\phi' = \frac{d\phi}{dt}); \tag{1.16}$$

(iii) *For each* $\eta \in X'$,

$$\frac{d}{dt} \langle u,\eta\rangle = \langle g,\eta\rangle, \tag{1.17}$$

in the scalar distribution sense, on (a,b).

Proof.

We suppose for simplicity that the interval $[a,b]$ is the interval $[0,T]$. A legitimate integration by parts shows that (i) implies (ii) and (iii); it remains to check that the property (iii) implies the property (ii) and that (ii) implies (i).

If (iii) is satisfied and $\phi \in \mathcal{D}((0,T))$, then by definition,

$$\int_0^T \langle u(t),\eta\rangle \, \phi'(t)dt = - \int_0^T \langle g(t),\eta\rangle \, \phi(t)dt \tag{1.18}$$

or

$$\left\langle \int_0^T u(t)\phi'(t)dt + \int_0^T g(t)\phi(t)dt, \eta \right\rangle = 0, \quad \forall \eta \in X',$$

so that (1.16) is satisfied.

Let us now prove that (ii) implies (i). We can reduce the general case to the case $g = 0$. To see this, we set $v = u - u_0$ with

$$u_0(t) = \int_0^t g(s)ds; \tag{1.19}$$

it is clear that u_0 is an absolutely continuous function and that $u'_0 = g$; hence (1.16) holds with u replaced by u_0 and

$$\int_0^T v(t)\phi'(t)dt = 0, \quad \forall \phi \in \mathcal{D}((0,T)). \tag{1.20}$$

The proof of (i) will be achieved if we show that (1.20) implies that v is a constant element of X.

Let ϕ_0 be some function in $\mathcal{D}((0,T))$, such that

$$\int_0^T \phi_0(t)dt = 1.$$

Any function ϕ in $\mathcal{D}((0,T))$ can be written as

$$\phi = \lambda\phi_0 + \psi', \quad \lambda = \int_0^T \phi(t)dt, \quad \psi\in\mathcal{D}((0,T)); \tag{1.21}$$

indeed since

$$\int_0^T (\phi(t) - \lambda\phi_0(t)dt = 0,$$

the primitive function of $\phi-\lambda\phi_0$ vanishing at 0, belongs to $\mathcal{D}((0,T))$, and ψ is precisely this primitive function. According to (1.20) and (1.21),

$$\int_0^T (v(t)-\xi)\phi(t)dt = 0, \quad \forall\phi\in\mathcal{D}((0,T)) \tag{1.21a}$$

where

$$\xi = \int_0^T v(s)\phi_0(s)ds.$$

To achieve the proof, it remains to show that (1.21a) implies that

$$v(t) = \xi \quad \text{a.e.},$$

i.e., that a function $w\in L^1(X)$ such that

$$\int_0^T w(t)\phi(t)dt = 0, \quad \forall\phi\in\mathcal{D}((0,T)), \tag{1.22}$$

is zero almost everywhere. This well-known result is proved by regularization: if $\tilde{w}$ is the function equal to w on $[0,T]$ and to 0 outside this interval, and if ρ_ε is an even regularizing function, then for ε small enough, $\rho_\varepsilon*\phi$ belongs to $\mathcal{D}((0,T))$, $\forall\phi\in\mathcal{D}((0,T))$, and

$$\int_0^T w(t)(\rho_\varepsilon*\phi)(t)dt = \int_{-\infty}^{+\infty} \tilde{w}(t)(\rho_\varepsilon*\phi)(t)dt = \int_{-\infty}^{+\infty} (\rho_\varepsilon*\tilde{w})(t)\phi(t)dt = 0.$$

Hence, for any $\eta > 0$ fixed, $\rho_\varepsilon*\tilde{w}$ is equal to 0 on the interval $[\eta,T-\eta]$, for ε small enough; as $\varepsilon\longrightarrow 0$, $\rho_\varepsilon*\tilde{w}$ converges to $\tilde{w}$ in $L^1(-\infty,+\infty;X)$. Thus w is zero on $[\eta, T-\eta]$; since $\eta > 0$ is arbitrarily small, w is zero on the whole interval $[0,T]$.

1.2 <u>The Existence and Uniqueness Theorem.</u>

Let Ω be a lipschitzian open bounded set in $\mathbb{R}^n$ and let $T > 0$ be fixed. We denote by Q the cylinder $\Omega \times (0,T)$. The linearized Navier-Stokes equations, are the evolution equations corresponding to the Stokes problem:

To find a vector function

$$u : \Omega \times [0,T] \longmapsto \mathbb{R}^n$$

and a scalar function

$$p : \Omega \times [0,T] \longmapsto \mathbb{R},$$

respectively equal to the velocity of the fluid and to its pressure, such that

$$\frac{\partial u}{\partial t} - \nu \Delta u + \text{grad } p = f \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1.23}$$

$$\text{div } u = 0 \quad \text{in} \quad Q, \tag{1.24}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \tag{1.25}$$

$$u(x,0) = u_0(x), \quad \text{in} \quad \Omega, \tag{1.26}$$

where the vector functions f and u_0 are given, f defined on $\Omega \times [0,T]$, u_0 defined on Ω; the equations (1.25) and (1.26) give respectively the boundary and initial conditions.

Let us suppose that u and p are classical solutions of (1.23) - (1.26), say $u \in \mathcal{C}^2(\overline{Q})$, $p \in \mathcal{C}^1(\overline{Q})$. If v denotes any element of $\mathcal{V}$, it is easily seen that

$$\left(\frac{\partial u}{\partial t}, v\right) + \nu((u,v)) = (f,v). \tag{1.27}$$

By continuity, the equality (1.27) holds also for each $v \in V$; we observe also that

$$\left(\frac{\partial u}{\partial t}, v\right) = \frac{d}{dt}(u,v).$$

This leads to the following weak formulation of the problem (1.23) - (1.26):

<u>For</u> f <u>and</u> u_0 <u>given,</u>

$$f \in L^2(0,T;V') \tag{1.28}$$

$$u_0 \in H, \tag{1.29}$$

<u>to find</u> u, <u>satisfying</u>

$$u \in L^2(0,T;V) \tag{1.30}$$

<u>and</u>

$$\frac{d}{dt}(u,v) + \nu((u,v)) = \langle f,v\rangle, \quad \forall v \in V. \tag{1.31}$$

$$u(0) = u_0. \tag{1.32}$$

If u belongs to $L^2(0,T;V)$ the condition (1.32) does not make sense in general; its meaning will be explained after the following two remarks:

(i) The spaces in (1.28) (1.29) (1.30) are the spaces for which existence and uniqueness will be proved; it is clear at least that a smooth solution u of (1.23) - (1.26) satisfies (1.30).

(ii) We cannot check right now that a solution of (1.30) - (1.32) is solution, in some weak sense, of (1.23) - (1.26); hence we postpone the investigation of this point until Section 1.5.

By (1.6) and (1.8), we can write (1.30) as

$$\frac{d}{dt}\langle u,v\rangle = \langle f-\nu Au,v\rangle, \quad \forall v \in V. \tag{1.33}$$

Since A is linear and continuous from V into V' and $u \in L^2(V)$, the function Au belongs to $L^2(V')$; hence $f-\nu Au \in L^2(V')$ and (1.33) and Lemma 1.1 show that

$$u' \in L^2(0,T;V') \tag{1.34}$$

and that u is a.e. equal to an absolutely continuous function from $[0,T]$ into V'. Any function satisfying (1.30) and (1.31) is, after modification on a set of measure zero, a continuous function from $[0,T]$ into V', and therefore the condition (1.32) makes sense.

Let us suppose again that f is given in $L^2(V')$ as in (1.28). If u satisfies (1.30) and (1.31), then as observed before u satisfies (1.34) and (1.33). According to Lemma 1.1 the equality (1.33) is itself equivalent to

$$u' + \nu Au = f. \tag{1.35}$$

Conversely if u satisfies (1.30), (1.34), and (1.35), then u clearly satisfies (1.31), $\forall v \in V$.

An alternate formulation of the weak problem is the following:

Given f *and* u_0 *satisfying* (1.28)-(1.29), *to find* u *satisfying*

$$u \in L^2(0,T;V), \quad u' \in L^2(0,T;V'), \tag{1.36}$$

$$u' + \nu Au = f, \quad \text{on } (0,T), \tag{1.37}$$

$$u(0) = u_0. \tag{1.38}$$

Any solution of (1.36) - (1.38) is a solution of (1.30) - (1.32) and conversely

Concerning the existence and uniqueness of solution of these problems, we will prove the following result.

Theorem 1.1.

For given f and u_0 which satisfy (1.28) and (1.29), there exists a unique function u which satisfies (1.36) - (1.38). Moreover

$$u \in \mathcal{C}([0,T];H). \tag{1.39}$$

The proof of the existence is given in Section 1.3, that of the uniqueness and of (1.39) in Section 1.4.

1.3 Proof of the Existence in Theorem 1.1.

We use the Faedo-Galerkin method. Since V is separable there exists a sequence $w_1,\cdots,w_m,\cdots$, which is free and total in V. For each m we define an approximate solution u_m of (1.37) or (1.31) as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t)w_i, \tag{1.40}$$

and

$$(u'_m,w_j) + \nu((u_m,w_j)) = \langle f,w_j\rangle, \quad j = 1,\cdots,m, \tag{1.41}$$

$$u_m(0)=u_{0m}, \tag{1.42}$$

where u_{0m} is, for example, the orthogonal projection in H of u_0 on the space spanned by $w_1,\cdots,w_m$ (1).

The functions g_{im}, $1 \leq i \leq m$, are scalar functions defined on $[0,T]$, and (1.41) is a linear differential system for these functions; indeed we have

$$\sum_{i=1}^{m} (w_i,w_j)g'_{im}(t) + \nu \sum_{i=1}^{m} ((w_i,w_j))g_{im}(t) = \langle f(t),w_j\rangle; \quad j = 1,\cdots,m \tag{1.43}$$

since the elements $w_1,\cdots,w_m$, are linearly independent, it is well-known that the matrix with elements (w_i,w_j) $(1 \leq i,j \leq m)$ is non-singular; hence by inverting this matrix we reduce (1.43) to a linear system with constant coefficients

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(1) u_{0m} can be any element of the space spanned by $w_1,\cdots,w_m$ such that $u_{0m} \longrightarrow u_0$ in the norm of H, as $m \longrightarrow \infty$.

$$(1.44)\quad g'_{im}(t) + \sum_{j=1}^{m} \alpha_{ij} g_{jm}(t) = \sum_{j=1}^{m} \beta_{ij} \langle f(t),w_j\rangle, \quad 1 \leq i \leq m,$$

where $\alpha_{ij},\beta_{ij} \in \mathbb{R}$.

The condition (1.42) is equivalent to m equations

$$(1.45)\qquad g_{im}(0) = \text{the } i^{\underline{th}} \text{ component of } u_{0m}.$$

The linear differential system (1.44) together with the initial conditions (1.45) defines uniquely the g_{im} on the whole interval $[0,T]$.

Since the scalar functions $t \mapsto \langle f(t),w_j\rangle$ are square integrable, so are the functions g_{im} and therefore, for each m

$$(1.46)\qquad u_m \in L^2(0,T;V), \quad u'_m \in L^2(0,T;V).$$

We will obtain a priori estimates independent of m for the functions u_m and then pass to the limit.

<u>A Priori Estimates</u>.

We multiply equation (1.41) by $g_{jm}(t)$ and add these equations for $j = 1,\cdots,m$. We get

$$(u'_m(t),u_m(t)) + \nu\|u_m(t)\|^2 = \langle f(t),u_m(t)\rangle \; .$$

But, because of (1.46),

$$2(u'_m(t),u_m(t)) = \frac{d}{dt}\,|u_m(t)|^2,$$

and this gives

$$(1.47)\qquad \frac{d}{dt}\,|u_m(t)|^2 + 2\nu\|u_m(t)\|^2 = 2\langle f(t),u_m(t)\rangle$$

The right-hand side of (1.47) is majorized by

$$2\|f(t)\|_{V'}\,\|u_m(t)\| \leq \nu\|u_m(t)\|^2 + \frac{1}{\nu}\,\|f(t)\|^2_{V'} \; .$$

Therefore

$$(1.48)\qquad \frac{d}{dt}\,|u_m(t)|^2 + \nu\|u_m(t)\|^2 \leq \frac{1}{\nu}\,\|f(t)\|^2_{V'}.$$

Integrating (1.48) from 0 to s, s > 0, we obtain in particular

$$(1.49)\quad |u_m(s)|^2 \leq |u_{0m}|^2 + \frac{1}{\nu}\int_0^s \|f(t)\|^2_{V'}dt \leq |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{V'}dt$$

Hence:

(1.50) $$\sup_{s\in[0,T]} |u_m(s)|^2 \leq |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2 dt.$$

The right-hand side of (1.50) is finite and independent of m; therefore

(1.51) <u>The sequence</u> u_m <u>remains in a bounded set of</u> $L^\infty(0,T;H)$.

We then integrate (1.48) from 0 to T and get

(1.52) $$|u_m(T)|^2 + \nu\int_0^T \|u_m(t)\|^2 dt \leq |u_{0m}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2 dt$$
$$\leq |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2 dt.$$

This shows that

(1.53) <u>The sequence</u> u_m <u>remains in a bounded set of</u> $L^2(0,T;V)$.

<u>Passage to the Limit</u>.

The a priori estimate (1.51) shows the existence of an element u in $L^\infty(0,T;H)$ and a subsequence $m'\longrightarrow\infty$, such that

(1.54) $u_{m'}$ converges to u, for the weak-star topology of $L^\infty(0,T;H)$;

(1.54) means that for each $v\in L^1(0,T;H)$,

(1.55) $$\int_0^T (u_{m'}(t)-u(t),v(t))dt \longrightarrow 0, \quad m'\longrightarrow\infty.$$

By (1.53) the subsequence $u_{m'}$ belongs to a bounded set of $L^2(0,T;V)$; therefore another passage to a subsequence shows the existence of some u_* in $L^2(0,T;V)$ and some subsequence (still denoted $u_{m'}$) such that

(1.56) $u_{m'}$ converges to u_*, for the weak topology of $L^2(0,T;V)$.

The convergence (1.56) means

$$\int_0^T \langle u_{m'}(t)-u_*(t),v(t)\rangle dt \longrightarrow 0, \quad \forall v\in L^2(0,T;V').$$

In particular, by (1.6),

(1.57) $$\int_0^T (u_{m'}(t),v(t))dt \longrightarrow \int_0^T (u_*(t),v(t))dt,$$

for each v in $L^2(0,T;H)$. By comparison with (1.55) we see that

(1.58) $$\int_0^T (u(t)-u_*(t),v(t))dt = 0,$$

for each v in $L^2(0,T;H)$; hence

(1.59) $$u = u_* \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

In order to pass to the limit in equations (1.41) and (1.42), let us consider scalar functions ψ continuously differentiable on $[0,T]$ and such that

(1.60) $$\psi(T) = 0$$

For such a function ψ we multiply (1.41) by $\psi(t)$, integrate in t and integrate by parts:

$$\int_0^T (u'_m(t),w_j)\psi(t)dt = -\int_0^T (u_m(t)\psi'(t),w_j)dt-(u_m(0),w_j)\psi(0).$$

Hence we find,

(1.61) $$-\int_0^T (u_m(t),\psi'(t)w_j)dt + \nu\int_0^T ((u_m(t),\psi(t)w_j))dt$$

$$= (u_{0m},w_j)\psi(0) + \int_0^T <f(t),w_j>dt.$$

The passage to the limit for $m = m' \longrightarrow \infty$ in the integrals on the left-hand side is easy using (1.54), (1.57), and (1.59); we observe also that

(1.62) $$u_{0m} \longrightarrow u_0, \text{ in } H, \text{ strongly.}$$

Hence we find in the limit

(1.63) $$-\int_0^T (u(t),\psi'(t)w_j)dt + \nu\int_0^T ((u(t),\psi(t)w_j))dt$$

$$= (u_0,w_j)\psi(0) + \int_0^T <f(t),w_j>\psi(t)dt.$$

The equality (1.63) which holds for each j, allows us to write by a linearity argument that

(1.64) $$-\int_0^T (u(t),v)\psi'(t)dt + \nu\int_0^T ((u(t),v))\psi(t)dt$$

$$= (u_0,v)\psi(0) + \int_0^T <f(t),v>\psi(t)dt,$$

for each v which is a finite linear combination of the w_j. Since each term of (1.64) depends linearly and continuously on v, for the norm of V, the equality (1.64) is still valid, by continuity, for each v in V.

Now, writing in particular (1.64) with $\psi = \phi \in \mathcal{D}((0,T))$, we find the following equality which is valid in the distribution sense on $(0,T)$:

$$\frac{d}{dt}(u,v) + \nu((u,v)) = \langle f,v\rangle, \quad \forall v \in V; \tag{1.65}$$

this is exactly (1.31). As proved before the statement of the Theorem 1.1, this equality and (1.59) imply that u' belongs to $L^2(0,T;V')$ and

$$u' + \nu Au = f. \tag{1.66}$$

Finally, it remains to check that $u(0) = u_0$ (the continuity of u is proved in Section 1.4). For this, we multiply (1.65) by $\psi(t)$, (the same ψ as before), integrate in t, and integrate by parts:

$$\int_0^T \frac{d}{dt}(u(t),v)\psi(t)dt = -\int_0^T (u(t),v)\psi'(t)dt + (u(0),v)\psi(0).$$

We get

$$\begin{aligned}-\int_0^T (u(t),v)\psi'(t)dt + \nu\int_0^T ((u(t),v))\psi(t)dt \\ = (u(0),v)\psi(0) + \int_0^T \langle f(t),v\rangle\psi(t)dt.\end{aligned} \tag{1.67}$$

By comparison with (1.64), we see that

$$(u_0-u(0),v)\psi(0) = 0,$$

for each $v \in V$, and for each function ψ of the type considered. We can choose such that $\psi(0) \neq 0$, and therefore

$$(u(0)-u_0,v) = 0, \quad \forall v \in V.$$

This equality implies that

$$u(0) = u_0$$

and achieves the proof of the existence.

1.4 Proof of the Continuity and Uniqueness.

This proof is based on the following lemma which is a particular case of a general theorem of interpolation of Lions-Magenes [1]:

Lemma 1.2.

Let V, H, V' be three Hilbert spaces, each space included in the following one as in (1.5), V' being the dual of V. If a function u belongs to $L^2(0,T;V)$ and its derivative u' belongs to $L^2(0,T;V')$, then u is almost everywhere equal to a function continuous from $[0,T]$ into H and we have the following equality, which holds in the scalar distribution sense on $(0,T)$

$$\frac{d}{dt}|u|^2 = 2\langle u',u\rangle \quad . \tag{1.68}$$

The equality (1.68) is meaningful since the functions

$$t \longmapsto |u(t)|^2, \quad t \longmapsto \langle u'(t),u(t)\rangle$$

are both integrable on $[0,T]$.

An alternate elementary proof of the lemma is given below.

If we assume this lemma, (1.39) becomes obvious and it only remains to check the uniqueness. Let us assume that u and v are two solutions of (1.36) - (1.38) and let $w = u-v$. Then w belongs to the same spaces as u and v, and

$$w' + \nu Aw = 0, \quad w(0) = 0. \tag{1.69}$$

Taking the scalar product of the first equality (1.69) with $w(t)$, we find

$$\langle w'(t),w(t)\rangle + \nu\|w(t)\|^2 = 0 \quad \text{a.e.}$$

Using then (1.68) with u replaced by w, we obtain

$$\frac{d}{dt}|w(t)|^2 + 2\nu\|w(t)\|^2 = 0$$

$$|w(t)|^2 \leq |w(0)|^2 = 0, \quad t\in[0,T],$$

and hence $u(t) = v(t)$ for each t.

Proof of Lemma 1.2.

The elementary proof of Lemma 1.2 which was announced before, is now given in the two following lemmas.

Lemma 1.3.

Under the assumptions of Lemma 1.2, the equality (1.68) is satisfied.

Proof.

By regularizing the function $\tilde{u}$ from $\mathbb{R}$ into V, which is equal to u on [0,T] and to 0 outside this interval, we easily obtain a sequence of functions u_m such that

(1.70) $\forall m$, u_m is infinitely differentiable from [0,T] into V,

(1.71) as $m \to \infty$,

$$u_m \to u \text{ in } L^2_{loc}(0,T;V)$$
$$u'_m \to u' \text{ in } L^2_{loc}(0,T;V').$$

Because of (1.6) and (1.70), the equality (1.68) for u_m is obvious:

(1.72) $$\frac{d}{dt}|u_m(t)|^2 = 2(u'_m(t),u_m(t)) = 2\langle u'_m(t),u_m(t)\rangle, \quad \forall m.$$

As $m \to \infty$, it follows from (1.71) that

$$|u_m|^2 \to |u|^2 \text{ in } L^1_{loc}(0,T)$$
$$\langle u'_m,u_m\rangle \to \langle u',u\rangle \text{ in } L^1_{loc}(0,T).$$

These convergences also hold in the distribution sense; therefore we are allowed to pass to the limit in (1.72) in the distribution sense; in the limit we find precisely (1.68).

Since the function

$$t \mapsto \langle u'(t),u(t)\rangle$$

is integrable on [0,T], the equality (1.68) shows us that the function u of Lemma 1.3 satisfies

(1.73) $$u \in L^\infty(0,T;H).$$

In the particular case of the function u satisfying (1.36) - (1.38), this was proved directly in Section 1.3.

According to Lemma 1.1, u is continuous from [0,T] into V'. Therefore, with this and (1.73), the following Lemma 1.4 shows us that u is weakly continuous from [0,T] into H, i.e.,

(1.74) $\forall v \in H$, the function $t \mapsto (u(t),v)$ is continuous.

Admitting temporarily this point we can achieve the proof of Lemma 1.2. We must prove that for each $t_0 \in [0,T]$,

(1.75) $$|u(t) - u(t_0)|^2 \longrightarrow 0, \quad \text{as} \quad t \longrightarrow t_0.$$

Expanding this term, we find

$$|u(t)|^2 + |u(t_0)|^2 - 2(u(t),u(t_0)).$$

When $t \longrightarrow t_0$, $|u(t)|^2 \longrightarrow |u(t_0)|^2$ since by (1.68),

$$|u(t)|^2 = |u(t_0)|^2 + 2\int_{t_0}^{t} \langle u'(s),u(s)\rangle ds;$$

because of (1.74)

$$(u(t),u(t_0)) \longrightarrow |u(t_0)|^2,$$

and (1.75) is proved.

The proof of Lemma 1.2 is achieved as soon as we prove the next lemma. This lemma is stated in a slightly more general form.

Lemma 1.4.

Let X and Y be two Banach spaces, such that

(1.76) $$X \subset Y$$

with a continuous injection.

If a function ϕ belongs to $L^\infty(0,T;X)$ and is weakly continuous with values in Y, then ϕ is weakly continuous with values in X.

Proof.

Perhaps replacing Y by the closure of X in Y, we can suppose that X is dense in Y. Hence the dense continuous imbedding of X into Y gives by duality a dense continuous imbedding of Y' (dual of Y) into X' (dual of X):

(1.77) $$Y' \subset X'.$$

By assumption, for each $\eta \in Y'$,

(1.78) $$\langle\phi(t),\eta\rangle \longrightarrow \langle\phi(t_0),\eta\rangle, \quad \text{as} \quad t \longrightarrow t_0, \ \forall t_0,$$

and we must prove that (1.78) is also true for each $\eta \in X'$.

We first prove that $\phi(t) \in X$ for each t and that

(1.79) $$\|\phi(t)\|_X \leq \|\phi\|_{L^\infty(0,T;X)}, \quad \forall t \in [0,T].$$

Indeed, by regularizing the function $\tilde{\phi}$ equal to ϕ on $[0,T]$ and to 0 outside this interval, we find a sequence of smooth functions ϕ_m from $[0,T]$ into X such that

$$\|\phi_m(t)\|_X \leq \|\phi\|_{L^\infty(X)}, \quad \forall m, \forall t \in [0,T]$$

and

$$\langle \phi_m(t), \eta \rangle \longrightarrow \langle \phi(t), \eta \rangle, \quad m \longrightarrow \infty, \quad \forall \eta \in Y'.$$

Since

$$|\langle \phi_m(t), \eta \rangle| \leq \|\phi\|_{L^\infty(X)} \|\eta\|_{X'}, \quad \forall m, \forall t,$$

we obtain in the limit

$$|\langle \phi(t), \eta \rangle| \leq \|\phi\|_{L^\infty(X)} \|\eta\|_{X'}, \quad \forall t \in [0,T], \quad \forall \eta \in Y'.$$

This inequality shows that $\phi(t) \in X$ and that (1.79) holds.

Finally let us prove (1.78) for η in X'. Since Y' is dense in X', there exists, for each $\varepsilon > 0$, some $\eta_\varepsilon \in Y'$ such that

$$\|\eta - \eta_\varepsilon\|_{X'} \leq \varepsilon.$$

We then write

$$\langle \phi(t) - \phi(t_0), \eta \rangle = \langle \phi(t) - \phi(t_0), \eta - \eta_\varepsilon \rangle + \langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle$$

$$|\langle \phi(t) - \phi(t_0), \eta \rangle| \leq 2\varepsilon \|\phi\|_{L^\infty(X)} + |\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle.$$

As $t \longrightarrow t_0$, since $\eta_\varepsilon \in Y'$, the continuity assumption implies that

$$\langle \phi(t) - \phi(t_0), \eta_\varepsilon \rangle \longrightarrow 0$$

and hence

$$\overline{\lim} \, |\langle \phi(t) - \phi(t_0), \eta \rangle| \leq 2\varepsilon \|\phi\|_{L^\infty(X)}$$

Since $\varepsilon > 0$ is arbitrarily small, the preceding upper limit is zero, and (1.78) is proved.

1.5 Miscellaneous Remarks.

We give in this section some remarks and complements to Theorem 1.1.

An Extension of Theorem 1.1.

Theorem 1.1 is a particular case of an abstract theorem, involving abstract spaces V,H, and an abstract operator A; see Lions-Magenes [1].

