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TOPICS IN NONLINEAR ANALYSIS

(retirage 1982)

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These notes represent most of the material covered in a graduate course taught at the University of Wisconsin, Madison in 1974-75.

Although some new techniques have since appeared (as homogenization or compensated compactness which I lectured more recently on) these notes have conserved the interesting property that some of the results are not written elsewhere. As these results are improvements of earlier ones that I mainly learned from J.L. LIONS in similar lectures I hope that these notes will provide an up to date basis so the reader can focus his (or her) efforts on more recent developments.

Orsay, November 1978

Contents :

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING COMPACTNESS METHOD

L. Tartar

I. Preliminaries: Functional Analysis

Distributions.

Let $\Omega^{\text{open}} \subseteq \mathbb{R}^N$. $\mathcal{D}(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω. $\mathcal{D}_K(\Omega) = \{f \in \mathcal{D}(\Omega): \text{supp}(f) \subset K\}$ if $K^{\text{compact}} \subseteq \Omega$. Thus $\mathcal{D}(\Omega) = \bigcup \mathcal{D}_K(\Omega)$. The topology one puts on $\mathcal{D}_K(\Omega)$ makes it a Fréchet space. It is not a Banach space.

A distribution T is a continuous linear functional on $\mathcal{D}(\Omega)$, in the following sense. If $\varphi_n \in \mathcal{D}_K(\Omega)$, i.e. $\varphi_n \to \varphi$ uniformly on K together with all derivatives, then $T(\varphi_n) \to T(\varphi)$.

$T(\varphi)$ is also denoted $\langle T, \varphi \rangle$.

Example: 1) Let $f \in L^1_{loc}(\varphi)$. Define

$$\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx \qquad \text{for any } \varphi \in \mathcal{D}(\Omega).$$

2) δ or δ_0 Dirac measure.

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad \text{for any } \varphi \in \mathcal{D}(\Omega).$$

$\langle \delta, \varphi \rangle$ cannot be given by $\langle f, \varphi \rangle$ for f a function.

$\mathcal{D}'(\Omega)$ denotes the class of distributions on $\mathcal{D}(\Omega)$.

Distributions can always be differentiated by the following definition:

$$\langle \frac{\partial T}{\partial x_i}, \varphi \rangle = -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle$$

The derivative is linear and continuous.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

Definition: If $\alpha \in C^{\infty}$, we can define the product of α and $T \in \mathcal{D}'(\Omega)$: $\langle \alpha T, \varphi \rangle = \langle T, \alpha\varphi \rangle$. Then

1) $\alpha(\beta T) = (\alpha\beta)T$,

2) $\frac{\partial}{\partial x_i}(\alpha T) = \frac{\partial \alpha}{\partial x_i} T + \alpha \frac{\partial T}{\partial x_i}$,

3) It is impossible to multiply two arbitrary distributions. For example δ and δ. (There is no multiplication which can be defined on $\mathcal{D}'(\Omega)$ which is associative.)

Example: (of a derivative of a distribution)

The Heaviside function is defined on $\mathbb{R}$ by $Y(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$. Then $\frac{d}{dx} Y = \delta$.

Convolution: If f, g are defined on $\mathbb{R}^N$, $f * g$ (the convolution of f and g) is defined by

$$[f*g](x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy$$

when the integral makes sense.

For distributions in general we have no convolutions, but for distributions with compact support we can extend convolution to be defined.

Definition: $T \in \mathcal{D}'(\Omega)$ has compact support $K \subset \Omega$ if $\langle T, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$: $\text{supp}(\varphi) \cap K = \phi$.

If $S, T \in \mathcal{D}'(\Omega)$ and T has compact support, define $\langle S * T, \varphi \rangle = \langle S_x, \langle T_y, \varphi(x+y) \rangle \rangle$. Here $\langle T_y, \varphi(x+y) \rangle$ is a function of x.

Properties: 1) Commutative

2) Associative when well-defined (If $S_1, S_2, S_3 \in \mathcal{D}'(\Omega)$ and two have compact support, $S_1 * (S_2 * S_3) = (S_1 * S_2) * S_3$.

3) $\delta * S = S$

4) $\frac{\partial}{\partial x_i} * S = \frac{\partial S}{\partial x_i}$

$(\frac{\partial}{\partial x_i} = \frac{\partial \delta}{\partial x_i} \; ; \; \langle \frac{\partial}{\partial x_i}, \varphi \rangle = - \frac{\partial \varphi}{\partial x_i}(0)),$

5) $\frac{\partial}{\partial x_i}(S * T) = \frac{\partial S}{\partial x_i} * T = S * \frac{\partial T}{* x_i}$.

<u>Corollary:</u> If S is a distribution, $T \in \mathcal{D}(\mathbb{R}^N)$, then $S * T$ is a C^∞ function.

6) $S * T$ is continuous (in some sense) ($S_n \to S$, $T_n \to T$ somehow, then $S_n * T_n \to S * T$.)

We will need the following

<u>Result:</u> Let $\rho \in \mathcal{D}(\mathbb{R})$, $\rho \geq 0$, $\int \rho = 1$. Define $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(\frac{x}{\epsilon})$. Then $\rho_\epsilon \to \delta$ as $\varepsilon \to 0$, and for all $S \in \mathcal{D}'(\Omega)$, $S * \rho_\epsilon \to S$ as $\epsilon \to 0$.

<u>Fourier Transform:</u>

Let $u : \mathbb{R}^N \to \mathbb{C}$. Define $\mathcal{F} u(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \langle x, \xi \rangle} u(x) dx$ for $\xi \in \mathbb{R}^N$. $\langle \xi, x \rangle = \sum_{j=1}^{n} x_i \xi_i$. Assume the integral makes sense. Define

$\bar{\mathcal{F}}[u] = \int_{\mathbb{R}^N} u(x) \, e^{2\pi i \langle x, \xi \rangle} dx$.

<u>Definition:</u> $S(\mathbb{R}^N)$ = space of C^∞ rapidly decreasing functions; that is $u \in S(\mathbb{R}^N)$ if u and all its derivatives converge to 0, as $|x| \to \infty$, more rapidly than any power of $\frac{1}{|x|}$. In other words, if given p a natural number, $|x|^p Du(x) \to 0$ as $|x| \to \infty$ for any derivative D.

Then $\mathcal{D}(\mathbb{R}^n) \subseteq S(\mathbb{R}^N)$. $S(\mathbb{R}^N)$ is a Frechet space; its dual space is $S'(\mathbb{R}^N)$.

Theorem: $\mathfrak{F}$ is well-defined on $S(\mathbb{R}^N)$, and $\mathfrak{F}$ takes S into S. Furthermore, 1) $\mathfrak{F}[\frac{\partial u}{\partial x_i}] = 2\pi i \xi_i \mathfrak{F}[u]$,

2) $\frac{\partial}{\partial \xi_i} \mathfrak{F}[u] = -2\pi \mathfrak{F}[x_i u]$,

3) $\mathfrak{F}$ is an isomorphism on S with inverse $\bar{\mathfrak{F}}$.

Definition: If $T \in S'(\mathbb{R}^N)$, that is, T is a continuous linear functional on $S(\mathbb{R}^N)$, we define $\mathfrak{F}[T]$ by

$$\langle \mathfrak{F}[T], \varphi \rangle = \langle T, \mathfrak{F}[\varphi] \rangle \quad \forall \varphi \in S(\mathbb{R}^N).$$

Example: If $T = 1$, $\mathfrak{F}[1](\xi) = \delta_0$, $\mathfrak{F}[\delta_0](\xi) = 1$.

Theorem: (Bessel-Parseval) If $u \in L^2(\mathbb{R}^N)$, then $\mathfrak{F}[u] \in L^2(\mathbb{R}^N)$ and $\|\mathfrak{F}[u]\|_{L^2} = \|u\|_{L^2}$.

This theorem can be classically proven by extending from $L^1 \cap L^2$, but it can also be proven by using the distribution definition of $\mathfrak{F}$.

Sobolev Spaces.

These are spaces of functions in $L^p(\Omega)$ together with some derivatives, for some $p : 1 \leq p \leq \infty$.

Definition: $C_c(\Omega)$: This is the space of continuous functions with compact support in Ω, under one of the possible norms $\|\cdot\|_p$ $1 \leq p \leq \infty$ defined as follows:

$$\begin{cases} \|u\|_p = (\int_\Omega |u|^p dx)^{1/p} & 1 \leq p < \infty \\ \|u\|_\infty = \max_{x \in \Omega} |u(x)| \end{cases}$$

Note: $C_c(\Omega)$ is not complete under these norms. One can consider $L^p(\Omega)$ as the completion of $C_c(\Omega)$ under the norm $\|u\|_p$ for $1 \leq p < \infty$. $L^\infty(\Omega)$

is defined in the usual way.

Then there is the following result:

Proposition: The dual space of $L(\Omega)^p$ is $L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$, for $1 \leq p < \infty$.

This is not true for $p = \infty$; however, C_c is weak - * dense in L^∞. It is not strongly dense in L^∞: continuous bounded functions form a closed subspace of L^∞.

The $L^p(\Omega)$ are Banach spaces for $1 \leq p \leq \infty$, and, for $1 < p < \infty$, $L^p(\Omega)$ is reflexive. Thus the weak topology on $L^p(\Omega)$, for $1 < p < \infty$, has the favorable property that bounded sets are weakly relatively compact.

L^∞ is the dual space of L^1, as mentioned. The weak-* topology on L^∞ has good properties also.

Definition: u_n converges weak-* to u in $L^\infty(\Omega)$ if

$$\int u_n f \to \int uf \qquad \text{for every} \quad f \in L^1(\Omega).$$

Bounded sets in $L^\infty(\Omega)$ are weak - * relatively compact. This does not hold for $L^1(\Omega)$.

Convolutions: If $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, then $f * g \in L^r(\mathbb{R}^N)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $1 \leq p, q, r \leq \infty$.

Ordinary Products: If $f \in L^p$, $g \in L^q$, $fg \in L^s$ where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. $1 \leq p, g, s \leq \infty$.

Inequalities: With regard to the above, there are the two standard inequalities:

1. Hölder's inequality: $\|fg\|_s \leq \|f\|_p \|g\|_q$
2. Young's inequality $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

Take $p = 1$, so that $q = r$. If $f \in L^1(\mathbb{R}^N)$, $\|f\|_1 \leq 1$, then convolution with f is an operation which takes $L^q(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ and is a contraction. Hence, if we take $f = \rho_\epsilon$ as before; $\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho(\frac{x}{\epsilon})$, where $\rho \geq 0$, $\int \rho = 1$, so $\|\rho_\epsilon * g\|_p \leq \|g\|_p$ for $g \in L^p(\mathbb{R}^N)$.

In fact, $\rho_\epsilon * g \to g$ in $L^p(\mathbb{R}^N)$ as $\epsilon \to 0$, for $1 \leq p < \infty$. Also, $\rho_\epsilon * g$ is very smooth for ρ smooth. To prove that $\rho_\epsilon * g \to g$ in $L^p(\mathbb{R}^N)$, take $g \in C_c(\mathbb{R}^N)$, which is dense in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$. (Hence the result is not proven for $p = \infty$. In general, it fails.)

The general idea of a Sobolev space is a space of functions $u \in L^p(\Omega)$ with derivatives in $L^q(\Omega)$ and boundary conditions of some sort. We shall consider here the case $p = 2$, in which the spaces will be Hilbert spaces-good for Fourier transform methods. For p or $q \neq 2$, the spaces will be Banach spaces.

<u>Definition (1):</u> $u \in H^m(\mathbb{R}^N)$ if $u \in L^2(\mathbb{R}^N)$ and all derivatives of u of orders $\leq m$ are also in $L^2(\mathbb{R}^N)$. The norm on $H^m(\mathbb{R}^N)$ is defined by:

$$\|u\|^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\mathbb{R}^N)} \qquad \alpha \text{ is a multi-index.}$$

<u>Proposition:</u> 1) $H^m(\mathbb{R}^N)$ is a Hilbert space.

<u>Proof:</u> $((u,v)) = \sum_{|\alpha| \leq m} \int_{\mathbb{R}^N} D^\alpha u \, \overline{D^\alpha v}$ is an inner product which gives the above norm, so $H^m(\mathbb{R}^N)$ is a pre-Hilbert space. The space is complete, under the norm $\|\cdot\|$: Take a Cauchy sequence $\{u_k\}_{k=1}^\infty$. Then if $|\alpha| \leq m$

$$\|D^\alpha u_\ell - D^\alpha u_k\|_{L^2} \leq \|u_\ell - u_k\| ,$$

so $\{D^\alpha u_\ell\}_{\ell=1}^\infty$ is Cauchy in $L^2(\mathbb{R}^N)$. Hence, $D^\alpha u_\ell \to V_\alpha \in L^2(\mathbb{R}^N)$, since $L^2(\mathbb{R}^N)$ is complete.

Note that differentiation is a continuous operation in $\mathcal{D}'(\mathbb{R}^N)$. That is, if $\langle T_n, \varphi\rangle \to \langle T,\varphi\rangle$ $\forall\varphi \in \mathcal{D}(\mathbb{R}^N)$, and $\langle \frac{\partial T_n}{\partial x_i},\varphi\rangle \to \langle S,\varphi\rangle$ $\forall \varphi \in \mathcal{D}(\mathbb{R}^N)$, then $S = \frac{\partial T}{\partial x_i}$.

(Since $\langle \frac{\partial T}{\partial x_i}, \varphi\rangle = -\langle T, \frac{\partial \varphi}{\partial x_i}\rangle = -\lim\langle T_n, \frac{\partial\varphi}{\partial x_i}\rangle$ which is

$$= \lim\langle \frac{\partial T_n}{\partial x_i},\varphi\rangle = \langle S,\varphi\rangle \ \forall\varphi \in (\mathbb{R}^N).)$$

If $u_\ell \to u$ in $L^2(\mathbb{R}^N)$, then $u_\ell \to u$ in $\mathcal{D}'(\mathbb{R}^N)$: that is

$$\int u_\ell\varphi \to \int u\varphi \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^N).$$

(In fact, this is true for $\varphi \in L^2(\mathbb{R}^N)$). Thus we have $D^\alpha u_\ell \to D^\alpha u$ in $\mathcal{D}'(\mathbb{R}^N)$. But we already have $D^\alpha u_\ell \to V_\alpha$ in $L^2(\mathbb{R}^N) \Longrightarrow D^\alpha u_\ell \to V_\alpha$ in $\mathcal{D}'(\mathbb{R}^N)$, so $V_\alpha = D^\alpha u$. ∎

Remark: We use a distribution result, which is to say, we went to a larger space.

Definition (2): $u \in H^m(\mathbb{R}^N)$ if $\mathcal{F}[u] = \hat{u}$ satisfies:

$$(1 + |\xi|^2)^{m/2}\hat{u}(\xi) \in L^2(\mathbb{R}^N) \quad (\text{or } (1 + |\xi|)^m \hat{u}(\xi) \in L^2(\mathbb{R}^2).)$$

Note that if $u \in L^2(\mathbb{R}^N)$, then $\hat{u} \in L^2(\mathbb{R}^N)$ and $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$.

Also,

$$\frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^N) \Longleftrightarrow \mathcal{F}[\frac{\partial u}{\partial x_j}] \in L^2(\mathbb{R}^N)$$

$$\Longleftrightarrow 2\, i\pi\, \xi_j\, \hat{u}(\xi) \in L^2(\mathbb{R}^N).$$

$D^\alpha u \in L^2(\mathbb{R}^N)$ for $|\alpha| \leq m \Longrightarrow P_\alpha(\xi)\hat{u}(\xi) \in L^2(\mathbb{R}^N)$, where P_α is a polynomial of degree $\leq m$. Thus we can obtain an equivalent norm. $H^m(\mathbb{R}^N)$, defined this way is thus complete since it is isometrically isomorphic to an L^2-space under different measure. This definition will be used for m real ≥ 0.

<u>Example:</u> $\Delta u = f$ gives $-4\pi^2 |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$.

<u>Properties of $H^m(\mathbb{R}^N)$:</u>

.1) $H^m(\mathbb{R}^N)$ is a Hilbert space

.2) We have multiplication by smooth functions.

If $\varphi \in C^m(\mathbb{R}^N)$, then $u \to \varphi u$ is a bounded linear operator from $H^m(\mathbb{R}^N)$ to $H^m(\mathbb{R}^N)$.

<u>Proof:</u> Apply definition (1) and Leibnitz's formula for the derivative of a product.

For definition (2) one can show: if $m \geq s$ then $u \to \varphi u$ is bounded from $H^s(\mathbb{R}^N)$ into itself. ∎

.3) $\mathcal{D}(\mathbb{R}^N)$ is dense in $H^m(\mathbb{R}^N)$.

<u>Proof:</u> Step (1) (truncation) If $u \in H^m(\mathbb{R}^N)$, let $\theta \in \mathcal{D}(\mathbb{R}^N)$ such that θ $\theta \equiv 1$ in a neighborhood of 0 . Define $\theta_k(x) = \theta(\frac{x}{k})$ so $\theta_k(x) \to 1$ as $k \to \infty$. Then $u_k(x) = \theta_k(x)u(x) \in H^m(\mathbb{R}^N)$ has compact support, and $u_k \to u$ in $L^2(\mathbb{R}^N)$ strongly by Lebesgue's dominated convergence theorem. Using Leibnitz's formula, $D^\alpha u_k \to D^\alpha u$.

<u>Remark:</u> Hence, the functions in $H^m(\mathbb{R}^N)$ with compact support are dense in $H^m(\mathbb{R}^N)$.

Step (2): (Regularization) Take $u \in H^m(\mathbb{R}^N)$ with compact support. Choose ρ as before: $\rho \in \mathcal{D}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho = 1$, and $\rho \geq 0$. Take $\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho(\frac{x}{\epsilon})$, and let $u_\epsilon = u * \rho_\epsilon$. Young's inequality then implies that $\|u_\epsilon\|_2 \leq \|\rho_\epsilon\|_1 \cdot \|u\|_2 = \|u\|_2$. Hence $\{u_\epsilon\}_{\epsilon \geq 0}$ is a bounded set in $L^2(\mathbb{R}^N)$.

Using the fact that $C_c(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, one can show that $u_\epsilon \to u$ strongly in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$. Then, $D(u * \rho_\epsilon) = Du * \rho_\epsilon \to Du$ in

$L^2(\mathbb{R}^N)$, so $u_\epsilon \to u$ in $H^m(\mathbb{R}^N)$ as $\epsilon \to 0$. ∎

Definition (3): $H^m(\mathbb{R}^N)$ = the completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm $\|u\|^2 = \sum_{|\alpha| \leq m} \|D^\alpha u\|_2^2$.

Remark: This holds true with $s > 0$, not just integral m. We have another property of $H^m(\mathbb{R}^N)$:

.4) (Continuity property of elements of $H^m(\mathbb{R}^N)$) If $s > \frac{N}{2}$, then $H^s(\mathbb{R}^N) \subseteq C^0(\mathbb{R}^N)$ and

$$\sup_{x \in \mathbb{R}^N} |u(x)| \leq C\|u\|$$

Proof: $u = \bar{\mathfrak{F}}[\hat{u}]$, and $\bar{\mathfrak{F}}$ takes $L^1(\mathbb{R}^N)$ into $C^0(\mathbb{R}^N)$. Thus, it will suffice to prove $\hat{u} \in L^1(\mathbb{R}^N)$.

$$\hat{u}(\xi) = (1 + |\xi|)^s \hat{u}(\xi) \frac{1}{(1+|\xi|)^s}$$, and we know that

$(1 + |\xi|)^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)$, so we need only that $(1 + |\xi|)^{-s} \in L^2(\mathbb{R}^N)$. But this is to say $\int_{\mathbb{R}^N} (1 + |\xi|)^{-2s} d\xi < \infty$, which holds if and only if $s > \frac{n}{2}$. ∎

Corollary: If $s > \frac{N}{2} + k$, k a positive integer, then $H^s(\mathbb{R}^N) \subseteq C^k(\mathbb{R}^N)$.

Remark: A derivative of order k takes elements of H^s into H^{s-k}.

If $s = \frac{N}{2} + \alpha$, $0 < \alpha < 1$, then $H^s(\mathbb{R}^N) \subseteq C^{0,\alpha}(\mathbb{R}^N)$. (The Lipschitz functions of order α;

$$\left| \frac{u(x) - u(y)}{|x-y|^\alpha} \right| \leq \text{constant.} \quad)$$

Proof: $u(x+h) = \bar{\mathfrak{F}}[\hat{u}(\cdot)e^{2i\pi\langle h,\cdot\rangle}](x)$, so

$$u(x+h) - u(x) = \bar{\mathfrak{F}}[\hat{u}(\cdot)(e^{2\pi i\langle h,\cdot\rangle} - 1)](x)$$

$$\Longrightarrow \|u(x+h) - u(x)\|_{C^0(\mathbb{R}^N)} \leq \|\hat{u}(\xi)[e^{2\pi i\langle h,\xi\rangle} - 1]\|_{L^1(\mathbb{R}^N)}$$

By Hölder's inequality, this is $\leq \| |\xi|^s \hat{u}(\xi) \|_{L^2} \cdot \| \frac{e^{2\pi i\langle h,\xi\rangle}-1}{\|\xi\|^s} \|_{L^2}$. The first factor is bounded. The second gives $\int_{\mathbb{R}^N} \frac{|e^{2\pi i\langle h,\xi\rangle}-1|^2}{|\xi|^{2s}} d\xi$. Let $\xi = \frac{1}{|h|}\eta$ and take h colinear to the first basis vector.

$$\int \frac{|e^{2\pi i\eta_1}-1|^2}{\frac{1}{|h|^{2s}}|\eta|^{2s}} \frac{1}{|h|^N} d\eta = |h|^{2s-N} \cdot C .$$ Placing this back in the above, we must take square roots. We get

$$\|u(x+h) - u(x)\|_{C^0(\mathbb{R}^N)} \leq C \cdot |h|^{\alpha} . \qquad \blacksquare$$

.5) L^p-properties of elements of $H^m(\mathbb{R}^N)$:

<u>Theorem:</u> (Sobolev-Peetre) If $0 \leq s < \frac{N}{2}$, then $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ where $\frac{1}{p} = \frac{1}{2} - \frac{s}{N} > 0$.

.6) Trace properties of elements of $H^s(\mathbb{R}^N)$:

<u>Problem:</u> Can we restrict u to a subspace of $\mathbb{R}^N$, say $\mathbb{R}^{N-1}$?

<u>Theorem:</u> The mapping $\mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^{N-1})$ given by $\varphi(x_1,\ldots,x_n) \to \varphi(x_1,\ldots,x_{n-1},0)$ (the <u>trace</u> of φ on $\mathbb{R}^{N-1}$) is continuous from $\mathcal{S}(\mathbb{R}^N)$ under the norm of $H^s(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^{N-1})$ under the norm of $H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})$, if $s > \frac{1}{2}$.

<u>Note:</u> If $s > \frac{N}{2}$, this process can be continued stepwise down to dimension 0 giving property 4).

<u>Proof:</u> Let $u \in \mathcal{S}(\mathbb{R}^N)$ and $v(x_1,\ldots,x_{n-1}) = u(x_1,\ldots,x_{n-1},0)$.

<u>Claim:</u> $\mathcal{F}[v](\xi_1,\ldots,\xi_{n-1}) = \int_{\mathbb{R}} \mathcal{F}[u](\xi_1,\ldots,\xi_{n-1},\xi_n)d\xi_n$.

The proof of this claim uses the fact that $\mathcal{F}(\delta) = 1$: that is $\langle \mathcal{F}\delta,\varphi\rangle = \langle \delta,\mathcal{F}[\varphi]\rangle = \mathcal{F}[\varphi](0) = \int_{\mathbb{R}^N}\varphi(x)dx$ for $\varphi \in \mathcal{S}(\mathbb{R}^N)$. This can also be expressed as

$$\psi(0) = \int \bar{\mathcal{F}}[\psi](x)\, dx .$$

Let $\xi' = (\xi_1, \ldots, \xi_{n-1})$. We want to prove that $(1 + |\xi'|)^{s-\frac{1}{2}} \mathfrak{F}[v](\xi') \in L^2(\mathbb{R}^{N-1})$.

Using the above,

$$|\mathfrak{F}[v](\xi')|^2 \leq \left(\int_{\mathbb{R}} |\mathfrak{F}[u](\xi_1, \ldots, \xi_{n-1}, \xi_n)\, d\xi_n\right)^2$$

Cauchy-Schwartz then gives

$$\leq \int_{\mathbb{R}} (1 + |\xi'| + |\xi_n|)^{2s} |\mathfrak{F}u|^2 d\xi_n \cdot \int_{\mathbb{R}} \frac{d\xi_n}{(1+|\xi'| + |\xi_n|)^{2s}} \quad .$$

Use the change of variable $\xi_n = (1 + |\xi'|)t$:

$$\int_{\mathbb{R}} \frac{(1 + |\xi'|)dt}{(1+|\xi'|)^{2s}(1+t)^{2s}} = \frac{a}{(1+|\xi'|)^{2s-1}} \implies$$

$$(1+|\xi'|)^{2s-1} |\mathfrak{F}[v](\xi')|^2 \leq a \int_{\mathbb{R}} (1+|\xi'| + |\xi_n|)^{2s} |\mathfrak{F}[u](\xi_1, \ldots, \xi_{n-1}, \xi_n)|^2 d\xi_n .$$

Integrate over ξ' in $\mathbb{R}^{N-1}$ to obtain the result. ∎

Example: Let $\Omega \subseteq \mathbb{R}^N$, f be defined on Ω. Find u such that

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

usually $u \in H^1(\Omega)$. $\partial\Omega$ is a manifold rather than a space of codimension 1.

Remark: Let X, Y be Hilbert spaces (Banach spaces). Then we can consider the space of all $u : \mathbb{R} \to X$ such that $u \in L^2(\mathbb{R}, X)$ and $\frac{du}{dt} : \mathbb{R} \to Y$ such that $\frac{du}{dt} \in L^2(\mathbb{R}; Y)$. (Here X is continuously imbedded in Y.)

Under such circumstances it is very important to know that $u(0)$ has a meaning:

Sobolev spaces in $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$.

Definition (4): If $\Omega^{\text{open}} \subset \mathbb{R}^N$, m is a nonnegative integer,

$$H^m(\Omega) = \{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \ \ \forall \alpha : |\alpha| \leq m \} .$$

Definition (5): $H^s(\Omega) = \{f|_\Omega : f \in H^s(\mathbb{R}^N)\}$ for $s \geq 0$.

Remark: These definitions are not always equivalent. They are equivalent if Ω is a "regular" open set.

We will study $H^m(\Omega)$ where $\Omega = \mathbb{R}^N_+$, using definition (4).

Properties: 0) If D^α is a derivative with $|\alpha| = k$, it maps $H^m(\Omega)$ into $H^{m-k}(\Omega)$.

1) We have multiplication by smooth functions exactly as before.

2) If $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is the space of functions which are the restrictions to $\mathbb{R}^N_+$ of elements of $\mathcal{D}(\mathbb{R}^N)$, (note that $f \in \mathcal{D}(\overline{\mathbb{R}^N_+})$ may no longer have compact support in $\mathbb{R}^N_+$) then $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $H^m(\mathbb{R}^N_+)$ for $m \geq 0$. However, $\mathcal{D}(\mathbb{R}^N_+)$ is not dense in $H^m(\mathbb{R}^N_+)$ for $m \geq 1$.

Proof: of 2) Step 1. Let $u \in H^m(\mathbb{R}^N_+)$, and $h > 0$. Define $u_h(x_1,\ldots,x_n) = u(x_1,\ldots,x_{n-1},x_n+h)$. Then as $h \to 0$, $u_h|_{\mathbb{R}^N_+} \to u$ in $H^m(\mathbb{R}^N_+)$. It is sufficient to prove this for $u \in L^2(\mathbb{R}^N_+)$.

Exercise: If $\varphi \in L^2(\mathbb{R}_+)$, $\varphi_\delta = \varphi(x+\delta)$ for $\delta > 0$, then $\varphi_\delta \to \varphi$ strongly in $L^2(\mathbb{R}^+)$ as $\delta \to 0$.

Clearly, $\|\varphi_\delta\|_2 \leq \|\varphi\|_2$. Prove the result for φ continuous first, then extend via the density of $C_c(\mathbb{R}_+)$ in $L^2(\mathbb{R}_+)$.

Step 2: fix $h > 0$.

Define $u_{\epsilon,h} = u_h * \rho_\epsilon$ defined in $\mathbb{R}^N_+$ for ϵ sufficiently small. Then $u_{\epsilon,h} \to u_h$ in $H^m(\mathbb{R}^N_+)$ as $\epsilon \to 0$.

We can replace u_h by $v_h = \theta_h \varphi_h$ where $\theta_h(x_N)$ is smooth and satisfies $\theta_h(x) = 0$ for $x \leq -\frac{h}{2}$, $\theta_h(x) = 1$ for $x \geq -\frac{h}{4}$. Then

$v_h \in H^m(\mathbb{R}^N)$, $v_h\big|_{\mathbb{R}^N_+} \to u$, and $v_h * \rho_\epsilon \to v_h$ in $H^m(\mathbb{R}^N)$. ∎

Another important property of $H^m(\mathbb{R}^N_+)$ is

3). Extension to $\mathbb{R}^N$.

Theorem: There exists a linear continuous map P from $H^m(\mathbb{R}^N_+)$ intc $H^m(\mathbb{R}^N)$ such that $P u\big|_{\mathbb{R}^N_+} = u \;\; \forall\, u \in H^m(\mathbb{R}^N_+)$. (Thus definition 4 and definition 5 are equivalent for $\mathbb{R}^N_+$.)

Proof: It is sufficient to define P on a dense subset of $H^m(\mathbb{R}^N_+)$; for example, $\mathcal{D}(\overline{\mathbb{R}^N_+})$.

Let

$$v(x_1,\ldots,x_{n-1},x_n) = \begin{cases} u(x_1,\ldots,x_{n-1},x_n) & \text{if } x_n > 0 \\ \sum_{k=0}^{m} a_k\, u(x_1,\ldots,x_{n-1},-kx_n) & \text{if } x_n < 0 \end{cases}$$

where we will choose the a_k (real) in an appropriate manner. v is a smooth function in $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$.

We must show that the traces are the same on both sides of $\mathbb{R}^{N-1}$. We need that $v, \frac{\partial v}{\partial x_N}, \ldots, \frac{\partial^{m-1} v}{\partial x_N^{m-1}}$ be continuous:

Because, if we have a function φ on $\mathbb{R}$ continuous, with $\frac{d\varphi}{dx} \in L^2(\mathbb{R}_+)$, $\frac{d\varphi}{dx} \in L^2(\mathbb{R}_-)$, then $\frac{d\varphi}{dx} \in L^2(\mathbb{R})$ if and only if φ is continuous at 0.

So we want to choose the a_k so that $\sum_{k=0}^{m} a_k(-k)^p = 1$ for $0 \le p \le m-1$. Solving these m equations in m unknowns, we get the proper a_k. ∎

Theorem: (Trace theorem) $H^m(\mathbb{R}^N) \to H^{m-\frac{1}{2}}(\mathbb{R}^{N-1})$ given by $u(x_1,\ldots,x_n) \to u(x_1,\ldots,x_{n-1},0)$ is called the trace. Also,

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \xrightarrow{\text{trace}} \mathcal{D}(\mathbb{R}^{N-1}).$$

Thus we have the following situation:

$$\begin{array}{ccccc} H^m(\mathbb{R}^N_+) & \xrightarrow{\ P\ } & H^m(\mathbb{R}^N) & \xrightarrow{\text{trace}} & H^{m-\frac{1}{2}}(\mathbb{R}^{N-1}) \\ & \searrow^{\text{Identity}} & \downarrow{\scriptstyle \text{Restriction}} & & \\ & & H^m(\mathbb{R}^N_+) & & \end{array}$$

If $u \in H^m(\mathbb{R}^N_+)$, $|\alpha| \leq m-1$, then $u \to D^{\alpha}(x_1, \ldots, x_{n-1}, 0)$ exists as a map: $H^m(\mathbb{R}^N_+) \to H^{m-|\alpha| - \frac{1}{2}}(\mathbb{R}^{N-1})$.

<u>Remark:</u> $\mathcal{D}(\mathbb{R}^N_+)$ is <u>not</u> dense in $H^m(\mathbb{R}^N_+)$ for $m \geq 1$, for: let $u_n \in \mathcal{D}(\mathbb{R}^N_+)$, $u_n \to u$ in $H^m(\mathbb{R}^N_+)$, $|\alpha| \leq m-1$. Then trace $D^{\alpha}u_n \to$ trace D^{α}_{u}. But $u_n \in \mathcal{D}(\mathbb{R}^N_+)$, so trace $D^{\alpha}u_n \equiv 0$. Hence trace $D^{\alpha}u = 0$.

<u>Definition:</u> $H^m_0(\mathbb{R}^N_+) = \{u \in H^m(\mathbb{R}^N_+) : \text{trace } D^{\alpha}u = 0 \quad |\alpha| \leq m-1\}$. So the closure of $\mathcal{D}(\mathbb{R}^N_+)$ is in $H^m_0(\mathbb{R}^N_+)$. Conversely

<u>Theorem:</u> If $u \in H^m_0(\mathbb{R}^N_+)$, there exists a sequence

$$\{u_k\}_{k=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^N_+) : u_k \to u \text{ in } H^m(\mathbb{R}^N_+) .$$

The proof uses:

<u>Hardy's Inequality:</u> Let v be defined on $[0,\infty)$, and be locally integrable. Let $w(t) = \frac{1}{t}\int_0^t v(s)ds$.

<u>Lemma:</u> If $t^{\alpha}v \in L^2_*(0,\infty) = L^2((0,\infty); \frac{dt}{t})$ with $\alpha < 1$, then $t^{\alpha}w \in L^2_*(0,\infty)$ and

$$\|t^{\alpha}w\|_{L^2_*} \leq \frac{1}{1-\alpha}\|t^{\alpha}v\|_{L^2_*} .$$

<u>Proof:</u> $C_c(0,\infty)$ is dense in $\{v : t^{\alpha}v \in L^2_*(0,\infty)\}$ so it is enough to prove it for $v \in C_c(0,\infty)$. Note that $(tw)' = v$. Multiply by $t^{2\alpha}w\frac{dt}{t}$ and integrate:

$$\int_0^{\infty} vt^{2\alpha}w\frac{dt}{t} = \int_0^{\infty}(w+tw')t^{2\alpha}w\frac{dt}{t} = \int_0^{\infty}|t^{\alpha}w|^2\frac{dt}{t} + \int_0^{\infty}t^{2\alpha}ww'\,dt .$$

Integrate the last term by parts:

$$\int_0^\infty t^{2\alpha} ww'dt = \frac{t^{2\alpha}|w|^2}{2}\Big|_0^\infty - \alpha\int_0^\infty t^{2\alpha}|w|^2\frac{dt}{t}$$

Now, $|w(t)| \leq \frac{C}{t}$ for t large enough, so the first term goes to 0. Hence

$$(1-\alpha)\int_0^\infty |t^\alpha w|^2\frac{dt}{t} = \int_0^\infty v\, t^{2\alpha}\, w\frac{dt}{t} \leq \|t^\alpha v\|_{L^2_*}\|t^\alpha w\|_{L^2_*} .$$

So $\quad (1-\alpha)\|t^\alpha w\|^2_{L^2_*} \leq \|t^\alpha v\|_{L^2_*}\|t^\alpha w\|_{L^2_*}$ which gives the desired result. ■

<u>Corollary:</u> If $u \in H^m(0,\infty)$, $u(0) = u'(0) = \ldots = u^{m-1}(0) = 0$, then $\frac{u}{t^m} \in L^2(0,\infty)$, and $\frac{u^{(k)}}{t^{m-k}} \in L^2(0,\infty)$ for $0 \leq k \leq m$.

<u>Proof:</u> $u(t) = \int_0^t u'(s)ds$ since $u(0) = 0$. Hardy's inequality with $\alpha = \frac{1}{2}$, $u' \in L^2$, gives $\frac{1}{t}\int_0^t u'(s)ds \in L^2 \Longrightarrow \frac{u(t)}{t} \in L^2$. Apply again, with $\frac{u'}{t} \in L^2$, $\alpha = -\frac{1}{2}$, $\frac{t^{-\frac{1}{2}}u(t)}{t} \in L^2_* \Longrightarrow \frac{u}{t^2} \in L^2$. Continuing, we can take $m-1$ derivatives. This completes the proof. ■

As a further illustration of the uses of the corollary, take $x_1,\ldots,x_{n-1}$ as parameters. Then

$$\int_{\mathbb{R}} \left|\frac{u}{x_n^m}\right|^2 dx_n \leq \int_{\mathbb{R}} \sum_{|\alpha|=m} |D^\alpha u|^2 dx_n$$

$$\left\|\frac{u}{x_n^m}\right\|_{L^2} \leq C\|u\|_{H_0^m}$$

if the trace of u is 0.

Hence,

$$\left\|\frac{D^\alpha u}{x_n^{m-|\alpha|}}\right\|_{L^2} \leq C\|u\|_{H_0^m} \qquad \text{for } |\alpha| \leq m .$$

<u>Approximation of u:</u> Let $\theta(x)$ be smooth with $\theta(x) = 0$ for $x \leq 1$ and $\theta(x) = 1$ for $x \geq 2$.

Let $\quad \theta_k(x) = \theta(kx)$.

Let $u_k(x_1,\ldots,x_n) = \theta_k(x_n)\, u(x_1,\ldots,x_n)$.

Then the claim is that $u_k \to u$ in H^m . (This is clear in L^2.)

For $i \neq n$ $$\frac{\partial u_k}{\partial x_i} = \theta_k \frac{\partial u}{\partial x_i} \xrightarrow{k\to\infty} \frac{\partial u}{\partial x_i}$$

For $i = n$ $$\frac{\partial u_k}{\partial x_n} = \theta_k \frac{\partial u}{\partial x_n} + k\, x_n \theta'(kx_n) \cdot \frac{u}{x_n} \xrightarrow{k\to\infty} \frac{\partial u}{\partial x_n}$$

the second term converging to 0 by the dominated convergence theorem. Similarly for higher derivatives so $u_k \to u$ in $H^m(\mathbb{R}^N_+)$. Then take a regularization $\rho_\varepsilon * u_k \to u_k$ and $\rho_\varepsilon * u_k$ is smooth. ∎

<u>Definition:</u> $H^m_0(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$.

<u>Sobolev spaces of negative order:</u> Recall that for $s \geq 0$

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^N)\} .$$

Similarly,

<u>Definition:</u> $H^s(\mathbb{R}^N) = \{u \in S'(\mathbb{R}^N) : (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^N|\}$ for $s < 0$.

<u>Theorem:</u> The dual space of $H^s(\mathbb{R}^N)$, where $s \geq 0$, is isometric to $H^{-s}(\mathbb{R}^N)$:

$$\sup_{0 \neq v \in H^s} \frac{|\int u\bar{v}|}{\|v\|_s} = \|u\|_{H^{-s}}$$

for $u \in L^2(\mathbb{R}^N)$.

<u>Proof:</u> $\int u\,\bar{v} = \int \hat{u}\,\bar{\hat{v}}$. Hence as $\|v\|^2_{H^s} = \int (1+|\xi|^2)^s |\hat{v}|^2 d\xi$

$$\sup_{\|v\|_{H^s} \leq 1} |\int \hat{u}\,\bar{\hat{v}}| \leq (\int (1+|\xi|^2)^{-s} |\hat{u}|^2)^{\frac{1}{2}}$$ by Cauchy-Schwartz. ∎

<u>Characterization of $H^{-m}(\mathbb{R}^N)$:</u> (m an integer).

$S \in H^{-m} \iff$ we can write (perhaps in more than one way)

$$S = \sum_{|\alpha| \leq m} D^\alpha \varphi_\alpha : \varphi_\alpha \in L^2(\mathbb{R}^N) .$$

Proof: If $S = \sum_{|\alpha| \leq m} D^\alpha \varphi_\alpha$, $\langle S, \psi\rangle = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} \langle \varphi_\alpha, D^\alpha \psi\rangle$,

so $|\langle S,\psi\rangle| \leq \sum_\alpha \|\varphi_\alpha\|_{L^2} \|D^\alpha\psi\|_{L^2} \leq C\|\psi\|_{H^m}$, so $S \in H^{-m}$.