If instead of (1.28) we assume that

$$f = f_1 + f_2, \quad f_1 \in L^2(0,T;V'), \quad f_2 \in L^1(0,T;H), \tag{1.80}$$

then all the conclusions of Theorem 1.1 are true with only one modification:

$$u' \in L^2(0,T;V') + L^1(0,T;H). \tag{1.81}$$

In the proof of the existence, we write after (1.47):

$$\begin{aligned}\frac{d}{dt}|u_m(t)|^2 + 2\nu\|u_m(t)\|^2 &\leq 2\|f_1(t)\|_{V'}\|u_m(t)\| + 2|f_2(t)|\,|u_m(t)| \\ &\leq \nu\|u_m(t)\|^2 + \frac{1}{\nu}\|f_1(t)\|^2_{V'} + |f_2(t)|\,\{1 + |u_m(t)|^2\}.\end{aligned} \tag{1.82}$$

Hence, in particular

$$\frac{d}{dt}\{1 + |u_m(t)|^2\} \leq \frac{1}{\nu}\|f_1(t)\|^2_{V'} + |f_2(t)|\{1 + |u_m(t)|^2\}. \tag{1.83}$$

Multiplying this by $\exp\{-\int_0^t |f_2(\sigma)|d\sigma\}$, we obtain

$$\frac{d}{dt}\{\exp(-\int_0^t |f_2(\sigma)|d\sigma)\cdot(1 + |u_m(t)|^2)\} \leq \frac{1}{\nu}\|f_1(t)\|^2_{V'}\exp(-\int_0^t |f_2(\sigma)|d\sigma).$$

Integrating this inequality from 0 to s, $s > 0$, we obtain a majoration similar to (1.50) which implies (1.51). Then integrating (1.82) from 0 to T we obtain (1.53).

The proof of the existence is then conducted exactly as in Section 1.3.

Concerning the derivative u', we have

$$u' = -\nu Au + f_1 + f_2 \in L^2(0,T;V') + L^1(0,T;H). \tag{1.84}$$

It is easy to see that Lemma 1.2 is also valid if

$$\begin{cases} u \in L^2(0,T;V) \cap L^\infty(0,T;H), \\ u' \in L^2(0,T;V') + L^1(0,T;H). \end{cases} \tag{1.85}$$

After noticing that, we can prove the uniqueness and the continuity of u, $u \in \mathcal{C}([0,T];H)$, exactly as in Section 1.4.

The Case Ω Unbounded.

For the evolution problem, when Ω is unbounded, the introduction of the space Y considered in the stationary unbounded case (Chapter I, Section 2.3) is no longer necessary. All the previous results hold if Ω is unbounded and V is equipped with the norm (1.9). Let us assume, in the most general case, that f satisfies (1.80). We have exactly the same results as in Theorem 1.1, if f satisfies (1.28), and the same results as in Remark 1.1 if f satisfies (1.80). The only difference is that we must replace (1.82) by

$$\begin{aligned}\frac{d}{dt}|u_m(t)|^2 + 2\nu\|u_m(t)\|^2 &\le 2\|f_1(t)\|_{V'} [\![u_m(t)]\!] + 2|f_2(t)|\ |u_m(t)| \\ &\le \nu\|u_m(t)\|^2 + \nu|u_m(t)|^2 + \frac{1}{\nu}\|f_1(t)\|^2_{V'} + |f_2(t)|\ (1+|u_m(t)|^2).\end{aligned} \tag{1.86}$$

Hence

$$\frac{d}{dt}\{1+|u_m(t)|^2\} \le (|f_2(t)| + \nu)\{1+|u_m(t)|^2\} + \frac{1}{\nu}\|f_1(t)\|^2_{V'}. \tag{1.87}$$

This inequality is then treated exactly as (1.83), to obtain (1.51). After that, integrating (1.86) from 0 to T, we obtain

$$\int_0^T \|u_m(t)\|^2 dt \le \text{Const}.$$

This majoration, together with (1.51), gives (1.53).

The proofs of the existence, the uniqueness, and the continuity are then exactly the same as before.

Interpretation of the Variational Problem.

We wish to make precise in what sense the function u defined by Theorem 1.1 is solution of the initial problem (1.23) - (1.26).

Proposition 1.1.

Under the assumptions of Theorem 1.1, there exists a distribution p on $Q = \Omega \times (0,T)$, such that the function u defined by Theorem 1.1 and p satisfy (1.23) in the distribution sense in Q; (1.24) is satisfied in the distribution sense too and (1.26) is satisfied in the sense

$$u(t) \to u_0, \text{ in } L^2(\Omega), \text{ as } t \to 0. \tag{1.88}$$

Proof.

The equality (1.24) is an easy consequence of $u \in L^2(0,T;V)$; (1.26) and (1.88) follow also immediately from Theorem 1.1; (1.25) is satisfied in a sense which depends on the trace theorems available for Ω since u is in $L^2(0,T;\mathbb{H}_0^1(\Omega))$.

To introduce the pressure, let us set

$$(1.89) \qquad U(t) = \int_0^t u(s)ds, \quad F(t) = \int_0^t f(s)ds.$$

It is clear that, at least,

$$U \in \mathcal{C}([0,T];V), \quad F \in \mathcal{C}([0,T];V').$$

Integrating (1.31), we see that

$$(1.90) \quad (u(t)-u_0,v) + \nu((U(t),v)) = \langle F(t),v\rangle, \quad \forall t \in [0,T], \quad \forall v \in V,$$

or

$$\langle u(t)-u_0 - \nu\Delta U(t)-F(t),v\rangle = 0, \quad \forall t \in [0,T], \quad \forall v \in V.$$

By an application of Proposition I.1.3, we find, for each $t \in [0,T]$, the existence of some function $P(t)$,

$$P(t) \in L^2(\Omega),$$

such that

$$(1.91) \qquad u(t)-u_0 - \nu\Delta U(t) + \text{grad } P(t) = F(t).$$

By Proposition I.1.2, the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $\mathbb{H}^{-1}(\Omega)$. Observing that

$$(1.92) \qquad \text{grad } P = F + \nu\Delta U - u + u_0,$$

we conclude that grad P belongs to $\mathcal{C}([0,T];\mathbb{H}^{-1}(\Omega))$ as the right-hand side of (1.92) does; therefore

$$(1.93) \qquad P \in \mathcal{C}([0,T];L^2(\Omega)).$$

This enables us to differentiate (1.91) in the t variable, in the distribution sense in $Q = \Omega \times (0,T)$; setting

$$(1.94) \qquad p = \frac{\partial P}{\partial t},$$

we obtain precisely (1.23).

We do not have in general any information on p better than (1.93)-(1.94). In the next proposition, we will get more regularity on p after assuming more regularity on the data f and u_0.

Some Results of Regularity.

Assuming that the data Ω, f, u_0, are sufficiently smooth, we can obtain as much regularity as desired for u and p. We will only prove a simple result of this type:

Proposition 1.2.

Let us assume that Ω is of class $\mathscr{C}^2$, that

$$f \in L^2(0,T;H) \tag{1.95}$$

and

$$u_0 \in V. \tag{1.96}$$

Then

$$u \in L^2(0,T;\mathbb{H}^2(\Omega)), \tag{1.97}$$

$$u' \in L^2(0,T;H), \quad \text{i.e.,} \quad u' \in \mathbb{L}^2(Q), \tag{1.98}$$

$$p \in L^2(0,T;H^1(\Omega)). \tag{1.99}$$

Proof.

The first point is to obtain (1.98); this is proved by getting another a priori estimate for the approximate solution u_m constructed by the Galerkin method.

Using the notation of Section 1.3, we multiply (1.41) by $g'_{jm}(t)$, and add these equalities for $j = 1, \cdots, m$; this gives

$$|u'_m(t)|^2 + \nu((u_m(t), u'_m(t))) = (f(t), u'_m(t))$$

or

$$2|u'_m(t)|^2 + \nu \frac{d}{dt} \|u_m(t)\|^2 = 2(f(t), u'_m(t)). \tag{1.100}$$

We then integrate (1.100) from 0 to T, and use the Schwarz inequality; we obtain:

$$2\int_0^T |u'_m(t)|^2 dt + \nu\|u_m(T)\|^2 = \nu\|u_{0m}\|^2 + 2\int_0^T (f(t), u'_m(t))dt$$

$$\leq \nu\|u_{0m}\|^2 + \int_0^T |f(t)|^2 dt + \int_0^T |u'_m(t)|^2 dt,$$

$$\int_0^T |u'_m(t)|^2 dt \leq \nu\|u_{0m}\|^2 + \int_0^T |f(t)|^2 dt. \tag{1.101}$$

The basis w_j used for the Galerkin method can be chosen so that $w_j \in V$ for each j and we can take

u_{0m} = the projection in V of u_0 on the space spanned by $w_1, \cdots, w_m$.

Therefore

(1.102) $$u_{0m} \longrightarrow u_0 \quad \text{in } V \text{ strongly as } m \to \infty,$$

and

(1.103) $$\|u_{0m}\| \leq \|u_0\|.$$

With these choices of the w_j and of u_{0m}, (1.101) shows that

(1.104) the u'_m <u>belong to a bounded set of</u> $L^2(0,T;H)$,

and (1.98) is proved.

We then come back to the equalities (1.23), (1.24), (1.25) and we apply the regularity theorem of the stationary case (Theorem I.2.4): For almost every t in $[0,T]$,

(1.105) $$\begin{cases} -\nu\Delta u(t) + \operatorname{grad} p(t) = f-u' \in L^2(0,T;\mathbb{L}^2(\Omega)) \\ \operatorname{div} u(t) = 0 \quad \text{in } \Omega \\ u(t) = 0 \quad \text{on } \partial\Omega \end{cases}$$

so that $u(t)$ belongs to $\mathbb{H}^2(\Omega)$ and $p(t)$ belongs to $\mathbb{H}^1(\Omega)$. Moreover, since the mapping

$$f(t)-u'(t) \longrightarrow \{u(t),p(t)\}$$

is linear continuous from $\mathbb{L}^2(\Omega)$ into $\mathbb{H}^2(\Omega) \times \dot{H}^1(\Omega)$, due to I(2.40), and since

$$f-u' \in L^2(0,T;\mathbb{L}^2(\Omega)),$$

it is clear that (1.97) and (1.99) are satisfied.

§2. COMPACTNESS THEOREMS.

The compactness theorems presented in Chapter II are not sufficient for the nonlinear evolution problems. Our goal now is to prove some compactness theorems which are appropriate for the nonlinear problems of the remainder of this chapter.

After a preliminary result given in Section 2.1, we prove in Section 2.2 a compactness theorem in the frame of Banach spaces. In Section 2.3 we prove two other compactness theorems in the frame of Hilbert spaces; one of them involves fractional derivatives in time of the functions.

Some related discrete form of these theorems will be studied later on.

2.1 A Preliminary Result.

The proof of the compactness theorems of the next two sections will be based on the following lemma.

Lemma 2.1.

Let X_0, X, *and* X_1 *be three Banach spaces such that*

$$X_0 \subset X \subset X_1 \tag{2.1}$$

the injection of X *into* X_1 *being continuous and:*

(2.2) *the injection of* X_0 *into* X *is compact.*

Then for each $\eta > 0$, *there exists some constant* c_η *depending on* η *(and on the spaces* X_0, X, X_1*) such that:*

$$\|v\|_X \leq \eta\|v\|_{X_0} + c_\eta\|v\|_{X_1}, \quad \forall v \in X_0. \tag{2.3}$$

Proof.

The proof is by contradiciton. Saying that (2.3) is not true amounts to saying that there exists some $\eta > 0$ such that for each c in $\mathbb{R}$,

$$\|v\|_X \geq \eta\|v\|_{X_0} + c\|v\|_{X_1},$$

for at least one v. Taking $c = m$, we obtain a sequence of elements v_m satisfying

$$|v_m|_X \geq \eta|v_m|_{X_0} + m|v_m|_{X_1}, \quad \forall m.$$

We consider then the normalized sequence

$$w_m = \frac{v_m}{\|v_m\|_{X_0}},$$

which satisfies

(2.4) $$\|w_m\|_X \geq \eta + m\|w_m\|_{X_1}, \quad \forall m.$$

Since $\|w_m\|_{X_0} = 1$, the sequence w_m is bounded in X and (2.4) shows that

(2.5) $$\|w_m\|_{X_1} \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty.$$

In addition, by (2.2) the sequence w_m is relatively compact in X; hence we can extract from w_m a subsequence w_μ strongly convergent in X; because of (2.5) the limit of w_μ must be 0, but this contradicts (2.4) as:

$$\|w_\mu\|_X \geq \eta > 0, \quad \forall \mu.$$

2.2 A Compactness Theorem in Banach Space.

Let X_0, X, X_1, be three Banach spaces such that

(2.6) $$X_0 \subset X \subset X_1,$$

where the injections are continuous and:

(2.7) $$X_i \text{ is reflexive}, \quad i = 0,1,$$

(2.8) $$\text{the injection } X_0 \longmapsto X \text{ is compact.}$$

Let $T > 0$ be a fixed finite number, and let α_0, α_1, be two finite numbers such that $\alpha_i > 1$, $i = 0,1$.

We consider the space

(2.9) $$\mathcal{Y} = \mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1)$$

(2.10) $$\mathcal{Y} = \{v \in L^{\alpha_0}(0,T;X_0),\ v' = \frac{dv}{dt} \in L^{\alpha_1}(0,T;X_1)\}.$$

The space $\mathcal{Y}$ is provided with the norm

(2.11) $$\|v\|_{\mathcal{Y}} = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v'\|_{L^{\alpha_1}(0,T;X_1)},$$

which makes it a Banach space. It is evident that

$$\mathcal{Y} \subset L^{\alpha_0}(0,T;X),$$

with a continuous injection. Actually we will prove that this injection is compact.

Theorem 2.1.

Under the assumptions (2.6) to (2.9) the injection of $\mathcal{Y}$ into $L^{\alpha_0}(0,T;X)$ is compact.

Proof.

(i) Let u_m be some sequence which is bounded in $\mathcal{Y}$. We must prove that this sequence contains a subsequence u_μ strongly convergent in $L^{\alpha_0}(0,T;X)$.

Since the spaces X_i are reflexive and $1 < \alpha_i < +\infty$, the spaces $L^{\alpha_i}(0,T;X_i)$, $i = 0,1$, are likewise reflexive, and hence $\mathcal{Y}$ is reflexive. Therefore, there exists some u in $\mathcal{Y}$ and some subsequence u_μ with

(2.12) $$u_\mu \longrightarrow u \text{ in } \mathcal{Y} \text{ weakly, as } \mu \longrightarrow \infty,$$

which means

(2.13) $$\begin{cases} u_\mu \longrightarrow u \text{ in } L^{\alpha_0}(0,T;X_0) \text{ weakly} \\ u'_\mu \longrightarrow u' \text{ in } L^{\alpha_1}(0,T;X_1) \text{ weakly.} \end{cases}$$

It suffices to prove that

(2.14) $$v_\mu = u_\mu - u \text{ converges to } 0 \text{ in } L^{\alpha_0}(0,T;X) \text{ strongly.}$$

(ii) The theorem will be proved if we show that

(2.15) $$v_\mu \longrightarrow 0 \text{ in } L^{\alpha_0}(0,T;X_1) \text{ strongly.}$$

In fact, due to Lemma 2.1, we have

$$\|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leq \eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_0)} + c_\eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_1)}$$

and since the sequence v_μ is bounded in $\mathcal{Y}$:

(2.16) $$\|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leq c\eta + c_\eta \|v_\mu\|_{L^{\alpha_0}(0,T;X_1)}$$

Passing to the limit in (2.16) we see by (2.15) that

(2.17) $$\overline{\lim_{\mu \longrightarrow \infty}} \|v_\mu\|_{L^{\alpha_0}(0,T;X)} \leq c\eta;$$

since $\eta > 0$ is arbitrarily small in Lemma 2.1, this upper limit is 0 and thus (2.14) is proved.

(iii) To prove (2.15) we observe that

(2.18) $$\mathcal{Y} \subset \mathcal{C}([0,T];X_1),$$

with a continuous injection; the inclusion (2.18) results from Lemma 1.1, and the continuity of the injection is very easy to check.

We infer from this, the majoration

(2.19) $$\|v_\mu(t)\|_{X_1} \le c, \quad \forall t \in [0,T], \ \forall \mu.$$

According to Lebesgue's Theorem, (2.15) is now proved if we show that, for almost every t in $[0,T]$,

(2.20) $$v_\mu(t) \longrightarrow 0 \ \text{ in } X_1 \ \text{ strongly, as } \mu \longrightarrow \infty.$$

We will prove (2.20) for $t = 0$; the proof would be similar for any other t. We write

$$v_\mu(0) = v_\mu(t) - \int_0^t v_\mu'(\tau)d\tau$$

and by integration

$$v_\mu(0) = \frac{1}{s}\left\{\int_0^s v_\mu(t)dt - \int_0^s \int_0^t v_\mu'(\tau)d\tau dt\right\}$$

Hence

(2.21) $$v_\mu(0) = a_\mu + b_\mu$$

with

(2.22) $$a_\mu = \frac{1}{s}\int_0^s v_\mu(t)dt, \quad b_\mu = -\frac{1}{s}\int_0^s (s-t)v'_\mu(t)dt.$$

For a given $\varepsilon > 0$, we choose s so that

$$\|b_\mu\|_{X_1} \le \int_0^s \|v'_\mu(t)\|_{X_1} dt \le \frac{\varepsilon}{2}.$$

Then, for this fixed s, we observe that, as $\mu \longrightarrow \infty$, $a_\mu \longrightarrow 0$ in X_0 weakly and thus in X_1 strongly; for μ sufficiently large

$$\|a_\mu\|_{X_1} \le \frac{\varepsilon}{2},$$

and (2.20), for $t = 0$, follows.

2.3 A Compactness Theorem Involving Fractional Derivatives.

The next compactness theorem is in the frame of Hilbert spaces and is based on the notion of fractional derivatives of a function.

Let us assume that X_0, X, X_1, are Hilbert spaces with

(2.23) $$X_0 \subset X \subset X_1,$$

the injections being continuous and

(2.24) the injection of X_0 into X is compact.

If v is a function from $\mathbb{R}$ into X_1, we denote by $\hat{v}$ its Fourier transform

(2.25) $$\hat{v}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} v(t)dt.$$

<u>The derivative in t of order γ of v is the inverse Fourier transform of</u> $(2i\pi\tau)^{\gamma}\hat{v}$ or

(2.26) $$\widehat{D_t^{\gamma}v}(\tau) = (2i\pi\tau)^{\gamma}\,\hat{v}(\tau).$$ [1]

For given $\gamma > 0$ [2], we define the space

(2.27) $$\mathcal{H}^{\gamma}(\mathbb{R};X_0,X_1) = \{v\in L^2(\mathbb{R};X_0), D_t^{\gamma}v \in L^2(\mathbb{R};X_1)\}.$$

This is a Hilbert space for the norm,

$$\|v\|_{\mathcal{H}^{\gamma}(\mathbb{R},X_0,X_1)} = \{\|v\|^2_{L^2(\mathbb{R};X_0)} + \| |\tau|^{\gamma}\hat{v}\|^2_{L^2(\mathbb{R};X_1)}\}^{\frac{1}{2}}.$$

We associate to any set $K\subset\mathbb{R}$, the subspace $\mathcal{H}^{\gamma}_K$ of $\mathcal{H}^{\gamma}$ defined as the set of functions u in $\mathcal{H}^{\gamma}$ with support contained in K:

(2.28) $$\mathcal{H}^{\gamma}_K(\mathbb{R};X_0,X_1) = \{u\in\mathcal{H}^{\gamma}(\mathbb{R};X_0,X_1),\ \text{support } u\subset K\}.$$

The compactness theorem can now be stated:

<u>Theorem 2.2.</u>

<u>Let us assume that</u> X_0,X,X_1 <u>are Hilbert spaces which satisfy</u> (2.23) <u>and</u> (2.24).

<u>Then for any bounded set</u> K <u>and any</u> $\gamma > 0$, <u>the injection of</u> $\mathcal{H}^{\gamma}_K(\mathbb{R};X_0,X_1)$ <u>into</u> $L^2(\mathbb{R};X)$ <u>is compact</u>.

<u>Proof</u>.

(i) Let γ and K be fixed, and let u_m be a bounded sequence in $\mathcal{H}^{\gamma}_K(\mathbb{R};X_0,X_1)$. We must show that u_m contains a subsequence strongly convergent in $L^2(\mathbb{R};X)$.

Since $\mathcal{H}^{\gamma}(\mathbb{R};X_0,X_1)$ is a Hilbert space, the sequence u_m contains a subsequence u_μ weakly convergent in this space to some element u. It is clear that u must belong to $\mathcal{H}^{\gamma}_K$ too; therefore, setting

$$v_\mu = u_\mu - u,$$

the sequence v_μ appears as a bounded sequence of $\mathcal{H}^{\gamma}_K(\mathbb{R};X_0,X_1)$, which converges weakly to 0 in $\mathcal{H}^{\gamma}$; this means

(2.29) $$v_\mu \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R};X_0) \quad \text{weakly}$$

---

(1) The definition (2.26) is consistent with the usual definition for γ an integer.

(2) In the applications, $0 < \gamma \leq 1$ in general.

(2.30) $$|\tau|^{\gamma}\hat{v}_{\mu} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R};X_1) \quad \text{weakly.}$$

The theorem is proved if we show that u_μ converges strongly to u in $L^2(\mathbb{R};X)$, that is to say

(2.31) $$v_\mu \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R};X) \quad \text{strongly.}$$

(ii) The second point of the proof is to show that (2.31) is proved if we prove that

(2.32) $$v_\mu \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R};X_1) \quad \text{strongly.}$$

Due to Lemma 2.1,

(2.33) $$\|v_\mu\|_{L^2(\mathbb{R};X)} \leq \eta\|v_\mu\|_{L^2(\mathbb{R};X_0)} + c_\eta\|v_\mu\|_{L^2(\mathbb{R};X_1)}$$

and since v_μ is bounded in $L^2(\mathbb{R};X_0)$,

(2.34) $$\|v_\mu\|_{L^2(\mathbb{R};X)} \leq c\eta + c_\eta\|v_\mu\|_{L^2(\mathbb{R};X_1)}.$$

If we assume (2.32), then, letting $\mu \longrightarrow \infty$ in (2.34), we obtain

$$\overline{\lim_{\mu\longrightarrow\infty}} \|v_\mu\|_{L^2(\mathbb{R};X)} \leq c\eta.$$

Since η is arbitrarily small in Lemma 2.1, this upper limit must be 0 and (2.31) follows.

(iii) Finally let us prove (2.32). According to the Parseval theorem,

(2.35) $$I_\mu = \int_{-\infty}^{+\infty} \|v_\mu(t)\|_{X_1}^2\, dt = \int_{-\infty}^{+\infty} \|\hat{v}_\mu(\tau)\|_{X_1}^2\, d\tau,$$

where $\hat{v}_\mu$ denotes the Fourier transform of v_μ. We must show that

(2.36) $$I_\mu \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow \infty.$$

For this, we write

$$I_\mu = \int_{|\tau|\leq M} \|\hat{v}_\mu(\tau)\|_{X_1}^2\, d\tau + \int_{|\tau|>M} (1+|\tau|^{2\gamma})\|\hat{v}_\mu(\tau)\|_{X_1}^2 \cdot \frac{d\tau}{(1+|\tau|^{2\gamma})}$$

$$\leq \frac{c}{1+M^{2\gamma}} + \int_{|\tau|\leq M} \|\hat{v}_\mu(\tau)\|_{X_1}^2\, d\tau,$$

since v_μ is bounded in $\mathcal{H}^\gamma$.

For $\varepsilon > 0$ given, we choose M so that

$$\frac{c}{1+M^{2\gamma}} \leq \frac{\varepsilon}{2} .$$

Hence

$$I_\mu \leq \int_{|\tau|\leq M} \|\hat{v}_\mu(\tau)\|^2_{X_1} d\tau + \frac{\varepsilon}{2} ,$$

and (2.36) is proved if we show that, for this fixed M,

$$J_\mu = \int_{|\tau|\leq M} \|\hat{v}_\mu(\tau)\|^2_{X_1} d\tau \longrightarrow 0, \quad \text{as} \quad \mu \longrightarrow \infty. \tag{2.37}$$

This is proved via the Lebesgue theorem. If χ denotes the characteristic function of K, then $v_\mu \chi = v_\mu$ and

$$\hat{v}_\mu(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} \chi(t) v_\mu(t) dt.$$

Thus

$$\|\hat{v}_\mu(\tau)\|_{X_1} \leq \|v_\mu\|_{L^2(\mathbb{R};X_1)} \|e^{-2i\pi t\tau} \chi\|_{L^2(\mathbb{R})} ,$$

$$\|\hat{v}_\mu(\tau)\|_{X_1} \leq \text{Const.} \tag{2.38}$$

On the other hand for each σ in X_0, and each fixed τ,

$$((\hat{v}_\mu(\tau),\sigma))_{X_0} = \int_{-\infty}^{+\infty} ((v_\mu(t), e^{-2i\pi t\tau} \chi(t)\sigma))_{X_0} dt,$$

and this goes to 0, as $\mu \longrightarrow \infty$, because of (2.29). The sequence $\hat{v}_\mu(\tau)$ converges to 0 weakly in X_0 and therefore strongly in X and X_1.

With this last remark and (2.38), the Lebesgue theorem implies (2.37).

Using the methods of the last theorem, we can prove another compactness theorem similar to Theorem 2.1. Nevertheless, this theorem is not contained in and does not contain Theorem 2.2.

<u>Theorem 2.3</u>.

<u>Under the hypotheses</u> (2.23) <u>and</u> (2.24), <u>the injection of</u> $\mathcal{Y}(0,T;2,1;X_0,X_1)$ (1) <u>into</u> $L^2(0,T;X)$ <u>is compact</u>.

<u>Proof</u>.

Let u_m be a bounded sequence in this space $\mathcal{Y}$; we denote by $\tilde{u}_m$ the function defined on the whole line $\mathbb{R}$, which is equal to u_m on $[0,T]$, and

---

(1) For the definition of this space see (2.9)-(2.10).

to 0 outside this interval. By Theorem 2.2, the result is proved if we show that the sequence $\tilde{u}_m$ remains bounded in the space $\mathcal{H}^\gamma(\mathbb{R};X_0,X_1)$, for some $\gamma > 0$.

Because of Lemma 1.1, each function u_m is, after modification on a set of measure 0, continuous from $[0,T]$ into X_1, and more precisely the injection of $\mathcal{Y}$ into $\mathcal{C}([0,T];X_1)$ is continuous.

It is classical that since $\tilde{u}_m$ has two discontinuities, at 0 and T, the distribution derivative of $\tilde{u}_m$ is given by

$$(2.39) \qquad \frac{d}{dt}\tilde{u}_m = \tilde{g}_m + u_m(0)\delta_0 - u_m(T)\delta_T,$$

where δ_0 and δ_T are the Dirac distributions at 0 and T, and

$$(2.40) \qquad g_m = u'_m = \text{the derivative of } u_m \text{ on } [0,T].$$

After a Fourier transformation, (2.39) gives

$$(2.41) \qquad 2i\pi\tau\hat{u}_m(\tau) = \hat{g}_m(\tau) + u_m(0) - u_m(T)\exp(-2i\pi\tau T),$$

where $\hat{g}_m$ and $\hat{u}_m$ denote the Fourier transform of $\tilde{g}_m$ and $\tilde{u}_m$ respectively.

Since the functions g_m remain bounded in $L^1(0,T;X_1)$, the functions $\tilde{g}_m$ remain bounded in $L^1(\mathbb{R};X_1)$ and the functions $\hat{g}_m$ are bounded in $L^\infty(\mathbb{R};X_1)$:

$$(2.42) \qquad \|\hat{g}_m(\tau)\|_{X_1} \leq \text{Const.}, \quad \forall m, \forall\tau \in \mathbb{R}.$$

We already pointed out that the injection of $\mathcal{Y}$ into $\mathcal{C}([0,T];X_1)$ is continuous; thus

$$\|u_m(0)\|_{X_1} \leq \text{Const.}, \quad \|u_m(T)\|_{X_1} \leq \text{Const.},$$

and (2.41) shows us that

$$(2.43) \qquad |\tau|^2 \, \|\hat{u}_m(\tau)\|^2_{X_1} \leq c, \quad \forall m, \forall\tau\in\mathbb{R}.$$

For γ fixed, $\gamma < \frac{1}{2}$, we observe that

$$|\tau|^{2\gamma} \leq c_0(\gamma)\,\frac{1+\tau^2}{1+|\tau|^{2(1-\gamma)}}, \quad \forall\tau\in\mathbb{R}.$$

Therefore

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{u}_m(\tau)\|^2_{X_1} d\tau \leq c_0(\gamma) \int_{-\infty}^{+\infty} \frac{1+\tau^2}{1+|\tau|^{2(1-\gamma)}} \|\hat{u}_m(\tau)\|^2_{X_1} d\tau$$

$$\leq c_1 \int_{-\infty}^{+\infty} \frac{d\tau}{1+|\tau|^{2(1-\gamma)}} + c_0(\gamma) \int_{-\infty}^{+\infty} \|\hat{u}_m(\tau)\|^2_{X_1} d\tau$$

by (2.43).

Since $\gamma < \frac{1}{2}$, the integral

$$\int_{-\infty}^{+\infty} \frac{d\tau}{1+|\tau|^{2(1-\gamma)}}$$

is convergent; on the other hand, by the Parseval equality, we see that

$$\int_{-\infty}^{+\infty} \|\dot{\hat{u}}_m(\tau)\|^2_{X_1} d\tau = \int_0^T \|u_m(t)\|^2_{X_1} dt,$$

and these integrals are bounded.

We conclude that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{u}_m(\tau)\|^2_{X_1} d\tau \leq c_2, \tag{2.44}$$

where c_2 depends on γ.

It is clear now that the sequence u_m is bounded in $\mathcal{H}^\gamma(\mathbb{R};X_0,X_1)$ and the proof is achieved.

Remark 2.1.

Assuming only that X_1 is a Hilbert space, X_0,X being Banach spaces, it can be proved in a similar way that the injection of

$$\mathcal{Y}(0,T;\alpha_0,1;X_0,X_1),$$

into $L^{\alpha_0}(0,T;X)$ is compact, α_0 any finite number, $\alpha_0 > 1$.

§3. EXISTENCE AND UNIQUENESS THEOREMS ($n \leq 4$).

This section is concerned with existence and uniqueness theorems for weak solutions of the full Navier-Stokes equations ($n \leq 4$). In Section 3.1 we give the variational formulation of these equations, following J. Leray, and we state an existence theorem for such solutions when the dimension $n \leq 4$. The proof of this theorem, due to J.L. Lions, is given in Section 3.2. It is based on the construction of an approximate solution by the Galerkin method; then a passage to the limit using, in particular, an a priori estimate on a fractional derivative in time of the approximate solution, and a compactness theorem contained in Section 2. An alternate proof based on a semi-discretization in time and valid in all dimensions is discussed in Section 4.

In Section 3.3 we develop the uniqueness theorem of weak solutions ($n = 2$). In the three-dimensional case there is a gap between the class of functions where existence is known, and the smaller classes where uniqueness is proved; an example of such a uniqueness theorem is developed in Section 3.4 ($n = 3$). In Section 3.5 we show in the two dimensional case the existence of more regular solutions, assuming more regularity on the data; a similar result holds in the three dimensional case for local solutions, that is to say solutions which are defined on some "small" interval of time, assuming that the data is sufficiently small.

3.1 An Existence Theorem in $\mathbb{R}^n$ ($n \leq 4$).

The notations are the usual ones, in particular those recalled at the beginning of Section 1.1; Ω is an open lipschitzian set which we suppose bounded for simplicity; the unbounded case is discussed in Remarks 3.1 and 3.2.

We recall [(1)] that since the dimension is less than or equal to 4, one can define on $\mathbb{H}_0^1(\Omega)$, and in particular on V, a trilinear continuous form b by setting

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_\Omega u_i (D_i v_j) w_j \, dx. \tag{3.1}$$

If $u \in V$, then

$$b(u,v,v) = 0, \quad \forall v \in \mathbb{H}_0^1(\Omega). \tag{3.2}$$

For u,v in V, we denote by $B(u,v)$ the element of V' defined by

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(1) Cf. Section 1.1, Chapter II.

$$\langle B(u,v),w\rangle = b(u,v,w), \quad \forall w \in V, \tag{3.3}$$

and we set

$$B(u) = B(u,u) \in V', \quad \forall u \in V. \tag{3.4}$$

In its classical formulation, the initial boundary value problem of the full Navier-Stokes equations is the following:

To find a vector function

$$u: \Omega \times [0,T] \longrightarrow \mathbb{R}^n$$

and a scalar function

$$p: \Omega \times [0,T] \longrightarrow \mathbb{R},$$

such that

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^{n} u_i D_i u + \text{grad } p = f \quad \text{in} \quad Q = \Omega \times (0,T), \tag{3.5}$$

$$\text{div } u = 0 \quad \text{in} \quad Q, \tag{3.6}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{3.7}$$

$$u(x,0) = u_0(x), \quad \text{in} \quad \Omega. \tag{3.8}$$

As before, the functions f and u_0 are given, respectively defined on $\Omega \times [0,T]$ and Ω.

Let us assume that u and p are classical solutions of (3.5) - (3.8), say $u \in \mathcal{C}^2(\overline{Q})$, $p \in \mathcal{C}^1(\overline{Q})$. Obviously $u \in L^2(0,T;V)$, and if v is an element of $\mathcal{V}$, one can check easily that

$$\frac{d}{dt}(u,v) + \nu((u,v)) + b(u,u,v) = \langle f,v\rangle. \tag{3.9}$$

By continuity, equation (3.9) will hold for each $v \in V$.

This suggests the following weak formulation of the problem (3.5) - (3.9) (cf. J. Leray [1], [2], [3]):

<u>Problem 3.1.</u>

<u>For</u> f <u>and</u> u <u>given with</u>

$$f \in L^2(0,T;V') \tag{3.10}$$

$$u_0 \in H, \tag{3.11}$$

<u>to find</u> u <u>satisfying</u>

(3.12) $$u \in L^2(0,T;V)$$

and

(3.13) $$\frac{d}{dt}(u,v) + \nu((u,v)) + b(u,u,v) = \langle f,v\rangle, \quad \forall v \in V$$

(3.14) $$u(0) = u_0.$$

If u merely belongs to $L^2(0,T;V)$, the condition (3.14) need not make sense. But if u belongs to $L^2(0,T;V)$ and satisfies (3.13), then we will show as in the linear case (using Lemma 1.1) that u is almost everywhere equal to some continuous function, so that (3.14) is meaningful.

Before showing this, we recall that we are considering the case $n \leq 4$; we will modify slightly the preceding formulation in higher dimensions (see Section 4.1).

Lemma 3.1.

We assume that the dimension of the space is $n \leq 4$ and that u belongs to $L^2(0,T;V)$.

Then the fucntion Bu defined by

$$\langle Bu(t),v\rangle = b(u(t),u(t),v), \quad \forall v \in V, \quad \text{a.e. in } t \in [0,T],$$

belongs to $L^1(0,T;V')$.

Proof.

For almost all t, $Bu(t)$ is an element of V', and the measurability of the function

$$t \in [0,T] \longmapsto Bu(t) \in V'$$

is easy to check. Moreover, since b is trilinear continuous on V,

(3.15) $$\|Bw\|_{V'} \leq c\|w\|^2, \quad \forall w \in V,$$

so that

$$\int_0^T \|Bu(t)\|_{V'}dt \leq c \int_0^T \|u(t)\|^2 dt < +\infty,$$

and the lemma is proved.

Now if u satisfies (3.12) - (3.13), then according to (1.6), (1.8), and the above lemma, one can write (3.13) as

$$\frac{d}{dt}\langle u,v\rangle = \langle f-\nu Au-Bu,v\rangle, \quad \forall v \in V.$$

Since Au belongs to $L^2(0,T;V')$, as in the linear case, the function $f-\nu Au-Bu$ belongs to $L^1(0,T;V')$. Lemma 1.1 implies then that

(3.16) $$\begin{cases} u' \in L^1(0,T;V') \\ u' = f - \nu Au - Bu, \end{cases}$$

and that u is almost everywhere equal to a function continuous from $[0,T]$ into V'. This remark gives a sense to (3.14).

An alternate formulation of the problem (3.12)- (3.14) is the following one:

Problem 3.2.

Given f and u, satisfying (3.10) - (3.11), to find u satisfying

(3.17) $$u \in L^2(0,T;V), \quad u' \in L^1(0,T;V'),$$

(3.18) $$u' + \nu Au + Bu = f \quad \text{on} \quad (0,T),$$

(3.19) $$u(0) = u_0.$$

We showed that any solution of Problem 3.1 is a solution of Problem 3.2; the converse is very easily checked and these problems are thus equivalent.

The existence of solutions of these problems is ensured by the following theorem (cf. J.L. Lions [1]).

Theorem 3.1.

The dimension is $n \leq 4$. Let there be given f and u_0 which satisfy (3.10) - (3.11). Then, there exists at least one function u which satisfies (3.17) - (3.19). Moreover,

(3.20) $$u \in L^\infty(0,T;H)$$

and u is weakly continuous from $[0,T]$ into H(1).

The proof of the existence of a u satisfying (3.20) is developed in Section 3.2; the continuity result is a direct consequence of (3.20), the continuity of u in V', and Lemma 1.4.

Remark 3.1.

(i) Theorem 3.1 also holds if we assume that

$$f = f_1 + f_2; \quad f_1 \in L^2(0,T;V'), \quad f_2 \in L^1(0,T;H).$$

For the corresponding modifications of the proof of the theorem, the reader is referred to Section 1.5.

(ii) Theorem 3.1 is also valid if Ω is unbounded; for the details, cf. Remark 3.2.

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(1) i.e., $\forall v \in H$, $t \mapsto (u(t),v)$ is a continuous scalar function.

3.2 Proof of Theorem 3.1.

(i) We apply the Galerkin procedure as in the linear case. Since V is separable and $\mathcal{V}$ is dense in V, there exists a sequence $w_1,\cdots,w_m,\cdots$ of elements of $\mathcal{V}$; which is free and total in V [1]. For each m we define an approximate solution u_m of (3.13) as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t)w_i \tag{3.21}$$

and

$$(u'_m(t),w_j) + \nu((u_m(t),w_j)) + b(u_m(t),u_m(t),w_j) = \langle f(t),w_j\rangle, \quad t\in[0,T], \quad j = 1,\cdots,m, \tag{3.22}$$

$$u_m(0) = u_{0m}, \tag{3.23}$$

where u_{0m} is the orthogonal projection in H of u_0 onto the space spanned by $w_1,\cdots,w_m$ [2].

The equations (3.22) form a nonlinear differential system for the functions $g_{1m},\cdots,g_{mm}$;

$$\sum_{i=1}^{m} (w_i,w_j)g'_{im}(t) + \nu \sum_{i=1}^{m} ((w_i,w_j))g_{im}(t) + \sum_{i,\ell=1}^{m} b(w_i,w_\ell,w_j)g_{im}(t)g_{\ell m}(t) = \langle f(t),w_j\rangle. \tag{3.24}$$

Inverting the nonsingular matrix with elements (w_i,w_j), $1 \leq i,j \leq m$, we can write the differential equations in the usual form

$$g'_{im}(t) + \sum_{j=1}^{m} \alpha_{ij}g_{jm}(t) + \sum_{j,k=1}^{m} \alpha_{ijk}g_{jm}(t)g_{km}(t) = \sum_{j=1}^{m} \beta_{ij}\langle f(t),w_j\rangle \tag{3.25}$$

where α_{ij}, α_{ijk}, $\beta_{ij}\in\mathbb{R}$.

The condition (3.23) is equivalent to the m scalar initial conditions

$$g_{im}(0) = \text{the } i^{th} \text{ component of } u_{0m}. \tag{3.26}$$

---

(1) The w_j are chosen in $\mathcal{V}$ for simplicity. With some technical modifications, we could take the w_j in V.

(2) We could take for u_{0m} any element of that space such that

$$u_{0m}\longrightarrow u_0, \quad \text{in } H, \quad \text{as } m\longrightarrow\infty.$$

The nonlinear differential system (3.25) with the initial condition (3.26) has a maximal solution defined on some interval $[0,t_m)$. If $t_m < T$, then $|u_m(t)|$ must tend to $+\infty$ as $t \longrightarrow t_m$; the a priori estimates we will prove later show that this does not happen and therefore $t_m = T$.

(ii) The first a priori estimates are obtained as in the linear case. We multiply (3.22) by $g_{jm}(t)$ and add these equations for $j = 1,\cdots,m$. Taking (3.2) into account, we get

$$(3.27) \qquad (u'_m(t),u_m(t)) + \nu\|u_m(t)\|^2 = \langle f(t),u_m(t)\rangle.$$

Then we write

$$\frac{d}{dt}|u_m(t)|^2 + 2\nu\|u_m(t)\|^2 = 2\langle f(t),u_m(t)\rangle \leq 2\|f(t)\|_{V'}\|u_m(t)\|$$
$$\leq \nu\|u_m(t)\|^2 + \frac{1}{\nu}\|f(t)\|^2_{V'},$$

so that

$$(3.28) \qquad \frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 \leq \frac{1}{\nu}\|f(t)\|^2_{V'}.$$

Integrating (3.28) from 0 to s we obtain, in particular,

$$|u_m(s)|^2 \leq |u_{0m}|^2 + \frac{1}{\nu}\int_0^s \|f(t)\|^2_{V'}dt \leq |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{V'}dt.$$

Hence

$$(3.29) \qquad \sup_{s\in[0,T]} |u_m(s)|^2 \leq u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{V'}dt$$

which implies that

(3.30) *The sequence* u_m *remains in a bounded set of* $L^\infty(0,T;H)$.

We then integrate (3.28) from 0 to T to get

$$|u_m(T)|^2 + \nu\int_0^T \|u_m(t)\|^2 dt \leq |u_{0m}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{V'}dt$$
$$\leq |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|^2_{V'}dt.$$

This estimate enables us to say that

(3.31) *The sequence* u_m *remains in a bounded set of* $L^2(0,T;V)$.

(iii) Let $\tilde{u}_m$ denote the function from $\mathbb{R}$ into V, which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. The Fourier transform of $\tilde{u}_m$ is denoted by $\hat{u}_m$.

In addition to the previous inequalities, which are similar to the estimates in the linear case, we want to show that

$$(3.32) \qquad \int_{-\infty}^{+\infty} |\tau|^{2\gamma} \, |\hat{u}_m(\tau)|^2 dt \leq \text{Const.}, \text{ for some } \gamma > 0.$$

Along with (3.31), this will imply that

(3.33) $\tilde{u}_m$ <u>belongs to a bounded set of</u> $\mathcal{H}^\gamma(\mathbb{R};V,H)$

and will enable us to apply the compactness result of Theorem 2.2.

In order to prove (3.32) we observe that (3.22) can be written [1]

$$(3.34) \qquad \frac{d}{dt}(\tilde{u}_m,w_j) = \langle\tilde{f}_m,w_j\rangle + (u_{0m},w_j)\delta_0-(u_m(T),w_j)\delta_T, \quad j = 1,\cdots,m$$

where δ_0, δ_T are the Dirac distributions at 0 and T and

$$(3.35) \qquad \begin{cases} f_m = f-\nu Au_m-Bu_m \\ \tilde{f}_m = f_m \text{ on } [0,T], \ 0 \text{ outside this interval.} \end{cases}$$

By the Fourier transform, (3.34) gives

$$(3.36) \qquad 2i\pi\tau(\hat{u}_m,w_j) = \langle\hat{f}_m,w_j\rangle + (u_{0m},w_j) - (u_m(T),w_j)\exp(-2i\pi T\tau),$$

$\hat{u}_m$(resp. $\hat{f}_m$) denoting the Fourier transform of $\tilde{u}_m$ (resp. $\tilde{f}_m$).

We multiply (3.35) by $\hat{g}_{jm}(\tau)$ (= Fourier transform of $\tilde{g}_{jm}$) and add the resulting equations for $j = 1,\cdots,m$; we get:

$$(3.37) \quad 2i\pi\tau|\hat{u}_m(\tau)|^2 = \langle\hat{f}_m(\tau),\hat{u}_m(\tau)\rangle + (u_{0m},\hat{u}_m(\tau)) - (u_m(T),\hat{u}_m(\tau))\exp(-2i\pi T\tau).$$

Because of inequality (3.15),

$$\int_0^T \|f_m(t)\|_{V'}dt \leq \int_0^T (\|f(t)\|_{V'} + \nu\|u_m(t)\| + c\|u_m(t)\|^2)dt,$$

and this remains bounded according to (3.31). Therefore

$$\sup_{\tau\in\mathbb{R}} \|\hat{f}_m(\tau)\|_{V'} \leq \text{Const.}, \ \forall m.$$

Due to (3.29),

[1] Compare this with the proof of Theorem 2.3.

$$|u_m(0)| \leq \text{Const.}, \quad |u_m(T)| \leq \text{Const.},$$

and we deduce from (3.37) that

$$|\tau| \ |\hat{u}_m(\tau)|^2 \leq c_2\|\hat{u}_m(\tau)\| + c_3 \ |\hat{u}_m(\tau)|$$

or

$$|\tau| \ |\hat{u}_m(\tau)|^2 \leq c_4\|\hat{u}_m(\tau)\|. \tag{3.38}$$

For γ fixed, $\gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \leq c_5(\gamma) \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \quad \forall\tau\in\mathbb{R}.$$

Thus

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \ |\hat{u}_m(\tau)|^2 d\tau \leq c_5(\gamma) \int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} \ |\hat{u}_m(\tau)|^2 d\tau$$

$$\leq \text{ (by (3.38))}$$

$$\leq c_6 \int_{-\infty}^{+\infty} \frac{\|\hat{u}_m(\tau)\| d\tau}{1+|\tau|^{1-2\gamma}} + c_7 \int_{-\infty}^{+\infty} \|\hat{u}_m(\tau)\|^2 d\tau.$$

Because of the Parseval equality and (3.31), the last integral is bounded as $m \to \infty$; thus (3.32) will be proved if we show that

$$\int_{-\infty}^{+\infty} \frac{\|\hat{u}_m(\tau)\|}{1+|\tau|^{1-2\gamma}} d\tau \leq \text{Const.}$$

By the Schwarz inequality and the Parseval equality we can estimate these integrals by

$$\left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}\right)^{\frac{1}{2}} \left(\int_0^T \|u_m(t)\|^2 dt\right)^{\frac{1}{2}},$$

which is finite since $\gamma < \frac{1}{4}$, and bounded as $m \to \infty$ by (3.31).

The proof of (3.32) and (3.33) is achieved.

(iv) The estimates (3.30) and (3.31) enable us to assert the existence of an element $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and a subsequence $u_{m'}$, such that

$$\begin{cases} u_{m'} \to u \text{ in } L^2(0,T;V) \text{ weakly, and in} \\ L^\infty(0,T;H) \text{ weak-star, as } m' \to \infty. \end{cases} \tag{3.40}$$

Due to (3.33) and Theorem 2.2, we also have

(3.41) $$u_{m'} \longrightarrow u \quad \text{in} \quad L^2(0,T;H) \quad \text{strongly.}$$

The convergence results (3.40) and (3.41) enable us to pass to the limit. We proceed essentially as in the linear case.

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply (3.22) by $\psi(t)$, integrate in t, and then integrate by parts. This leads to the equation

$$\begin{aligned} &- \int_0^T (u_m(t),\psi'(t)w_j)dt + \nu \int_0^T ((u_m(t),w_j\psi(t)))\, dt \\ (3.42)\qquad &+ \int_0^T b(u_m(t),u_m(t),w_j\psi(t))dt = (u_{0m},w_j)\psi(0) + \int_0^T \langle f(t),w_j\psi(t)\rangle dt. \end{aligned}$$

Passing to the limit with the sequence m' is easy for the linear terms; for the nonlinear term we apply the next lemma, Lemma 3.2. In the limit we find that the equation

$$\begin{aligned} &- \int_0^T (u(t),v\psi'(t))dt + \nu \int_0^T ((u(t),v\psi(t)))dt \\ (3.43)\qquad &+ \int_0^T b(u(t),u(t),v\psi(t))dt = (u_0,v)\psi(0) + \int_0^T \langle f(t),v\psi(t)\rangle dt, \end{aligned}$$

holds for $v = w_1, w_2, \cdots$; by linearity this equation holds for $v =$ any finite linear combination of the w_j, and by a continuity argument (3.43) is still true for any $v \in V$.

Now writing, in particular, (3.43) with $\psi = \phi \in \mathscr{D}((0,T))$, we see that u satisfies (3.13) in the distribution sense.

Finally, it remains to prove that u satisfies (3.14). For this we multiply (3.13) by ψ, and integrate. After integrating by parts the first term, we get

$$\begin{aligned} &- \int_0^T (u(t),v\psi(t))dt + \nu \int_0^T ((\dot{u}(t),v\psi(t)))dt \\ (3.44)\qquad &+ \int_0^T b(u(t),u(t),v\psi(t))dt = (u(0),v)\psi(0) + \int_0^T \langle f(t),v\psi(t)\rangle dt. \end{aligned}$$

By comparison with (3.43),

$$(u(0)-u_0,v)\psi(0) = 0.$$

We can choose ψ with $\psi(0) = 1$; thus

$$(u(0)-u_0,v) = 0, \quad \forall v \in V,$$

and (3.14) follows. ∎

The proof of Theorem 3.1 will be complete once we prove the following lemma.

Lemma 3.2.

If u_μ converges to u in $L(0,T;V)$ weakly and $L^2(0,T;H)$ strongly, then for any vector function w with components in $\mathcal{C}^1(\overline{Q})$,

$$\text{(3.45)} \qquad \int_0^T b(u_\mu(t),u_\mu(t),w(t))dt \longrightarrow \int_0^T b(u(t),u(t),w(t))dt.$$

Proof.

We write

$$\int_0^T b(u_\mu,u_\mu,w)dt = -\int_0^T b(u_\mu,w,u_\mu)dt = -\sum_{i,j=1}^n \int_0^T \int_\Omega (u_\mu)_i(D_i w_j)(u_\mu)_j dxdt.$$

These integrals converge to

$$-\sum_{i,j=1}^n \int_0^T \int_\Omega u_i(D_i w_j)u_j dxdt = -\int_0^T b(u,w,u)dt = \int_0^T b(u,u,w)dt,$$

and the lemma is proved. ∎

Remark 3.2.

When Ω is unbounded, we prove (3.30) and (3.31) as we did in Section 1.5 for the linear case. Then (3.32) and (3.33) follow in the same way as before. The main difference lies in the fact that the injection of V into H is no longer compact.

Nevertheless we can extract a subsequence $u_{m'}$, which satisfies (3.40). Then, for any ball $\mathcal{O}$ included in Ω, the injection of $H^1(\mathcal{O})$ into $L^2(\mathcal{O})$ is compact and (3.33) shows that:

$$\text{(3.46)} \qquad u_m|_{\mathcal{O}} \text{ belongs to a bounded set of } \mathcal{W}^\gamma(\mathbb{R};\mathbb{H}^1(\mathcal{O}),\mathbb{L}^2(\mathcal{O})), \ \forall \mathcal{O}.$$

Then Theorem 2.2 implies that

$$u_{m'}|_{\mathcal{O}} \longrightarrow u|_{\mathcal{O}} \text{ in } \mathbb{L}^2(\mathcal{O}) \text{ strongly}, \ \forall \mathcal{O},$$

which means

(3.47) $$u_{m'} \longrightarrow u \quad \text{in } \mathbb{L}^2_{loc}(\Omega) \quad \text{strongly.}$$

In particular, for a fixed j,

$$u_{m'}|_{\Omega'} \longrightarrow u|_{\Omega'} \quad \text{in } \mathbb{L}^2(\Omega') \quad \text{strongly,}$$

where Ω' denotes the support of w_j, and this suffices to pass to the limit in (3.42). ∎

3.3 Regularity and Uniqueness ($n = 2$).

When the dimension of the space is $n = 2$, the solution of (3.17) - (3.19) whose existence is ensured by Theorem 3.1 satisfies some further regularity property and is actually unique.

The proof of these results is based on the following lemmas.

Lemma 3.3.

If $n = 2$, *for any open set* Ω

(3.48) $$\|v\|_{L^4(\Omega)} \leq 2^{\frac{1}{4}} \|v\|^{\frac{1}{2}}_{L^2(\Omega)} \|\text{grad } v\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)}, \quad \forall v \in H^1_0(\Omega).$$

Proof.