Conversely, if $u \in H^m$, the map $u \overset{\pi}{\to} \{(D^\alpha u) : |\alpha| \leq m\}$ is a map from $H^m \overset{\pi}{\to} (L^2(\mathbb{R}^N))^\beta$. ($\beta$ = # of α : $|\alpha| \leq m$.) If S is a continuous linear functional on $\pi(H^m(\mathbb{R}^N))$, which is a closed subspace of $(L^2(\mathbb{R}^N))^\beta$. By Hahn-Banach's theorem, $\exists\, T \in (L^2(\mathbb{R}^N))^\beta : T\big|_{\pi(H^m(\mathbb{R}^N))} = S$. Thus

$T(v_\alpha) = \sum_{|\alpha| \leq m} (\varphi_\alpha, v_\alpha)_{L^2}$ where $\varphi_\alpha \in L^2(\mathbb{R}^N)$. Hence

$$S(u) = \sum_{|\alpha|\leq m} (\varphi_\alpha, D^\alpha u) = \Big(\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \varphi_\alpha, u\Big)$$

for $u \in \mathcal{D}(\mathbb{R}^N)$. Since $\mathcal{D}(\mathbb{R}^N)$ is dense in $H^m(\mathbb{R}^N)$, we have the desired result. ∎

Definition: $H^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$, $L^2(\Omega)$ being identified with its dual.

Characterization: A distribution S is an element of $H^{-m}(\Omega) \iff S = \sum_{|\alpha|\leq m} D^\alpha \varphi_\alpha$ where $\varphi_\alpha \in L^2(\Omega)$. (The same proof works.)

Remark: The dual space of $H^m(\mathbb{R}_+^N)$ is a subset of the distributions with support in $\overline{\mathbb{R}_+^N}$, for $m \geq 1$.

Theorem: (Compactness). If u_k is a bounded sequence in $H^1(\mathbb{R}^N)$ and all the u_n have support in some common compact set in $\mathbb{R}^N$, then we can extract a subsequence converging strongly in $L^2(\mathbb{R}^N)$.

Remark: $H_0^1(\Omega) \subseteq L^2(\Omega)$, and, by this theorem, if Ω is open and bounded, the injection is compact.

The theorem is <u>not</u> true without the condition on the support of the u_k, as the following shows: Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \neq 0$. Let $\varphi_k(x_1,\ldots,x_n) = \varphi(x_1+k, x_2,\ldots,x_n)$. Then $\|\varphi_k\| = \|\varphi\|$, $\varphi_k \to 0$ weakly in L^2, that is,

$$\int \varphi_k f \to 0 \ \forall\, f \in L^2(\mathbb{R}^N) .$$

But $\varphi_n \not\to 0$ strongly, by $\|\varphi_k\| \to \|\varphi\| \neq 0$.

<u>Proof:</u> (Of the compactness theorem). Extract a weakly convergent subsequence, $\{u_k\} \subseteq L^2(\mathbb{R}^N)$.

We want to extract a Cauchy sequence in $L^2(\mathbb{R}^N)$, <u>or</u> extract a Cauchy sequence from $\{\mathfrak{F}[u_k]\}$.

We know that $\int (1 + |\xi|^2)|\hat{u}_k(\xi)|^2 d\xi \leq C$ (this is just $\|u_k\|^2_{H^1}$). Hence, it is easily seen that

$$\int_{|\xi| \geq r} |\hat{u}_k(\xi)|^2 d\xi \leq \frac{C}{(1+r^2)} .$$

Thus

$$\overline{\lim_{k,\ell\to\infty}} \int_{|\xi| \geq r} |\hat{u}_k - \hat{u}_\ell|^2 d\xi \leq \frac{C'}{(1+r^2)} .$$

Take $|\xi| \leq r$.

$$\hat{u}_k(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i\langle \xi, x\rangle} u_k(x)dx$$

Say $\operatorname{supp}(u_k) \subseteq K$ for all $k = 1,2,\ldots$. Then

$$\hat{u}_k(\xi) = \int_K e^{-2\pi i\langle \xi, x\rangle} u_k(x)dx \to \int_K e^{-2\pi i\langle \xi, x\rangle} u(x)dx$$

since $e^{-2\pi i\langle \xi, x\rangle} \in L^2(K)$ and $u_k \to u$ in $L^2(K)$ weak. So $\hat{u}_k(\xi) \to \hat{u}(\xi)$ for $|\xi| \leq r$

$$|\hat{u}_k(\xi)| \leq \int_K |u_k(\xi)|dx \leq \|u_k\|_{L^2} \cdot \sqrt{\operatorname{meas}(K)}$$

by Cauchy-Schwartz, so $\hat{u}_k \to \hat{u}$ strongly on $|\xi| \leq r$ by the dominated convergence theorem.

So we have $\lim_{k,\ell\to\infty} \int_{|\xi|\le r} |\hat{u}_k - \hat{u}_\ell|^2 d\xi = 0$, and

$$\overline{\lim_{k,\ell\to\infty}} \int_{|\xi|\ge r} |\hat{u}_k - \hat{u}_\ell|^2 d\xi \le \frac{C'}{(1+r^2)} .$$

Take $r\to\infty$, and we have $\|\hat{u}_k - \hat{u}_\ell\|^2_{L^2} \to 0$ as $k,\ell\to\infty$. Thus we get strong convergence. ∎

<u>Sobolev spaces on Ω, an open set in $\mathbb{R}^N$:</u> Recall

<u>Definition (4):</u> $H^m(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}$.

<u>Definition (5):</u> $H^m(\Omega) = \{u : \exists\, v \in H^m(\mathbb{R}^N) \text{ and } v|_\Omega = u\}$.

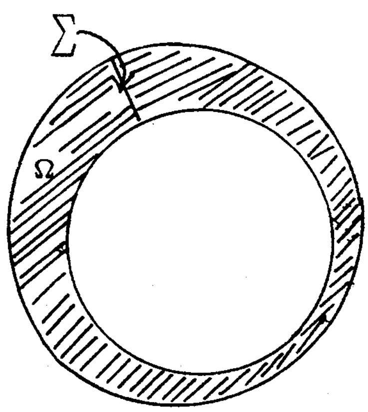
For Definition (4), the norm is $\|u\| = (\sum_{|\alpha|\le m} \|D^\alpha u\|^2_{L^2})^{\frac{1}{2}}$, while for Definition (5) we define the norm to be

$$\|u\| = \operatorname{Inf}\{\|v\|_{H^m(\mathbb{R}^N)} : v \in H^m(\mathbb{R}^N) \text{ and } v|_\Omega = u\} .$$

<u>Note that</u> u may be in $X = H^m(\Omega)$ as defined by Definition (4) yet <u>not</u> in $Y = H^m(\Omega)$ as defined by Definition (5), for general Ω .

<u>Example:</u> Let Ω be as shown:

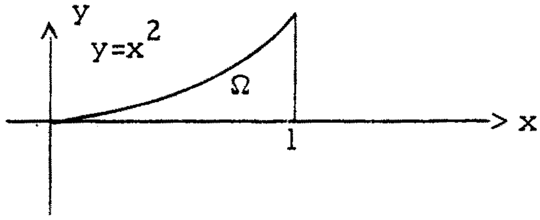
Note: $\sum \not\subset \Omega$



Let u have different values approaching $\sum$ from different sides. Then u could be in X , but not in Y .

Definition: By "regular open set" is meant a manifold with boundary such that, at each point of the boundary, the open set is on only one side.

Example: Take $u = x^{\alpha}$. Is $u \in H^1(\Omega)$?



$$\|u\|^2_{L^2} = \iint_{\Omega} |x|^{2\alpha} dxdy = \int_0^1 |x|^{2\alpha+r} dx < \infty \quad \text{if} \quad 2\alpha+r > -1 .$$

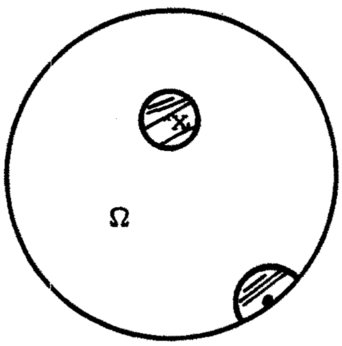
Similarly, $u' \in L^2(\Omega)$ if $2(\alpha-1) + r > -1$. Hence, $u \in H^1(\Omega)$ if $2\alpha+r > 1$. Note that α can be negative if r is large.

Theorem: (Sobolev) If $v \in H^1(\mathbb{R}^2)$, then $v \in L^p(\mathbb{R}^2)\ \forall\ p$. Now,

$$\|u\|^p_{L^p(\Omega)} = \int_{\Omega} |x|^{p\alpha} < \infty_{\forall p} \implies p\alpha+r > -1\ \forall\ p \implies \alpha \geq 0 .$$

Referring again to the above, if $u \in Y$, then $u \in L^p(\Omega)\ \forall\ p$. Hence $x^{\alpha} \in Y \implies \alpha \geq 0$. But we have seen above that $x^{\alpha} \in X$ for $\alpha < 0$ if r is large. Hence, again in this case, $X \neq Y$.

This indicates that our domain Ω must be such that a nbd of a point $x \in \Omega \cup \partial\Omega$ can be taken to "resemble" a nbd in $\mathbb{R}^N$, or else in $\mathbb{R}^N_+$. If $\partial\Omega$ is smooth, this is true. One then has $X = Y$.



This is done by covering Ω with open sets Ω_i and using a C^m-partition of unity to concentrate on the Ω_i separately.

II. Navier-Stokes Equations

The Physical Problem:

We have a fluid in $\mathbb{R}^3$, whose velocity is $\vec{u}(x,t)$.

1) Conservation laws.

a) Conservation of mass. Let $\rho(x,t)$ be the density of the fluid at position $x \in \mathbb{R}^3$ and time t. We have a certain mass of fluid at Ω_{t_0} which has moved to $\Omega_{t_0+\delta_t}$ at $t_0 + \delta t$.

Then conservation of mass gives

$$\frac{d}{dt} \iiint_{\Omega_t} \rho(x,t)dx = 0 .$$

Also, $\frac{d}{dt} \iiint_{\Omega_t} \rho(x,t)dx = \iiint_{\Omega_{t_0}} \frac{\partial \rho}{\partial t}(x,t)dt + \iint_{\partial\Omega_{t_0}} \rho(x,t)\vec{u} \cdot \vec{n}\, d\sigma$

where $\vec{n}$ is the exterior normal to $\partial\Omega_{t_0}$. Transform the surface integral into a volume integral over Ω_{t_0}, and we obtain:

$$\iiint_{\Omega_{t_0}} \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho(x,t)\, \vec{u}_i)dx .$$

Thus we obtain the equation

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \vec{u}_i) = 0 .$$

b) Conservation of momentum: Suppose we have exterior forces, applied to Ω_{t_0}. Then this law gives

$$\frac{d}{dt}\left(\iiint_{\Omega_{t_0}} \rho\, \vec{u}\, dx \right) = \text{forces} . \qquad \text{(these are known)} .$$

Call the given forces in Ω $\vec{f}(x,t)$. Suppose we have constraints (i.e. viscosity), which are tensors σ_{ij}. Then we obtain, using

$$F_i = \sum_j \sigma_{ij} n_j$$

$$\frac{d}{dt}\left(\iiint_{\Omega_{t_0}} \rho \vec{u}\, dx\right) = \iiint_{\Omega_t} \vec{f}\, dx + \overrightarrow{\left(\iint_{\partial\Omega} \sum_j \sigma_{ij} n_j d\sigma\right)} \;.$$

Writing this last as a volume integral, we obtain the following equations:

(combining also the equations from a))

$$\rho\left[\frac{\partial \vec{u}_i}{\partial t} + \frac{\partial \vec{u}_i}{\partial x_j} \vec{u}_j\right] = f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \;.$$

c) <u>Conservation of angular momentum:</u> We get

$$\frac{d}{dt} \iiint (\rho \vec{u} \Lambda \vec{x}) dx = \iiint \vec{f} \Lambda \vec{x} dx + \iint (\vec{F}) \Lambda \vec{x}\, d\sigma \;.$$

Using the preceeding equalities, we obtain $\sigma_{ji} = \sigma_{ij}$.

d) <u>Conservation of energy:</u> Let $e(x,t)$ be the density of internal energy. (temperature) Then

$$\frac{d}{dt} \iiint \rho \cdot \left(\tfrac{1}{2}|\vec{u}|^2 + e\right) dx = \iiint \vec{f} \cdot \vec{u}\, dx + \iiint \left(\overrightarrow{\sum_j \sigma_{ij} n_j}\right) \vec{u}\, dx$$

$$+ \iiint \rho w\, dx - \iint_{\partial\Omega} \vec{q} \cdot \vec{n}\, d\sigma \;.$$

The term ρw represents the creation of energy as by chemical reaction: w is given. The last term represents convection. This all gives:

$$\rho\left[\frac{\partial e}{\partial t} + \sum_i \frac{\partial e}{\partial x_i} \vec{u}\right] = \rho w - \sum_i \frac{\partial q_i}{\partial x_i} + \sum_{j,i} \sigma_{ij} \frac{\partial u_i}{\partial x_j} \;.$$

2) <u>Next we have to consider the constitutive equations.</u>

These give relations between ρ, σ_{ij}, $\vec{u}_i$, $\vec{q}_i$, and e .

$$\sum_{i,j} \sigma_{ij} \frac{\partial \vec{u}_i}{\partial x_j} \geq 0 \qquad \text{(energy lost by viscosity).}$$

Write as a symmetric form $\sum_{i,j} \sigma_{ij} D_{ij}$ with $D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Assuming that we have an incompressible viscous fluid, $\rho = \rho_0$ a constant, and $\sigma_{ij} = -p(x)\delta_{ij} + 2\nu D_{ij}$ where p is the pressure. This gives us the Navier-Stokes equations. $\rho = \rho_0$ in the first equation above gives

$$1^\circ. \qquad \sum_j \frac{\partial \vec{u}_j}{\partial x_j} = 0 .$$

Considering the second equation,

$$\sum_j \frac{\partial}{\partial x_j}\sigma_{ij} = -\frac{\partial p}{\partial x_i} + \nu \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \vec{u}_i}{\partial x_j} + \frac{\partial \vec{u}_j}{\partial x_i}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \nu \Delta u_i + \frac{\partial}{\partial x_i}\sum_j \frac{\partial \vec{u}_j}{\partial x_j}$$

$$\overset{\shortparallel}{0} \text{ by } 1^\circ .$$

Hence, we obtain three equations:

$$2^\circ. \qquad \rho_0\left[\frac{\partial \vec{u}_i}{\partial t} + \sum_j \frac{\partial \vec{u}_i}{\partial x_j}\vec{u}_j\right] - \nu \Delta \vec{u}_i = f_i - \frac{\partial p}{\partial x_i} .$$

We want to solve the system for $x \in \Omega$, $t \in [0,T]$ $u_i(x,0)$ given; for $x \in \partial\Omega$ $u_i(x,t) = 0$ (or a given function of t. This is harder.)

We will consider $\Omega \subseteq \mathbb{R}^2$.

Kinetic energy at time t:

$$\frac{\rho_0}{2} \iiint_\Omega \sum_i |\vec{u}_i|^2 dx$$

It is natural to consider the space of all $\vec{u}$: $\vec{u}_i \in L^2(\Omega)$ (so that kinetic energy is finite) and $\vec{u}$ is bounded in time. The energy lost by viscosity between 0 and T is

$$\int_0^T \iint_\Omega \sum_{ij}(-p\delta_{ij} + 2\nu D_{ij})\, d_{ij} \approx C\int_0^T \int_\Omega \sum_{ij} \left|\frac{\partial \vec{u}_i}{\partial x_j}\right|^2 dxdt .$$

Consider the function space $\chi = \{\vec{u}_i : \frac{\partial u_i}{\partial x_i} \in L^2(\Omega)\}$. Let $K = \{\vec{u}_i \in L^2(\Omega)\}$. We will ask that $\vec{u} \in L^2(0,T;\chi)$, $\vec{u} \in L^\infty(0,T;K)$.

Mathematical formulation and function spaces:

Let Ω be a bounded regular open set in $\mathbb{R}^2$. Let

$\mathcal{V} = \{\varphi = (\varphi_1,\varphi_2) : \varphi_i \in \mathcal{D}(\Omega),\ \operatorname{div}\varphi = \sum_{i=1}^{2} \frac{\partial\varphi_i}{\partial x_i} = 0\}$. Define on $\mathcal{V}$ two norms:

$$|\varphi|_H^2 = \sum_{i=1}^{n} \|\varphi_i\|^2_{L^2(\Omega)} \qquad (n=2)$$

$$\|\varphi\|_V^2 = \sum_{i,j=1}^{n} \left\|\frac{\partial\varphi_i}{\partial x_j}\right\|^2_{L^2(\Omega)}$$

Let H = the completion of $\mathcal{V}$ with respect to $|\cdot|_H$.

V = the completion of $\mathcal{V}$ with respect to $\|\cdot\|_V$

<u>Proposition 1:</u> $V = \{u : u_i \in H_0^1(\Omega)$ and $\operatorname{div} u = 0\}$.

<u>Proof:</u> First, we show $V \subseteq \{u : u_i \in H_0^1(\Omega),\ \operatorname{div} u = 0\}$. Take a Cauchy sequence $u_n \in \mathcal{V}$ such that $\|u_n - u_m\|_V \to 0$. We will use

<u>Poincare's Inequality:</u> If Ω is bounded, $\exists\, c \geq 0$: for $u \in \mathcal{D}(\Omega)$,

$$\int_\Omega |u|^2 dx \leq c \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx .$$

<u>Proof:</u> $0 \leq \int_\Omega \left|\frac{\partial u}{\partial x_1} + a u\right|^2 dx = \int_\Omega \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + 2au\frac{\partial u}{\partial x_1} + a^2|u|^2\right)dx$

$\Longrightarrow \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 dx \geq \int_\Omega \left(\frac{\partial a}{\partial x_1} - a^2\right)|u|^2 dx$ after integration by part

$\exists\, a : \frac{\partial a}{\partial x_1} - a^2 \geq \alpha > 0$ on Ω. ∎

Using this inequality, $\{u_n\}$ is a Cauchy sequence in $H^1(\Omega)$, so $\{u_n\}$ converges to an element of $\overline{\mathcal{D}(\Omega)} \subset H^1(\Omega)$, that is, $H_0^1(\Omega)$. div is a bounded linear map $(H^1(\Omega))^n \to L^2(\Omega)$ so the limit u of $\{u_n\}$ in $H_0^1(\Omega)$ has $\operatorname{div} u = 0$. This proves one inclusion.

The proof of the other inclusion is more difficult and will be given later. ∎

Proposition 2: $H = \{u \mid u_i \in L^2(\Omega), \sum_{i=1}^{n} u_i(x)\eta_i(x) = 0$ on $\partial\Omega$ where η is the normal to $\partial\Omega$, and $\operatorname{div} u = 0$ in the sense of distributions$\}$.

Remark: Here we use a difficult theorem (for general Ω). If $u_i \in L^2(\Omega)$ and $\operatorname{div} u \in L^2(\Omega)$, one can define trace $u \cdot n = \sum_i u_i \eta_i \in H^{-\frac{1}{2}}(\partial\Omega)$ (so as to extend u smoothly.)

For an element of V the velocity at a point of $\partial\Omega$ is 0. For an element of H it is tangent to $\partial\Omega$ and may be $\neq 0$.

Remark: If $\varphi \in (\mathcal{D}(\Omega))^n$, $\operatorname{div}\varphi = 0$, then $\sum_i \langle \varphi_i, \frac{\partial p}{\partial x_i} \rangle_{\mathcal{D},\mathcal{D}'} = 0$ for $p \in \mathcal{D}'(\Omega)$ since

$$\sum_i \langle \varphi_i, \frac{\partial p}{\partial x_i} \rangle = -\langle \sum_i \frac{\partial \varphi_i}{\partial x_i}, p \rangle$$

$$= -\langle \operatorname{div}\varphi, p \rangle .$$

Back to the Navier-Stokes equations:

We will take $\rho = \rho_0 = 1$. Then the equations 2° become

$$\frac{\partial u_i}{\partial t} + \sum_j \frac{\partial u_i}{\partial x_j} u_j - \nu u_i = f_i - \frac{\partial p}{\partial x_i} .$$

Suppose $\varphi \in \mathcal{V}$. Take the inner product of φ_i and the above equation and sum over i, use the two propositions. Then,

$$\frac{\partial}{\partial t} \sum_i (u_i, \varphi_i) + \left(\sum_{ij} \frac{\partial u_i}{\partial x_j} u_j, \varphi_i \right) - \nu \sum_i (\Delta u_i, \varphi_i) = \sum_i (f_i, \varphi_i) .$$

Suppose $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then u satisfies the Navier-Stokes equation (1° and 2°) if and only if $\forall\, v \in V$

$$\frac{\partial}{\partial t}(u,v)_H + \left(\sum_{ij} \frac{\partial u_i}{\partial x_j} u_j, v_i \right) + \nu((u,v))_V = (f,v)_H .$$

Functional spaces related to the Navier-Stokes equation.