It suffices to prove (3.48) for $v \in \mathcal{D}(\Omega)$. For such a v, we write

$$v^2(x) = 2 \int_{-\infty}^{x_1} v(\xi_1, x_2) D_1 v(\xi_1, x_2) d\xi_1,$$

and therefore

(3.49) $$v^2(x) \leq 2 v_1(x_2),$$

where

(3.50) $$v_1(x_2) = \int_{-\infty}^{+\infty} |v(\xi_1, x_2)| \, |D_1 v(\xi_1, x_2)| d\xi_1.$$

Similarly

(3.51) $$v^2(x) \leq 2 v_2(x_1),$$

where

(3.52) $$v_2(x_1) = \int_{-\infty}^{+\infty} |v(x_1, \xi_2)| \, |D_2 v(x_1, \xi_2)| d\xi_2$$

and thus

$$\int_{\mathbb{R}^2} v^4(x)dx \le 4 \int_{\mathbb{R}^2} v_1(x_2)v_2(x_1)dx \le 4 \left(\int_{-\infty}^{+\infty} v_1(x_2)dx_2\right)\left(\int_{-\infty}^{+\infty} v_2(x_1)dx_1\right)$$

$$\le 4 \|v\|^2_{L^2(\mathbb{R}^2)} \|D_1 v\|_{L^2(\mathbb{R}^2)} \|D_2 v\|_{L^2(\mathbb{R}^2)}$$

$$\le 2\|v\|^2_{L^2(\mathbb{R}^2)} \|\text{grad } v\|_{\mathbb{L}^2(\mathbb{R}^2)}.$$

■

Lemma 3.4.

If $n = 2$,

(3.53) $$|b(u,v,w)| \le 2^{\frac{1}{2}} |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| \ |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}}, \ \forall u,v,w \in \mathbb{H}_0^1(\Omega).$$

If u belongs to $L^2(0,T;V) \cap L^\infty(0,T;H)$, then Bu belongs to $L^2(0,T;V')$ and

(3.54) $$\|Bu\|_{L^2(0,T;V')} \le 2^{\frac{1}{2}} |u|_{L^\infty(0,T;H)} \|u\|_{L^2(0,T;V)}.$$

Proof.

By repeated application of the Schwarz and Hölder inequalities we find:

$$|b(u,v,w)| \le \sum_{i,j=1}^{2} \int_\Omega |u_i(D_i v_j)w_j| dx$$

$$\le \sum_{i,j=1}^{2} \|u_i\|_{L^4(\Omega)} \|D_i v_j\|_{L^2(\Omega)} \|w_j\|_{L^4(\Omega)}$$

$$\le \left(\sum_{i,j=1}^{2} \|D_i v_j\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{2} \|u_i\|^2_{L^4(\Omega)}\right)^{\frac{1}{2}} \left(\sum_{j=1}^{2} \|w_j\|^2_{L^4(\Omega)}\right)^{\frac{1}{2}}.$$

Due to (3.48),

$$\sum_{i=1}^{2} \|u_i\|^2_{L^4(\Omega)} \le 2^{\frac{1}{2}} \sum_{i=1}^{2} \left(\|u_i\|_{L^2(\Omega)} \|\text{grad } u_i\|_{\mathbb{L}^2(\Omega)}\right)$$

$$\le 2^{\frac{1}{2}} |u| \ \|u\|.$$

With a similar inequality for w, we finally get (3.53).

If u,v,w belong to V, the relation

$$b(u,v,w) = -b(u,w,v)$$

gives another estimate of b:

(3.55) $$|b(u,v,w)| \le 2^{\frac{1}{2}} |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} |v|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} \|w\|, \ \forall u,v,w \in V.$$

In particular,

$$|b(u,u,v)| \le 2^{\frac{1}{2}} |u| \, \|u\| \, \|v\|, \quad \forall u,v \in V, \tag{3.56}$$

and hence

$$\|Bu\|_{V'} \le 2^{\frac{1}{2}} |u| \, \|u\|, \quad \forall u \in V. \tag{3.57}$$

If now $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $Bu(t)$ belongs to V' for almost every t and the estimate

$$\|Bu(t)\|_{V'} \le 2^{\frac{1}{2}} |u(t)| \, \|u(t)\| \tag{3.58}$$

shows that Bu belongs to $L^2(0,T;V')$ and implies (3.54). ∎

We can now state and prove the main result (cf. J.L. Lions and G. Prodi [1]).

Theorem 3.2.

In the two-dimensional case, the solution u of Problems 3.1-3.2 given by Theorem 3.1 is unique. Moreover u is almost everywhere equal to a function continuous from $[0,T]$ into H and

$$u(t) \longrightarrow u_0, \quad \text{in } H, \quad \text{as } t \longrightarrow 0. \tag{3.59}$$

Proof.

(i) We first prove the result of regularity.

According to (3.18) and Lemma 3.4,

$$u' = f - \nu Au - Bu,$$

and since each term in the right-hand side of this equation belongs to $L^2(0,T;V')$, u' also belongs to $L^2(0,T;V')$; this remark improves (3.17):

$$u' \in L^2(0,T;V'). \tag{3.60}$$

This improvement of (3.17) enables us to apply Lemma 1.2, which states exactly that u is almost everywhere equal to a function continuous from $[0,T]$ into H. Thus

$$u \in \mathscr{C}([0,T];H) \tag{3.61}$$

and (3.59) follows easily.

We recall also that Lemma 1.2 asserts that for any function u in $L^2(0,T;V)$ which satisfies (3.60), the equation below holds:

$$\frac{d}{dt} |u(t)|^2 = 2\langle u'(t), u(t) \rangle. \tag{3.62}$$

This result will be used in the proof of uniqueness which we will start now.

(ii) Let us assume that u_1 and u_2 are two solutions of (3.17) - (3.19), and let $u = u_1 - u_2$. As shown before, u_1, u_2, and thus u, satisfy (3.60).

The difference $u = u_1 - u_2$ satisfies

$$u' + \nu A u = -Bu_1 + Bu_2 \tag{3.63}$$

$$u(0) = 0 \quad . \tag{3.64}$$

We take a.e. in t the scalar product of (3.63) with $u(t)$ in the duality between V and V'. Using (3.62), we get

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 2b(u_2(t),u_2(t),u(t)) - 2b(u_1(t),u_1(t),u(t)). \tag{3.65}$$

Because of (3.2) the right-hand side of this equality is equal to

$$-2b(u(t),u_2(t),u(t)).$$

With (3.53) we can majorize this expression by

$$2^{\frac{3}{2}}|u(t)|\,\|u(t)\|\,\|u_2(t)\| \leq 2\nu\|u(t)\|^2 + \frac{1}{\nu}|u(t)|^2\|u_2(t)\|^2.$$

Putting this in (3.65) we find

$$\frac{d}{dt}|u(t)|^2 \leq \frac{1}{\nu}|u(t)|^2\|u_2(t)\|^2.$$

Since the function $t \mapsto \|u_2(t)\|^2$ is integrable, this shows that

$$\frac{d}{dt}\{\exp(-\frac{1}{\nu}\int_0^t \|u_2(s)\|^2 ds) \cdot |u(t)|^2\} \leq 0.$$

By integration and (3.64), we have

$$|u(t)|^2 \leq 0, \quad \forall t \in [0,T].$$

Thus

$$u_1 = u_2,$$

and the solution is unique. ∎

Remark 3.3.

As a consequence of (3.48) the (unique) solution of the Navier-Stokes equations satisfies

$$u \in \mathbb{L}^4(Q) \quad (n = 2). \tag{3.66}$$

∎

Remark 3.4.

Theorem 3.2 covers both the bounded and unbounded cases; there are no differences in the proofs of the two cases.

3.4 About Regularity and Uniqueness ($n = 3$).

The results of Section 3.3 cannot be extended to higher dimensions due to lack of information concerning the regularity of the weak solutions given by Theorem 3.1.

Nevertheless, we will prove some further regularity properties of a solution, which are weaker than those of the two-dimensional case. We then give an uniqueness theorem in a class of functions for which the existence is not known; this result shows, nevertheless, how an improvement in the information concerning the regularity of weak solutions leads to uniqueness.

The result similar to Lemma 3.3 is

Lemma 3.5.

If $n = 3$, for any open set Ω:

$$\|v\|_{L^4(\Omega)} \leq 2^{\frac{1}{2}} \|v\|^{\frac{1}{4}}_{L^2(\Omega)} \|\text{grad } v\|^{\frac{3}{4}}_{\mathbb{L}^2(\Omega)}, \quad \forall v \in H^1_0(\Omega). \tag{3.67}$$

Proof.

We only have to prove (3.67) for $v \in \mathcal{D}(\Omega)$. For such a v, by application of (3.48), we write

$$\int_{\mathbb{R}^3} v^4(x)dx \leq 2 \int_{-\infty}^{+\infty} \left\{\left(\int_{\mathbb{R}^2} v^2 dx_1 dx_2\right)\left(\int_{\mathbb{R}^2} \sum_{i=1}^{2} (D_i v)^2 dx_1 dx_2\right)\right\} dx_3 \tag{3.68}$$

$$\leq 2 \left(\operatorname*{Sup}_{x_3} \cdot \int_{\mathbb{R}^2} v^2 dx_1 dx_2\right)\left(\sum_{i=1}^{2} \|D_i v\|^2_{L^2(\mathbb{R}^3)}\right).$$

But

$$v^2(x) = 2 \int_{-\infty}^{x_3} v(x_1, x_2, \xi_3) D_3 v(x_1, x_2, \xi_3) d\xi_3$$

$$\leq 2 \int_{-\infty}^{+\infty} |v(x_1, x_2, \xi_3)| \; |D_3 v(x_1, x_2, \xi_3)| d\xi_3$$

and hence

$$\operatorname*{Sup}_{x_3} \int_{\mathbb{R}^2} v^2 dx_1 dx_2 \leq 2 \int_{\mathbb{R}^3} |v| \; |D_3 v| dx \leq 2\|v\|_{L^2(\mathbb{R}^3)} \|D_3 v\|_{L^2(\mathbb{R}^3)}.$$

With this inequality we deduce from (3.68) that

$$\int_{\mathbb{R}^3} v^4(x)\,dx \le 4\|v\|_{L^2(\mathbb{R}^3)}\|D_3 v\|_{L^2(\mathbb{R}^3)}\left(\sum_{i=1}^{2}\|D_i v\|^2_{L^2(\mathbb{R}^3)}\right)$$

$$\le 4\|v\|_{L^2(\mathbb{R}^3)}\left(\sum_{i=1}^{3}\|D_i v\|_{L^2(\mathbb{R}^3)}\right)^{\frac{3}{2}}$$

and (3.67) follows. ∎

Theorem 3.3.

If $n = 3$, the solution u of (3.17) - (3.19) given by Theorem 3.1 satisfies

(3.69) $$u \in L^{\frac{8}{3}}(0,T;\mathbb{L}^4(\Omega))$$

(3.70) $$u' \in L^{\frac{4}{3}}(0,T;V').$$

Proof.

For almost every t, according to (3.67),

(3.71) $$\|u(t)\|_{\mathbb{L}^4(\Omega)} \le c_0 |u(t)|^{\frac{1}{4}}\|u(t)\|^{\frac{3}{4}}.$$

The function on the right-hand side belongs to $L^{\frac{8}{3}}(0,T)$, and thus also the function on the left-hand side.

By use of the Hölder inequality, we derived in Chapter II the bound

(3.72) $$|b(u,u,v)| = |b(u,v,u)| \le c_1\|u\|^2_{\mathbb{L}^4(\Omega)}\|v\|, \quad \forall u,v \in V$$ [(1)].

Therefore, if $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$, Bu belongs to $L^{\frac{4}{3}}(0,T;V')$ since

(3.73) $$\|Bu(t)\|_{V'} \le c_1\|u(t)\|^2_{\mathbb{L}^4(\Omega)}$$

(3.74) $$\|Bu(t)\|_{V'} \le c_2|u(t)|^{\frac{1}{2}}\|u(t)\|^{\frac{3}{2}}, \quad \text{a.e.}$$ ∎

In the two dimensional case we established that any solution of (3.17) - (3.19) satisfies (3.60) and (3.66) and this was the property which essentially enabled us to prove uniqueness. For $n = 3$, (3.60) and (3.66) are replaced by the weaker results (3.69) - (3.70).

We show now that there is at most one solution in a smaller class of functions than the class in which we obtained existence.

Theorem 3.4.

If $n = 3$, there is at most one solution of Problem 3.2 such that

---

(1) This inequality with c depending on n holds for any dimension of space.

(3.75) $$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

(3.76) $$u \in L^8(0,T;\mathbb{L}^4(\Omega)).$$

Such a solution would be continuous from $[0,T]$ _into_ H.

Proof.

(i) The inequalities (3.72) - (3.73) imply that if u satisfies (3.76) then

(3.77) $$Bu \in L^2(0,T;V') \quad \text{(at least)}.$$

Therefore if u satisfies (3.75) - (3.76) and (3.18), then

(3.78) $$u' \in L^2(0,T;V')$$

and according to Lemma 1.2, u is almost everywhere equal to a continuous function from $[0,T]$ into H.

(ii) By the Hölder inequality and (3.67),

$$|b(u,u,v)| \leq c_0 \|u\|_{\mathbb{L}^4(\Omega)} \|v\|_{\mathbb{L}^4(\Omega)} \|u\|,$$

(3.79) $$|b(u,u,v)| \leq c_1 |u|^{\frac{1}{4}} \|u\|^{\frac{7}{4}} \|v\|_{\mathbb{L}^4(\Omega)}.$$

(iii) Let us assume that u_1 and u_2 are two solutions of (3.17) - (3.19) which satisfy (3.75) - (3.76), and let $u = u_1 - u_2$.

As in the proof of Theorem 3.2 one can show that

(3.80) $$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 2b(u(t),u(t),u_2(t)).$$

We then bound the right-hand side, according to (3.79), by

$$2c_1 |u(t)|^{\frac{1}{4}} \|u(t)\|^{\frac{7}{4}} \|u_2(t)\|_{\mathbb{L}^4(\Omega)} \leq \nu\|u(t)\|^2 + c_2 |u(t)|^2 \|u_2(t)\|^8_{\mathbb{L}^4(\Omega)}.$$

We get

$$\frac{d}{dt}|u(t)|^2 \leq c_2 \|u_2(t)\|^8_{\mathbb{L}^4(\Omega)} |u(t)|^2.$$

Since the function $t \mapsto |u_2(t)|^8_{\mathbb{L}^4(\Omega)}$ is integrable, we can complete the proof as done for Theorem 3.2. ▪

Remark 3.5.

The preceding proof is valid for Ω bounded or unbounded. ▪

Remark 3.6.

There are many similar results of uniqueness which can be proved by assuming

(3.75) $$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

(3.76) $$u \in L^8(0,T;\mathbb{L}^4(\Omega)).$$

<u>Such a solution would be continuous from</u> $[0,T]$ <u>into</u> H.

<u>Proof</u>.

(i) The inequalities (3.72) - (3.73) imply that if u satisfies (3.76) then

(3.77) $$Bu \in L^2(0,T;V') \quad \text{(at least)}.$$

Therefore if u satisfies (3.75) - (3.76) and (3.18), then

(3.78) $$u' \in L^2(0,T;V')$$

and according to Lemma 1.2, u is almost everywhere equal to a continuous function from $[0,T]$ into H.

(ii) By the Hölder inequality and (3.67),

$$|b(u,u,v)| \leq c_0 \|u\|_{\mathbb{L}^4(\Omega)} \|v\|_{\mathbb{L}^4(\Omega)} \|u\|,$$

(3.79) $$|b(u,u,v)| \leq c_1 |u|^{\frac{1}{4}} \|u\|^{\frac{7}{4}} \|v\|_{\mathbb{L}^4(\Omega)}.$$

(iii) Let us assume that u_1 and u_2 are two solutions of (3.17) - (3.19) which satisfy (3.75) - (3.76), and let $u = u_1 - u_2$.

As in the proof of Theorem 3.2 one can show that

(3.80) $$\frac{d}{dt} |u(t)|^2 + 2\nu \|u(t)\|^2 = 2b(u(t),u(t),u_2(t)).$$

We then bound the right-hand side, according to (3.79), by

$$2c_1 |u(t)|^{\frac{1}{4}} \|u(t)\|^{\frac{7}{4}} \|u_2(t)\|_{\mathbb{L}^4(\Omega)} \leq \nu \|u(t)\|^2 + c_2 |u(t)|^2 \|u_2(t)\|^8_{\mathbb{L}^4(\Omega)}.$$

We get

$$\frac{d}{dt} |u(t)|^2 \leq c_2 \|u_2(t)\|^8_{\mathbb{L}^4(\Omega)} |u(t)|^2.$$

Since the function $t \mapsto |u_2(t)|^8_{\mathbb{L}^4(\Omega)}$ is integrable, we can complete the proof as done for Theorem 3.2. ∎

<u>Remark 3.5</u>.

The preceding proof is valid for Ω bounded or unbounded. ∎

<u>Remark 3.6</u>.

There are many similar results of uniqueness which can be proved by assuming some other properties of regularity. For example (cf. Lions [2] p. 84), there is uniqueness in any dimension if, in place of (3.76), u satisfies

$$u \in L^s(0,T;\mathbb{L}^r(\Omega)) \tag{3.81}$$

with

$$\begin{cases} \frac{2}{s} + \frac{n}{r} \leq 1 & \text{if } \Omega \text{ is bounded,} \\ \frac{2}{s} + \frac{n}{r} = 1 & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{3.82}$$

3.5 More Regular Solutions.

Our purpose in this section is to prove that by assuming more regularity on the data, we can obtain more regular solutions in the two dimensional case. In the three dimensional case the existence of such more regular solutions is only proved if we assume that the given data u_0, f, are "small enough" or that ν is large enough.

3.5.1 The Two-Dimensional Case.

Theorem 3.5.

We assume that $n = 2$ *and that*

$$f \text{ and } f' \in L^2(0,T;V'), \quad f(0) \in H \tag{3.83}$$

$$u_0 \in \mathbb{H}^2(\Omega) \cap V. \tag{3.84}$$

Then the unique solution of Problem 3.2 *given by Theorems* 3.1 *and* 3.2 *satisfies*

$$u' \in L^2(0,T;V) \cap L^\infty(0,T;H). \tag{3.85}$$

Proof.

(i) We return to the Galerkin approximation used in the proof of Theorem 3.1. We need only show that this approximate solution also satisfies the two a priori estimates:

$$\begin{cases} u'_m \text{ remains in a bounded set of} \\ L^2(0,T;V) \cap L^\infty(0,T;H). \end{cases} \tag{3.86}$$

In the limit (3.86) implies (3.85).

Since $u_0 \in V \cap \mathbb{H}^2(\Omega)$, we can choose u_{0m} as the orthogonal projection in $V \cap \mathbb{H}^2(\Omega)$ of u_0 onto the space spanned by $w_1, \cdots, w_m$; then

$$(3.87)\qquad \begin{cases} u_{0m} \longrightarrow u_0 \quad \text{in } \mathbb{H}^2(\Omega), \quad \text{as} \quad m \to \infty, \\ \|u_{0m}\|_{\mathbb{H}^2(\Omega)} \leq \|u_0\|_{\mathbb{H}^2(\Omega)}. \end{cases}$$

(ii) We multiply (3.22) by $g'_{jm}(t)$ and add the resulting equations for $j = 1,\cdots,m$; this gives

$$(3.88)\qquad |u'_m(t)|^2 + \nu((u_m(t),u'_m(t))) + b(u_m(t),u_m(t),u'_m(t)) = \langle f(t),u'_m(t)\rangle.$$

In particular, at time $t = 0$,

$$(3.89)\qquad |u'_m(0)|^2 = (f(0),u'_m(0)) + \nu(\Delta u_{0m},u'_m(0)) - b(u_{0m},u_{0m},u'_m(0))$$

so that

$$(3.90)\qquad |u'_m(0)| \leq |f(0)| + \nu|\Delta u_{0m}| + |Bu_{0m}|.$$

It is clear from (3.87) that

$$|\Delta u_{0m}| \leq c_0\|u_{0m}\|_{\mathbb{H}^2(\Omega)} \leq c_0\|u_0\|_{\mathbb{H}^2(\Omega)}.$$

u_{0m} we have, by the Hölder inequality,

$$|b(u,u,v)| \leq c_1\|u\|_{\mathbb{L}^4(\Omega)}|\text{grad } u|_{\mathbb{L}^4(\Omega)}|v|$$

$$\leq \text{(by (3.48) and the Sobolev inequality)}$$

$$\leq c_2\|u\|\ \|u\|_{\mathbb{H}^2(\Omega)}|v|, \quad \forall u \in \mathbb{H}^2(\Omega), \quad \forall v \in \mathbb{L}^2(\Omega)$$

and hence

$$(3.91)\qquad |Bu_{0m}| \leq c_2\|u_{0m}\|\ \|u_{0m}\|_{\mathbb{H}^2(\Omega)} \leq \text{(by (3.87))} \leq c_2\|u_0\|^2_{\mathbb{H}^2(\Omega)}.$$

Finally (3.90) and the above estimates show that

(3.92) $\quad u'_m(0)$ <u>belongs to a bounded set of</u> H.

(iii) We are allowed to differentiate (3.22) in the t variable, and since f satisfies (3.83), we get

$$(3.93)\qquad (u''_m,w_j) + \nu((u'_m,w_j)) + b(u'_m,u_m,w_j) + b(u_m,u'_m,w_j) = \langle f',w_j\rangle, \quad j = 1,\cdots,m.$$

We multiply (3.93) by $g'_{jm}(t)$ and add the resulting equations for $j = 1,\cdots,m$; we find (taking (3.2) into account):

$$(3.94)\qquad \frac{d}{dt}|u'_m(t)|^2 + 2\nu\|u'_m(t)\|^2 + 2b(u'_m(t),u_m(t),u'_m(t)) = 2\langle f'(t),u'_m(t)\rangle.$$

By Lemma 3.4,

$$2b(u_m'(t),u_m(t),u_m'(t) \leq 2^{\frac{3}{2}}|u_m'(t)|\ \|u_m'(t)\|\ \|u_m(t)\|$$

$$\leq \nu\|u_m'(t)\|^2 + \frac{2}{\nu}\|u_m(t)\|^2\ |u_m'(t)|^2.$$

Thus, we deduce from (3.94) that

$$(3.95)\quad \frac{d}{dt}|u_m'(t)|^2 + \frac{\nu}{2}\|u_m'(t)\|^2 \leq \frac{2}{\nu}|f'(t)|^2_{V'} + \phi_m(t)|u_m'(t)|^2$$

where

$$\phi_m(t) = \frac{2}{\nu}\|u_m(t)\|^2.$$

Then, by the usual method of the Gronwall inequality,

$$\frac{d}{dt}\{|u_m'(t)|^2 \exp(-\int_0^t \phi_m(s)ds)\} \leq \frac{2}{\nu}|f'(t)|^2_{V'}$$

whence

$$(3.96)\quad |u_m'(t)|^2 \leq \{|u_m'(0)|^2 + \frac{2}{\nu}\int_0^t |f'(s)|^2_{V'}ds\} \exp\int_0^t \phi_m(s)ds.$$

Since the functions u_m remain in a bounded set of $L^2(0,T;V)$ (cf. (3.31)) and because of (3.92), the right-hand side of (3.96) is uniformly bounded in $s\in[0,T]$ and m:

(3.97) u_m' <u>belongs to a bounded set of</u> $L^\infty(0,T;H)$.

With (3.97) we infer easily from (3.95) that the u_m' remain in a bounded set of $L^2(0,T;V)$.

The proof is achieved. ∎

<u>Theorem 3.6</u>.

The assumptions are those of Theorem 3.5 and we assume moreover that Ω is a bounded set of class $\mathscr{C}^2$ and that

$$(3.98)\qquad f\in L^\infty(0,T;H).$$

Then the function u satisfies

$$(3.99)\qquad u\in L^\infty(0,T;\mathbb{H}^2(\Omega)).$$

<u>Proof</u>.

(i) We write (3.18) in the form

$$(3.100)\qquad \nu((u(t),v)) = (g(t),v),\ \forall v\in V,$$

where

(3.101) $$g(t) = f(t)-u'(t)-Bu(t).$$

The proof is now based on two successive applications of Proposition I.2.2.

(ii) Since $u \in L^\infty(0,T;V)$ and

(3.102) $$|b(u(t),u(t),v)| \leq c_0 \|u(t)\|_{\mathbb{L}^4(\Omega)} \|u(t)\| \; \|v\|_{\mathbb{L}^4(\Omega)}$$
$$\leq c_1 \|u(t)\|^2 \; \|v\|_{\mathbb{L}^4(\Omega)},$$

we have

$$Bu \in L^\infty(0,T;\mathbb{L}^{\frac{4}{3}}(\Omega)).$$

Thus $(f-u' \in L^\infty(0,T;H))$,

(3.103) $$g \in L^\infty(0,T;\mathbb{L}^{\frac{4}{3}}(\Omega)).$$

Proposition I.2.2 then implies that

$$u \in L^\infty(0,T;\mathbb{W}^{2,\frac{4}{3}}(\Omega)).$$

By the Sobolev theorem, $W^{2,\frac{4}{3}}(\Omega) \subset L^\infty(\Omega)$, and hence

$$u \in \mathbb{L}^\infty(Q).$$

(iii) We can now improve (3.103). We replace (3.102) by the inequality

$$|b(u(t),u(t),v)| \leq c_2 \|u\|_{\mathbb{L}^\infty(Q)} \|u(t)\| \; \|v\|$$

which shows that

$$Bu \in L^\infty(0,T;H).$$

This implies that

$$g \in L^\infty(0,T;H)$$

and another application of Proposition I.2.2 gives

$$u \in L^\infty(0,T;\mathbb{H}^2(\Omega)).$$

Remark 3.7.

By repeated application of Proposition I.2.2 it is now easy to prove, exactly as in Proposition II.1.1, that if Ω is of class $\mathscr{C}^\infty$, u_0 is given in $\mathscr{C}^\infty(\Omega)$, and f is given in $\mathscr{C}^\infty(\overline{Q})$, then the solution u is in $\mathscr{C}^\infty(\overline{Q})$. By the same methods, intermediate regularity properties can be obtained with suitable hypotheses on the data. ∎

3.5.2 The Three Dimensional Case.

We will prove for $n = 3$ some regularity properties similar to those obtained for $n = 2$, but in the present case this will be done only by assuming that the data are "small."

In the next theorem we denote by c some constant such that

$$(3.104) \qquad |b(u,v,w)| \leq c\|u\| \, \|v\| \, \|w\|, \; \forall u,v,w \in V.$$

Theorem 3.7.

We assume that $n = 3$ and that there are given f and u_0 satisfying

$$(3.105) \qquad u_0 \in \mathbb{H}^2(\Omega) \cap V$$

$$(3.106) \qquad f \in L^{\infty}(0,T;H), \quad f' \in L^1(0,T;H)$$

and a further condition given in the course of the proof which is satisfied if ν is large enough or if f and u_0 are "small enough" [(1)].

Then there exists a unique solution of Problem 3.2 which satisfies moreover

$$(3.107) \qquad u' \in L^2(0,T;V) \cap L^{\infty}(0,T;H).$$

Proof.

(i) To begin with, we observe that uniqueness is merely a consequence of Theorem 3.4, because such a solution will satisfy

$$(3.108) \qquad u \in L^{\infty}(0,T;V)$$

and then $V \subset \mathbb{L}^4(\Omega)$ implies (see (3.76)) that

$$(3.109) \qquad u \in L^{\infty}(0,T;\mathbb{L}^4(\Omega)).$$

(ii) Some of the steps of the existence proof are the same as in Theorem 3.4: we use the Galerkin method of Theorem 3.1, and we choose the basis and u_m so that (3.87) holds. The estimates (3.90), (3.91), and thus (3.92) still hold:

$$(3.110) \qquad |u_m'(0)| \leq d_1 = |f(0)| + \nu c_0 \|u_0\|_{\mathbb{H}^2(\Omega)} + c_1 \|u_0\|^2_{\mathbb{H}^2(\Omega)}.$$

We derive in the same fashion equation (3.94) and using (3.104) we deduce now from it:

$$(3.111) \qquad \frac{d}{dt} |u_m'(t)|^2 + 2(\nu - c\|u_m(t)\|) \, \|u_m'(t)\|^2 \leq 2|f'(t)| \, |u_m'(t)|.$$

(iii) There results from (3.28) and (3.29) that

---

(1) Like (3.115).

$$(3.112) \qquad \nu\|u_m(t)\|^2 \leq \frac{1}{\nu}\|f(t)\|^2_{V'} - 2(u_m(t),u_m'(t))$$

$$\leq \frac{1}{\nu}\|f(t)\|^2_{V'} + 2|u_m(t)|\ |u_m'(t)|$$

$$\leq \frac{d_2}{\nu} + 2(|u_0|^2 + \frac{Td_2}{\nu})^{\frac{1}{2}}\ |u_m'(t)|$$

where

$$(3.113) \qquad d_2 = \|f\|^2_{L^\infty(0,T;V')} .$$

Using (3.110), we infer from (3.112) that, at time $t = 0$,

$$(3.114) \qquad \nu\|u_m(0)\|^2 \leq \frac{d_2}{\nu} + 2d_1(|u_0|^2 + \frac{Td_2}{\nu})^{\frac{1}{2}} = d_3.$$

The hypothesis mentioned in the statement of the theorem is that

$$(3.115) \qquad d_4 = \frac{d_2}{\nu} + (1+d_1^2)\ (|u_0|^2 + \frac{Td_2}{\nu})^{\frac{1}{2}} \exp(\int_0^T |f'(s)|ds) < \frac{\nu^3}{c^2} .$$

Since $d_3 \leq d_4$, we get as a consequence of (3.114) - (3.115)

$$\nu\|u_m(0)\|^2 \leq d_3 \leq d_4 < \frac{\nu^3}{c^2}$$

and then

$$\nu - c\|u_m(0)\| > 0.$$

We deduce from this inequality that $\nu - c\|u_m(t)\|$ remains positive on some interval with origin 0. We denote by T_m the first time $t \leq T$ such that

$$\nu - c\|u_m(T_m)\| = 0$$

or, if this does not happen, $T_m = T$.
Then

$$(3.116) \qquad \nu - c\|u_m(t)\| \geq 0, \quad 0 \leq t \leq T_m.$$

(iv) With (3.116) we deduce from (3.111) that

$$\frac{d}{dt}|u_m'(t)|^2 \leq 2|f'(t)|\ |u_m'(t)|,$$

$$\frac{d}{dt}(1+|u_m'(t)|^2) \leq |f'(t)|\ (1+|u_m'(t)|^2).$$

Thus

$$\frac{d}{dt}\{(1+|u_m'(t)|^2)\exp(-\int_0^t |f'(s)|ds)\} \leq 0,$$

$$1+|u_m'(t)|^2 \leq (1+|u_m'(0)|^2)\exp(\int_0^t |f'(s)|ds),$$

and, by (3.110),

(3.117) $$1+|u_m'(t)|^2 \leq (1+d_1^2)\exp(\int_0^T |f'(s)|ds), \quad 0 \leq t \leq T_m.$$

From (3.112), (3.115), and (3.117) we get

$$\nu\|u_m(t)\|^2 \leq d_4, \quad 0 \leq t \leq T_m,$$

(3.118) $$\nu - c\|u_m(t)\| \geq \nu - c\sqrt{\frac{d_4}{\nu}} > 0, \quad 0 \leq t \leq T_m.$$

Then $T_m = T$, and (3.111) implies

$$\frac{d}{dt}|u_m'(t)|^2 + 2(\nu - c\sqrt{\frac{d_4}{\nu}})\|u_m'(t)\|^2 \leq 2|f'(t)|\;|u_m'(t)|, \quad 0 \leq t \leq T,$$

and we easily deduce from this relation that

(3.119) u_m' remains in a bounded set of $L^2(0,T;V)\cap L^\infty(0,T;H)$.

The existence is proved. ∎

As in the two-dimensional case, we also have

Theorem 3.8.

With the assumptions of Theorem 3.7, and if we assume moreover that Ω is of class $\mathscr{C}^2$, the function u satisfies

(3.120) $$u \in L^\infty(0,T;\mathbb{H}^2(\Omega)).$$

Proof.

We write (3.18) in the form,

$$\nu((u(t),v)) = (g(t),v), \quad \forall v \in V,$$

with

$$g(t) = f(t) - u'(t) - Bu(t).$$

Since $f-u' \in L^\infty(0,T;H)$, Proposition I.2.2 gives (3.120) provided we show that

(3.121) $Bu \in L^\infty(0,T;H)$ (and hence $g \in L^\infty(0,T;H)$).

This result is also obtained by repeated application of Proposition I.2.2 and various estimates on the form b.

By the Hölder inequality, we have:

$$|b(u(t),u(t),v)| \leq c_0 \|u(t)\|_{\mathbb{L}^6(\Omega)} \|u(t)\| \; |v|_{\mathbb{L}^3(\Omega)}$$

(3.122) $$|b(u(t),u(t),v)| \leq c_1 \|u(t)\|^2 \; \|v\|_{\mathbb{L}^3(\Omega)}.$$

We deduce from (3.122) (and $u \in L^\infty(0,T;V)$), that

$$Bu \in L^\infty(0,T;\mathbb{L}^{\frac{3}{2}}(\Omega)), \quad g \in L^\infty(0,T;\mathbb{L}^{\frac{3}{2}}(\Omega)).$$

Proposition I.2.2 implies that

$$u \in L^\infty(0,T;W^{2,\frac{3}{2}}(\Omega))$$

and, in particular (since $W^{2,\frac{3}{2}}(\Omega) \subset L^8(\Omega)$ for example, since $n = 3$),

$$u \in L^\infty(0,T;\mathbb{L}^8(\Omega)).$$

Using again the Hölder inequality we estimate b by

$$b(u(t),u(t),v) \leq c_0 \|u(t)\|_{\mathbb{L}^8(\Omega)} \|u(t)\| \; \|v\|_{\mathbb{L}^{\frac{8}{3}}(\Omega)}.$$

Hence

$$Bu, g \in L^\infty(0,T;\mathbb{L}^{\frac{8}{5}}(\Omega))$$

and by Proposition I.2.2,

(3.123) $$u \in L^\infty(0,T;W^{2,\frac{8}{5}}(\Omega)) \subset \mathbb{L}^\infty(\Omega \times [0,T]).$$

With (3.123) and $u \in L^\infty(0,T;V)$, the proof of (3.121) is easy, and thus Theorem 3.8 is proved. ∎

Remark 3.8.

The same remark about regularity as in Remark 3.7 holds. ∎

Introduction of the Pressure ($n \leq 4$).

For introducing the pressure, let us set

$$U(t) = \int_0^t u(s)ds, \quad \beta(t) = \int_0^t Bu(s)ds, \quad F(t) = \int_0^t f(s)ds.$$

If u is a solution of (3.17) - (3.19) then, for any $n \leq 4$,

(3.124) $$U, \beta, F \in \mathscr{C}([0,T];V').$$

Integrating (3.18), we see that

(3.125) $\nu((U(t),v)) = \langle g(t),v\rangle, \quad \forall v\in V, \forall t\in[0,T],$

with

$$g(t) = F(t)-\beta(t)-u(t)+u_0, \quad g\in\mathcal{C}([0,T];V').$$

By application of Proposition I.1.3, we get for each $t\in[0,T]$, the existence of some function $P(t)$,

$$P(t)\in L^2(\Omega),$$

such that

$$-\nu\Delta U(t) + \text{grad } P(t) = g(t)$$

or

$$u(t)-u_0-\nu\Delta U(t) + \beta(t) + \text{grad } P(t) = F(t).$$

According to Proposition I.1.2, the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $\mathbb{H}^{-1}(\Omega)$. Observing that

$$\text{grad } P = g-\nu\Delta U,$$

we conclude that grad P belongs to $\mathcal{C}([0,T];\mathbb{H}^{-1}(\Omega))$ and therefore

(3.127) $P\in\mathcal{C}([0,T];L^2(\Omega)).$

This enables us to differentiate (3.126) in the distribution sense in $Q = \Omega\times(0,T)$; setting

(3.128) $p = \dfrac{\partial P}{\partial t},$

we obtain

(3.129) $\dfrac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{n} u_i D_i u + \text{grad } p = f, \quad \text{in } Q.$

The pressure appears in general as a distribution on Q defined by (3.127) - (3.128). Under the assumptions of Theorems 3.6 (n = 2) or 3.8 (n = 3), the application of Proposition I.2.2 shows also that

(3.130) $p\in L^\infty(0,T;H^1(\Omega)).$ ∎

§4. ALTERNATE PROOF OF EXISTENCE BY SEMI-DISCRETIZATION.

Our goal now is to give an alternate proof of the existence of weak solutions of the Navier-Stokes equations which will be valid in any number of space dimensions. An approximate solution is constructed by semi-discretization in t, and we then pass to the limit using compactness arguments.

In Section 4.1 we reformulate the problem in a way which is appropriate in any dimension and we state the existence results; Section 4.2 describes the construction of the approximate solution; Sections 4.3 and 4.4 deal with the a priori estimates and the passage to the limit.

4.1 Statement of the Problem.

Before giving the existence theorem in the higher dimensions we must reformulate the problem of weak solutions. As in the stationary case, if $n > 4$, the form b is not trilinear continuous on V and a statement such as (3.10)-(3.14) does not make sense since the $b(u(t),u(t),v)$ term in (3.13) is perhaps not defined.

For this purpose we introduce again (see Chapter II, Section 1.2) the spaces V_s :

$$V_s = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega)\cap\mathbb{H}^s(\Omega), \quad s \geq 1. \tag{4.1}$$

The spaces $\mathbb{H}_0^1(\Omega)\cap\mathbb{H}^s(\Omega)$ and V_s are endowed with the usual Hilbert norm of $\mathbb{H}^s(\Omega)$:

$$\|u\|_{\mathbb{H}^s(\Omega)} = \{\sum_{[j]\leq s} |D^j u|^2\}^{\frac{1}{2}} \quad (s \text{ integer}). \tag{4.2}$$

Obviously $(s \geq 1)$,

$$V_s \subset V \tag{4.3}$$

with a continuous injection and V_s is dense in V.

The form b is defined on $V\times V\times V_s$, provided $s \geq \frac{n}{s}$; more precisely:

Lemma 4.1.

The form b is trilinear continuous on $V\times V\times V_s$ if $s \geq \frac{n}{2}$ and

$$|b(u,v,w)| \leq c|u|\ \|v\|\ \|w\|_{V_s} \tag{4.4}$$ (1)

Proof.

For $u,v,w\in\mathcal{V}$, the Hölder inequality gives

---

(1) Any dimension, Ω bounded or not.

$$|b(u,v,w)| = |b(u,w,v)| \leq \sum_{i,j=1}^{n} \|u_i\|_{L^2(\Omega)} \|D_i w_j\|_{L^n(\Omega)} \|v_j\|_{L^{\frac{2n}{n-2}}(\Omega)}$$

$$\leq \text{(by the Sobolev inequality } H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega))$$

$$\leq c_0 |u| \, \|v\| \sum_{i,j=1}^{n} \|D_i w_j\|_{L^n(\Omega)} .$$

Since $s \geq \frac{n}{2}$, $H^{s-1}(\Omega)$ is included in $L^q(\Omega)$ where

(4.5) $$\frac{1}{q} = \frac{1}{2} - \frac{s-1}{n}, \quad q \geq n.$$

If $w \in V_s$ then $D_i w_j$ belongs to $H^{s-1}(\Omega)$ and to $L^q(\Omega)$; $D_i w_j$ belonging to $L^q(\Omega) \cap L^2(\Omega)$ implies that $D_i w_j \in L^n(\Omega)$ too and

$$\|D_i w_j\|_{L^n(\Omega)} \leq c_1 \|w\|_{V_s}$$

so that

(4.6) $$|b(u,v,w)| \leq c_2 |u| \, \|v\| \, \|w\|_{V_s} .$$

This estimate shows that we can extend by continuity the form b from $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$ onto $V \times V \times V_s$, and even $H \times V \times V_s$, by (4.6).

<u>Lemma 4.2.</u>

If u belongs to $L^2(0,T;V) \cap L^\infty(0,T;H)$ then Bu belongs to $L^2(0,T;V_s')$ for $s \geq \frac{n}{2}$.

<u>Proof.</u>

By the definition of B and because of (4.4),

$$|\langle Bu(t),v\rangle| = |b(u(t),u(t),v)| \leq c|u(t)| \, \|u(t)\| \, \|v\|_{V_s}, \quad \forall v \in V_s;$$

hence

(4.7) $$\|Bu(t)\|_{V_s'} \leq c|u(t)| \, \|u(t)\| \quad \text{for a.a. } t \in [0,T]$$

and the lemma is proved. ■

In all dimensions of space, we can give the following weak formulation of the Navier-Stokes problem:

<u>Problem 4.1.</u>

For f and u_0 given such that

(4.7) $$f \in L^2(0,T;V'),$$

(4.8) $$u_0 \in H,$$

<u>to find</u> u <u>satisfying</u>

(4.9) $$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

(4.10) $$\frac{d}{dt}(u,v) + \nu((u,v)) + b(u,u,v) = \langle f,v \rangle, \quad \forall v \in V_s \quad (s \geq \frac{n}{2})$$

(4.11) $$u(0) = u_0.$$

If u satisfies (4.9) and (4.10), then

$$\frac{d}{dt}\langle u,v \rangle = \langle g,v \rangle, \quad \forall v \in V_s$$

with

$$g = f - Bu - \nu Au.$$

Due to Lemma 4.2, Bu belongs to $L^2(0,T;V_s')$ and since $f-\nu Au$ belongs to $L^2(0,T;V')$,

(4.12) $$g \in L^2(0,T;V_s').$$

Lemma 1.1 then implies that

(4.13) $$\begin{cases} u' \in L^2(0,T;V_s') \\ u' = f - \nu Au - Bu; \end{cases}$$

therefore u is almost everywhere equal to a continuous function from [0,T] into V_s' and (4.11) makes sense.

An alternate formulation of Problem 4.1 is the following one:

<u>Problem 4.2.</u>

<u>Given</u> f <u>and</u> u_0, <u>satisfying</u> (4.7)-(4.8), <u>to find</u> u <u>satisfying</u>

(4.14) $$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad u' \in L^2(0,T;V_s') \ (s \geq \frac{n}{2}),$$

(4.15) $$u' + \nu Au + Bu = f \quad \text{on} \quad]0,T[,$$

(4.16) $$u(0) = u_0.$$

The formulations (4.9) - (4.11) and (4.14) - (4.16) are equivalent.

The existence of solutions of these problems is given by the following theorem which implies Theorem 3.1:

<u>Theorem 4.1.</u>

<u>Let there be given</u> f <u>and</u> u_0 <u>which satisfy</u> (4.7)-(4.8). <u>Then, there</u>

exists at least one solution u *of Problem* 4.2. *Moreover* u *is weakly continuous from* $[0,T]$ *into* H.

This theorem is proved in Sections 4.2 and 4.3; the weak continuity in H is a direct consequence of (4.14) and Lemma 1.4.

4.2 *The Approximate Solutions.*

Let N be a integer which will later go to infinity and set

$$k = T/N. \tag{4.17}$$

We will define recursively a family of elements of V, say $u^0, u^1, \cdots, u^N$, where u^m will be in some sense an approximation of the function u we are looking for, on the interval $mk < t < (m+1)k$.

We define first the elements $f^1, \cdots, f^N$ of V':

$$f^m = \frac{1}{k}\int_{(m-1)k}^{mk} f(t)dt, \quad m = 1,\cdots,N; \quad f^m \in V'. \tag{4.18}$$

We begin with

$$u^0 = u_0, \quad \text{\underline{the given initial data};} \tag{4.19}$$

then when $u^0,\cdots,u^{m-1}$ are known, we define u^m as an element of V which satisfies

$$\frac{u^m - u^{m-1}}{k} + \nu A u^m + B u^m = f^m; \tag{4.20}$$

u^m depends on k; for simplification we denote them u^m in lieu of u_k^m.

The existence of such a u^m is asserted by Lemma 4.3, whose proof is postponed to the end of this section.

Lemma 4.3.

For each fixed k *and each* $m \geq 1$, *there exists at least one* u^m *satisfying* (4.20) *and moreover*

$$|u^m|^2 - |u^{m-1}|^2 + |u^m - u^{m-1}|^2 + 2k\nu\|u^m\|^2 \leq 2k\langle f^m, u^m\rangle. \tag{4.21}$$

For each fixed k (or N), we associate to the elements $u^1,\cdots,u^N$, the following approximate functions:

$$u_k: [0,T) \mapsto V, \quad u_k(t) = u^m, \quad t \in [(m-1)k, mk), \; m = 1,\cdots,N \tag{4.22}$$

$$(4.23)\qquad \begin{array}{l} w_k: \ [0,T] \longmapsto H, \ w_k \ \underline{\text{is continuous, linear on}} \\ \underline{\text{each interval}} \ [(m-1)k, mk] \ \underline{\text{and}} \ w_k(mk) = u^m, \ m = 0,\cdots,N. \end{array}$$

In Section 4.3 we will give a priori estimates of these functions; we will then pass to the limit $k \to 0$ (Section 4.3).

Proof of Lemma 4.3.

The equation (4.20) must be understood in a space larger than V', for example in a space V'_s, $s \geq \frac{n}{2}$. It is equivalent to

$$(4.24)\qquad (u^m,v) + k\nu((u^m,v)) + kb(u^m,u^m,v) = \langle u^{m-1}+kf^m,v\rangle, \ \forall v \in V_s.$$

We proceed by the Galerkin method, essentially as for Theorem II.1.2.

We choose a sequence of elements $w_1,\cdots,w_i,\cdots$, of V_s which is free and total in V_s and thus in V. For each r, by application of Lemma 1.4, we prove the existence of an element ϕ_r (depending on r,k,m):

$$(4.25)\qquad \phi_r = \sum_{i=1}^{r} \xi_{i,r} w_i,$$

$$(4.26)\qquad (\phi_r,v) + k\nu((\phi_r,v)) + kb(\phi_r,\phi_r,v) = \langle u^{m-1}+kf^m,v\rangle, \forall v \in Sp(w_1,\cdots,w_r)^{(1)}.$$

We must then get an a priori estimate independent of r, and pass to the limit $r \to \infty$. (k and m are fixed in this proof).

Taking $v = \phi_r$ in (4.26) we get

$$(4.27)\qquad (\phi_r-u^{m-1},\phi_r) + k\nu\|\phi_r\|^2 = k\langle f^m,\phi^r\rangle.$$

Now

$$(4.28)\qquad 2(a-b,a) = |a|^2-|b|^2 + |a-b|^2, \ \forall a,b \in H,$$

so that (4.27) gives

$$(4.29)\qquad \begin{aligned} |\phi_r|^2 + |\phi_r-u^{m-1}|^2 + 2k\nu\|\phi_r\|^2 &= |u^{m-1}|^2 + 2k\langle f^m,\phi_r\rangle \\ &\leq |u^{m-1}|^2 + 2k\|f^m\|_{V'}|\phi_r| \leq |u^{m-1}|^2 + \nu k\|\phi_r\|^2 + \frac{k}{\nu}\|f^m\|^2_{V'} . \end{aligned}$$

Hence

$$(4.30)\qquad |\phi_r|^2 + |\phi_r-u^{m-1}|^2 + k\nu\|\phi_r\|^2 \leq |u^{m-1}|^2 + \frac{k}{\nu}\|f^m\|^2_{V'} .$$

---

(1) $Sp(w_1,\cdots,w_r)$ = the space spanned by $w_1,\cdots,w_r$.

The inequality (4.30) shows that the sequence ϕ_r remains bounded in V as $r \longrightarrow \infty$. Therefore we can extract from ϕ_r a subsequence $\phi_{r'}$ such that

$$(4.31) \qquad \phi_{r'} \longrightarrow \phi \text{ in } V \text{ weakly, as } r' \longrightarrow \infty.$$

By standard arguments we then pass to the limit in (4.26) and prove that $\phi = u^m$ satisfies (4.24).

It remains to establish (4.21). This would be obvious if we could take $v = u^m$ in (4.24); since $u^m \notin V_s$ in general we proceed instead by passage to the limit. We pass to the lower limit in (4.29), noting that the norm is lower semicontinuous for the weak topology:

$$|\phi|^2 \leq \varliminf_{r' \longrightarrow \infty} |\phi_{r'}|^2, \quad \|\phi\|^2 \leq \varliminf_{r' \longrightarrow \infty} \|\phi_{r'}\|^2.$$

The proof is complete. ∎

4.3 A Priori Estimates.

Lemma 4.4.

$$(4.32) \qquad |u^m|^2 \leq d_1, \quad m = 1,\cdots,N,$$

$$(4.33) \qquad k \sum_{m=1}^{N} \|u^m\|^2 \leq \frac{1}{\nu} d_1,$$

$$(4.34) \qquad \sum_{m=1}^{N} |u^m - u^{m-1}|^2 \leq d_1,$$

where d_1 depends only on the data:

$$(4.35) \qquad d_1 = |u_0|^2 + \frac{1}{\nu}\int_0^T \|f(s)\|_{V'}^2 ds.$$

Proof.

As mentioned in the proof of Lemma 4.3, we cannot take the scalar product of (4.20) by u^m, at least for $n > 4$ ($Bu^m \notin V'$). But (4.21) will play the same role as the equation we would obtain by this prodedure.

We majorize the right-hand side of (4.21) by

$$2k\|f^m\|_{V'} \|u^m\| \leq k\nu\|u^m\|^2 + \frac{k}{\nu}\|f^m\|_{V'}^2,$$

and we obtain

(4.36) $|u^m|^2 - |u^{m-1}|^2 + |u^m - u^{m-1}|^2 + k\nu\|u^m\|^2 \le \frac{k}{\nu}\|f^m\|^2_{V'}$, $m = 1,\cdots,N.$

Adding the equalities (4.36) for $m = 1,\cdots,N$, we find

(4.37) $$|u^N|^2 + \sum_{m=1}^{N} |u^m - u^{m-1}|^2 + \nu k \sum_{m=1}^{N} \|u^m\|^2 \le |u_0|^2 + \frac{k}{\nu} \sum_{m=1}^{N} \|f^m\|^2_{V'} .$$

Adding the equalities (4.36) for $m = 1,\cdots,r$, and dropping the terms $\|u^m\|^2$, we get

(4.38) $$|u^r|^2 \le |u_0|^2 + \frac{k}{\nu}\sum_{m=1}^{r} \|f^m\|^2_{V'} \le |u_0|^2 + \frac{k}{\nu}\sum_{m=1}^{N} \|f^m\|^2_{V'} , \quad r = 1,\cdots,N.$$

The lemma is now a consequence of (4.37)-(4.38) and of a majoration of the right-hand side of these inequalities given in the next lemma. ∎

<u>Lemma 4.5.</u>
<u>Let</u> f^m <u>be defined by</u> (4.18). <u>Then</u>

(4.39) $$k \sum_{m=1}^{N} \|f^m\|^2_{V'} \le \int_0^T \|f(t)\|^2_{V'} dt.$$

<u>Proof.</u>
Due to the Schwarz inequality,

$$\|f^m\|^2_{V'} = \frac{1}{k^2} \|\int_{(m-1)k}^{mk} f(t)dt\|^2_{V'} \le \frac{1}{k}\int_{(m-1)k}^{mk} \|f(t)\|^2 dt.$$

Then (4.39) follows by summation of these inequalities for $m = 1,\cdots,N$. ∎

The last a priori estimate is the following:

<u>Lemma 4.6.</u>
<u>The sum</u> $k \sum_{m=1}^{N} \|\frac{u^m - u^{m-1}}{k}\|^2_{V'_s}$ <u>is bounded independently of</u> k.

<u>Proof.</u>
Taking the norm in (4.20) we obtain

$$\|\frac{u^m - u^{m-1}}{k}\|_{V'_s} \le \|f^m\|_{V'_s} + \nu\|Au^m\|_{V'_s} + \|Bu^m\|_{V'_s} \le c_1\{\|f^m\|_{V'} + \|u^m\|_V\} + \|Bu^m\|_{V'_s} ,$$

$$\|\frac{u^m - u^{m-1}}{k}\|^2_{V'_s} \le c_2\{\|f^m\|^2_{V'} + \|u^m\|^2_V + \|Bu^m\|^2_{V'_s}\}.$$

From (4.4) and (4.32) we get

$$\|Bu^m\|^2_{V'_s} \leq c_3 |u^m|^2 \; \|u^m\|^2 \leq c_4 \|u^m\|^2 .$$

We finally have

$$k \sum_{m=1}^{N} \left\|\frac{u^m - u^{m-1}}{k}\right\|^2_{V'_s} \leq c_5 k \sum_{m=1}^{N} (\|f^m\|^2_{V'} + \|u^m\|^2),$$

and we finish the proof using (4.33) and Lemma 4.5. ▪

It is interesting now to interpret the above in terms of the approximate functions:

Lemma 4.7.