1. Let $\chi(\mathbb{R}^n) = \{u \in H^{-1}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in H^{-1}(\mathbb{R}^N),\ i = 1,\dots,n\}$

Recall that $u \in H^{-1}(\mathbb{R}^N) \iff \frac{\hat{u}(\xi)}{(1+|\xi|^2)^{\frac{1}{2}}} \in L^2(\mathbb{R}^N)$.

Lemma 1: $\chi(\mathbb{R}^N) = L^2(\mathbb{R}^N)$.

Proof: $u \in \chi \iff \frac{\hat{u}(\xi)}{(1+|\xi|^2)^{\frac{1}{2}}}$ and $\frac{|\xi_i \hat{u}(\xi)|}{(1+|\xi|^2)^{\frac{1}{2}}} \in L^2(\mathbb{R}^N)$.

$$\iff \frac{(1+|\xi|)\,\hat{u}(\xi)}{(1+|\xi|^2)^{\frac{1}{2}}} \in L^2(\mathbb{R}^N) \iff \hat{u} \in L^2(\mathbb{R}^N)$$
$$\iff u \in L^2(\mathbb{R}^N).$$

2. $\chi(\mathbb{R}^N_+) = \{u \in H^{-1}(\mathbb{R}^N_+) : \frac{\partial u}{\partial x_i} \in H^{-1}(\mathbb{R}^N_+),\ i = 1,\dots,n\}$.

Recall that $f \in H^{-1}(\Omega)$ implies $f = f_0 + \sum \frac{\partial f_i}{\partial x_i}$ for $f_i \in L^2(\Omega)$.

Note that $\|f\|_{H^{-1}} = \sup_{\substack{u \in H_0^1 \\ \|u\| \le 1}} (f,u)$.

Lemma 2: $\overline{\mathcal{D}(\mathbb{R}^N_+)}$ is dense in $\chi(\mathbb{R}^N_+)$.

Proof: Take $u \in \chi(\mathbb{R}^N_+)$, let $u_h(x_1,\dots,x_n) = u(x_1,\dots,x_{n-1},x_n+h)$ if u is a function. For general $u \in \chi(\mathbb{R}^N_+)$, define u_h as follows: $\langle u_h,\varphi\rangle = \langle u,\varphi_{-h}\rangle$ for $\varphi \in H_0^1(\mathbb{R}^N_+)$. Note that the two definitions coincide if u is a function. Then $u_h \to u$ in $\chi(\mathbb{R}^N_+)$ as $h \to 0$.

Then approach u_h by $u_h * \rho_\epsilon \big|_{\mathbb{R}^N_+} \to u_h$ in $\chi(\mathbb{R}^N_+)$. ∎

Lemma 3: $\exists$ a continuous extension from $\chi(\mathbb{R}^N_+) \to \chi(\mathbb{R}^N)$.

Proof: Define P on $\overline{\mathcal{D}(\mathbb{R}^N_+)}$ as follows:

Define $Q : H^1(\mathbb{R}^N) \to H_0^1(\mathbb{R}^N_+)$ by

$$(Qu)(x_1,\ldots,x_n) = \begin{cases} 0 & \text{if } x_n < 0 \\ u(x_1,\ldots,x_n) + \sum_{j=1}^{2} a_j u(x_1,\ldots,x_{n-1},-jx_n) & x_n > 0 \end{cases}$$

where $1 + \sum_{j=1}^{2} a_j = 0$. (This ensures $Qu = 0$ on $x_n = 0$). Now,

$$Q\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} Qu \text{ if } i \neq n .$$
$$= \frac{\partial}{\partial x_n} Ru \text{ if } i = n$$

where

$$(Ru)(x_1,\ldots,x_n) = \begin{cases} 0 & \text{if } x_n < 0 \\ u(x_1,\ldots,x_n) + \sum_{j=1}^{2} \frac{a_j}{-j} u(x_1,\ldots,x_{n-1},-jx_n) \end{cases}$$

$R: H^1(\mathbb{R}^N) \to H_0^1(\mathbb{R}_+^N)$ so we require $1 + \sum_{j=1}^{2} \frac{a_j}{-j} = 0$. Extend Q onto $H^1(\mathbb{R}^N)$ by continuity;

Let $P = {}^tQ$ (formal adjoint)

$$P: H_0^1(\mathbb{R}_+^N)' \to H^1(\mathbb{R}^N)'$$

That is, $P: H^{-1}(\mathbb{R}_+^N) \to H^{-1}(\mathbb{R}_N)$.

We then have the properties that

$$\frac{\partial}{\partial x_i} P = P \frac{\partial}{\partial x_i} \qquad i \neq n$$

$$\text{Since } \langle \frac{\partial}{\partial x_i} P u, \varphi \rangle = -\langle Pu, \frac{\partial \varphi}{\partial x_i} \rangle = -\langle u, Q\frac{\partial \varphi}{\partial x_i} \rangle$$
$$= -\langle u, \frac{\partial}{\partial x_i} Q\varphi \rangle$$
$$= \langle \frac{\partial u}{\partial x_i}, Q\varphi \rangle = \langle P\frac{\partial u}{\partial x_i}, \varphi \rangle$$

for $\varphi \in H^1(\mathbb{R}^N)$.

Hence $\frac{\partial u}{\partial x_i} \in H^{-1}(\mathbb{R}_+^N) \implies \frac{\partial}{\partial x_i} Pu = P\frac{\partial u}{\partial x_i} \in H^{-1}(\mathbb{R}^N)$.

$\frac{\partial}{\partial x_n} P = {}^tR \frac{\partial}{\partial x_n}$ where ${}^tR : H^{-1}(\mathbb{R}_+^N) \to H^{-1}(\mathbb{R}^N)$. If $\frac{\partial u}{\partial x_n} \in H^{-1}(\mathbb{R}_+^N)$, then

$\frac{\partial}{\partial x_n} Pu \in H^{-1}(\mathbb{R}^N)$. Finally, we must prove P is an extension

$$H^{-1}(\mathbb{R}^N_+) \xrightarrow{P} H^{-1}(\mathbb{R}^N) \xrightarrow{\pi} H^{-1}(\mathbb{R}^N_+)$$
$$H^1_0(\mathbb{R}^N_+) \xleftarrow{Q} H^1(\mathbb{R}^N) \xleftarrow{\sim} H^1_0(\mathbb{R}^N_+)$$

$\sim$ is the map: $\tilde{u}$ is extended by 0 on $\mathbb{R}^N_-$.

π = transpose of $\sim$ = restriction to $\mathbb{R}^N_+$.

We need only note that πP = identity $\Leftrightarrow$ $Q\sim$ = identity which is true. ∎

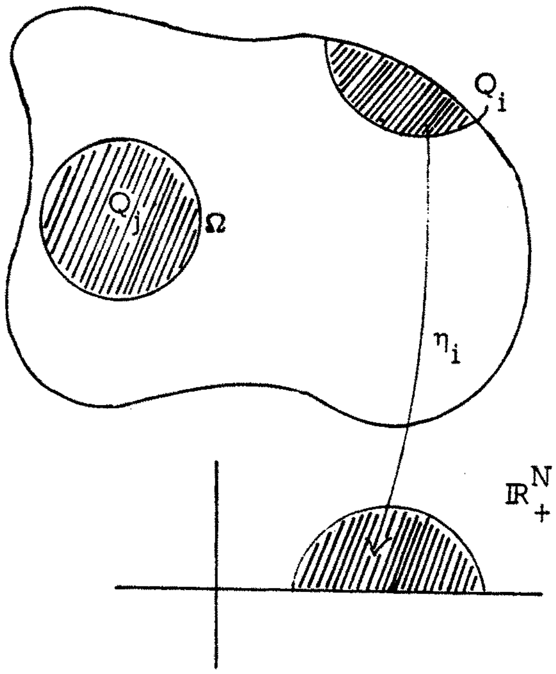
Lemma 4: $\chi(\mathbb{R}^N_+) = L^2(\mathbb{R}^N_+)$.

Proof: $u \in \chi(\mathbb{R}^N_+) \Rightarrow Pu \in \chi(\mathbb{R}^N) = L^2(\mathbb{R}^N)$ by Lemma 1. Hence $\pi P u = u \in L^2(\mathbb{R}^N_+)$. ∎

Lemma 5: Let Ω be a regular, open, bounded set. Then $\chi(\Omega) = L^2(\Omega)$.

Proof: Write $u = \sum \theta_i u$ where θ_i are a smooth partition of unity subordinate to a given covering of Ω.

Now, if $\theta_j \in \mathcal{D}(\Omega)$, $\theta_j u \in \chi(\mathbb{R}^N) = L^2(\mathbb{R}^N)$ so $\theta_j u \in L^2(\Omega)$ (we can extend $\theta_j u$ onto $\mathbb{R}^N$ as 0 outside the support of θ_j.). If $\theta_i \in \mathcal{D}(\bar{\Omega})$ (take a function η_i that has 2 bdd derivatives as does η_i^{-1}.)

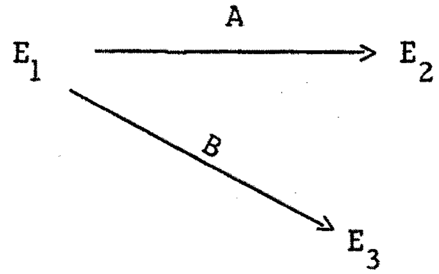


Then $\theta_i u \in \chi(\Omega) \overset{\eta_i}{\rightarrow} (\theta_i u) \circ \eta^{-1} \in \chi(\mathbb{R}^N_+) \Rightarrow (\theta_i u)\circ\eta^{-1} \in L^2(\mathbb{R}^N_+)$, by Lemma 4. Map back via η_i to get the result, since η_i preserves L^2. ∎

<u>Lemma 6:</u> $L^2(\Omega) \xrightarrow{\text{grad}} (H^{-1}(\Omega))^N$ given by $u \rightarrow (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$ has closed range in $(H^{-1}(\Omega))^N$. (Ω is as in Lemma 5.)

<u>Proof:</u> For the proof, we require the following lemma by Peetre. Let E_1, E_2, and E_3 be Banach spaces and suppose we have:

($\approx$ = equivalent norm)



such that $\|u\|_{E_1} \approx \|Au\|_{E_2} + \|Bu\|_{E_3}$, and suppose B is compact. Then Ker(A) is of finite dimension in E_1, and Im(A) is closed in E_2. The proof of this lemma is left to the reader.

For the proof of Lemma 6, take $E_1 = L^2(\Omega)$, $E_2 = (H^{-1}(\Omega))^N$, $E_3 = H^{-1}(\Omega)$, A = grad, and B the injection. Note that the injection $H^1_0 \rightarrow L^2$ is compact, so the dual injection $L^2(\Omega) \rightarrow H^{-1}$ is also.

To satisfy the hypotheses, we must show $\|u\|_{L^2} \approx |\text{grad}u|_{(H^{-1})^N} + \|u\|_{H^{-1}}$. We know that $|\text{grad } u|_{(H^{-1})^N} + |u|_{H^{-1}} \leq C|u|_{L^2}$. Since we have Banach spaces, we need to prove that $|\text{grad } u|_{(H^{-1})^N} + |u|_{H^{-1}}$ is a complete norm on L^2.

Take a Cauchy sequence $\{u_n\}$ in the above norm. Then $\{\text{grad } u_n\}$ is a Cauchy sequence in $(H^{-1})^N$, $\{u_n\}$ is Cauchy in H^{-1}. H^{-1} is complete

$$\Rightarrow \quad \begin{matrix} u_n \rightarrow u \text{ in } H^{-1} \\ \text{grad } u_n \rightarrow \text{grad } u \in (H^{-1})^N \end{matrix} \quad \Rightarrow u \in \chi(\Omega) = L^2.$$

This completes the hypotheses of the verification of Peetre's lemma. ∎

Lemma 7: If $f \in (H^{-1}(\Omega))^N$ and $\langle f, \varphi\rangle = 0 \ \forall \varphi \in (H_0^1(\Omega))^N$ such that $\operatorname{div} \varphi = 0$, then $f = \operatorname{grad}(p)$ where $p \in L^2(\Omega)$.

Proof: In $(H^{-1}(\Omega))^N$ take $Y = \{\operatorname{grad}(p) : p \in L^2(\Omega)\}$. In $(H_0^1(\Omega))^N$, take take $Z = \{u \mid \operatorname{div} u = 0\}$. Then the lemma states that if $f \in Z^{\perp}$, $f \in Y$. By Lemma 6 Y is closed in $(H^{-1}(\Omega))^N$, so it is enough to prove $Y^{\perp} = Z$. Then $Z^{\perp} = Y^{\perp\perp} = \bar{Y} = Y$. Now, $u \in Y^{\perp}$ if $(u, \operatorname{grad} p) = 0 \ \forall p \in L^2$. But for $p \in \mathcal{D}(\Omega)$, $(u, \operatorname{grad} p) = -(\operatorname{div} u, p)$ so $u \in Y^{\perp} \Rightarrow \operatorname{div} u = 0$. Hence $u \in Z$, and $Y^{\perp} = Z$. ■

Lemma 8: If $p \in L^2(\Omega)$ and $\int_\Omega p \, dx = 0$, then $\exists\, u \in (H_0^1(\Omega))^N$ such that $\operatorname{div} u = p$. (Ω connected.) We can choose u depending continuously and linearly on p.

Proof: $\operatorname{grad}: L^2(\Omega) \to (H^{-1}(\Omega))^N$ has closed range by Lemma 6. Let $N = \{u : \operatorname{grad} u = 0\} = \{u = \text{constant}\}$. Then $L^2(\Omega)/N \underset{\text{isomorphic}}{\cong}$ Image(grad).

The adjoint of this map is

$$(H_0^1(\Omega))^N \big/ \{u : \operatorname{div} u = 0\} \xrightarrow{\ \operatorname{div}\ } L^2(\Omega)/m$$

where $m = \{u : u \perp \text{constants}\} = \{u : \int_\Omega u = 0\}$. $Y = \operatorname{grad}(L^2)$ is closed in $(H^{-1}(\Omega))^N$,

$$Y' = (H_0^1(\Omega))^N \big/ Y^{\perp} = (H_0^1(\Omega))^N \big/ Z \quad \text{where } Z \text{ is } \{u : \operatorname{div} u = 0\}.$$

Now, if $u \in H_0^1$, $p \in L^2$, $(\operatorname{grad} p, u) = -(p, \operatorname{div} u)$ apply the surjectivity of the above map to complete the proof of the lemma. ■

Lemma 9: If $f \in (H^{-1}(\Omega))^N$ and $(f, \varphi) = 0 \ \forall \varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{div} \varphi = 0$, then $f = \operatorname{grad} p$ where $p \in L^2$.

Proof: Let Ω_n be an increasing sequence of connected regular open sets such that $\overline{\Omega_n} \subset \Omega$ and $\bigcup \Omega_n = \Omega$.

Take $u \in (H_0^1(\Omega_n))^N$ such that $\operatorname{div} u = 0$. Then for small ε $\rho_\varepsilon * u \in (\mathcal{D}(\Omega))^N$ and $\operatorname{div}(\rho_\varepsilon * u) = \rho_\varepsilon * \operatorname{div} u = 0$. So $(f,u) = \lim_{\varepsilon \to 0}(f, \rho_\varepsilon * u) = 0$. By Lemma 7 applied to Ω_n $f|_{\Omega_n} = \operatorname{grad} p_n$ with $p_n \in L^2(\Omega_n)$.

Since $(p_{n+1} - p_n)$ is a constant on Ω_n we can suppose that $p_{n+1} = p_n$ on Ω_n, and we have proved that $f = \operatorname{grad} p$ with $p \in L^2_{loc}(\Omega)$.

Then we prove the lemma if Ω is regular and strictly star shaped with respect to a point (taking this point as origin this means that $\theta \bar{\Omega} \subset \Omega$ for $0 \le \theta < 1$). If $u \in (H_0^1(\Omega))^N$ and $\operatorname{div} u = 0$ we define $u_\theta(x) = u(\frac{x}{\theta})$ which still has divergence 0 but has compact support in Ω and can be approached by $\rho_\varepsilon * u_\theta$. Since $(f,u) = \lim_{\theta \to 1} \lim_{\varepsilon \to 0} (f, \rho_\varepsilon * u_\theta) = 0$ we have $f = \operatorname{grad} q$ with $q \in L^2(\Omega)$ by Lemma 7.

In the general case each point of $\partial\Omega$ has a neighborhood ω connected regular and strictly star shaped. So $f = \operatorname{grad} q$ on ω with $q \in L^2(\omega)$. Since $p|_\omega - q$ is constant on ω we have $p \in L^2(\omega)$ and this gives $p \in L^2(\Omega)$. ■

Lemma 10: $\{\varphi \in (\mathcal{D}(\Omega))^N ; \operatorname{div}\varphi = 0\}$ is dense in $\{\varphi \in (H_0^1)^N : \operatorname{div}\varphi = 0\}$.

Proof: Take $f \in \{\varphi \in (\mathcal{D}(\Omega))^N : \operatorname{div}\varphi = 0\}^\perp$ and $f \in (H^{-1}(\Omega))^N$. By Lemma 9 $f = \operatorname{grad} p$ where $p \in L^2(\Omega)$, $\Rightarrow f \in \{\varphi \in (H_0^1)^N : \operatorname{div}\varphi = 0\}^\perp$ by Lemma 7.

Hence $\{\varphi \in (\mathcal{D}(\Omega))^N : \operatorname{div}\varphi = 0\} \supseteq \{\varphi \in (H_0^1)^N : \operatorname{div}\varphi = 0\}$. It is trivial that the other inclusion holds. ■

Recall now that $\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega))^N : \operatorname{div}\varphi = 0\}$ and $V = \{u \in (H_0^1(\Omega))^N : \operatorname{div} u = 0\}$ and that $\|u\|^2 = \sum_{i,j} \|\frac{\partial u_i}{\partial x_j}\|^2_{L^2(\Omega)}$. So Lemma 10 proves the second inclusion in Proposition 1.

Let H = the closure of $\mathcal{V}$ in $(L^2(\Omega))^N$.

Claim: $H = \{u \in (L^2(\Omega))^N : \operatorname{div} u = 0$ and $u \cdot \eta|_{\partial\Omega} = 0\}$ (η is the normal vector to $\partial\Omega$).

Lemma 11. Let $\chi = \{u \in (L^2(\Omega))^N : \operatorname{div} u \in L^2(\Omega)\}$, and define

$$\|u\|_\chi^2 = \sum_i \|u_i\|^2_{L^2(\Omega)} + \|\operatorname{div} u\|^2_{L^2(\Omega)} .$$

Then $(\mathcal{D}(\bar{\Omega}))^N$ is dense in χ.

Proof: Similar to that of a previous lemma. ■

Lemma 12: If $u \in \chi$, $\varphi \in H_0^1(\Omega)$, then $(\operatorname{div} u, \varphi) + (u, \operatorname{grad}\varphi) = 0$.

Proof: For smooth functions, integrate by parts. Then use density arguments. ■

Lemma 13: $u \to$ trace of $u.n$ on $\partial\Omega$ extends to a continuous map from χ into $H^{-\frac{1}{2}}(\partial\Omega) = H^{\frac{1}{2}}(\partial\Omega)'$.

Proof: Let ψ be the trace on $\partial\Omega$ of a function φ in $H^1(\Omega)$. ($\psi \in H^{\frac{1}{2}}(\partial\Omega)$.)

For $u \in \chi$ we have a linear map: $L(\psi) = (\operatorname{div} u, \varphi) + (u, \operatorname{grad}\varphi)$ which depends only on ψ. That is, $\varphi_1|_{\partial\Omega} = \varphi_2|_{\partial\Omega} = \psi \Rightarrow \varphi_1 - \varphi_2 \in H_0^1(\Omega)$, so by Lemma 2,

$$(\operatorname{div} u, \varphi_1) + (u_1 \operatorname{grad} \varphi_1) = (\operatorname{div} u, \varphi_2) + (u, \operatorname{grad} \varphi_2) .$$

If $u \in (\mathcal{D}(\bar{\Omega}))^N$, then $L(\psi) = \int_{\partial\Omega} (\vec{u} \cdot \vec{n})\, \psi \, d\Sigma$ since

$$\begin{aligned}(\operatorname{div} u, \varphi) + (u, \operatorname{grad} \varphi) &= \sum_i \int_\Omega \left(\frac{\partial u_i}{\partial x_i} \varphi + u_i \frac{\partial \varphi}{\partial x_i}\right) dx \\ &= \sum_i \int_\partial (u_i \varphi)\, n_i \, d\Sigma \\ &= \int_{\partial\Omega} \psi \sum_i u_i n_i \, d\Sigma .\end{aligned}$$

Since there is a continuous map: $\psi \in H^{\frac{1}{2}}(\partial\Omega) \xrightarrow{\text{lift}} \varphi \in H^1(\Omega)$ with $\varphi\big|_{\partial\Omega} = \psi$.