The functions u_k and w_k are in a bounded set of $L^2(0,T;V) \cap L^\infty(0,T;H)$; w'_k is bounded in $L^2(0,T;V'_s)$ and

(4.40) $$u_k - w_k \longrightarrow 0 \quad \text{in} \quad L^2(0,T;H) \quad \text{as} \quad k \longrightarrow \infty.$$

Proof.

The estimations on u_k and w_k are just interpretations of (4.32)-(4.33) and Lemma 4.6; (4.40) is a consequence of (4.34) and the next lemma. ▪

Lemma 4.8.

(4.41) $$|u_k - w_k|_{L^2(0,T;H)} = \sqrt{\frac{k}{3}} \left(\sum_{m=1}^{N} |u^m - u^{m-1}|^2\right)^{\frac{1}{2}}.$$

Proof.

$$w_k(t) - u_k(t) = \frac{(t-mk)}{k} (u^m - u^{m-1}) \quad \text{for} \quad (m-1)k \leq t \leq mk,$$

$$\int_{(m-1)k}^{mk} |w_k(t) - u_k(t)|^2 dt = \frac{k}{3} |u^m - u^{m-1}|^2 \quad ,$$

and we find (4.41) by summation. ▪

4.4 Passage to the Limit.

Due to Lemma 4.7, we can extract from u_k a subsequence $u_{k'}$ such that

(4.42) $$u_{k'} \longrightarrow u \quad \text{in} \quad L^2(0,T;V) \quad \text{weakly,}$$
$$\text{in} \quad L^\infty(0,T;H) \quad \text{weak-star.}$$

---

(1) Strictly speaking, if $u_0 \notin V$, $w_k(t) \notin V$ for $0 \leq t < k$; in this case we simply replace $L^2(0,T;V)$ by $L^2_{loc}(]0,T];V)$ whenever we are considering the functions w_k.

We want to prove that u is a solution of (4.14) - (4.16); we need a strong convergence result for the u_k, in order to pass to the limit in (4.20). The functions w_k will play, for this, a useful auxiliary role.

We can choose the subsequence k', so that

(4.43) $$w_{k'} \longrightarrow u_* \quad \text{in } L^2(0,T;V) \text{ weakly,} \quad \text{in } L^\infty(0,T;H) \text{ weak-star,}$$

(4.44) $$\frac{dw_{k'}}{dt} \longrightarrow u_*' \quad \text{in } L^2(0,T;V_s') \text{ weakly.}$$

Because of (4.40), $u = u_*$.

Theorem 2.1 shows us that

(4.45) $$w_{k'} \longrightarrow u \quad \text{in } L^2(0,T;H);$$

thus by (4.40),

(4.46) $$u_{k'} \longrightarrow u \quad \text{in } L^2(0,T;H).$$

The equations (4.20) can be interpreted as

(4.47) $$\frac{dw_k}{dt} + \nu A u_k + B u_k = f_k,$$

with f_k defined by

$$f_k(t) = f^m, \quad (m-1)k \le t < mk, \quad m = 1,\cdots,N.$$

Because of (4.42), (4.46), and Lemmas 3.2 and 4.2,

$$Bu_{k'} \longrightarrow Bu \quad \text{in } L^2(0,T;V_s') \text{ weakly.}$$

By Lemma 4.9 below,

$$f_k \longrightarrow f \quad \text{in } L^2(0,T;V');$$

therefore we can pass to the limit in (4.47), and we find

$$u' + \nu A u + B u = f.$$

Due to (4.43),(4.44) and Lemma 4.1,

$$\langle w_{k'}(t),\sigma\rangle \longrightarrow \langle u(t),\sigma\rangle, \quad \forall \sigma \in V_s', \forall t \in [0,T];$$

since $w_{k'}(0) = u_0$, we get

$$u(0) = u_0.$$

We have proved that u satisfies (4.14) - (4.16); the proof of Theorem 4.1 will be complete once we prove

Lemma 4.9.

(4.48) $$f_k \longrightarrow f \quad \text{in} \quad L^2(0,T;V'), \quad \text{as} \quad k \longrightarrow 0.$$

Proof.

We observe that the transformation

$$f \longmapsto f_k$$

is a linear averaging mapping in $L^2(0,T;V')$; this mapping is continuous by Lemma 4.5 which enables us to assert

(4.49) $$\|f_k\|_{L^2(0,T;V')} \leq \|f\|_{L^2(0,T;V)}.$$

Therefore, instead of proving (4.48) for any f in $L^2(0,T;V')$ we need only to prove it for f in a dense subspace of $L^2(0,T;V')$; for an f in $\mathcal{C}([0,T];V')$ the result is elementary and we skip its proof. ∎

Remark 4.1.

Summing the equations (4.21) for $m = 1,\cdots,r$, and dropping the terms $|u^m-u^{m-1}|^2$, we get

(4.50) $$|u^r|^2 + 2k\nu \sum_{m=1}^{r} \|u^m\|^2 \leq |u_0|^2 + 2k \sum_{m=1}^{r} \langle f^m,u^m\rangle.$$

The relation (4.50) can be interpreted as

(4.51) $$|u_k(t)|^2 + 2\nu \int_0^{t_k} \|u_k(s)\|^2 ds \leq |u_0|^2 + \int_0^{t_k} \langle f_k(s),u_k(s)\rangle ds,$$

where

(4.52) $$t_k = (m+1)k, \quad \text{for} \quad mk \leq t < (m+1)k.$$

For each fixed t, $u_k(t)$ is bounded in H independently of k and t; as $k' \longrightarrow 0$, $u_{k'}(t)$ converges to $u(t)$ in V'_s weakly; therefore [(1)]

(4.53) $$u_{k'}(t) \longrightarrow u(t) \quad \text{in } H \text{ weakly, as } k' \longrightarrow 0, \ \forall t \in [0,T].$$

We then pass to the lower limit in (4.51) (t fixed, $k' \longrightarrow 0$), using (4.42) and (4.53). This leads to the energy inequality:

(4.54) $$|u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 ds \leq |u_0|^2 + 2\int_0^t \langle f(s),u(s)\rangle ds, \quad \forall t \in [0,T].$$

If $n = 2$, using (3.62), it is easy to prove directly the energy equality:

(4.55) $$|u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 ds = |u_0|^2 + 2\int_0^t \langle f(s),u(s)\rangle ds, \ \forall t \in [0,T].$$ ∎

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(1) Proof by contradiction.

§5. DISCRETIZATION OF THE NAVIER-STOKES EQUATIONS:

I. General Stability and Convergence Theorems.

This section is concerned with a general discussion of the discretization of the evolution Navier-Stokes equations. We study here a full discretization of the equations, both in the space and time variables:

1) The discretization in the space variables appears through the introduction of an external approximation of the space V; for example, one of the approximations (APX 1) to (APX 4), corresponding either to finite differences or finite elements. Actually these particular examples will be discussed in more detail in reference [9].

2) For the discretization in the time variables, we propose, among many natural and classical schemes, four schemes with two levels in time (fully implicit scheme, Cranck-Nicholson scheme, a scheme implicit in the linear part and explicit in its nonlinear part, and a scheme of explicit type).

After the description of the scheme under consideration we proceed to study the stability of these schemes. The problem of stability is the terminology in Numerical Analysis for the problem of getting a priori estimates on the approximate solutions. It is classical that the discretization in both space and time of evolution equations can lead to unstable or conditionally stable schemes: the approximate solutions are unbounded unless the discretization parameters satisfy some restriction. We discuss in full detail the numerical stability of the four schemes considered. To our knowledge the methods used here are non-classical methods for studying the stability of nonlinear equations. The study of nonlinear instability is a difficult problem; our study here, based on the energy method, leads only to sufficient conditions for stability; the stability conditions which are obtained seem close to being necessary, but the problem of necessary conditions of stability is not studied at all in the text.

The last subject treated in this section is the convergence of the schemes. Two general convergence theorems in suitable spaces are proved for the different schemes. The proof of convergence depends on discrete compactness methods. Owing to the lack of uniqueness of weak solutions in the three dimensional case, the convergence results obtained in the two and three dimensional cases are different, and better, of course, if $n = 2$.

The repartition of this material throughout subsequent subsections is the following: In Subsection 5.1 we describe the general type of discretization and the numerical schemes which will be studied. In Subsections 5.2, 5.3, and 5.4, we successively study the stability of Schemes 5.1 and 5.2 (Subsection 5.2), 5.3 (Subsection 5.3) and 5.4 (Subsection 5.4). Subsection 5.5 deals with auxiliary a priori estimates of a rather technical character (involving fractional derivatives in time of the approximate functions). Subsection 5.6 contains the description of the consistency hypotheses, the statement of the general convergence theorems, and the proofs of these theorems. ■

The application of these results to specific approximations of the space V will be treated in reference [9]. There we will also study practical methods for the resolution of the discrete problems. In fact, a more complete account of the numerical study of the nonlinear evolution Navier-Stokes equations is given therein and other methods of approximation include the fractional step or projection method and the artificial compressibility method.

From now on we restrict ourselves to the "concrete" dimensions of space, $n = 2$ and $n = 3$. ■

5.1 Description of the Approximation Schemes.

From now on we will be concerned with the approximation of the solutions of the Navier-Stokes equations in the two and three dimensional cases exclusively, Ω being bounded. For simplicity we suppose that the given data, u_0, f, satisfy

$$f \in L^2(0,T;H), \tag{5.1}$$

and, as before,

$$u_0 \in H. \tag{5.2}$$

Theorems 3.1 and 3.2 ensure us that there exists a unique solution of Problem 3.2 if $n = 2$, and that there exists at least one such solution if $n = 3$.

Let there be given a stable and convergent external approximation of the space V, say $\{(V_h, p_h, r_h)_{h\in\mathcal{H}}, (\overline{\omega}, F)\}$; the V_h are assumed to be finite dimensional spaces. This approximation could be any of the approximations (APX 1),$\cdots$,(APX 4), that were described in Chapter I. For simplicity we assume that

$$V_h \subset \mathbb{L}^2(\Omega), \quad \forall h \in \mathcal{H}, \tag{5.3}$$

a condition which is realized by all the previous approximations. The space V_h is therefore equipped with two norms: the norm $|\cdot|$ induced by $\mathbb{L}^2(\Omega)$ and its own norm $\|\cdot\|_h$. Since V_h is finite dimensional, these norms must be equivalent; the quotient of the two norms is bounded by a constant which may depend on h. Therefore we assume more precisely that

(5.4) $$|u_h| \leq d_0 \| u_h \|_h, \quad \forall u_h \in V_h,$$

d_0 independent of h, and

(5.5) $$\| u_h \|_h \leq S(h) \; |u_h|, \quad \forall u_h \in V_h.$$

The constant $S(h)$, which usually depends on h, plays an important role in the study of the stability of the numerical approximation; for this reason $S(h)$ is sometimes called the <u>stability constant</u>. Usually $S(h) \longrightarrow +\infty$, as $h \longrightarrow 0$.

Let there be given a trilinear continuous form on V_h, say $b_h(u_h,v_h,w_h)$, which satisfies

(5.6) $$b_h(u_h,v_h,v_h) = 0 \quad \forall u_h, v_h \in V_h,$$

(5.7) $$|b_h(u_h,v_h,w_h)| \leq d_1 \| u_h \|_h \| v_h \|_h \| w_h \|_h,$$

$$\forall u_h, v_h, w_h \in V_h, \quad (d_1 \text{ independent of } h),$$

and some further properties which will be announced when needed (i.e., when discussing the stability and the convergence of the schemes).

Let us divide the interval $[0,T]$ into N intervals of equal length k:

(5.8) $$k = T/N.$$

As in Section 4, we associate with k and the functions f, the elements $f^1, \cdots, f^N$:

(5.9) $$f^m = \frac{1}{k} \int_{(m-1)k}^{mk} f(t)dt, \quad m = 1, \cdots, N; \quad f^m \in \mathbb{L}^2(\Omega).$$

We will describe and study four basic schemes chosen from among a large class of interesting and sometimes classical schemes which have been proposed for the Navier-Stokes equations.

For all the four schemes we define recursively for each h and k a family of elements $u_h^0, \cdots, u_h^N$, of V_h. Actually these elements depend on h,k (and the data), and should be denoted u_{hk}^m; nevertheless, for simiplicity we do not emphasize this double dependence.

In each of the four schemes, we start the recurrence with

(5.10) u_h^0 = <u>the orthogonal projection of</u> u_0 <u>onto</u> V_h, <u>in</u> $\mathbb{L}^2(\Omega)$;

this definition makes sense by (5.3) and we immediately observe that

(5.11) $$|u_h^0| \leq |u_0|, \quad \forall h.$$

Scheme 5.1.

When $u_h^0,\cdots,u_h^{m-1}$, are known, u_h^m is the solution in V_h of

(5.12) $$\frac{1}{k}(u_h^m-u_h^{m-1},v_h)+\nu((u_h^m,v_h))_h+b_h(u_h^{m-1},u_h^m,v_h)=(f^m,v_h),\quad \forall v_h\in V_h.$$

Scheme 5.2.

When $u_h^0,\cdots,u_h^{m-1}$, are known, u_h^m is the solution in V_h of

(5.13) $$\frac{1}{k}(u_h^m-u_h^{m-1},v_h)+\frac{\nu}{2}((u_h^{m-1}+u_h^m,v_h))_h+\frac{1}{2}b_h(u_h^{m-1},u_h^{m-1}+u_h^m,v_h)$$
$$=(f^m,v_h),\quad \forall v_h\in V_h.$$

Scheme 5.3.

When $u_h^0,\cdots,u_h^{m-1}$, are known, u_h^m is the solution in V_h of

(5.14) $$\frac{1}{k}(u_h^m-u_h^{m-1},v_h)+\nu((u_h^m,v_h))_h+b_h(u_h^{m-1},u_h^{m-1},v_h)=(f^m,v_h),\quad \forall v_h\in V_h.$$

Scheme 5.4.

When $u_h^0,\cdots,u_h^{m-1}$, are known, u_h^m is the solution in V_h of

(5.15) $$\frac{1}{k}(u_h^m-u_h^{m-1},v_h)+\nu((u_h^{m-1},v_h))_h+b_h(u_h^{m-1},u_h^{m-1},v_h)=(f_h^m,v_h),\ \forall v_h\in V_h.$$

For all the schemes, the equation defining u_h^m is equivalent to a linear equation of the form

(5.16) $$a_h(u_h^m,v_h)=L_h(v_h),\quad \forall v_h\in V_h,$$

L_h depends on m, a_h depends on m for Schemes 5.1 and 5.2, but not in the case of Schemes 5.3 and 5.4.

We observe, in all cases, that

(5.17) $$a_h(v_h,v_h)\geq\frac{1}{k}|v_h|^2,$$

and therefore the existence and uniqueness of the solution of (5.16) is a consequence of the Projection Theorem (Theorem I.2.2).

Remark 5.1.

(i) The computation of u_h^m requires the inversion of a matrix;

- the matrix is positive definite, nonsymmetric, and depends on m for Schemes 5.1 and 5.2,

- the matrix is positive definite, symmetric, and does not depend on m for Schemes 5.3 and 5.4.

(ii) Scheme 5.1 is the standard fully implicit scheme; Scheme 5.2 is an interpretation of the classical Cranck-Nicholson scheme. Scheme 5.3 is a partly implicit scheme, implicit only in the linear part of the operator.

(iii) Scheme 5.4 is an explicit scheme or more precisely an interpretation of the so-called explicit schemes; this terminology is justified by the fact that this type of scheme usually gives u_h^m explicitly, that is to say without inverting any matrix. In the present case, due to the discrete condition $\text{div}\, u = 0$ built in the space V_h, the determination of u_h^m necessitates an inversion of a matrix. This restricts considerably the interest of this scheme, but we considered it of interest nevertheless.

(iv) Besides this discussion on the type of scheme, the reader is referred to reference [9] for practical methods of computation of the u_h^m. ■

Remark 5.2: Related Schemes.

(i) A related form of Schemes 5.1 and 5.2 is a nonlinear form of these schemes:

Scheme 5.1'.

$$\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) + \nu((u_h^m, v_h))_h + b_h(u_h^m, u_h^m, v_h) = (f^m, v_h), \quad \forall v_h \in V_h. \tag{5.18}$$

Scheme 5.2'.

$$\begin{aligned}\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) &+ \frac{\nu}{2}((u_h^{m-1} + u_h^m, v_h))_h + \frac{1}{4} b_h(u_h^{m-1} + u_h^m, u_h^{m-1} + u_h^m, v_h) \\ &= (f^m, v_h), \quad \forall v_h \in V_h.\end{aligned} \tag{5.19}$$

(ii) A related form of Scheme 5.3 is a Cranck-Nicholson scheme implicit in its linear part:

Scheme 5.3'.

$$\begin{aligned}\frac{1}{k}(u_h^m - u_h^{m-1}, v_h) &+ \frac{\nu}{2}((u_h^{m-1} + u_h^m, v_h))_h + b_h(u_h^{m-1}, u_h^{m-1}, v_h) \\ &= (f^m, v_h), \quad \forall v_h \in V_h.\end{aligned} \tag{5.20}$$

(iii) These Schemes could be studied by exactly the same methods as Schemes 5.1 - 5.4. ■

5.2 Stability of Schemes 5.1 and 5.2.

The problem is to prove some a priori estimates on the approximate solution.

5.2.1 Scheme 5.1.

Lemma 5.1.

The solutions u_h^m of (5.12) remain bounded in the following sense:

(5.21) $$|u_h^m|^2 \le d_2, \quad m = 0,\cdots,N,$$

(5.22) $$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \le d_2,$$

(5.23) $$k \cdot \sum_{m=1}^{N} \|u_h^m\|^2 \le \frac{1}{\nu} d_2,$$

where

(5.24) $$d_2 = |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 ds.$$

Proof.

We take $v_h = u_h^m$ in (5.12). Due to (5.6) and the identity

(5.25) $$2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2, \quad \forall\, a,b \in \mathbb{L}^2(\Omega),$$

we obtain

(5.26) $$\begin{aligned} |u_h^m|^2 - |u_h^{m-1}|^2 + |u_h^m - u_h^{m-1}|^2 + 2k\nu \|u_h^m\|_h^2 &= 2k(f^m, u_h^m) \\ &\le 2k|f^m|\, |u_h^m| \le \text{(by (5.4))} \\ &\le 2kd_0 |f^m|\, \|u_h^m\|_h \\ &\le k\nu \|u_h^m\|_h^2 + \frac{kd_0^2}{\nu} |f^m|^2. \end{aligned}$$

Hence

(5.27) $$|u_h^m|^2 - |u_h^{m-1}|^2 + |u_h^m - u_h^{m-1}|^2 + k\nu \|u_h^m\|_h^2 \le \frac{kd_0^2}{\nu} |f^m|^2, \quad m = 1,\cdots,N.$$

Adding these inequalities for $m = 1,\cdots,N$, we get

$$(5.28)\qquad |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + k\nu \sum_{m=1}^{N} \|u_h^m\|_h^2 \le |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{N} |f^m|^2 .$$

One can check as in Lemma 4.5 that

$$(5.29)\qquad k \sum_{m=1}^{N} |f^m|^2 \le \int_0^T |f(s)|^2 ds;$$

thus, by (5.11) and (5.29), it follows that the right-hand side of (5.28) is bounded by

$$(5.30)\qquad d_2 = |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 ds.$$

This proves (5.22) and (5.23).

We then add the inequalities (5.27) for $m = 1,\cdots,r$, dropping some positive terms, we get

$$|u_h^r|^2 \le |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2$$
$$\le \text{(due to the above)} \le d_2;$$

(5.21) is proved too. ▩

5.2.2 Scheme 5.2.

Lemma 5.2.

The solutions u_h^m of (5.13) remain bounded in the following sense:

$$(5.31)\qquad |u_h^m|^2 \le d_2, \quad m = 1,\cdots,N,$$

$$(5.32)\qquad k \sum_{m=1}^{N} \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 \le \frac{d_2}{\nu},$$

with the same d_2 as (5.24).

Proof.

We take $v_h = u_h^m + u_h^{m-1}$ in (5.13). Due to (5.6) we find

$$|u_h^m|^2 - |u_h^{m-1}|^2 + 2k\nu \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2$$
$$= k(f^m, u_h^m + u_h^{m-1})$$
$$(5.33)$$
$$\le 2kd_0 |f^m| \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h$$
$$\le k\nu \left\| \frac{u_h^m + u_h^{m-1}}{2} \right\|_h^2 + k \frac{d_0^2}{\nu} |f^m|^2 .$$

Therefore

$$(5.34)\qquad |u_h^m|^2 + k\nu \left\| \frac{u_h^m+u_h^{m-1}}{2} \right\|_h^2 \le |u_h^{m-1}|^2 + \frac{kd_0^2}{\nu} |f^m|^2 .$$

We add these relations for $m = 1,\cdots,N$ and get

$$|u_h^N|^2 + k\nu \sum_{m=1}^{N} \left\| \frac{u_h^m+u_h^{m-1}}{2} \right\|_h^2$$

$$\le |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{N} |f^m|^2$$

$$\le \text{(as before)} \le d_2 .$$

This proves (5.32); adding then the relations (5.34) for $m = 1,\cdots,r$, and dropping the unnecessary terms, we find

$$|u_h^r|^2 \le |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2 \le d_2 ;$$

this implies (5.31). ∎

5.2.3 Stability Theorems.

We recall first a definition:

Definition 5.1.

An infinite set of functions $\mathcal{E}$ is called $L^p(0,T;X)$ stable if and only if $\mathcal{E}$ is a bounded subset of $L^p(0,T;X)$.

It is interesting to deduce from the previous estimations some stability results.

In order to state these results, we introduce the approximate functions u_h

$$(5.35)\qquad u_h : [0,T) \longmapsto V_h ,$$

$$(5.36)\quad \begin{cases} u_h(t) = u_h^m, & (m-1)k \le t < mk \quad \text{(Scheme 5.1)} \\ u_h(t) = \dfrac{u_h^m+u_h^{m-1}}{2}, & (m-1)k \le t < mk \quad \text{(Scheme 5.2)}, \quad m = 1,\cdots,N. \end{cases}$$

Due to Lemmas 5.1 and 5.2,

$$(5.37)\qquad \sup_{t\in[0,T]} |u_h(t)| \le \sqrt{d_2} ,$$

$$(5.38)\qquad \int_0^T \|u_h(t)\|_h^2 dt \le \frac{d_2}{\nu} .$$

Since the prolongation operators $p_h \in \mathcal{L}(V_h, F)$ are stable, we have

$$(5.39) \quad \|p_h u_h\|_F \leq d_3 \|u_h\|_h, \quad \forall u_h \in V_h, \quad (d_3 \text{ independent of } h).$$

We infer from (5.38) that

$$\int_0^T \|p_h u_h(t)\|_F^2 dt \leq \frac{d_3^2 d_2}{\nu} .$$

These remarks enable us to state the stability theorem:

Theorem 5.1.

The functions u_h, $h \in \mathcal{H}$, *corresponding to Schemes* 5.1 *and* 5.2 *are unconditionally* $L^\infty(0,T;\mathbb{L}^2(\Omega))$ *stable; the functions* $p_h u_h$ *are unconditionally* $L^2(0,T;F)$ *stable.* ■

Remark 5.2.

The majoration (5.22), and similar majorations for the other schemes which we will give later on, does not correspond to stability results but will be technically useful for the proof of the convergence of the scheme.

For the same majoration for Scheme 5.2, see Subsection 5.4.3. ■

5.3 Stability of Scheme 5.3.

We infer from (5.5) and (5.7) that

$$(5.40) \quad |b_h(u_h,u_h,v_h)| \leq d_1 \|u_h\|_h^2 \|v_h\|_h \leq d_1 S^2(h) |u_h| \; \|u_h\|_h \; |v_h|, \quad \forall u_h, v_h \in V_h.$$

Sometimes this relation can be improved and this means an important improvement of some restrictive conditions of stability which will appear later on in this section; for this reason we will assume that

$$(5.41) \qquad |b_h(u_h,u_h,v_h)| \leq S_1(h) \; |u_h| \; \|u_h\|_h \; |v_h|, \quad \forall u_h, v_h \in V_h,$$

where at least

$$(5.42) \qquad S_1(h) \leq d_1 S^2(h).$$

5.3.1 A Priori Estimates.

Lemma 5.3.

We assume that k *and* h *satisfy*

$$(5.43) \qquad kS_1^2(h) \leq d', \quad kS^2(h) \leq d''\ ^{(1)},$$

---

(1) In practice, one of these relations should be a consequence of the other (this depends on the explicit values of S and S_1).

where d' and d'' are some constants depending on the data and are estimated in the course of the proof.

Then, the u_h^m given by (5.14) remain bounded in the following sense:

(5.44) $$|u_h^m| \leq d_4, \quad m = 0,\cdots,N$$

(5.45) $$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leq d_4,$$

(5.46) $$k \sum_{m=1}^{N} \|u_h^m\|_h^2 \leq d_4,$$

where d_4 is some constant depending only on the data, d', and d''.

Proof.

We write (5.14) with $v_h = u_h^m$. Using again (5.25), we obtain the relation

(5.47) $$|u_h^m|^2 - |u_h^{m-1}|^2 + |u_h^m - u_h^{m-1}|^2 + 2k\nu\|u_h^m\|_h^2$$
$$= -2kb_h(u_h^{m-1}, u_h^{m-1}, u_h^m) + 2k(f^m, u_h^m).$$

Due to (5.6) the right-hand side of (5.47) is equal to

$$-2kb_h(u_h^{m-1}, u_h^{m-1}, u_h^m - u_h^{m-1}) + 2k(f^m, u_h^m);$$

this expression is less than (cf. (5.4) and (5.41)):

$$2kS_1(h)|u_h^{m-1}|\ \|u_h^{m-1}\|_h\ |u_h^m - u_h^{m-1}| + 2kd_0|f^m|\ \|u_h^m\|_h\ ,$$

$$k\nu\|u_h^m\|_h^2 + 2k^2S_1^2(h)|u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2 + \frac{1}{2}\ |u_h^m - u_h^{m-1}|^2 + \frac{kd_0^2}{\nu}\ |f^m|^2.$$

Therefore

(5.48) $$|u_h^m|^2 - |u_h^{m-1}|^2 + \frac{1}{2}\ |u_h^m - u_h^{m-1}|^2 + k\nu\|u_h^m\|_h^2$$
$$-2k^2S_1^2(h)|u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2 \leq \frac{kd_0^2}{\nu}\ |f^m|^2.$$

We add these inequalities for $m = 1,\cdots,r$;

(5.49) $$|u_h^r|^2 + \frac{1}{2}\sum_{m=1}^{r} |u_h^m - u_h^{m-1}|^2 + k\nu\sum_{m=1}^{r} \|u_h^m\|_h^2$$
$$-2k^2S_1^2(h)\sum_{m=2}^{r} |u_h^{m-1}|^2\ \|u_h^{m-1}\|_h^2 \leq \lambda_m,$$

$$(5.50) \qquad \lambda_m = |u_h^0|^2 + \frac{kd_0^2}{\nu} \sum_{m=1}^{r} |f^m|^2 + 2k^2 S_1^2(h) |u_h^0|^2 \, \|u_h^0\|_h^2.$$

Let us assume that

$$(5.51) \qquad 2kS_1^2(h)\lambda_N \leq \nu-\delta, \quad \text{for some fixed } \delta, \quad 0 < \delta < \nu.$$

If this inequality holds, it is easy to show recursively that

$$(5.52) \quad |u_h^r|^2 + \frac{1}{2} \sum_{m=1}^{r} |u_h^m - u_h^{m-1}|^2 + k\delta \sum_{m=1}^{r} \|u_h^m\|_h^2 \leq \lambda_r, \quad r = 1,\cdots,N.$$

Indeed the relation (5.48) written with $m = 1$, shows us that (5.52) is true for $r = 1$. Let us assume then that (5.52) is valid up to the order $r-1$, and let us show this relation for the integer r.