This proves that for $u \in \chi$, $u \cdot \eta \in (H^{\frac{1}{2}}(\partial\Omega))'$

$$= H^{-\frac{1}{2}}(\partial\Omega) . \qquad \blacksquare$$

<u>Lemma 14:</u> If $u \in (L^2(\Omega))^N$ satisfies $(u, \operatorname{grad} \varphi) = 0 \ \forall \varphi \in H^1(\Omega)$, then $\operatorname{div} u = 0$ and $u \cdot \eta = 0$ on $\partial\Omega$.

<u>Proof:</u> First choose $\varphi \in \mathcal{D}(\Omega)$. By definition of derivatives,

$$\left(\frac{\partial u_i}{\partial x_i}, \varphi\right) = -\left(u_i, \frac{\partial \varphi}{\partial x_i}\right)$$

so

$$(\operatorname{div} u, \varphi) = -(u, \operatorname{grad} \varphi) = 0 \quad \forall \varphi \implies \operatorname{div} u = 0$$

in the sense of distributions, so, in fact, $u \in \chi$. We know

$$0 = (\operatorname{div} u, \varphi) + (u, \operatorname{grad} \varphi) = \int_{\partial\Omega} u \cdot \eta \, \varphi \, d\Sigma \qquad \forall \varphi \in H^{\frac{1}{2}}(\partial\Omega)$$

so $u \cdot n = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)$. $\blacksquare$

From here, prove the density theorem by orthogonality arguments. So $\mathcal{V}$ is dense in H. $\blacksquare$

<u>Question:</u> What is V' ?

$V \underset{\text{dense}}{\subseteq} H$. Identify H and its dual.

(The scalar product in H is $\sum_i \int u_i v_i = (u,v)$).

If $L \in V'$, then $v \to Lv \in \mathbb{R}$ is a continuous linear map. Since V is closed in $(H_0^1(\Omega))^N$, we can extend L to $\tilde{L} : (H_0^1(\Omega))^N \to \mathbb{R}$: i.e. $\exists f_1, \ldots, f_n \in H^{-1}(\Omega)$ such that

$$\tilde{L}(v) = \sum_i (v_i, f_i)_{H_0^1, H^{-1}} \quad \text{for} \quad v \in (H_0^1(\Omega))^N$$

and for $v \in V$, $\tilde{L}(v)$ so defined coincides with $L(v)$.

Two such expressions for $\tilde{L}$ coincide on V if and only if $f_i = g_i + \frac{\partial p}{\partial x_i}$ for some $p \in L^2(\Omega)$. ($\{f_i\}$ and $\{g_i\}$ are the different expressions for $\tilde{L}$.)

$$(f_i - g_i) \in (H^{-1}(\Omega))^N \text{ and } (f-g,v) = 0 \quad \forall v \in V.$$

We have seen that this holds $\Leftrightarrow$ f-g = grad p where $p \in L^2(\Omega)$.

Hence $V' = (H^{-1}(\Omega))^N / \{\text{grad } p : p \in L^2(\Omega)\}$

By definition

$$\|f\|_{V'} = \sup_{\substack{\|u\|_V \le 1 \\ (u \in V)}} \sum_i (f_i, u_i)_{H^{-1}, H_0^1}$$

Note that $V \underset{\text{dense}}{\subseteq} H \underset{\text{dense}}{\subseteq} V'$.

Henceforth, $|\cdot|$ will denote the norm in H. $\|\cdot\|$ will denote $\|\cdot\|_V$ and $\|\cdot\|_* = \|\cdot\|_{V'}$.

Then for $u \in V$, $|u| \le C\|u\|$

for $u \in H$, $\|u\|_* \le C|u|$.

<u>Back to Navier-Stokes Equations:</u>

Remember that $\rho = 1$. Then the problem can be stated as follows:

$$\frac{\partial u_i}{\partial t} - \nu\Delta u_i + \sum_j \frac{\partial u_i}{\partial x_j} u_j = f_i - \frac{\partial p}{\partial x_i} \quad \text{in } \Omega \times]0,T[$$

$$u_i(x,0) = u_{0,i} \quad (\text{given: } u_0 \in H)$$

$$u_i(x,t) = 0 \quad \text{if } x \in \partial\Omega$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times]0,T[$$

Usually $f \equiv 0$. It is natural to seek u in $L^2(0,T;V) \cap L^\infty(0,T;H)$. ($L^2(0,T;V)$ so that the energy lost to viscosity is finite; $L^\infty(0,T;H)$ so that the kinetic energy of the system remains bounded.)

We will prove an existence theorem, (uniqueness for $N = 2$) with $f \in L^2(0,T;V')$ or $L^2(0,T;H)$.

__Notation:__ 1) $a(u,v) = ((u,v)) = \sum_{i,j} \int \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx = \sum_i \int (-\Delta u_i, v_i) dx$ for $u,v \in V$. This is a continuous bilinear form on V.

2) $b(u,v,w) = \sum_{i,j} \int u_j \frac{\partial v_i}{\partial x_j} w_i dx$ for $u,v,w \in V$.

Remember that Ω is bounded.

If we are in the case $N = 2$, we have

__Lemma 1:__ b is a trilinear continuous form on $V \times V \times V$ and $b(u,v,w) + b(u,w,v) = 0 \ \forall\, u,v,w \in V$. In particular, $b(u,v,v) = 0 \ \forall\, u,v \in V$.

__Proof:__ Use $V \subseteq (L^4(\Omega))^2$ for $N \leq 4$ (Sobolev's Imbedding theorem, which can be more generally stated as

$H^m(\Omega) \subseteq L^p(\Omega)$ for $\frac{1}{p} = \frac{1}{2} - \frac{m}{N}$ for $m < \frac{N}{2}$.)

__Lemma 2:__ $\|\varphi\|_{L^4(\mathbb{R}^2)} \leq C \|\varphi\|^{\frac{1}{2}}_{H^1(\mathbb{R}^2)} \|\varphi\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}$ for all $\varphi \in H^1(\mathbb{R}^2)$.

Hence $H^1 \subseteq L^4$ for $N = 2$, and $u \in V \Rightarrow u \in (H^1_0(\Omega))^2 \Rightarrow u \in (H^1(\mathbb{R}^2))^2$ (extended by 0). Next,

$$|b(u,v,w)| \leq \sum_{i,j} \int_\Omega |u_j \frac{\partial v_i}{\partial x_j} w_i| dx .$$

Then use Hölder's inequality

$$\leq \sum_{i,j} \|u_i\|_{L^4} \left\|\frac{\partial v_i}{\partial x_j}\right\|_{L^2} \|w_i\|_{L^4} \quad \text{since} \quad \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{2} = 1 .$$

Using the lemma,

$$|b(u,v,w)| \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\| \quad |w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}$$

$$\leq C\|u\| \, \|v\| \, \|w\|$$

$$b(u,v,w) + b(u,w,v) = \sum_{i,j} \int_\Omega u_j \left[\frac{\partial v_i}{\partial x_j} w_i + \frac{\partial w_i}{\partial x_j} v_i \right] dx$$

$$= \sum_{i,j} \int_\Omega u_j \frac{\partial}{\partial x_j} (v_i \, w_i) dx.$$

If $u \in \mathcal{V}$ then u is smooth: integrate by parts,

$$= -\sum_{i,j} \int_\Omega \frac{\partial u_j}{\partial x_j} v_i w_i dx = -\sum_i \int_\Omega v_i w_i \sum_j \frac{\partial u_j}{\partial x_j} dx = 0$$

since $\sum_j \frac{\partial u_j}{\partial x_j} = 0$. Since $\mathcal{V}$ is dense in V and b is continuous, $b(u,v,w) + b(u,w,v) = 0 \;\; \forall \; \underline{u \in V}$, $v \; w \in V$. This completes the proof of Lemma 1.

<u>Proof:</u> (of Lemma 2) We will prove this if $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

$$\varphi^2(x_1,x_2) = \int_{y=-\infty}^{y=x_2} \frac{\partial}{\partial x_2} \varphi^2(x_1,y)dy$$

by the fundamental theorem of calculus

$$= -\int_{y=-\infty}^{y=x_2} 2\varphi(x_1,y) \frac{\partial \varphi}{\partial x_2} (x_1,y)dy .$$

Apply Cauchy-Schwarz , and we have

$$|\varphi(x_1,x_2)|^2 \leq 2\left(\int_{-\infty}^{x_2} |\varphi(x_1,y)|^2 dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{x_2} \left|\frac{\partial \varphi}{\partial x_2}(x_1,y)\right|^2 dy\right)^{\frac{1}{2}}$$

(we may as well let the upper limits be ∞.) If we do the same for x_2, we get

$$|\varphi(x_1,x_2)|^2 \leq 2\left(\int_{-\infty}^{\infty} |\varphi(y,x_2)|^2 dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \left|\frac{\partial \varphi}{\partial x_1}(y,x_2)\right|^2 dy\right)^{\frac{1}{2}}$$

multiplying, we get

$|\varphi(x_1,x_2)|^4 \le \lambda(x_1)\mu(x_2)$ where $\lambda(x_1)$ and $\mu(x_2)$ are the right halves of the respective inequalities above. Integrating

$$\iint_{\mathbb{R}^2} |\varphi(x_1,x_2)|^4 dx_1 dx_2 \le \left(\int_{\mathbb{R}} \lambda(x_1)dx_1\right)\left(\int_{\mathbb{R}} \mu(x_2)dx_2\right).$$

By Cauchy-Schwarz,

$$\int_{\mathbb{R}} \lambda(x_1)dx_1 \le 2\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(x_1,y)|^2 dy dx_1\right)^{\frac{1}{2}}\left(\iint_{\mathbb{R}\mathbb{R}}\left|\frac{\partial\varphi}{\partial x_2}(x_1,y)\right|^2 dy dx_1\right)^{\frac{1}{2}}$$

$$= 2\|\varphi\|_{L^2(\mathbb{R}^2)} \left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\mathbb{R}^2)} .$$

Similarly, we get a bound on μ. Hence,

$$\|\varphi\|^4_{L^4(\mathbb{R}^2)} \le 4\|\varphi\|^2_{L^2(\mathbb{R}^2)} \left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(\mathbb{R}^2)} \left\|\frac{\partial\varphi}{\partial x_2}\right\|_{L^2(\mathbb{R}^2)}$$

$$\le 2\|\varphi\|^2_{L^2(\mathbb{R}^2)}\left[\left\|\frac{\partial\varphi}{\partial x_1}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\frac{\partial\varphi}{\partial x_2}\right\|^2_{L^2(\mathbb{R}^2)}\right]$$

$$\le 2\|\varphi\|^2_{L^2(\mathbb{R}^2)} \|\varphi\|^2_{H^1(\mathbb{R}^2)} \qquad ■$$

<u>Definition:</u> $B: V\times V \to V'$ is defined by $(B(u,v),w) = b(u,v,w)$ $\forall\, u,v,w \in V$. By Lemma 1, $B(u,v)$ is continuous bilinear from $V \times V$ into V'

$$A : V \to V'$$

is defined by $(A(u),v) = a(u,v)$ $\forall\, u,v \in V$. A is continuous and linear from V into V'

$$(A(u),u) = \|u\|^2 .$$

<u>Lemma 3:</u> If $u,v \in L^2(0,T;V) \cap L^\infty(0,T;H)$, then $B(u,v) \in L^2(0,T;V')$.

<u>Proof:</u> Use $\|B(u,v)\|_* \le C\|u\| \;\|v\|$ to obtain $B(u,v) \in L^1(0,T;V')$.

$(B(u,v),w) = -(B(u,w),v)$ by Lemma 1. $|(B(u,v),w)| \le C\|u\|_{L^4}\|w\|\,\|v\|_{L^4}$, so $\|B(u,v)\|_* \le C\|u\|_{L^4}\|v\|_{L^4}$.

Hence, we obtain

$$\|B(u,v)\|_* \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}} .$$

Now,

$$|u| \in L^\infty(0,T) \text{ and } \|u\| \in L^2(0,T) \implies$$

$(u)^{\frac{1}{2}}\|u\|^{\frac{1}{2}} \in L^4(0,T)$ (that is $L^2(0,T;V) \cap L^\infty(0,T;H) \subseteq L^4(0,T;L^4(\Omega))$). ∎

Remark: The above was for $N = 2$.

For $N = 3$, $B(u,v) \in L^{4/3}(0,T;V')$

For $N = 4$, $B(u,v) \in L^1(0,T;V')$.

Theorem: If $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and u satisfies the Navier-Stokes equations with $f \in L^2(0,T;V')$, then $\frac{du}{dt} \in L^2(0,T;V')$ and $(\frac{du}{dt},\varphi) + \nu(A(u),\varphi) + (B(u,u),\varphi) = (f,\varphi)\ \forall\ \varphi \in V$. a.e.t.

Moreover, $u \in C^0(0,T;H)$; u is continuous from $[0,T] \to H$, and $u(0) = u_0$.

Proof: $u \in L^2(0,T;V) \cap L^\infty(0,T;H) \implies Au + B(u,u) - f \in L^2(0,T;V')$. Take $\varphi \in \mathcal{V}$, $\psi \in \mathcal{D}(]0,T[)$. Consider the equation

$$\frac{\partial u}{\partial t} + \nu Au + B(u,u) = f - \text{grad} p$$

in the sense of distributions on $\Omega \times]0,T[$. If we apply $\varphi(x)\psi(t)$, we obtain

$$-\iint_{\Omega\times]0,T[} u\frac{\partial}{\partial t}(\varphi(x)\psi(t))dxdt + \iint_{\Omega\times]0,T[} \nu \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial \varphi_i}{\partial x_j}\psi(t)dxdt$$

$$+ \int_0^T\int_\Omega (B(u,u),\varphi)\psi(t)dxdt = \int_0^T (f,\varphi)_{H^{-1},H_0^1}\psi(t)dt$$

because the last term $\langle -\text{grad } p, \varphi(x)\psi(t)\rangle$ is 0 as $\text{div}\,\varphi = 0$. Thus, after rewriting, we get for all $\varphi \in V$ (by density)

$$-\int_0^T (u,\varphi)_H \frac{\partial\psi}{\partial t}dt + \int_0^T \nu\, a(u,\varphi)\psi(t)dt + \int_0^T (B(u,u),\varphi)_{V',V}\psi(t)dt$$
$$= \int_0^T (f,\varphi)_{V',V}\psi(t)dt.$$

Let $f - \nu Au - B(u,u) = g \in L^2(0,T;V')$. Rewrite:

$$-\int_0^T (u,\varphi)_H \frac{\partial\psi}{\partial t}dt = \int_0^T (g,\varphi)_{V',V}\psi(t)dt.$$

Notice that $t \to (u,\varphi)$ is a good function of t. ($\in L^\infty(0,T)$.)

In the sense of distributions:

$$\frac{d}{dt}(u,\varphi)_H = (g,\varphi)_{V',V}.$$

So u satisfies $\frac{d}{dt}(u,\varphi) = (g,\varphi) = (f,\varphi) - \nu(A(u),\varphi) - (B(u,u),\varphi)$. This is the desired result.

$$|(\frac{\partial u}{\partial t},\varphi)| \leq |(g,\varphi)| \leq \|g\|_* \|\varphi\| \Longrightarrow \|\frac{\partial u}{\partial t}\|_* \leq \|g(t)\|_* \in L^2(0,T).$$

Converse: If $S_i = \frac{\partial u_i}{\partial t} - \nu\Delta u_i + \sum_j u_j \frac{\partial u_i}{\partial x_j} - f_i$, the new formulation gives, for $\varphi \in \mathcal{D}(\Omega)$, that (S,φ) is a distribution in t, and, if $\varphi \in \mathcal{V}$, then $(S,\varphi) = 0$. From this one can show that $S = \text{grad } p$ where $P \in \mathcal{D}'(\Omega \times]0,T[)$.∎

The meaning of $\frac{\partial u}{\partial t}$:

Definition 1: u is a distribution in (x,t) $\Longrightarrow$ $\frac{\partial u}{\partial t}$ is also a distribution in (x,t).

Definition 2: $u \in L^2(0,T;X)$. Then $\frac{\partial u}{\partial t}$ will be a vectorial distribution in X. (S is a vectorial distribution on $]0,T[$ into X - a Banach space, if we have a mapping $\langle S,\varphi\rangle$ from $\varphi \in \mathcal{D}(0,T) \to X$ which is linear and continuous.

Example: $u \in L^1_{loc}(0,T;X)$, $\langle S,\varphi\rangle = \langle u,\varphi\rangle = \int_0^T \underset{X}{u(t)}\, \underset{\mathbb{R}}{\varphi(t)}dt \in X$.

Then the derivative is defined by

$$\langle \frac{\partial S}{\partial t}, \varphi\rangle = -\langle S, \frac{\partial\varphi}{\partial t}\rangle .$$

We will often be interested in spaces like

$$E = \{u \in L^p(0,T;X) : \frac{du}{dt} \in L^q(0,T;Y)\}$$

where $1 < p,q < \infty$ and $X \subseteq Y$ densely, X,Y are Banach spaces.

This means $\exists\ f \in L^q(0,T;Y) : \forall\ \varphi \in \mathcal{D}(0,T)$,

$$\int_0^T -u(t)\frac{d\varphi}{dt}(t)dt = \int_0^T f(t)\varphi(t)dt \quad .$$
$$\nwarrow X \qquad\qquad \nwarrow Y$$

<u>Theorem:</u> (Density) Let X,Y be Banach spaces, with $X \underset{\text{dense}}{\subseteq} Y$. Let E be as above with $\|u\|_E = \|u\|_{L^p(0,T;X)} + \|\frac{du}{dt}\|_{L^p(0,T;Y)}$

Then $\mathcal{D}([0,T];X)$ is dense in E, where $\mathcal{D}([0,T];X)$ are the smooth functions from $[0,T]$ into X.

We give a proof for <u>$p = q$</u>.

<u>Proof:</u> 1) If $a \in C^1([0,T])$, then $u \to au$ is a linear map from E into E. $\frac{d}{dt}(au) = a\frac{du}{dt} + \frac{da}{dt}u$, and the first term is the product of a bounded function and one in $L^p(0,T;Y)$, while the second term is the product of a bounded function and one in $L^p(0,T;X) \subseteq L^p(0,T;Y)$.

Choose $a,b \in \mathcal{D}([0,T])$, $0 \leq a \leq 1$, $0 \leq b \leq 1$: $a+b = 1$ on $[0,T]$. Let $\text{supp}(a) \subseteq [0, \frac{2}{3}T]$, while $\text{supp}(b) \subseteq [\frac{1}{3}T,T]$. Now approach au and bu. (Thus approaching $au+bu=u$.)

$$v = au \in \{f \in L^p(0,\infty;X) : \frac{df}{dt} \in L^p(0,\infty;Y)\}$$

($au = 0$ for (T,∞).)

Let $v_h(t) = v(t+h)$ $t \geq -h$. Then $v_h \to v$ in E. Regularize by $p_\epsilon^* * v_h \to v_h$, where p is a smooth function with values in X. ∎

<u>Theorem:</u> (Trace) $u \to u(0)$ is continuous from E into Y.

<u>Proof:</u> By applying density: we need to show that it is continuous from

$$\mathcal{D}([0,T];X) \to Y \qquad (\text{in } \|\cdot\|_E) .$$

We need to find C : $\|u(0)\|_Y \leq C[\|u\|_{L^p(0,T;X)} + \|\frac{du}{dt}\|_{L^p(0,T;Y)}]$

for all $u \in \mathcal{D}([0,T];X)$.

Pick $a(t)$ smooth so that $a(0) = 1$ and $a(t) = 0$.

Let $v = au$. $\quad v(0) = -\int_0^T \frac{dv}{dt} dt$. $\qquad (v(T) = 0)$.

Then by Hölder's inequality

$$\|u(0)\|_Y = \|v(0)\|_Y \leq T^{1/p'} \|\frac{dv}{dt}\|_{L^p(0,T;Y)} \leq C[\|u\|_{L^p} + \|\frac{du}{dt}\|_{L^p}] . \blacksquare$$

<u>Corollary:</u> $E \subseteq C^0([0,T];Y)$ if u is smooth. The same estimate gives

$$\|u\|_{C^0([0,T];Y)} = \max_{0 \leq s \leq T} \|u(s)\|_Y \leq C\|u\|_E .$$

(Note: C depends on T.)

<u>The case for Hilbert spaces:</u>

Suppose $V \underset{\text{dense}}{\subseteq} H \underset{\text{dense}}{\subseteq} V'$ where H is a Hilbert space.

<u>Theorem:</u> $E = \{u \in L^2(0,T;V) : \frac{du}{dt} \in L^2(0,T;V')\} \subseteq C^0([0,T];H)$.

<u>Proof:</u> If u is smooth, we want to prove

$$|u(0)|_H \leq C\|u\|_E .$$

Put $v = au$, where a is as before. Then remark:

$$\frac{d}{dt} |v|_H^2 = 2(v, \frac{dv}{dt}) .$$

So
$$|v(T)|^2 - |v(0)|^2 = 2\int_0^T (v, \frac{dv}{dt}) dt$$

and, as $v(T) = 0$,

$$|v(0)|^2 = -2 \int_0^T (v, \frac{dv}{dt}) dt \leq \{\int_0^T \|v(t)\| \, \|\frac{dv}{dt}\|_* dt$$

$$\leq \int_0^T (\|v\|^2 + \|\frac{dv}{dt}\|_*^2 dt = \|v\|_E^2 \leq C\|u\|_E^2 . \qquad \blacksquare$$

Remark: If $f \in L^2(0,T;V')$, $u_0 \in H$ and u is in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and satisfies the Navier-Stokes equations, then $\frac{du}{dt} \in L^2(0,T;V')$,

$$\frac{d}{dt}(u,v) + \nu\, a(u,v) + b(u,u,v) = (f,v) \qquad \forall\, v \in V ,$$

$u \in C^0([0,T];H)$, so $u(0) = u_0$ has a meaning.

We know $a(u,v)$ is bilinear and continuous from $V \times V \to \mathbb{R}$ and $a(u,u) = \|u\|^2$, and that $b(u,v,w)$ is a trilinear continuous form on V, with $b(u,v,w) = -b(u,w,v)$, $(\Longrightarrow b(u,v,v) = 0)$ with

$$|b(u,v,w)| \le C\,|u|_H^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\|\, |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}} .$$

Remember that we needed $N \le 4$ for b to have the first group of properties, while the last estimate depended on $N = 2$.

Next, recall that we defined A and B so that $a(u,v) = (A(u),v)$ and $b(u,v,w) = B(u,v),w)$ $\forall\, u,v,w \in V$. We proved (with $N = 2$) that if $u,v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ then $B(u,v) \in L^2(0,T;V')$. (For $N = 3$; instead of L^2, we got $L^{4/3}$, while $N = 4$ gave L^1.)

Thus, we can state the following:

Theorem: If $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ solves the Navier-Stokes equations, then $\frac{du}{dt} + \nu A(u) + B(u,u) = f$ ($\leftarrow$ each term is in $L^2(0,T;V')$.)

$u(0) = u_0$.

(Note that we still are keeping $N = 2$:)

If $u \in L^2(0,T;V)$, and $\frac{du}{dt} \in L^2(0,T;V')$, then $u \in C^0(0,T;H)$ and moreover, $|u(t)|^2$ is absolutely continuous, $\frac{d}{dt}|u(t)|^2 = 2(\frac{du}{dt},u) \in L^1(0,T)$.

Since this is true for smooth functions by $|u(t)|^2 - |u(s)|^2 = 2\int_s^t (\frac{du}{dt},u)d\tau$, we will utilize the density of smooth functions.

Theorem (Uniqueness): If $f \in L^2(0,T;V')$ and $u_0 \in H$ then there is at most one solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

Proof: Let u_1, u_2 be two solutions. Then

$$\frac{d}{dt}(u_1-u_2) + \nu A(u_1-u_2) + B(u_1,u_1) - B(u_2,u_2) = 0$$

and $(u_1-u_2)(0) = 0$. Take the inner product of the above with u_1-u_2 to obtain

$$\tfrac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 + (B(u_1,u_1) - B(u_2,u_2),u_1-u_2) = 0 .$$

Now, $(B(u_1,u_1) - B(u_2,u_2),u_1-u_2) = (B(u_1-u_2,u_1),u_1-u_2) + (B(u_2,u_1-u_2),u_1-u_2)$.

But $(B(u_2,u_1-u_2),u_1-u_2)$ is of the form $(B(u,v),v) = b(u,v,v) = 0$.

$\Rightarrow$ We have that $|(B(u_1,u_1) - B(u_2,u_2),u_1-u_2)| \leq C\|u_1\|\,|u_1-u_2|\,\|u_1-u_2\|$ (from the above consideration and our estimates on b.)

Thus we obtain,

$$\begin{cases} \tfrac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \leq C\|u_1\|\,|u_1-u_2|\,\|u_1-u_2\| \\ |u_1 - u_2|(0) = 0 . \end{cases}$$

Use Young's inequality $(ab \leq \nu a^2 + \frac{b^2}{4\nu})$ in the right hand side to conclude

$$\tfrac{1}{4}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \leq \nu\|u_1-u_2\|^2 + \frac{C}{4\nu}\|u_1\|^2|u_1-u_2|^2$$

and so

$$\begin{cases} \frac{d}{dt}|u_1-u_2|^2 \leq C'\|u_1\|^2|u_1-u_2|^2 \\ |u_1-u_2|(0) = 0 . \end{cases}$$

Since $u_1 \in L^2$, $\|u_1\|^2 \in L^1$. Apply the following:

Lemma: (Gronwall's inequality) If $\varphi \geq 0$ satisfies

$$\frac{d\varphi}{dt} \leq \lambda(t)\varphi \qquad \lambda(t) \geq 0 \quad \lambda \in L^1(0,T),$$

then

$$\varphi(t) \leq \varphi(0)\exp\left(\int_0^t \lambda(s)ds\right) .$$

Proof: $\frac{d}{dt}[\varphi(t)\exp(-\int_0^t \lambda(s)ds)] \leq 0$. ∎

Using the lemma, $|u_1 - u_2| \equiv 0$. ∎

Theorem: (Existence for Navier-Stokes). If $f \in L^2(0,T;V')$ and $u_0 \in H$ then there exists a (unique) solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

Proof: We will use the Galerkin method.

First step: Choose a basis for V: that is, a set $\{w_1, w_2, \ldots\} \subseteq V$ such that $\forall m$, $\{w_1, \ldots, w_m\}$ is a linearly independent set, and the subspace spanned by $\{w_1, \ldots\}$ is dense in V .

Note: Such a basis exists $\Longleftrightarrow$ V is separable. Later, we will require other, "special" properties of our basis.

Second step: Study the approximate problem: that is, to find $u_n(t) = \sum_{i=1}^{n} g_i(t) w_i$ satisfying

$$\begin{cases} \left(\frac{du_n}{dt} + \nu A(u_n) + B(u_n, u_n) - f, w_i\right) = 0 \quad \text{for } i = 1, \ldots, n. \\ u_n(0) = u_{0n} \in \text{span}\{w_1, \ldots, w_n\} \quad \text{where} \quad u_{on} \to u \end{cases}$$

strongly in H as $n \to \infty$.

Remark: The space spanned by $\{w_i\}_{i=1}^{\infty} \underset{\text{dense}}{\subseteq} H$ and also is dense in V'.

The approximate problem is a system of ordinary differential equations for $g_1, \ldots, g_n$:

$$\sum_{i=1}^{n} \frac{dg_i(t)}{dt}(w_j, w_i) + G_j(t, g_1, \ldots, g_n) = 0 .$$

The matrix with entries (w_j, w_i) is invertible, since $\{w_i\}_{i=1}^{n}$ is linearly independent. Apply the inverse to obtain

$$\frac{dg_j}{dt} = \varphi_j(t, g_1, \ldots, g_n).$$

If the w_i are orthonormal, then the system was already in this form, since

$(w_j, w_i) = \delta_{ji}$. Using the Cauchy-Lipschitz existence theorem, we get solutions $u_n(t)$ to the approximate problem on some interval $[0, t_n) \subseteq [0, T]$.

Third step: We must show $t_n = T$ for $n = 1, 2, \ldots$ where $0 < T < \infty$. To do this, we must show

$$\max_{0 \leq x < t_n} |u_n(s)| \leq C < \infty \quad \text{(where } C \text{ may depend on } n\text{)}$$

so that $|u_n(t)|$ is bounded for $t \to t_n$.

Multiply the ith equation of the system by $g_i(t)$ and sum over i:

$$\left(\frac{du_n}{dt} + \nu A(u_n) + B(u_n, u_n) - f, u_n \right) = 0 \qquad (u_n = \sum g_i w_i).$$

Then,

$$\left(\frac{du_n}{dt}, u_n \right) = \tfrac{1}{2} \frac{d}{dt} |u_n|^2, \quad \nu(A(u_n), u_n) = \nu \|u_n\|^2,$$

and $(B(u_n, u_n), u_n) = 0$. This gives the estimate

$$\tfrac{1}{2} \frac{d}{dt} |u_n|^2 + \nu \|u_n\|^2 \leq \|f\|_* \|u_n\|.$$

Use Young's inequality to get $\leq \frac{\nu}{2} \|u_n\|^2 + \frac{1}{2\nu} \|f\|_*^2$, and so obtain

$$\frac{d}{dt} |u_n|^2 + \nu \|u_n\|^2 \leq \frac{1}{\nu} \|f\|_*^2.$$

Integrate from 0 to t where $0 \leq t < t_n$, and get

$$|u_n(t)|^2 + \nu \int_0^t \|u_n(s)\|^2 ds \leq |u_n(0)|^2 + \frac{1}{\nu} \int_0^t \|f(s)\|_*^2 ds$$

$\Longrightarrow |u_n(t)|^2 \leq |u_{0n}|^2 + \frac{1}{\nu} \int_0^T \|f(s)\|_*^2 ds$. This estimate is independent of t, so $t_n = T$ for all n.

Fourth step: Obtain sufficiently many estimates independently of n, to show $u_n \rightharpoonup u$, for some subsequence.

Since $u_{on} \to u_0$ in H, $|u_{on}| \leq C$ independently of n. This gives

$$\max_{0 \leq t \leq T} |u_n(s)|^2 + \int_0^T \|u_n(s)\|^2 ds \leq C \text{ independently of } n.$$

This means $\{u_n\}_{n=1}^{\infty} \subseteq K_1$, a bounded set in $L^\infty(0,T;H)$ and $\{u_n\}_{n=1}^{\infty} \subseteq K_2$

a bounded set in $L^2(0,T;V)$.

We can extract a subsequence u_m such $u_m \to u$ in $L^\infty(0,T;H)$ weak* and in $L^2(0,T;V)$ weak. This means that

$$\int_0^T (u_m,\varphi)_H dt \to \int_0^T (u,\varphi)_H dt \quad \forall \varphi \in L^1(0,T;H)$$

and

$$\int_0^T (u_m,\varphi)_{V,V'} dt \to \int_0^T (u,\varphi)_{V,V'} dt \quad \forall \varphi \in L^2(0,T;V')$$

$$(Au_n,w_i) \to (Au,w_i) \quad \text{in} \quad L^2(0,T) \text{ weakly}$$

$$(\frac{du_n}{dt},w_i) = \frac{d}{dt}(u_n,w_i)$$

$$(u_n,w_i) \to (u,w_i) \text{ in } L^\infty(0,T) \text{ weak}^* \text{ and in } L^2(0,T) \text{ weak.}$$

Hence $\frac{d}{dt}(u_n,w_i) \to \frac{d}{dt}(u,w_i)$ in the sense of distributions.

$\{B(u_n,u_n)\}_{n=1}^\infty$ is contained in a bounded set of $L^2(0,T;V')$ since

$$\|B(u,v)\|_{L^2(0,T;V')} \le C(\|u\|_{L^2(0,T;V)} + |u|_{L^\infty(0,T;H)}) .$$
$$(\|v\|_{L^2(0,T;V)} + |v|_{L^\infty(0,T;H)}) .$$

Hence we can extract a subsequence $B(u_m,u_m) \to g$ weakly in $L^2(0,T;V')$.

<u>Problem:</u> There is no a priori reason to believe $g = B(u,u)$ since B is not weakly continuous. ■

<u>Claim:</u>

$$u \text{ satisfies } \begin{cases} \frac{du}{dt} + \nu A(u) + g = f \\ u(0) = u_0 \end{cases}$$

(and so $\frac{du}{dt} \in L^2(0,T;V')$ since $\frac{du}{dt} = f - \nu A(u) - g$.

<u>Proof:</u> Let φ be a smooth function in $[0,T]$ that satisfies $\varphi(T) = 0$. Multiply by φ and integrate by parts. Then we obtain,

$$-\int_0^T (u_n, w_i)\varphi'(t)dt - (u_{0n}, w_i)\varphi(0) + \nu \int_0^T (A(u_n), w_i)\varphi(t)dt$$

$$+ \int_0^T (B(u_n, u_n), w_i)\varphi(t)dt = \int_0^T (f, w_i)\varphi(t)dt .$$

Each term has a limit as $n \to \infty$ (for a subsequence $\{u_n\}$ as chosen previously.). Taking the limit,

$$-\int_0^T (u, w_i)\varphi'(t)dt - (u_0, w_i)\varphi(0) + \nu \int_0^T (A(u), w_i)\varphi(t)dt +$$

$$+ \int_0^T (g, w_i)\varphi(t)dt = \int_0^T (f, w_i)\varphi(t)dt .$$

Let $h = f - \nu A(u) - g$. Then $h \in L^2(0,T;V')$.

$$-\int_0^T (u, w_i)\varphi'(t)dt - (u_0, w_i)\varphi(0) = \int_0^T (h, w_i)\varphi(t)dt .$$

Take $\varphi \in \mathcal{D}(]0,T[)$, so that $\varphi(0) = \varphi(T) = 0$, and we get

$$\frac{d}{dt}(u, w_i) = (h, w_i)$$

in the sense of distributions. Recall that the span of $\{w_i\}_{i=1}^{\infty}$ is dense in V. Since

$$(-\int_0^T u(t)\varphi'(t)dt - \int_0^T h(t)\varphi(t)dt, w_i) = 0$$

$\forall$ i, this implies

$$-\int_0^T u(t)\varphi'(t)dt - \int_0^T h(t)\varphi(t)dt = 0 .$$

Hence $\langle \frac{du}{dt}, \varphi\rangle - \langle h, \varphi\rangle = 0 \;\forall\; \varphi \in \mathcal{D}([0,T])$, so

$$\frac{du}{dt} = h \in L^2(0,T;V').$$

If a function $u \in L^2(0,T;V)$ and $\frac{du}{dt} \in L^2(0,T;V')$ then we know $u \in C^0(0,T;H)$. Take φ smooth as before: $\varphi(0) = 1$. Then

$$\int_0^T \frac{du}{dt}\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt + u(T)\varphi(T) - u(0)\varphi(0) ,$$

so $(u(0)\varphi(0), w_i) = (u_0\varphi(0), w_i) \;\forall\; i, \implies u(0) = u_0$. ∎

Theorem: If $\{\frac{du_n}{dt}\}_{n=1}^{\infty} \subseteq K_3$, a bounded set in $L^2(0,T;V')$, then $g = B(u,u)$.

Lemma: (Compactness) If $\{v_n\}$ is contained in a bounded set of $L^2(0,T;V)$, $\{\frac{dv_n}{dt}\}$ is contained in a bounded set in $L^2(0,T;V')$, and the injection $V \hookrightarrow H$ is compact, then $\{v_n\}$ is contained in a compact set in $L^2(0,T;H)$.

We will apply the lemma, then give a proof later. However, for now we remark: If Ω is bounded the injection $V \hookrightarrow H$ is compact. We know, if Ω is bounded, that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. If $w_n \rightharpoonup w$ weakly in V, then $(w_n)_i \rightharpoonup w_i$ weakly H_0^1, so $(w_n)_i \to w_i$ strongly in L^2, therefore. Hence $w_n \to w$ strongly in H.

Now we give a proof of the theorem.

Proof: Using the lemma, we can extract another subsequence, calling it $\{u_m\}$ such that

$$u_m \to u \text{ strongly in } L^2(0,T;H).$$

Then we can use the following continuity property of B: If $u_m \rightharpoonup u$ weakly in $L^2(0,T;V)$, $u_m \to u$ strongly in $L^2(0,T;H)$, then

$$B(u_m,u_m) \to B(u,u) \text{ weakly}$$

in $L^2(0,T;V')$. To see this, we need to show that, for $v \in L^2(0,T;V)$,

$$\int_0^T (B(u_m,u_m),v)dt \to \int_0^T (B(u,u),v)dt .$$

However,

$$\int_0^T (B(u_m,u_m),v)dt = -\int_0^T (B(u_m,v),u_m)dt$$

by the properties of B. Recall that

$$\int_0^T (B(u_m,v),u_m)dt = \int_0^T \int_\Omega (u_m)_j \frac{\partial v_i}{\partial x_j}(u_m)_i \, dxdt .$$

$(u_m)_i \rightharpoonup u_i$ weakly in $L^4(0,T;L^4(\Omega))$.

(Since $L^2(0,T;V) \cap L^\infty(0,T;H) \subseteq L^4(0,T;L^4(\Omega))$.)

$(u_m)_i \to u_i$ strongly in $L^2(0,T;L^2(\Omega))$

So $(u_m)_i(u_m)_j$ is bounded in $L^2(0,T;L^2(\Omega)$ and $(u_m)_i(u_m) \to u_i u_j$ strongly in $L^1(0,T;L^1(\Omega))$. Hence $(u_m)_i(u_m)_j \rightharpoonup u_i u_j$ weakly in $L^2(0,T;L^2(\Omega))$. ∎

How can we obtain $\{\frac{du_n}{dt}\}$ contained in a bounded set in $L^2(0,T;V')$?

We choose a special basis $\{w_i\}_{i=1}^\infty$.

Let $h_n = f - \nu A(u_n) - B(u_n,u_n)$. Then $\{h_n\}$ is contained in a bounded set in $L^2(0,T;V')$.

$$\left(\frac{du_n}{dt}, w_i\right) = (h_n, w_i) \qquad i = 1,2,\ldots,n.$$

$\frac{du_n}{dt}$ = Proj of h_n on the span of $\{w_1,\ldots,w_n\}$ with the norm of H.

A good choice for $\{w_i\}_{i=1}^\infty$ would be to take $\{w_i\}_{i=1}^\infty$ as orthogonal in V and V'. This gives

$$\left\|\frac{du_n}{dt}\right\|_* \le \|h_n\|_*.$$

For example, take for $\{w_i\}_{i=1}^\infty$ the eigenvectors of the symmetric operator A or A^{-1}.

(Riesz's theory). $A^{-1}: V' \to V$ is linear, so it maps $H \to H$.

($H \overset{inj}{\to} V' \overset{A^{-1}}{\to} V \overset{inj}{\to} H$.) Since $V \subsetneq H$ is compact, A^{-1} is compact as an operator from H into H. A^{-1} is symmetric, so the Riesz theory implies the existence of $\{w_i\}_{i=1}^\infty$, an orthonormal basis for H with $A^{-1}w_i = \mu_i w_i$ where $\mu_i \to 0$. Alternatively, $Aw_i = \frac{1}{\mu_i} w_i = \lambda_i w_i$ where $\lambda_i \to \infty$.

Then $((w_i,w_j))_V = (Aw_i,w_j)_H = (\lambda_i w_i, w_j)_H = \begin{cases} 0 & i \ne j \\ \lambda_i & i = j \end{cases}$

$$((w_i,w_j))_{V'} = (A^{-1}w_i,w_j) = \begin{cases} 0 & i \ne j \\ \frac{1}{\lambda_i} & i = j \end{cases}$$

Thus $\|w_i\|_V = \sqrt{\lambda_i}$, $\|w_i\|_{V'} = \frac{1}{\sqrt{\lambda_i}} = \sqrt{\mu_i}$.

This is the required "special basis", and the above remarks complete the proof of existence for the solution of the Navier-Stokes equation. ∎

Remark: $u_m \to u$ weakly implies u is a solution. Since we earlier proved uniqueness, this proves that the entire sequence converges weakly to u. ∎

Theorem: (Compactness) Let $B_0 \subseteq B_1 \subseteq B_2$ be three Banach spaces with continuous injection $B_1 \subsetneqq B_2$ and compact injection $B_0 \subsetneqq B_1$. If $\{u_n\}$ is contained in a bounded set in $L^P(0,T;B_0)$, $\{\frac{du_n}{dt}\}$ contained in a bounded set in $L^P(0,T;B_2)$, with $T < \infty$, and $1 \leq P < \infty$, then $\{u_n\}$ is contained in a compact set in $L^P(0,T;B_1)$.