We observe that, by assumption,

$$(5.53) \qquad |u_h^m|^2 \leq \lambda_m \leq \lambda_N, \quad m = 1,\cdots,r-1;$$

therefore, by (5.51),

$$\begin{aligned} 2k^2 S_1^2(h) &\sum_{m=2}^{r} |u_h^{m-1}|^2 \, \|u_h^{m-1}\|_h^2 \\ &\leq 2k^2 S_1^2(h)\lambda_N \sum_{m=2}^{r} \|u_h^{m-1}\|_h^2 \\ &\leq 2k^2 S_1^2(h)\lambda_N \sum_{m=1}^{r} \|u_h^m\|_h^2 \\ &\leq k(\nu-\delta) \sum_{m=1}^{r} \|u_h^m\|_h^2. \end{aligned}$$

Putting this majoration into (5.49), we get (5.52) for the integer r.

The proof is complete if we show that a condition of the type (5.43) ensures (5.51).

According to a majoration used in Lemmas 5.1 and 5.2 (see (5.11), (5.29))

$$\begin{aligned} \lambda_N &\leq |u_0|^2 + \frac{d_0^2}{\nu} \int_0^T |f(s)|^2 ds + 2k^2 S_1^2(h) |u_0|^2 \, \|u_h^0\|_h^2 \\ &\leq \text{(see (5.5) and (5.24))} \\ &\leq d_2 + 2k^2 S_1^2(h) S^2(h) |u_0|^2. \end{aligned}$$

Hence, if (5.43) is satisfied,

$$2kS_1^2(h)\lambda_N \leq 2d'(d_2+2d'd''|u_0|^2)$$

and this is certainly bounded by $\nu-\delta$ if d' and d'' are sufficiently small:

$$2d'(d_2+2d'd''|u_0|^2) \leq \nu-\delta. \tag{5.54}$$

The proof is complete. ∎

5.3.2 The Stability Theorem.

We define for the Scheme 5.3 the approximate functions u_h by:

$$u_h : [0,T) \longrightarrow V_h \tag{5.55}$$

$$u_h(t) = u_h^m, \quad (m-1)k \leq t < mk, \quad m = 1,\cdots,N.$$

We infer from (5.39), (5.44), (5.45) that if (5.43) holds then

$$\sup_{t\in[0,T]} |u_h(t)| \leq \sqrt{d_4} \ ,$$

$$\int_0^T \|p_h u_h(t)\|_F^2 dt \leq d_3 d_4,$$

and thus

Theorem 5.2.

The functions u_h and $p_h u_h$, $h \in \mathcal{H}$, corresponding to the Scheme 5.3 are respectively $L^\infty(0,T;\mathbb{L}^2(\Omega))$ and $L^2(0,T;F)$ stable, provided k and h remain connected by (5.43). ∎

Definition 5.2.

Conditions such as (5.43) are called stability conditions. They are sufficient conditions ensuring the stability of the scheme. A scheme is called conditionally or unconditionally stable according to whether such a condition occurs or not in proving stability. ∎

5.4 Stability of Scheme 5.4.

5.4.1 A Priori Estimates.

Lemma 5.4.

We assume that k and h satisfy

$$kS^2(h) \leq \frac{1-\delta}{4\nu}, \quad \text{for some } \delta, \ 0<\delta<1, \tag{5.56}$$

and

$$kS_1^2(h) \leq \frac{\nu\delta}{8d_5} \tag{5.57}$$

<u>where</u>

(5.58) $$d_5 = |u_0|^2 + (\frac{d_0^2}{\nu} + 4T) \int_0^T |f(s)|^2 ds.$$

<u>Then the</u> u_h^m <u>given by</u> (5.15) <u>remain bounded in the following sense</u>:

(5.59) $$|u_h^m|^2 \leq d_5, \quad m = 1,\cdots,N$$

(5.60) $$k \sum_{m=1}^{N} \|u_h^m\|_h^2 \leq \frac{2d_5}{\delta\nu}$$

(5.61) $$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leq \frac{d_5}{\delta}(2-\delta) + 4T \int_0^T |f(s)|^2 ds.$$

<u>Proof</u>.

We replace v_h by u_h^{m-1} in (5.15); due to the identity

(5.62) $$2(a-b,b) = |a|^2 - |b|^2 - |a-b|^2, \quad \forall a,b \in \mathbb{L}^2(\Omega),$$

we find

$$\begin{aligned} |u_h^m|^2 - |u_h^{m-1}|^2 - |u_h^m - u_h^{m-1}|^2 + 2k\nu\|u_h^{m-1}\|_h^2 \\ = 2k(f^m, u_h^m) \\ \leq 2kd_0|f^m| \, \|u_h^m\|_h \\ \leq \nu k\|u_h^m\|_h^2 + \frac{kd_0^2}{\nu}|f^m|^2 \quad , \end{aligned}$$

(5.63) $$|u_h^m|^2 - |u_h^{m-1}|^2 - |u_h^m - u_h^{m-1}|^2 + k\nu\|u_h^{m-1}\|_h^2 \leq \frac{kd_0^2}{\nu}|f^m|.$$

The difference with the preceding is that the term $|u_h^m - u_h^{m-1}|^2$ on the left-hand side is affected with a minus sign and so, we must majorize it.

In order to majorize $|u_h^m - u_h^{m-1}|^2$, we write (5.15) with $v = u_h^m - u_h^{m-1}$. This gives

(5.64) $$\begin{aligned} 2|u_h^m - u_h^{m-1}|^2 &= -2k\nu((u_h^{m-1}, u_h^m - u_h^{m-1}))_h \\ &\quad -2kb_h(u_h^{m-1}, u_h^{m-1}, u_h^m - u_h^{m-1}) + 2k(f^m, u_h^m - u_h^{m-1}). \end{aligned}$$

We successively majorize all the terms on the right-hand side, using repeatedly (5.5), (5.41), and the Schwarz inequality:

$$-2k\nu((u_h^{m-1},u_h^m-u_h^{m-1}))_h \leq 2k\nu\|u_h^{m-1}\|_h \, \|u_h^m-u_h^{m-1}\|_h$$

$$\leq 2k\nu S(h)\|u_h^{m-1}\|_h \, |u_h^m-u_h^{m-1}|_h$$

$$\leq \frac{1}{4}\,|u_h^m-u_h^{m-1}|_h^2 + 4k^2\nu^2 S^2(h)\|u_h^{m-1}\|_h^2,$$

$$-2kb_h(u_h^{m-1},u_h^{m-1},u_h^m-u_h^{m-1})\cdot$$

$$\leq 2kS_1(h)|u_h^{m-1}| \, \|u_h^{m-1}\|_h \, |u_h^m-u_h^{m-1}|$$

$$\leq \frac{1}{4}\,|u_h^m-u_h^{m-1}|^2 + 4k^2S_1^2(h)|u_h^{m-1}|^2 \, \|u_h^{m-1}\|_h^2 \quad ,$$

$$2k(f^m,u_h^m-u_h^{m-1}) \leq 2k|f^m| \, |u_h^m-u_h^{m-1}|$$

$$\leq \frac{1}{4}\,|u_h^m-u_h^{m-1}|^2 + 4k^2|f^m|^2.$$

Therefore (5.64) becomes

$$|u_h^m-u_h^{m-1}|^2 \leq 4k^2\nu^2S^2(h)\|u_h^{m-1}\|_h + 4k^2S_1^2(h)|u_h^{m-1}|^2 \, \|u_h^{m-1}\|_h^2 + 4k^2|f^m|^2$$

(5.65) $$\leq \text{(by (5.56))}$$

$$\leq k\nu(1-\delta)\|u_h^{m-1}\|_h^2 + 4k^2S_1^2(h)|u_h^{m-1}|^2 \, \|u_h^{m-1}\|_h^2 + 4k^2|f^m|^2;$$

for (5.63) we then have

(5.66)
$$|u_h^m|^2-|u_h^{m-1}|^2 + k(\nu\delta-4kS_1^2(h)|u_h^{m-1}|^2)\|u_h^{m-1}\|_h^2$$

$$\leq k(\frac{d_0^2}{\nu} + 4k)|f^m|^2$$

$$\leq \text{(since } k \leq T)$$

$$\leq k(\frac{d_0^2}{\nu} + 4T)|f^m|^2.$$

Summing these inequalities for $m = 1,\cdots,r$, we arrive at

(5.67) $$|u_h^r|^2 + k\sum_{m=1}^{r}(\nu\delta-4kS_1^2(h)|u_h^{m-1}|^2)\|u_h^{m-1}\|_h^2 \leq \mu_r,$$

where

(5.68) $$\mu_r = |u_h^0|^2 + k(\frac{d_0^2}{\nu} + 4T)\sum_{m=1}^{r}|f^m|^2.$$

Using (5.57) we will now prove recursively that

$$(5.69)\qquad |u_h^r|^2 + \frac{k\nu\delta}{2}\sum_{m=1}^{r}\|u_h^{m-1}\|_h^2 \le \mu_r,\quad r = 1,\cdots,N.$$

We observe first that

$$(5.70)\qquad \mu_r \le \mu_N = |u_h^0|^2 + k(\frac{d_0^2}{\nu} + 4T)\sum_{m=1}^{N}|f^m|^2$$

$$\le |u_0|^2 + (\frac{d_0^2}{\nu} + 4T)\int_0^T |f(s)|^2 ds = d_5.$$

The relation (5.69) is obvious for $r = 1$; writing (5.66) for $m = 1$ and using (5.57) we get

$$|u_h^1|^2 + k\nu\delta\|u_h^0\|_h^2 \le |u_h^0|^2 + k(\frac{d_0^2}{\nu} + 4T)|f^1|^2 + 4k^2S_1^2(h)|u_h^0|^2\ \|u_h^0\|_h^2 \le \mu_1 + \frac{\nu\delta}{2}\|u_h^0\|_h^2,$$

which is (5.69) for $r = 1$.

Assuming now that the relation (5.69) holds up to the order $r-1$, we will prove it at the order r. In fact by the recurrence hypothesis,

$$(5.71)\qquad |u_h^{r-1}|^2 \le \mu_{r-1} \le \mu_N \le \text{(by (5.70))} \le d_5.$$

Hence (5.67) gives

$$(5.72)\qquad |u_h^r|^2 + k\nu\delta\sum_{m=1}^{r}\|u_h^{m-1}\|_h^2 \le \mu_r + 4k^2S_1^2(h)d_5\sum_{m=1}^{r}\|u_h^{m-1}\|_h^2$$

$$\le \mu_r + \frac{k\nu\delta}{2}\sum_{m=1}^{r}\|u_h^{m-1}\|_h^2,$$

and (5.69) at the order r follows.

It remains to prove (5.61). For this we return to (5.65); using (5.56),(5.57), we get

$$(5.73)\qquad |u_h^m - u_h^{m-1}|^2 \le k\nu(1 - \frac{\delta}{2})\|u_h^{m-1}\|_h^2 + 4kT|f^m|^2.$$

By summation and using (5.29), we find (5.61).

5.4.2 The Stability Theorem.

We now set

$$(5.74)\qquad u_h : [0,T) \longmapsto V_h$$

$$(5.75)\qquad u_h(t) = u_h^{m-1},\quad (m-1)k \le t < mk,\quad m = 1,\cdots,N,$$

and we have

Theorem 5.3.

The functions u_h and $p_h u_h$, $h \in \mathcal{H}$, corresponding to Scheme 5.4 are respectively $L^\infty(0,T;\mathbb{L}^2(\Omega))$ and $L^2(0,T;F)$ stable, provided k and h remain connected by (5.56) - (5.58).

5.5 A Complementary Estimate for Scheme 5.2.

Using the techniques extensively applied in Sections 5.3 and 5.4, we can complete Section 5.2 by giving, in the case of Scheme 5.2, an estimation similar to the estimation

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leq \text{Const.} \tag{5.76}$$

that we proved for Schemes 5.1, 5.3, and 5.4. As mentioned in Remark 5.2, these estimations will be useful for the proof of the convergence.

Lemma 5.5.

The u_h^m defined by (5.13) (Scheme 5.2) satisfy:

$$\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 \leq d(1 + kS^4(h)), \tag{5.77}$$

where d denotes a constant depending only on the data.

Proof.

We take $v_h = 2k(u_h^m - u_h^{m-1})$ in (5.13) and obtain

$$2|u_h^m - u_h^{m-1}|^2 = -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 - kb_h(u_h^{m-1}, u_h^m + u_h^{m-1}, u_h^m - u_h^{m-1}) + 2k(f^m, u_h^m - u_h^{m-1})$$

$$\leq \text{(by (5.40) and (5.5))}$$

$$\leq -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 + kd_1 S^2(h)|u_h^{m-1}|\ \|u_h^m + u_h^{m-1}\|_h\ |u_h^m - u_h^{m-1}|$$
$$+ 2k|f^m|\ |u_h^m - u_h^{m-1}|$$

$$\leq -k\nu\|u_h^m\|_h^2 + k\nu\|u_h^{m-1}\|_h^2 + \frac{1}{2}|u_h^m - u_h^{m-1}|^2 + \frac{k^2}{2} d_1^2 S^4(h)|u_h^{m-1}|^2\|u_h^m + u_h^{m-1}\|_h^2$$
$$+ \frac{1}{2}|u_h^m - u_h^{m-1}|^2 + 2kT|f^m|^2.$$

Thus

$$\sum_{m=1}^{N} |u_h^m-u_h^{m-1}|^2 \le k\nu\|u_h^0\|_h^2 + 2kT \sum_{m=1}^{N} |f^m|^2 + \frac{k^2}{2} d_1^2 S^4(h) \sum_{m=1}^{N} |u_h^{m-1}|^2 \|u_h^m+u_h^{m-1}\|_h^2$$

$$\le \text{(by (5.5),(5.11),(5.29),(5.31),(5.32))}$$

$$\le k\nu S^2(h)|u_0|^2 + 2T\int_0^T |f(s)|^2 ds + \frac{2}{\nu} d_1^2 d_2^2 k S^4(h).$$

The proof is complete. ∎

5.6 Other A Priori Estimates.

In order to prove strong convergence results we will establish some further a priori estimates concerning the fractional derivatives in t of approximate functions. This Section 5.6 is essentially a technical section which is used in Section 5.7 where the convergence of the schemes is proved. ∎

For all the four schemes we define w_h, a function from $\mathbb{R}$ into V_h, by:

(5.78) w_h is a continuous function from $\mathbb{R}$ into V_h, linear on each interval $[mk,(m+1)k]$, and $w_h(mk) = u_h^m$, $m = 0,\cdots,N-1$; $w_h = 0$ outside the interval $[0,T]$.

Lemma 5.6.

Assuming the same stability conditions as in Theorems 5.1, 5.2, 5.3, [(1)] the Fourier transform $\hat{w}_h$ of w_h satisfies

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{w}_h(\tau)|^2 d\tau \le \text{Const.}, \quad \text{for} \quad 0 < \gamma < \frac{1}{4} \tag{5.79}$$

where the constant depends on γ and on the data.

Proof.

The four equations (5.12) - (5.15) can be interpreted as

$$\frac{d}{dt}(w_h(t),v_h) = ((g_h(t),v_h))_h, \quad \forall v_h \in V_h, t \in (0,T), \tag{5.80}$$

where the function g_h satisfies

$$\int_0^T \|g_h(t)\|_h dt \le \text{Const.} \tag{5.81}$$

For example, for Scheme 5.1, g_h is defined by

(1) No condition for Schemes 5.1, 5.2; conditions (5.43) for Scheme 5.3; conditions (5.56)-(5.57) for Scheme 5.4.

$$((g_h(t),v_h))_h = (f^m,v_h) - b_h(u_h^{m-1},u_h^m,v_h) - \nu((u_h^m,v_h))_h,$$

$$\forall v_h \in V_h, \quad (m-1)k \leq t < mk.$$

Inequality (5.81) follows from (5.4), (5.7), and the previous a priori estimates:

$$\|g_h(t)\|_h \leq d_0|f^m| + d_1\|u_h^{m-1}\|_h \|u_h^m\|_h + \nu\|u_h^m\|_h ,$$

$$\int_0^T \|g_h(t)\|_h dt \leq k \sum_{m=1}^{N} (d_0|f^m| + d_1\|u_h^{m-1}\|_h \|u_h^m\|_h + \nu\|u_h^m\|_h);$$

the right-hand side of this relation is bounded according to Lemma 5.1.

Let us infer (5.79) from (5.80)-(5.81). Extending g_h by 0 outside $[0,T]$ we get a function $\tilde{g}_h$ such that the following equality holds on the whole t line:

$$(5.82) \quad \frac{d}{dt}(w_h(t),v_h) = ((\tilde{g}_h(t),v_h))_h + (u_h^0,v_h)\delta_0 - (u_h^N,v_h)\delta_T, \quad \forall v_h \in V_h,$$

where δ_0,δ_T denote the Dirac distribution at 0 and T.

By taking the Fourier transform, we then have

$$-2i\pi\tau(\hat{w}_h(\tau),v_h) = ((\hat{g}_h(\tau),v_h))_h + (u_h^0,v_h) - (u_h^N,v_h)\exp(-2i\pi\tau T);$$

($\hat{g}_h$ = Fourier transform of $\tilde{g}_h$).

Putting $v_h = \hat{w}_h(\tau)$ and then taking absolute values we get

$$2\pi|\tau| \; |\hat{w}_h(\tau)|^2 \leq \|\hat{g}_h(\tau)\|_h \|\hat{w}_h(\tau)\|_h + c_1|\hat{w}_h(\tau)|,$$

since u_h^0 and u_h^N remain bounded.

Due to (5.81) we also have

$$\|\hat{g}_h(\tau)\|_h \leq \int_0^T \|g_h(t)\|_h dt \leq \text{Const.} = c_2,$$

and, finally,

$$(5.83) \qquad |\tau| \; |\hat{w}_h(\tau)|^2 \leq c_3\|\hat{w}_h(\tau)\|_h.$$

For fixed γ, $\gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \leq c_4(\gamma)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \quad \forall \tau \in \mathbb{R}.$$

Hence

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{w}_h(\tau)|^2 d\tau \le c_4(\gamma) \int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} |\hat{w}_h(\tau)|^2 d\tau$$

$$\le \text{(by (5.83))}$$

$$\le c_4(\gamma) \int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2 d\tau + c_5 \int_{-\infty}^{+\infty} \frac{\|\hat{w}_h(\tau)\|_h}{1+|\tau|^{1-2\gamma}} d\tau \tag{5.84}$$

$$\le \text{(by the Schwarz inequality)}$$

$$\le c_4 \int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2 d\tau + c_5 \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}\right)^{\frac{1}{2}}$$

$$\cdot \left(\int_{-\infty}^{+\infty} \|\hat{w}_h(\tau)\|_h^2 \, d\tau\right)^{\frac{1}{2}} .$$

The integral

$$\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}$$

is finite for $\gamma < \frac{1}{4}$. Therefore the right-hand side of the last inequality is finite and bounded due to the Parseval relation and the previous estimations:

$$\int_{-\infty}^{+\infty} |\hat{w}_h(\tau)|^2 d\tau \le d_0^2 \int_{-\infty}^{+\infty} \|\hat{w}_h(\tau)\|_h^2 d\tau = d_0^2 \int_0^T \|w_h(t)\|_h^2 dt \le \text{Const.} \tag{5.85}$$

The lemma follows. ▪

5.7 Convergence of the Numerical Schemes.

Our aim is to prove the convergence of Schemes 5.1 to 5.4, in some sense which will be described later on. We first state the consistency and compactness properties on the discretized data which are required to ensure the convergence. We then state and prove the convergence results.

5.7.1 Consistency and Compactness Hypotheses.

The subsequent hypotheses will be easier to state after this lemma:

Lemma 5.7.

Let $\{(V_h, p_h, r_h)_{h\in\mathcal{H}}, (\overline{\omega}, F)\}$ *denote a stable and convergent external approximation of* V. *Let us assume that for some sequence* $h'\to 0$, *a family of functions*

$$u_{h'}: [0,T] \mapsto V_{h'},$$

<u>satisfies</u>

(5.86) $$p_{h'}u_{h'} \longrightarrow \phi \quad \text{in} \quad L^2(0,T;F) \quad \text{weakly, as} \quad h' \longrightarrow 0.$$

<u>Then for almost every</u> t, $\phi(t) = \overline{\omega}u(t)$, <u>and</u>

(5.87) $$u = \overline{\omega}^{-1}\phi \in L^2(0,T;V).$$

<u>Proof</u>.

Let us denote by θ some function in $\mathscr{D}((0,T))$. It is easily checked that

$$\int_0^T p_{h'}u_{h'}(t)\theta(t)dt \longrightarrow \int_0^T \phi(t)\theta(t)dt,$$

as h' goes to 0. But condition (C2) of Definition I.3.6 [(1)] shows us that, under these circumstances,

$$\int_0^T \phi(t)\theta(t)dt \in \overline{\omega}V.$$

Since by definition, $\overline{\omega}V$ is isomorphic to V, $\overline{\omega}V$ is a closed subspace of F; taking now a sequence of functions θ_ε converging to the Dirac distribution at the point s, $s \in (0,T)$, we see that for almost every s in $[0,T]$,

$$\int_0^T \phi(t)\theta_\varepsilon(t)dt \longrightarrow \phi(s) \quad \text{in} \quad F,$$

and hence

$$\phi(s) \in \overline{\omega}V \quad \text{a.e.}$$

Then, as $\overline{\omega}$ is an isomorphism, $\overline{\omega}^{-1}\phi$ is defined and belongs to $L^2(0,T;V)$. ∎

The preceding lemma was quite general, but, in the present situation we assumed that

(5.88) $$V_h \subset \mathbb{L}^2(\Omega), \quad \forall h;$$

therefore it can happen that for some sequence $h' \longrightarrow 0$,

$$u_{h'} \longrightarrow u \quad \text{in} \quad L^2(0,T;\mathbb{L}^2(\Omega)) \quad \text{weakly},$$

$$p_{h'}u_{h'} \longrightarrow \phi \quad \text{in} \quad L^2(0,T;F) \quad \text{weakly}.$$

By Lemma 5.6, $\phi = \overline{\omega}u_*, u_* \in L^2(0,T;V)$. Without further information we cannot assert that $u = u_*$. But, actually this will be proved for each approximation considered:

---

(1) Definition of the approximation of a normed space in the general frame.

(5.89) <u>Let</u> $u_{h'}$ <u>be a sequence of functions from</u> $[0,T]$ <u>into</u> $V_{h'}$ <u>such that, as</u> $h' \longrightarrow 0$,

$$u_{h'} \longrightarrow u \text{ in } L^2(0,T;\mathbb{L}^2(\Omega)) \text{ weakly,}$$

$$p_{h'}u_{h'} \longrightarrow \phi \text{ in } L^2(0,T;F) \text{ weakly.}$$

<u>Then</u>

$$u \in L^2(0,T;V) \text{ and } \phi = \overline{\omega}u.$$

Besides (5.89), the consistency hypotheses are now the following [1]:

(5.90) <u>Let</u> $v_{h'}, w_{h'}$, <u>be two sequences of functions from</u> $[0,T]$ <u>into</u> $V_{h'}$, <u>such that, as</u> $h' \longrightarrow 0$,

$$p_{h'}v_{h'} \longrightarrow \overline{\omega}v \text{ in } L^2(0,T;F) \text{ weakly,}$$

$$p_{h'}w_{h'} \longrightarrow \overline{\omega}w \text{ in } L^2(0,T;F) \text{ (strongly).}$$

<u>Then, as</u> $h' \longrightarrow 0$,

$$\int_0^T ((v_{h'}(t), w_{h'}(t)))_{h'} dt \longrightarrow \int_0^T ((v(t), w(t)))dt.$$

(5.91) <u>Let</u> $u_{h'}, v_{h'}$, <u>be two sequences of functions from</u> $[0,T]$ <u>into</u> $V_{h'}$ <u>such that, as</u> $h' \longrightarrow 0$,

$$p_{h'}u_{h'} \longrightarrow \overline{\omega}u \text{ in } L^2(0,T;F) \text{ weakly,}$$

$$u_{h'} \longrightarrow u \text{ in } \mathbb{L}^2(Q) \text{ strongly, } Q = \Omega \times (0,T),$$

<u>and</u>

$$p_{h'}v_{h'} \longrightarrow \overline{\omega}v \text{ in } L^2(0,T;F) \text{ weakly.}$$

<u>Then as</u> $h' \longrightarrow 0$,

$$\int_0^T b_{h'}(u_{h'}(t), v_{h'}(t), \psi(t)r_{h'}w_{h'})dt \longrightarrow \int_0^T b(u(t), v(t), \psi(t)w)dt,$$

<u>for each scalar valued function</u> $\psi \in L^\infty(0,T)$ <u>and each</u> $w \in \mathcal{V}$.
<u>If moreover a sequence of functions</u> $\psi_{k'}$ <u>is given with</u>

$$\psi_{k'} \longrightarrow \psi \text{ in } L^\infty(0,T), \text{ as } k' \longrightarrow 0,$$

[1] Compare with the stationary case, (3.7),(3.8),Chapter I; (3.4),(3.5),(3.7), Chapter II.

then, as $h'\longrightarrow 0$, $k'\longrightarrow 0$,

$$\int_0^T b_{h'}(u_{h'}(t),v_{h'}(t),\psi_{k'}(t)r_{h'}w_{h'})dt \longrightarrow \int_0^T b(u(t),v(t),\psi(t)w)dt.$$ ■

In order to prove strong convergence results as those needed in (5.91) ($u_{h'}\longrightarrow u$ in $\mathbb{L}^2(Q)$ strongly) we will assume the following:

(5.92) Let $v_{h'}$ denote a sequence of functions from $\mathbb{R}$ into $V_{h'}$, with support in $[0,T]$ and such that

$$\int_0^T \|v_{h'}(t)\|^2_{h'}dt \le \text{Const.},$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{v}_{h'}(\tau)|^2 d\tau \le \text{Const.}, \quad \text{for some} \quad 0 < \gamma,$$

where $\hat{v}_h$ is the Fourier transform of v_h.