Lemma: $\forall \epsilon > 0 \ \exists C(\epsilon) : \|a\|_{B_1} \leq \epsilon \|a\|_{B_0} + C(\epsilon)\|a\|_{B_2}$, $\forall a \in B_0$.

Proof: (By contradiction) Otherwise, $\exists \epsilon > 0 : \forall n \ \exists a_n \in B_0$ with $\|a_n\|_{B_1} > \epsilon_0\|a_n\|_{B_0} + n\|a_n\|_{B_2}$. We can take $\|a_n\|_{B_1} = 1$, so that

$$1 > \epsilon_0\|a_n\|_{B_0} + n\|a_n\|_{B_2} \ . \Longrightarrow \|a_n\|_{B_0} < \frac{1}{\epsilon_0}$$

and $\|a_n\|_{B_2} < \frac{1}{n}$.

Thus, $\{a_n\}$ is contained in a bounded set in B_0, hence contained in a compact set in B_1. Therefore, we can extract a subsequence $a_m \to a_\infty$ strongly in B_1. Clearly $\|a_\infty\|_{B_1} = 1$. But $a_n \to 0$ strongly in B_2, so, by the continuity of the injection, $a_\infty = 0$ - Contradiction. ∎

Proof: (Compactness theorem). We know $\{u_n\}$ is bounded in $L^P(0,T;B_0)$, and $\{\frac{du_n}{dt}\}$ is bounded in $L^P(0,T;B_2)$. We want to extract a Cauchy sequence in $L^P(0,T;B_1)$.

First step: It is sufficient to extract a Cauchy sequence in $L^P(0,T;B_2)$, since the lemma gives

$$\|a\|_{B_1}^P \leq \epsilon \|a\|_{B_0}^P + C(\epsilon)\|a\|_{B_2}^P \quad \forall a \in B_0 .$$

That gives

$$\|V_n - V_m\|_{L^P(0,T;B_1}^P \leq \epsilon \|V_n - V_m\|_{L^P(0,T;B_0}^P + C(\epsilon)\|V_n - V_m\|_{L^P(0,T;B_2)}^P$$

and the first term on the right is bounded, while the second goes to zero, if $\{V_n\}$ is Cauchy in $L^P(0,T:B_2)$. Thus we would get

$$\overline{\lim_{n,m\to\infty}} \|V_n - V_m\|_{L^P(0,T;B_1)}^P \leq \epsilon \text{ constant } \forall \epsilon > 0 ,$$

so $\{V_n\}$ would be Cauchy in $L^P(0,T;B_1)$.

<u>Second Step:</u> Let θ be smooth on $[0,T]$: $\theta(0) = 1$ $\theta(T) = 0$. Then $\{\theta u_n\}$ and $\{(1-\theta)u_n\}$ satisfy the same hypotheses as does $\{u_n\}$. The two new sequences are handled analogously: we will work with $\{\theta u_n\}$ now, assuming $u_n(t) = 0$ for $t \geq T$, decompose u_n:

$$u_n(t) = \underbrace{\left(\frac{1}{h}\int_t^{t+h} u_n(s)ds\right)}_{a_n(t)} + \underbrace{\left(\frac{1}{h}\int_t^{t+h}(u_n(t) - u_n(s))ds\right)}_{b_n(t)} \text{ with } h > 0$$

$$\|b_n(t)\|_{B_2} \leq \frac{1}{h}\int_t^{t+h}\|u_n(t)-u_n(s)\|_{B_2} ds ,$$

and

$$\|u_n(t)-u_n(s)\|_{B_2} \leq \int_t^s \left\|\frac{du_n}{dt}(s)\right\|_{B_2} ds \leq |s-t|^{1/P'}\left(\int_t^s \left\|\frac{du_n}{dt}\right\|^P\right)^{1/P}$$

by Hölder's inequality. Thus $\|u_n(t) - u_n(s)\|_{B_2} \leq C\, h^{1/P}$, and so $\|b_n(t)\|B_2 \leq C\, h^{1/P'}$. Hence, $\|b_n\|_{L^P(0,T;B_2)} \leq C'\, h^{1/P'}$. If we can extract a Cauchy sequence from $\{a_n\}$, then we would have

$$\overline{\lim_{n,m\to\infty}} \|u_n - u_m\|_{L^P(0,T;B_2)} \leq C\, h^{1/P'} ,$$

and so, letting $h\to 0$, and using a diagonal procedure, we could extract a

Cauchy subsequence in $L^P(0,T;B_2)$ from $\{u_n\}$.

Fix $h > 0$. $\{a_n\}$ is contained in a bounded set in $C^0(0,T;B_0)$:

$$\|a_n(t)\|_{B_0} \le \frac{1}{h} h^{1/P'} \|u_n\|_{L^P(0,T;B_0)} \le \text{constant.}$$

$$\frac{da_n}{dt} = \frac{1}{h}(u_n(t+h) - u_n(t)) = \frac{1}{h}\int_t^{t+h} \frac{du_n}{dt}(0)d\sigma \text{ , so } \left\|\frac{da_n}{dt}\right\|_{B_2} \le \text{Constant.}$$

Hence $\{a_n\}$ is a uniformly bounded equicontinuous family in $C^0(0,T;B_2)$: $\{a_n\}$ takes values in a bounded in B_0, which is in a compact set in B_2. Apply the theorem of Arzela-Ascoli to extract a strongly convergent subsequence in $C^0(0,T;B_2)$ and so in $L^P(0,T;B_2)$. ∎

Properties of the solution of the Navier-Stokes Equation:

So far we have: If $u_0 \in H$, $f \in L^2(0,T;V')$, $\exists!\, u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ with $\frac{du}{dt} \in L^2(0,T;V')$ which satisfies

$$(1) \qquad \begin{aligned} &\frac{du}{dt} + \nu A(u) + B(u,u) = f \\ &u(0) = u_0 \ . \end{aligned}$$

Theorem: (Continuous dependence on u_0 and f.) If u_1 satisfies (1) with u_{01}, f_1, u_2 satisfies (1) with u_{02}, f_2, then

$$\|u_1-u_2\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le C(u_2)[|u_{01}-u_{02}|_H + \|f_1-f_2\|_{L^2(0,T;V')}].$$

Proof:

$$\frac{d}{dt}(u_1-u_2) + \nu A(u_1-u_2) + B(u_1,u_1) - B(u_2,u_2) = f_1-f_2$$

$$(u_1-u_2)(()) = u_{01} - u_{02} \ .$$

Take the inner product with $u_1 - u_2$:

$$\tfrac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \le \|f_1-f_2\|_*\|u_1-u_2\| + |B(u_1,u_1)-B(u_2,u_2),u_1-u_2)| .$$

As before,

$$|(B(u_1,u_1) - B(u_2,u_2),u_1,u_2)| \le |(Bu_1,u_1-u_2),u_1-u_2)| + |B(u_1-u_2,u_2),u_1-u_2)| = |B(u_1-u_2,u_2),u_1-u_2)|$$
$$\le C|u_1-u_2|\,\|u_1-u_2\|\,\|u_2\|.$$

Hence,

$$\tfrac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \le \|f_1-f_2\|_*\|u_1-u_2\| + C|u_1-u_2|\,\|u_1-u_2\|\,\|u_2\|.$$

Using Young's inequality on the right-hand side,

$$\le \frac{\nu}{4}\|u_1-u_2\|^2 + \frac{1}{\nu}\|f_1-f_2\|_*^2 + \frac{\nu}{4}\|u_1-u_2\|^2 + \frac{C}{\nu}\|u_2\|^2|u_1-u_2|^2$$

and so

$$\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \le \frac{2}{\nu}\|f_1-f_2\|_* + \frac{2C}{\nu}\|u_2\|^2|u_1-u_2|^2 \ .$$

Now, $\|u_2\|^2 \in L^1(0,T)$? Use Gronwall's inequality:

<u>First step:</u> Omit $\nu\|u_1-u_2\|^2$. Then let $\varphi(t) = |u_1(t) - u_2(t)|^2$. The above inequality becomes

$$\varphi'(t) < a(t) + \lambda(t)\,\varphi(t),$$

where $a, \lambda \in L^1(0,T)$. Let

$$\psi(t) = \varphi(t)\exp(-\int_0^t -\lambda(s)ds).$$

Then

$$\psi'(t) = (\varphi'-\lambda\varphi)\exp(-\int_0^t \lambda(s)ds) \le a(t)\exp(-\int_0^t \lambda(s)ds) \le a(t).$$

Integrate:

$$\psi(t) \le \psi(0) + \int_0^t a(s)ds \ \le \varphi(0) + \|a\|_{L^1(0,T)}$$

Thus,

$$\varphi(t) \le (\varphi(0) + \|a\|_{L^1(0,T)})\exp(\|\lambda\|_{L^1(0,T)})\ .$$

If u_2 is fixed, $\max|u_1-u_2| \le C(|u_{01} - u_2|^2 + \|f_1-f_2\|^2_{L^2(0,T;V')})$
where C depends on u_2

Second Step: Integrate the original inequality on $[0,T]$.

$$\nu\int_0^T \|u_1-u_2\|^2 dt \le |u_{01}-u_{02}|^2 + \int_0^T (a(t)+\lambda(t)\varphi(t)dt \le C.$$

(something of the same type as above.) Therefore,

$$\|u_1-u_2\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le C(|u_{01}-u_{02}| + \|f_1-f_2\|_{L^2(0,T;V')}).$$

C again depends on u_2. ∎

Questions:

1) If f does not depend on t, is there a solution which does not depend on t? (This is the stationary problem: given $f \in V'$, find $u \in V$: $\nu A(u) + B(u,u) = f$.)

2) For $f(t) \equiv f$, what is the asymptotic behavior of solutions? Does the solution with u_0 initial point converge to a stationary solution?

3) If $f \in L^\infty(0,\infty;V')$, is u bounded on $(0,\infty)$ in some space?

4) If $T < \infty$, does a periodic solution exist?

$$\begin{cases} \frac{du}{dt} + \nu A(u) + B(u,u) = f & f \in L^2(0,T;V') \\ u(0) = u(T) \end{cases}$$

5) If f is periodic of period T, does the solution converge to a periodic solution?

6) Regularity of the solution: If $f \in L^2(0,T;V)$, $u_0 \in V$, is the solution $u \in L^\infty(0,T;V)$?

Theorem: (Bounded solutions) Let $u_0 \in H$, $f \in L^2_{loc}(0,\infty;V')$ such that

$$\int_t^{t+1} \|f(s)\|_*^2 ds \le C \quad \forall\, t \in [0,\infty).$$

Then the solution u satisfies $u \in L^\infty(0,\infty;H)$ and

$$\int_t^{t+1} \|u(s)\|^2 ds \le C'.$$

Proof: We know that there is a solution u on each bounded interval. u satisfies

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 \leq \|f\|_*\|u\| \leq \frac{\nu}{2}\|u\|^2 + \frac{1}{2\nu}\|f\|_*^2 .$$

Thus,

$$(*) \quad \frac{d}{dt}|u|^2 + \nu\|u\|^2 \leq \frac{1}{\nu}\|f\|_*^2 .$$

Integrate (*) from t to t+1:

$$|u(t+1)|^2 + \nu\int_t^{t+1}\|u(s)\|^2 ds \leq |u(t)|^2 + \frac{1}{\nu}\int_t^{t+1}\|f(s)\|_*^2 ds.$$

If we can prove $|u(t)| \leq C$, we will have established that

$$\nu\int_t^{t+1}\|u(s)\|^2 ds \leq C_1 .$$

The injection $V \subseteq H$ is continuous. This implies that $\exists \gamma > 0$: $\|u\| \geq \gamma|u|$, so (*) $\Longrightarrow$

$$\frac{d}{dt}|u|^2 + \nu\gamma^2|u|^2 \leq \frac{1}{\nu}\|f\|_*^2 .$$

Let $\varphi(t) = |u(t)|^2$. Then

$$\begin{cases} \varphi'(t) + \epsilon\, \varphi(t) = g(t) \\ \varphi(0) = \text{given.} \end{cases}$$

The exact solution is

$$\varphi(t) = \varphi(0)e^{-\epsilon t} + \int_0^t e^{-\epsilon(t-s)}g(s)ds .$$

Integrate from t to t+1 and put $\psi(t) = \int_t^{t+1}\varphi(s)ds$

$$\begin{cases} \psi'(t) + \epsilon\, \psi = \int_t^{t+1} g(s)ds \leq C \\ \psi(0) \text{ given.} \end{cases}$$

Hence $\psi(t) \leq \psi(0)e^{-\epsilon t} + \frac{C}{\epsilon}$ if $\int_t^{t+1} g(s)ds \leq C$.

So $\int_t^{t+1}\varphi(s)ds$ is bounded.

If $t < s < t+1$, integrate from t to s.

$$\varphi(s) - \varphi(t) + (\text{something} \geq 0) \leq \int_t^s g(s)ds \leq C ,$$

so $s-1 < t < s \Longrightarrow \varphi(s) \leq \varphi(t) + C$. Integrate from $t = s-1$ to $t = s$:

$$\varphi(s) \leq \int_{s-1}^s \varphi(t)dt + C \leq C' .$$

Applying this with $\varepsilon = \nu\gamma^2$ we have $g(t) \leq \frac{1}{\nu} \| f(t) \|_*^2$ and we have an estimate

$$|u|^2_{L^\infty(0,\infty;H)} \leq C|u_0|^2 + C \sup_t \cdot \int_t^{t+1} \| f(s) \|_*^2 ds. \qquad \blacksquare$$

If φ satisfies $\varphi' + \epsilon\, \varphi = g$, $\int_{t+1}^t g(s)ds \leq C$, then

$$\limsup |\varphi| \leq C \limsup \int_t^{t+1} g(s)ds.$$

Indeed since $\varphi(0)\, e^{-\epsilon t} \to 0$ as $t \to \infty$. We have proved that if $\int_t^{t+1} f(s)ds \leq \alpha$, then $\limsup_{t\to\infty} |\varphi| \leq C\alpha$.

If $\int_t^{t+1} g(s)ds \leq \alpha$ for $t \geq N$, let $\tilde{\varphi}(t) = \varphi(N+t)$. Prove that $\lim\sup |\tilde{\varphi}| \leq C\alpha$. This gives rise to the following:

<u>Corollary 1:</u> If $\int_t^{t+1} \| f \|_*^2 ds \to 0$ as $t \to \infty$, then $|u(t)| \to 0$ and $\int_t^{t+1} \| u(s) \|^2 ds \to 0$ as $t \to \infty$.

<u>Corollary 2:</u> If $f \in L^\infty(0,\infty,V')$, then $u \in L^\infty(0,\infty,H)$.

<u>Question:</u> If $f(t) \to f_\infty \in V'$, does u converge to something in H?

We will give a positive answer if f_∞ has small norm. For this we will begin with an existence and uniqueness theorem for stationary solutions for small data.

Let C_0 be a constant such that

(1) $$|(B(u,v),w)| \leq C_0 \| u \| \, \| v \| \, \| w \| \qquad \text{(This requires } N \leq 4.)$$

<u>Theorem:</u> If $f_0 \in V'$, with $\| f_0 \|_* < \frac{\nu^2}{C_0}$, then there exists a unique solution u_0 of

$$\nu A(u_0) + B(u_0,u_0) = f_0, \quad \text{and} \quad \| u_0 \| \leq \frac{1}{\nu} \| f_0 \|_* .$$

For the proof we will use the following lemma:

Lemma: (Lax-Milgram) If V is a real Hilbert space and $a(u,v)$ is a continuous bilinear form on $V \times V$ satisfying $a(u,u) \geq \alpha \|u\|^2$ $(\alpha > 0)$ $\forall\, u \in V$, then a defines a map $\bar{A} : V \to V'$, given by $(\bar{A}u, v) = a(u,v)$, which is an isomorphism, and:

$$\|\bar{A}^{-1} f\|_V \leq \frac{1}{\alpha} \|f\|_{V'} . \qquad \blacksquare$$

Given $u \in V, f \in V'$, consider the following problem: Find $\bar{u} \in V$ satisfying $\nu A(\bar{u}) + B(u,\bar{u}) = f$. This is linear in u.

Let $a(\bar{u},v) = (\nu A(\bar{u}) + B(u,\bar{u}),v)$. Then $A(\bar{u},\bar{u}) = \nu(A(u),u) + (B(u,\bar{u}),\bar{u}) \geq \nu \|\bar{u}\|^2$. Then by Lax-Milgram's Lemma $\exists\ \bar{u} = T(u.f) : \|\bar{u}\| \leq \frac{1}{\nu}\|f\|_*$.

Problem: Find a fixed point of $u \to T(u,f)$. (f is fixed). Observe that T maps $\|u\| \leq \frac{1}{\nu}\|f\|_*$ into itself, since it maps all of V into that set.

First method: With $\|f_0\|_* < \dfrac{\nu^2}{C_o}$.

In this case, T is a strict contraction on $\{u : \|u\| \leq \frac{1}{\nu} \|f_0\|_*\}$. Hence there is a unique fixed point in this ball. This is seen as follows:

$$\bar{u} = T(u,f_0) \qquad \nu A(\bar{u}) + B(u,\bar{u}) = f_0$$

$$\bar{v} = T(v,f_0) \qquad \nu A(\bar{v}) + B(v,\bar{v}) = f_0 .$$

Take the difference, $\bar{u}-\bar{v}$, and its inner product with the difference of the two equations:

$$\nu \|\bar{u}-\bar{v}\|^2 = (B(v,\bar{v}) - B(u,\bar{u}),\bar{u}-\bar{v})$$

$$= (B(v,\bar{v}-\bar{u}),\bar{v}-\bar{u}) + (B(v-u,\bar{u}),\bar{v}-\bar{u})$$

$= (B(v-u,\bar{u}),\bar{v}-\bar{u}) \leq C_0 \|v-u\|\,\|\bar{u}\|\,\|\bar{v}-\bar{u}\|$. T will thus be a strict contraction since $\dfrac{C_0}{\nu}\|\bar{u}\| \leq \dfrac{C_0}{\nu^2}\|f_0\|_* < 1$.

Second Method: T is continuous, (T has Lipschitz constant $\frac{C_0}{\nu^2} \|f_0\|_*$.) and compact. Hence we can apply Schauder's fixed point theorem. (We will explain this later.) ∎

Theorem: (Convergence at ∞) Let $f_\infty \in V'$, $\|f_\infty\|_* < \frac{\nu^2}{C_0}$, $u_0 \in H$, and let $f \in L^2_{loc}(0,\infty;V')$ such that

$$\int_t^{t+1} \|f(s) - f_\infty\|_*^2 \, ds \to 0 \text{ as } t \to \infty .$$

Then $|u(t) - u_\infty| \to 0$, $\int_t^{t+1} \|u(s) - u_\infty\|^2 ds \to 0$, where u_∞ is the unique solution of $\nu A(u_\infty) + B(u_\infty, u_\infty) = f_\infty$.

Proof: $u_\infty \in V$, $t \to u_\infty(t) = u_\infty$ satisfies

$$\frac{du_\infty}{dt} + \nu A(u_\infty) + B(u_\infty, u_\infty) = f_\infty ,$$

since $\frac{du_\infty}{dt} = 0$. Now, $\frac{du}{dt} + \nu A(u) + B(u,u) = f$. Subtract, take the inner product with $u-u_\infty$, and we get

$$\tfrac{1}{2}\frac{d}{dt}|u-u_\infty|^2 + \nu\|u-u_\infty\|^2 \leq \|f-f_\infty\|_*\|u-u_\infty\| + C_0\|u_\infty\|\,\|u-u_\infty\|^2$$

in the same manner as before.

Remark that $C_0\|u_\infty\| \leq \frac{C_0}{\nu}\|f_\infty\|_* < \nu$. Let $\nu - C_0\|u_\infty\| = \epsilon > 0$.