Then such a sequence $v_{h'}$ is relatively compact in $\mathbb{L}^2(Q)$.

In particular, one can extract from $v_{h'}$ a subsequence (still denoted $v_{h'}$) with

$p_{h'}v_{h'}\longrightarrow \bar{\omega}v$ in $L^2(0,T;F)$ weakly,

$v_{h'}\longrightarrow v$ in $\mathbb{L}^2(Q)$ strongly. ■

5.7.2 The Convergence Theorems.

The convergence theorems are stated differently according to the dimension of the space ($n = 2$ or 3) and to the scheme considered.

We recall that we associated with the elements u_h^m a function u_h

$$u_h: [0,T]\longrightarrow V_h,$$

defined slightly differently for the four schemes (see (5.36), (5.55), (5.75)) [(1)]

(5.93) for $(m-1)k \le t < mk$ $(m = 1,\cdots,N)$,

$$u_h(t) = \begin{cases} u_h^m & \text{(Schemes 5.1 and 5.3)} \\ \frac{1}{2}(u_h^m + u_h^{m-1}) & \text{(Scheme 5.2)} \\ u_h^{m-1} & \text{(Scheme 5.4).} \end{cases}$$

(1) We emphasize that u_h depends on h and k; only for reasons of simplicity have we denoted this function by u_h instead of u_{hk}.

We have:

Theorem 5.4.

The dimension of the space is $n = 2$ and the assumptions are (5.1) to (5.7), (5.9), (5.10), (5.41), and (5.89) to (5.92). We denote by u the unique solution of Problem 3.1.

The following convergence results hold, as h and $k \longrightarrow 0$,

(5.94) $u_h \longrightarrow u$ in $\mathbb{L}^2(Q)$ strongly, $L^\infty(0,T;\mathbb{L}^2(\Omega))$ weak-star,

(5.95) $$p_h u_h \longrightarrow \overline{\omega}u \text{ in } L^2(0,T;F) \text{ weakly,}$$

provided:

(i) Scheme 5.1: no condition,

(ii) Scheme 5.2,

(5.96) $$kS^2(h) \longrightarrow 0,$$

(iii) Scheme 5.3: (5.43) is satisfied,

(iv) Scheme 5.4: (5.56)-(5.57) are satisfied.

Remark 5.3.

(i) For Schemes 5.1 and 5.2 it can be proved, without any further hypotheses, that

(5.97) $$p_h u_h \longrightarrow \overline{\omega}u \text{ in } L^2(0,T;F) \text{ strongly, as } h,k \longrightarrow 0.$$

(ii) The same results hold for the other schemes provided we assume moreover that

(5.98) $$kS_1^2(h) \longrightarrow 0 \text{ and } kS^2(h) \longrightarrow 0 \text{ (Schemes 5.3 and 5.4).}$$

(iii) The hypothesis (5.96) used in the proof of the convergence of Scheme 5.2 is probably unnecessary since the scheme is unconditionally $L^2(0,T;F)$ and $L^\infty(0,T;\mathbb{L}^2(\Omega))$ stable. ∎

Theorem 5.5.

The dimension of the space is $n = 3$ and, otherwise, the assumptions are the same as in Theorem 5.4.

Then, there exists some sequence $h', k' \longrightarrow 0$, such that

(5.99) $$u_{h'} \longrightarrow u \text{ in } \mathbb{L}^2(Q) \text{ strongly,}$$

(5.100) $$u_{h'} \longrightarrow u \text{ in } L^\infty(0,T;\mathbb{L}^2(\Omega)) \text{ weak-star,}$$

(5.101) $$p_{h'} u_{h'} \longrightarrow \overline{\omega}u \text{ in } L^2(0,T;F) \text{ weakly,}$$

where u *is some solution of Problem* 3.1.

For any other sequence $h',k'\longrightarrow 0$, *such that the convergences* (5.99) *to* (5.101) *hold,* u *must be some solution of Problem* 3.1. ∎

Remark 5.4.

We are not able to prove that the whole sequence converges due to lack of uniqueness of solution for Problem 3.1.

We also cannot prove strong convergence in $L^2(0,T;F)$ due to lack of an energy equality for the exact problem (Problem 3.1) (for $n = 3$ we only have an energy inequality; see Remark 4.1).

The two theorems are proved in the remainder of this Section 5.7; we will prove Theorem 5.4 with full details for Scheme 5.1 (including (5.97)) and in the other cases we will only sketch the proofs which are actually very similar.

5.7.3 *Proof of Theorem 5.4* (*Scheme* 5.1).

According to the stability theorem (Theorem 5.1), and to (5.89), there exists a sequence $h',k'\longrightarrow 0$, such that

$$(5.102)\qquad \begin{aligned} &u_{h'}\longrightarrow u \quad \text{in} \quad L^\infty(0,T;\mathbb{L}^2(\Omega)) \quad \text{weak-star,}\\ &p_{h'}u_{h'}\longrightarrow \bar{\omega}u \quad \text{in} \quad L^2(0,T;F) \quad \text{weakly,} \end{aligned}$$

for some u in $L^2(0,T;V)\cap L^\infty(0,T;H)$.

Let us consider the piecewise linear function w_h introduced in Section 5.6 (see (5.78)). By Lemma 5.6 and the estimations on the u_h^m, we have

$$|w_h|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} \le \text{Const.}\ ,$$

$$\|p_h w_h\|_{L^2(0,T;F)} \le \text{Const.}\ ,$$

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}|\hat{w}_h(\tau)|^2 d\tau \le \text{Const.}\ .$$

Hence, according to (5.92), the sequence $h',k'\longrightarrow 0$ can be chosen so that

$$(5.103)\qquad \begin{aligned} &w_{h'}\longrightarrow w \quad \text{in} \quad L^\infty(0,T;\mathbb{L}^2(\Omega)) \quad \text{weak-star ,}\\ &w_{h'}\longrightarrow w \quad \text{in} \quad L^2(0,T;\mathbb{L}^2(\Omega)) \quad \text{strongly,}\\ &p_{h'}w_{h'}\longrightarrow \bar{\omega}w \quad \text{in} \quad L^2(0,T;F) \quad \text{weakly,} \end{aligned}$$

where $w\in L^2(0,T;V)\cap L^\infty(0,T;H)$.

We now observe that:

Lemma 5.8.

(5.104) $u_h - w_h \longrightarrow 0$ in $L^2(0,T;\mathbb{L}^2(\Omega))$ strongly as h and $k \longrightarrow 0$.

Thus

$$w = u \tag{5.105}$$

and

$$u_{h'} \longrightarrow u \text{ in } L^2(0,T;\mathbb{L}^2(\Omega)) \text{ strongly,} \tag{5.106}$$

as h' and $k' \longrightarrow 0$.

Proof.

Exactly as in Lemma 4.8, we check that

$$|u_h - w_h|_{L^2(0,T;\mathbb{L}^2(\Omega))} = \sqrt{\frac{k}{3}}\left(\sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2\right)^{\frac{1}{2}}. \tag{5.107}$$

Then (5.104) follows from the majoration (5.22); (5.105), (5.106) are obvious consequences of (5.102), (5.103), and (5.104).

The next point is to prove that u is a solution of Problem 3.1.

Lemma 5.9.

The function u appearing in (5.102), (5.103), (5.105) is a solution of Problem 3.1.

Proof.

We easily interpret (5.12) in the following way:

$$\frac{d}{dt}(w_h(t),v_h) + \nu((u_h(t),v_h))_h + b_h(u_h(t-k),u_h(t),v_h) \tag{5.108}$$

$$= (f_k(t),v_h), \quad \forall t \in [0,\underline{T}], \forall v_h \in V_h,.$$

where

$$f_k(t) = f^m, \quad (m-1)k \le t < mk. \tag{5.109}$$

Let v be any element in $\mathcal{V}$ and let us take $v_h = r_h v$ in (5.108). Let ψ be a continuously differentiable scalar function on $[0,T]$, with

$$\psi(T) = 0. \tag{5.110}$$

We multiply (5.108) (where $v_h = r_h v$) by $\psi(t)$, integrate in t, and integrate the first term by parts to get:

$$-\int_0^T (w_h(t),\psi'(t)r_hv)dt + \nu\int_0^T ((u_h(t),\psi(t)r_hv))_h dt$$

(5.111) $$+\int_0^T b_h(u_h(t-k),u_h(t),\psi(t)r_hv)dt$$

$$= (u_h^0,r_hv)\psi(0) + \int_0^T (f_k(t),\psi(t)r_hv)dt.$$

We now pass to the limit in (5.111) with the sequence $h',k'\longrightarrow 0$ using essentially (5.90), (5.91), (5.102), (5.103), and Lemma 5.8; we recall also that

(5.112) $r_hv\longrightarrow\bar{\omega}v$ in F (strongly),

(5.113) $u_h^0\longrightarrow u_0$ in $\mathbb{L}^2(\Omega)$ (strongly) [1],

(5.114) $f_k\longrightarrow f$ in $L^2(0,T;\mathbb{L}^2(\Omega))$ (see Lemma 4.9).

We find in the limit

$$-\int_0^T (u(t),\psi'(t)v)dt + \nu\int_0^T ((u(t),\psi(t)v))dt + \int_0^T b(u(t),u(t),\psi v)dt$$

(5.115)

$$= (u_0,v)\psi(0) + \int_0^T (f(t),v)\psi(t)dt.$$

We infer from this equality that u is a solution of Problem 3.1, exactly as we did in the proof of Theorem 3.1 after (3.43). ∎

Since the solution of Problem 3.1 is unique (see Theorem 3.2), a contradiction argument that we have already used very often shows that

(5.116) The convergences (5.102), (5.103) hold for the whole family $h,k\longrightarrow 0$.

This completes the proof of Theorem 5.4. ∎

5.7.4 Proof of (5.97).

For the sake of completeness we will also prove (5.97). In order to prove this point we need a preliminary result which is quite general and interesting by itself.

---

(1) We recall the proof of (5.113); due to (5.11) it suffices to prove this for $u_0\in\mathcal{V}$ and in this case

$$|u_h^0-u_0| \le |r_hu_0-u_0| \le \|p_hr_hu_0-\bar{\omega}u_0\|_F\longrightarrow 0.$$

Lemma 5.10.

Let $\{(V_h, p_h, r_h)_{\mathcal{H}}, (\bar{\omega}, F)\}$ be a stable and convergent external approximation of V. For a given element v of $L^2(0,T;V)$, one can define for each $h \in \mathcal{H}$ a function $v_h^+ \in L^2(0,T;V_h)$ such that

$$p_h v_h^+ \longrightarrow \bar{\omega} v \text{ in } L^2(0,T;F) \text{ as } h \longrightarrow 0.$$

Proof.

The proof is essentially the same as that of Proposition I.3.1.

The result is obvious if v is a step function; since the step functions are dense in $L^2(0,T;V)$, the result follows in the general case by an argument of double passage to the limit as in Proposition I.3.1.

Lemma 5.11.

The dimension of the space is $n = 2$; then for Scheme 5.1

$$p_h u_h \longrightarrow \bar{\omega} u \text{ in } L^2(0,T;F) \text{ (strongly)}, \tag{5.117}$$

as h and $k \longrightarrow 0$.

Proof.

We consider the expression

$$X_h = |u_h^N - u(T)|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2\nu \int_0^T \|u_h(t) - u_h^+(t)\|_h^2 dt,$$

with u_h^+ defined as in Lemma 5.10.

According to (5.21) (Lemma 5.1),

$$|u_h^N| \leq \text{Const.};$$

hence there exists a sequence $h', k' \longrightarrow 0$, with

$$u_{h'}^N \longrightarrow \chi \text{ in } \mathbb{L}^2(\Omega) \text{ weakly.} \tag{5.118}$$

We temporarily assume that

$$(\chi - u(T), v) = 0, \quad \forall v \in H. \tag{5.119}$$

We then prove that

$$X_{h'} \longrightarrow 0.$$

Actually, we consider

$$X_h = X_h^1 + X_h^2 + X_h^3,$$

where

$$X_h^1 = |u(T)|^2 + 2\nu \int_0^T \|u_h^+(t)\|_h^2 dt \quad \to |u(T)|^2 + 2\nu \int_0^T \|u(t)\|^2 dt$$

(by Lemma 5.10 and (5.90)),

$$X_h^2 = -2(u_h^N, u(T)) - 4\nu \int_0^T ((u_h(t), u_h^+(t))_h \longrightarrow -2|u(T)|^2 - 4\nu \int_0^T \|u(t)\|^2 dt$$

(by Lemma 5.10, (5.90) and (5.118)-(5.119), we recall that $u(T) \in H$), and

(5.120)
$$X_h^3 = |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2\nu \int_0^T \|u_h(t)\|_h^2 dt$$
$$= |u_h^N|^2 + \sum_{m=1}^{N} |u_h^m - u_h^{m-1}|^2 + 2k \sum_{m=1}^{N} \|u_h^m\|_h^2.$$

By summation of the equalities (5.26) for $m = 1, \cdots, N$, we get

(5.121)
$$X_h^3 = |u_h^0|^2 + 2k \sum_{m=1}^{N} (f^m, u_h^m)$$
$$= |u_h^0|^2 + 2 \int_0^T (f_k(t), u_h(t) dt.$$

It is then clear that

$$X_h^3 \longrightarrow |u_0|^2 + 2 \int_0^T (f(t), u(t)) dt, \quad \text{as} \quad h, k \longrightarrow 0.$$

Hence

(5.122)
$$X_{h'} \longrightarrow |u_0|^2 + 2 \int_0^T (f(t), u(t)) dt - |u(T)|^2 - 2\nu \int_0^T \|u(t)\|^2 dt,$$

and this limit is 0 due to (4.55).

By a contradiction argument we show as well that the whole family X_h converges to 0:

$$X_h \longrightarrow 0, \quad \text{as} \quad h, k \longrightarrow 0.$$

In particular

$$\int_0^T \|u_h(t) - u_h^+(t)\|_h^2 dt \longrightarrow 0,$$

and

$$\int_0^T \|p_h u_h(t) - \bar{\omega} u(t)\|_F^2 dt \le c\{\int_0^T \|u_h(t) - u_h^+(t)\|_h^2 dt + \int_0^T \|p_h u_h^+(t) - \bar{\omega} u(t)\|_F^2 dt\} \longrightarrow 0$$

and (5.117) follows.

It remains to prove (5.119).

By summation of (5.12) for $m = 1,\cdots,N$, we get

$$(u_h^N-u_h^0,v_h) + k\nu \sum_{m=1}^{N} ((u_h^m,v_h))_h + k \sum_{m=1}^{N} b_h(u_h^{m-1},u_h^m,v_h) = k \sum_{m=1}^{N} (f^m,v_h).$$

Taking $v_h = r_h v$, $v \in \mathcal{V}$, we easily pass to the limit and get

$$(\chi-u_0,v) + \nu \int_0^T ((u(t),v))dt + \int_0^T b(u(t),u(t),v)dt = \int_0^T (f(t),v)dt, \quad \forall v \in \mathcal{V}.$$

But since we deduce by integration of (3.13) a similar equation with χ replaced by $u(T)$, we conclude that

$$(\chi-u(T),v) = 0, \quad \forall v \in \mathcal{V},$$

which implies (5.119) by density.

The proof of Lemma 5.11 is complete. ∎

5.7.5 <u>Proof of Theorems 5.4 and 5.5 (other cases)</u>.

For Schemes 5.2, 5.3, 5.4 and in the case $n = 2$, the proof is very similar to the above one, using the corresponding a priori estimates.

For Scheme 5.2, we introduced the condition (5.96) as a sufficient condition to prove (5.104); more precisely, in this case, the analog of (5.104) is a consequence of (5.77), (5.107) and (5.96). For Schemes 5.3, 5.4, the stability conditions (5.43), (5.56), (5.57) merely ensure that u_h and $p_h u_h$ remain bounded in the suitable spaces.

For the proof of (5.97), the condition (5.98) appears as follows:

- For Scheme 5.4 the "natural" expression similar to X_h in Lemma 5.11 is

$$Y_h = |u_h^N-u(T)|^2 - \sum_{m=1}^{N} |u_h^m-u_h^{m-1}|^2 + 2\nu \int_0^T \|u_h(t)-u_h^+(t)\|_h^2 dt;$$

in order to deduce (5.97) from the fact that $Y_h \longrightarrow 0$, it suffices that

$$\sum_{m=1}^{N} |u_h^m-u_h^{m-1}|^2 \longrightarrow 0,$$

and this is a consequence of (5.98).

- For Scheme 5.3, we consider the same expression X_h, but due to some terms involving b_h (see (5.47)), the expression X_h^3 is not as simple as (5.121); (5.98) shows that the supplementary terms of X_h^3 involving b_h converge to 0.

If $n = 3$, we observe first that the stability conditions are not explicitly mentioned in the statement of Theorem 5.5. Actually it is with the help of these conditions, and the a priori estimates that they imply, that we prove the existence of a subsequence $h',k' \longrightarrow 0$, such that (5.99) - (5.101) hold ; that u is a solution of Problem 3.1 is proved exactly as before. ∎

COMMENTS AND BIBLIOGRAPHY

Chapter I

Section 1 contains a preliminary study of the basic spaces V and H; the trace theorem is proved by the methods of J.L. Lions-E. Magenes, see ref. [1]. The characterization of $H^{\perp}$ was given by O.A. Ladyzhenskaya [1] for $n = 3$; the proof presented here is slightly simplified and extended to all dimensions.

We have not given any systematic study nor review concerning the Sobolev spaces. We restrict ourselves to recalling properties of these spaces when needed (Sections 1.1 of Chapters I and II in particular). As mentioned in the text, the reader is referred for proofs and further material to J.L. Lions [1], J.L. Lions and E. Magenes [1], J. Nečas [1], L. Sobolev [1], among other references.

The variational formulation of Stokes equations was first introduced (in the general frame of the nonlinear evolution case) by J. Leray [1], [2], [3], for the study of weak or turbulent solutions of the Navier-Stokes equations. The existence of a solution of the Stokes variational problem is easily obtained by the Classical Projection Theorem, whose proof is recalled for the sake of completeness. The study of the nonvariational Stokes problem, and the regularity of solutions is based on the paper of Cattabriga [1] (if $n = 3$) and on the paper of Agmon-Douglis-Nirenberg [1] on elliptic systems (any dimension); the results are recalled without proofs.

The concept of approximation of a normed space and variational problem was studied in particular by J.P. Aubin [1], J. Céa [1]; the presentation followed here is that of R. Temam [8]. The discrete Poincaré Inequality (Section 3.3) and the approximation of V by finite differences are in J. Céa [1]. The approximation of V by conforming finite elements was first studied and used by M. Fortin [2]; our description of the approximations (APX 2), (APX 3), (APX 4) (conforming finite elements), follows essentially M. Fortin [2]. In this reference one can also find many results of computations using this type of discretization; see also J.P. Thomasset [1], Borsenberger [1]. The material related to the nonconforming finite elements for the approximation of divergence free vector functions is due to Crouzeix, R. Glowinsky, P.A. Raviart, and the author. Other aspects of the subject (nonconforming finite elements of higher degree and more refined error estimates) can be found in Crouzeix and P.A. Raviart [1]; for numerical experiments, see Borsenberger [1].

For other applications of finite elements in fluid mechanics, see J.E. Hirsh [1]. Concerning the general theory of finite elements, let us mention the synthesis works of I. Babushka and A.K. Aziz [1], P.A. Raviart [2], G. Strang [1], and the proceedings edited by I. Babushka [1]. For more references on finite elements (in general situations) the reader is referred to the bibliography of these works. The description of finite element methods given here is almost completely self-contained; we only assume a few specific results whose proofs would necessitate an introduction of tools quite remote from our previous work.

After discretization of the Stokes problem, we have to solve a finite dimensional linear problem where the unknown is an element u_h of a finite dimensional space V_h. There are then two possibilities:

a) either the space V_h possesses a natural and simple basis, such that the problem reduces to a linear system with a sparse matrix for the components of u_h in this basis; in this case we solve the problem by resolution of this linear system.

b) or, if not, the finite dimensional problem is not so simple to solve (ill-conditioned or non-sparse matrix), even if it possesses a unique solution. In this case, appropriate algorithms must be introduced in order to solve these problems; this is the purpose of Section 5.

The algorithms described in Section 5 were introduced in the frame of optimization theory and economics in Arrow-Hurwicz-Uzawa [1]; the application of these procedures to problems of hydrodynamics is studied in J. Céa and R. Glowinsky [1], M. Fortin [2], M. Fortin, R. Peyret, and R. Temam [1]. See in D. Bégis [1], M. Fortin [2], an experimental investigation of the optimal choice of the parameter ρ (or ρ and α); a theoretical resolution of this problem in a very particular case is given in Crouzeix [2].

The approximation of incompressible fluids by slightly compressible fluids, as in Section 6, has been studied by J.L. Lions [4] and R. Temam [2]. The full asymptotic development of u_ε given here is new; for further investigations of this point see M.C. Pélissier [1].

Chapter II

Section 1 develops a few standard results concerning existence and uniqueness of solutions of the nonlinear stationary Navier-Stokes equations. We follow essentially O.A. Ladyzhenskaya [1] and J.L. Lions [2]. A more complete discussion of the regularity of solutions and the theory of hydrodynamical potentials can be found in O.A. Ladyzhenskaya [1]; for regularity, see also H. Fujita [1]. The stationary Navier-Stokes equations in an unbounded domain has been studied by R. Finn [1] – [5]; R. Finn and D.R. Smith [1], [2].

Section 2 gives discrete Sobolev inequalities and compactness theorems. The principle of the proofs in the case of finite differences parallels the corresponding proofs in the continuous case (see, for instance, J.L. Lions [1], J.L. Lions-E. Magenes [1]). The proof of discrete Sobolev inequalities has never been published before, the proof of the discrete compactness theorem can be found in P.A. Raviart [1]. For conforming finite elements the proofs are much simpler: in particular, for the discrete compactness theorem, the problem is reduced by a simple device to the continuous case; these results are new.

The discussion of the discretization of the stationary Navier-Stokes equations follows the principles developed in Chapter I. The general convergence theorem is similar to that of Chapter I and the same type of discretizations of V are considered; the differences lie in the lack of uniqueness of solutions of the exact problem. The numerical algorithms of Subsection 3.3 has been introduced and tested in Fortin-Peyret-Temam [1].

We did not establish here nonuniqueness results for the stationary Navier-Stokes equations (1). The nonuniqueness of stationary solutions of the Navier-Stokes and related equations has been proved with the aid of bifurcation theory and topological degree theory. The main results in this direction are due to P.H. Rabinowitz [2] and W. Velte [1], [2]. In [2], W. Velte proves the nonuniqueness of solutions of the Taylor problem and the situation is very similar to the problems for which existence is proved in Section 1, although not identical. For other applications of bifurcation theory see in particular, J.B. Keller and S. Antman [1], P.H. Rabinowitz [1], [3].

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(1) This proof will be included in R. Temam [9].

Chapter III

The existence and uniqueness results for the linearized Navier-Stokes equations (Section 1) are a special case of general results of existence and uniqueness of solutions of linear variational equations (see for instance, J.L. Lions-E. Magenes [1], vol. 2). In order to be self-contained, we have given an elementary proof of some technical results which usually are established as easy consequences of deeper results {Lemma 1.1 which is more natural in the frame of vector valued distribution theory (L. Schwartz [2]) or Lemma 1.2 which can be proved by interpolation methods (J.L. Lions-E. Magenes [1])}.

Theorem 2.1 is one of the standard compactness theorems used in the theory of nonlinear evolution equations. Other compactness theorems are proved and used in J.L. Lions [2].

The existence and uniqueness results related to the nonlinear Navier-Stokes equations and given in Sections 3 and 4 are now classical and are in the line of the early works of J. Leray [1], [2], [3]; see E. Höpf [1], [2], O.A. Ladyzhenskaya [1], J.L. Lions [2], [3], J.L. Lions and G. Prodi [1], Serrin [3]. Further results on the regularity of solutions and the study of existence of classically differentiable solutions of the Navier-Stokes equations can be found in the 2nd edition of O.A. Ladyzhenskaya [1]. For the analyticity of solutions, see C. Foias and G. Prodi [1], H. Fujita and K. Masuda [1], C. Kahane [1], K. Masuda [1], J. Serrin [3].

Let us mention also two recent and completely different approaches to the existence and uniqueness theory that we did not treat here. The first one is that of E.B. Fabes, B.F. Jones, and N.M. Riviere [1] based on singular integral operator methods and giving existence and uniqueness results in L^p spaces. The other one is the method of D.G. Ebin and J. Marsden [1] connecting the Navier-Stokes initial value problem with the geodesics of a Riemannian manifold and thus using the methods of global analysis.

The material of Section 5 containing a discussion of the stability and convergence of simple discretization schemes for the Navier-Stokes equations is essentially new; a similar study for different equations or different schemes was done in R. Temam [2], [3], [4]. Stability and convergence of some unconditionally stable one step schemes are given in O.A. Ladyzhenskaya [5].

A more complete treatment of the numerical analysis of the Navier-Stokes equations is included in R. Temam [9] (in particular, the practical application of the schemes in Section 5, the fractional step or projection method, and the artificial compressibility method). Among many other references on the subject, see A.J. Chorin [1], [2], [3], C.K. Chu and G. Johansson [1], C.K. Chu, K.W. Morton, and K.V. Roberts [1], M. Fortin, R. Peyret, and R. Temam [1], M. Fortin [1], M. Fortin and R. Temem [1], O.A. Ladyzhenskaya and V.I. Rivkind [1], G. Marshall [1], [2], C.S. Peskin [1], [2].

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