Then

$$\tfrac{1}{2}\frac{d}{dt}|u-u_\infty|^2 + \epsilon\|u-u_\infty\|^2 \leq \|f-f_\infty\|_*\|u-u_\infty\|$$
$$\leq \frac{\epsilon}{2}\|u-u_\infty\|^2 + \frac{1}{2\epsilon}\|f-f_\infty\|_*^2 ,$$

and so

$$\frac{d}{dt}|u-u_\infty|^2 + \epsilon\|u-u_\infty\|^2 \leq C\|f-f_\infty\|_*^2 .$$

Thus,

$$\limsup_{t\to\infty}\left(|u-u_\infty|^2 + \int_t^{t+1}\|u-u_\infty\|^2 ds\right) \leq C \limsup_{t\to\infty}\left(\int_t^{t+1}\|f-f_\infty\|^2 ds\right) .$$

So $|u-u_\infty| \to 0$ and $\int_t^{t+1}\|u-u_\infty\|^2 \to 0$.

This implies $\int_t^{t+1} \|\frac{du}{dt}\|_*^2 ds \to 0$ as $t \to \infty$.

This proves also that $\int_t^{t+1} u(s) \to \infty$ in V, because

$$\|\int_t^{t+1}(u(s)-u_\infty)ds\| \leq \int_t^{t+1}\|u(s)-u_\infty\|^2 ds \leq (\int_t^{t+1}\|u-u_\infty\|^2 ds)^{\frac{1}{2}} \to 0. \quad \blacksquare$$

Stationary solutions: the general case

Let $f \in V'$. The problem is to find $u \in V$ such that

(1) $\nu A(u) + B(u,u) = f$. All solutions of (1) satisfy

$$\nu\|u\|^2 = \nu(A(u),u) + (B(u,u),u) = (f,u) \leq \|f\|_*\|u\| .$$

Thus $\|u\| \leq \frac{1}{\nu}\|f\|_*$. We have defined $T:V \to V$ by $u = Tv$ is the solution of the linear problem $\nu A(u) + B(v,u) = f$. T maps V into

$$\{u:\|u\| \leq \frac{1}{\nu}\|f\|_*\} .$$

We want to find a fixed point of T. (We have achieved this for $\|f\|_*$ small.) We will use a fixed point theorem. There are two main possibilities:

Theorem: (Brouwer) If K is a nonempty compact, convex set in $\mathbb{R}^N$, and $S:K \to K$ is continuous, then S has at least one fixed point.

Theorem: (Schauder) The same result holds if K is a compact, convex set in a Banach space E and S is continuous, $S:K \to K$.

First Method: Use Schauder's theorem. The difficulty is to find the set K.

Let $K_1 = \{u:\|u\| \leq \frac{1}{\nu}\|f\|_*\}$. Then $T:K_1 \to K_1$, but K_1 is not compact. We must show $T(K_1)$ is compact, and use K = the closure of the convex hull of $T(K_1)$, $K = \overline{\text{conv}(T(K_1))}$.

Second Method: Use Brouwer's Theorem in an approximating problem.

Consider the Galerkin basis $w_1, w_2, \ldots$. For $n \in \mathbb{N}$, find $u_n = \sum_{i=1}^n g_i w_i$ satisfying $(\nu A(u_n) + B(u_n,u_n) - f, w_i) = 0$ for $i = 1,2,\ldots,n$.

Let T_n be the map given by $u_n = T_n v_n$ where u_n solves $(\nu A(u_n) + B(v_n,u_n) - f, w_i)$ for $i = 1,2,\ldots,n$. Multiply by $g_i w_i$ and sum, and get $\|u_n\| \le \frac{1}{\nu}\|f\|_*$.

T_n maps $\{u \in \text{span}\,\{w_1,\ldots,w_n\}: \|u\| \le \frac{1}{\nu}\|f\|_*\}$ into itself. T_n is continuous. (It has Lipschitz constant $\frac{C_0\|f\|_*}{\nu^2}$.) Hence, by Brouwer's theorem, $\exists$ a fixed point u_n. All u_n, $n = 1,2,\ldots,$ belong to $\{u \in V: \|u\| \le \frac{1}{\nu}\|f\|_*\}$. Extract a subsequence, call it $\{u_n\}$, such that $u_n \rightharpoonup u$ weakly in V; $u_n \to u$ strongly in H, since the injection $V \subsetneq H$ is compact. This implies $B(u_n,u_n) \rightharpoonup B(u,u)$ weakly in V' by a previous result. (Remember, $(B(u_n,u_n),v) = -(B(u_n,v),u_n) \to -B(u,v),u) = (B(u,u),v)$.)

Thus u satisfies $(\nu A(u) + B(u,u) - f, w_i) = 0 \;\forall\, i$, and so is a stationary solution. ∎

<u>Remark:</u> If $|(B(u,v),u)| \le C_*\|u\|^2\|V\|$, then we have uniqueness if $\|f\|_* < \frac{\nu^2}{C_*}$.

<u>Periodic solutions:</u>

Let $0 < T < \infty$, $f \in L^2(0,T;V')$. Is there a solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$\text{(1)} \quad \begin{cases} u' + \nu A(u) + B(u,u) = f \\ u(0) = u(T). \end{cases}$$

<u>Remark:</u> We have the a priori estimate

$$\|u\|_{L^2(0,T;V)} \le \frac{1}{\nu}\|f\|_{L^2(0,T;V')}$$

Proof: If u solves (1), $u_0 \in H$. Then $\frac{du}{dt} \in L^2(0,T;V')$. We can take the inner product with u :

$$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \|f\|_*\|u\|.$$

Integrate from 0 to T,

$$\int_0^T \frac{d}{dt}|u|^2 dt = |u(T)|^2 - |u(0)|^2 = 0 .$$

Hence

$$\nu\int_0^T \|u\|^2\, dt \le \int_0^T \|f\|_*\|u\|\, dt.$$

Use Cauchy-Schwarz:

$$\le \left(\int_0^T \|u\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \|f\|_*^2\right)^{\frac{1}{2}}$$

which gives the desired result. ∎

Let $\gamma > 0$, so that $\|V\| \ge \gamma|V|_H$.

Lemma: We have the a priori estimate

$$|u_0| \le \varphi(\gamma,T)\|f\|_{L^2(0,T;V')}$$

Proof: $$\tfrac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \|f\|_*\|u\| \le \frac{\nu}{2}\|u\|^2 + \frac{1}{2\nu}\|f\|_*^2$$

$$\Longrightarrow \frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{1}{\nu}\|f\|_*^2 .$$

Therefore

$$\frac{d}{dt}|u|^2 + \nu\gamma^2|u|^2 \le \frac{1}{\nu}\|f\|_*^2 .$$

This estimate gives

$$|u(t)|^2 \le |u_0|^2 e^{-\nu\gamma^2 t} + \int_0^t e^{-\nu\gamma^2(t-s)}\frac{1}{\nu}\|f(s)\|_*^2 ds .$$

(Obtained by $\frac{d}{dt}(e^{\nu\gamma^2 t}|u|^2) < \frac{1}{\nu}\|f\|_*^2 e^{\nu\gamma^2 t}$ and integration.)

For $t = T$, this gives

$$|u(T)|^2 \le e^{-\nu\gamma^2 T}|u_0|^2 + C\|f\|^2_{L^2(0,T;V')}$$

(where C depends on T.) This implies, since $u_0 = u(T)$, that

$$|u_0|^2 \leq \frac{1}{1 - e^{-\nu^2 T}} C\|f\|^2_{L^2(0,T;V')}$$ ∎

<u>Theorem:</u> (existence of periodic solutions) If $f \in L^2(0,T;V')$ there exists at least one periodic solution $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

<u>Proof:</u> By proving the existence of a fixed point of the map $S:H \to H$ given by $Su_0 = u(T)$ where u satisfies

$$\begin{cases} u' + \nu A(u) + B(u,u) = f \\ u(0) = u_0 \end{cases} .$$

First, we derive some estimates on S. Multiplying by u

$$\tfrac{1}{2} \frac{d}{dt}|u|^2 + \nu\|u\|^2 \leq \|f\|_*\|u\|$$

and get

$$\frac{d}{dt}|u|^2 + \nu\gamma^2|u|^2 \leq \frac{1}{\nu}\|f\|_*^2 .$$

This gives as before

$$|u(T)|^2 \leq e^{-\nu\gamma T}|u_0|^2 + \frac{T}{\nu}\|f\|^2_{L^2(0,T;V')} .$$

If $\quad |u_0|^2 \leq \frac{1}{1-e^{-\nu^2 T}} \frac{T}{\nu}\|f\|^2_{l^2(0,T;V')} \stackrel{def}{\equiv} m^2$, then $|u(T)|^2 \leq m^2$.

Hence S maps $\{u \in H: |u| \leq m\}$ into itself.

<u>First Method:</u> Use the Schauder fixed point theorem. We must show S is compact. To do that, we must prove regularity theorems.

<u>Second Method:</u> Use the Galerkin method.

Let $w_1, w_2, \ldots$ be a Galerkin basis, $u_n = \sum_{i=1}^{n} g_i(t)w_i$ satisfy

$$\begin{cases} \frac{du_n}{dt} + \nu A(u_n) + B(u_n,u_n) - f, w_i) = 0 \quad i = 1,..,n \\ u_n(0) = u_0 \in \mathrm{Span}\{w_1,\ldots,w_n\} . \end{cases}$$

Let $S_n(u_0) = u_n(T)$. Then the above estimates show that

$$|u_n(0)| \le m \Longrightarrow |u_n(T)| = |S_n(u_n(0))| \le m .$$

Use the Brouwer's fixed point theorem to find a fixed point for the map S_n.

The solutions $\{u_n\}$ corresponding to these fixed points are all contained in a bounded set in $L^2(0,T;V)$ and $L^\infty(0,T;H)$, since

$$\|u_n\|_{L^2(0,T;V)} \le \frac{1}{\nu}\|f\|_{L^2(0,T;V')} \quad \text{and} \quad |u_n(0)| \le m .$$

If the w_i are a <u>special basis</u>, then $\{\frac{du_n}{dt}\}$ is contained in a bounded set in $L^2(0,T;V')$.

Now, if $u_n \to u$ weakly in $L^2(0,T;V)$, weakly* in $L^\infty(0,T;H)$, then $B(u_n,u_n) \to B(u,u)$ weakly in $L^2(0,T;V')$, so u satisfies

$$\left(\frac{du}{dt} + \nu A(u) + B(u,u) - f, w_i\right) = 0 \quad \forall i.$$

Hence u is a solution; since all the u_n are periodic of period T, so is u. That is

$$u_n(0) \to u(0) \text{ weakly in } H, \quad u_n(T) \to u(T) \text{ weakly in } H .$$

This gives the existence of a periodic solution. ∎

<u>Uniqueness of Periodic Solutions:</u>

Let $|(B(u,v),u)| \le C_1 |u| \, \|u\| \, \|v\|$.

<u>Theorem:</u> If $\|f\|_{L^2(0,T;V')} < \frac{\nu^2\gamma\sqrt{T}}{C_1}$ there is a unique periodic solution.

<u>Proof:</u> If u_1 and u_2 are solutions,

$$\frac{d}{dt}(u_1-u_2) + \nu A(u_1-u_2) + B(u_1,u_1) - B(u_2,u_2) = 0$$

$$(u_1-u_2)(0) = (u_1-u_2)(T) .$$

Take the inner product with $u_1 - u_2$: Then use

$$(B(u_1,u_1) - B(u_2,u_2), u_1-u_2) = (B(u_1-u_2,u_1),u_1-u_2) + (B(u_2,u_1-u_2),u_1-u_2)$$

So $|B(u_1,u_1) - B(u_2,u_2), u_1-u_2)| \le C_1 |u_1-u_2| \, \|u_1-u_2\| \, \|u_1\|$

Hence

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \le C_1|u_1-u_2|\,\|u_1-u_2\|\,\|u_1\|$$

$$\le \epsilon \, \|u_1-u_2\|^2 + \frac{C_1^2\|u_1\|^2|u_1-u_2|^2}{4\epsilon}$$

(where ϵ is to be chosen.)

Since $\|u_1-u_2\| \ge \gamma|u_1-u_2|$,

$$\frac{1}{2}\frac{d}{dt}|u_1-u_2|^2 + (\nu-\epsilon)\gamma^2|u_1-u_2|^2 \le \frac{C_1^2\|u_1\|^2}{4\epsilon}|u_1-u_2|^2$$

Then if $\varphi(t) = |u_1(t) - u_2(t)|^2$, the above becomes

$$\frac{d}{dt}\varphi(t) + 2\lambda(t)\varphi(t) \le 0$$

where

$$\lambda(t) = (\nu-\epsilon)\gamma^2 - \frac{C_1^2\|u_1(t)\|^2}{4\epsilon}$$

which can be written

$$\frac{d}{dt}\left(\varphi(t)\, e^{2\int_0^t \lambda(s)ds}\right) \le 0$$

Therefore, $\varphi(T)\, e^{2\int_0^T \lambda(s)ds} \le \varphi(0)$: and so, if $e^{2\int_0^T \lambda(s)ds} > 1$, then

$\varphi(0) \le 0 \implies \varphi(0) = 0$ so $\varphi \equiv 0$ Now

Now,

$$\int_0^T \lambda(s)ds = (\nu-\epsilon)\gamma^2 T - \frac{C_1^2}{4\epsilon}\|u_1\|^2_{L^2(0,T;V)}$$

We know that $\|u\|_{L^2(0,T;V)} \le \frac{1}{\nu}\|f\|_{L^2(0,T;V')}$

Thus, we have uniqueness if

$$(\nu-\epsilon)\gamma^2 T > \frac{C_1^2}{4\epsilon}\,\frac{\|f\|^2_{L^2(0,T;V')}}{\nu^2} \ge \frac{C_1^2}{4\epsilon}\|u\|^2_{L^2(0,T;V)}$$

This is possible if $\|f\|_{L^2(0,T;V')} < \frac{\nu^2\gamma\sqrt{T}}{C_1}$ by taking $\varepsilon = \frac{\nu}{2}$

Theorem: (Convergence) If $\|\bar{f}\|_{L^2(0,T;V')} < \frac{\nu^2\gamma\sqrt{T}}{C_1}$ and $\bar{f}$ is periodic of of period T, let $f \in L^2_{loc}(0,\infty;V')$ satisfy

$$\int_t^{t+T} \|f(s)-\bar{f}(s)\|_*^2 ds \to 0 \text{ as } t \to \infty,$$

and let $u_0 \in H$. Then $|u-\bar{u}| \to 0$ as $t \to \infty$, and

$$\int_t^{t+T} \|u-\bar{u}\|^2 ds \to 0 \text{ as } t \to \infty ,$$

where $\bar{u}$ is the (unique) periodic solution.

Proof: $\frac{d\bar{u}}{dt} + \nu A(\bar{u}) + B(\bar{u},\bar{u}) = \bar{f} \qquad \bar{u}(0) = \bar{u}(T)$.

$\frac{du}{dt} + \nu A(u) + B(u,u) = f \qquad u(0) = u_0$

Subtract and take the inner product with $u - \bar{u}$. Then

$$\tfrac{1}{2}\frac{d}{dt}|u-\bar{u}|^2 + \nu\|u-\bar{u}\|^2 \le \|f-\bar{f}\|_*\|u-\bar{u}\| + C_1|u-\bar{u}|\,\|u-\bar{u}\|\,\|\bar{u}\|$$

$$\le \eta\|u-\bar{u}\|^2 + \frac{1}{4\eta}\|f-\bar{f}\|_*^2 + \epsilon\|u-\bar{u}\|^2 + \frac{C_1^2|u-u|^2\|\bar{u}\|^2}{4\epsilon}$$

where $\eta + \epsilon < \nu$. Apply the estimate $\|u-\bar{u}\| \ge \gamma|u-\bar{u}|$:

$$\tfrac{1}{2}\frac{d}{dt}|u-\bar{u}|^2 + (\nu-\epsilon-\eta)\gamma^2|u-\bar{u}|^2 \le \frac{C_1^2|u-\bar{u}|^2\|\bar{u}\|^2}{4\epsilon} + \frac{1}{4\eta}\|f-\bar{f}\|_*^2$$

Let $\varphi(t) = |u(t)-\bar{u}(t)|^2$. Then φ satisfies

(*) $\frac{d}{dt}\varphi(t) + 2\lambda(t)\varphi(t) \le \psi(t)$ where $\lambda(t) = (\nu-\epsilon-\eta)\gamma^2 - C_1^2\frac{\|\bar{u}\|^2}{4\epsilon}$ and

$\psi(t) = \frac{1}{2\eta}\|f-\bar{f}\|_*^2$. By the last theorem we know, with $\psi = 0$, $\exists\, \epsilon : \int_0^T \lambda(s)ds > 0$

This was with $\eta=0$ but with η small enough and ϵ optimal, this will still be true. Now λ is periodic of period T, and we know that

$$\int_t^{t+T}\psi(s)ds \to 0 \quad \text{as } t \to \infty \quad \text{by the hypothesis on } f-\bar{f} .$$

Let $a = \exp(2\int_0^T \lambda(s)ds) > 1$ and $b = \max_{0\le t\le T}(\exp(2\int_0^t \lambda(s)ds))$.

Then by (*)

$$\frac{d}{dt}(\varphi(t)\exp(2\int_0^t \lambda(s)ds)) \le \psi(t)\exp(2\int_0^t\lambda(s)ds) \le b\,\psi(t) .$$

Integrating, $a\varphi(T) - \varphi(0) \leq b \int_0^T \psi(s)ds$. Change the origin and use the fact that λ is periodic, and get

$$a\,\varphi((n+1)T) \leq \varphi(nT) + b\int_{nT}^{(n+1)T} \psi(s)ds .$$

Thus,

$$a > 1 \implies \varphi(nT) \to 0 \text{ as } n \to \infty .$$

Remark: If $a\,x_{n+1} = x_n + y_n$, $\overline{\lim_{n\to\infty}}\, x_n \leq \frac{1}{a-1} \overline{\lim_{n\to\infty}}\, y_n$ $(a > 1.)$
Because if $y_n \leq m$ for large m then $a(x_{n+1} - \frac{m}{a-1}) \leq (x_n - \frac{m}{a-1})$ for large n so $\overline{\lim}(x_n - \frac{m}{a-1}) \leq 0$.

Therefore, $|u-\bar{u}| \to 0 \implies \int_t^{t+T} \|u-\bar{u}\|^2 ds \to 0$ as $t \to \infty$ by integrating the equation.

Regularity of the solution of the Navier-Stokes Equation.

Part 1, Regularity in time:

The equation is
$$\begin{cases} u' + \nu A(u) + B(u,u) = f \\ u(0) = u_0 . \end{cases}$$

We have existence: If $f \in L^2(0,T;V')$, $u_0 \in H$, then a solution u exists, with $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $u' \in L^2(0,T;V')$.

Theorem: If f and f' are in $L^2(0,T;V')$, (so that $f \in C^0(0,T;V')$) and $u_0 \in V$ with $f(0) - \nu A(u_0) - B(u_0,u_0) \in H$, then $u' \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{cases} u'' + \nu A(u') + B(u,u') + B(u',u) = f' \\ u'(0) = f(0) - \nu A(u_0) - B(u_0,u) . \end{cases}$$

Thus, $u'' \in L^2(0,T;V')$ so $u \in C^0(0,T;V) \cap C^1(0,T;H)$.

Proof: Via the Galerkin method. Let $w_1, w_2, \ldots$ be a "basis" for V. Find $u_n(t) = \sum_{i=1}^n g_i(t) w_i$ satisfying

$$
(1) \quad \begin{cases} \left(\dfrac{du_n}{dt} + \nu A(u_n) + B(u_n,u_n) - f - \epsilon_n, w_i\right) = 0 & i = 1,\dots,n \\ u_n(0) = u_{0n} \in \mathrm{Span}\,\{u_1,\dots,w_n\}\,. \end{cases}
$$

where $\epsilon_n = \nu A(u_{0n}) + B(u_{0n},u_{0n}) - \nu A(u_0) - B(u_0,u_0) \in V'$ and $u_{0n} \to u_0$ strongly in V. Then $\epsilon_n \to 0$ strongly in V'.

We have the following estimates on u_n: Take the inner product with g_i and sum over i. Then,

$$
\begin{cases} \frac{1}{2}\dfrac{d}{dt}|u_n|^2 + \nu\|u_n\|^2 \le \|f\|_*\|u\| + \|\epsilon_u\|_*\|u_n\| \\ \qquad |u_n(0)| = |u_{0n}| \le C\,. \end{cases}
$$

Thus, $\{u_n\}$ is contained in a bounded set in $L^2(0,T;V)$ and $L^\infty(0,T;H)$.

Let $v_n = \dfrac{du_n}{dt}$, $v_0 = f(0) - \nu A(u_0) + B(u_0,u_0) \in H$. Differentiate to obtain

$$
(2) \quad \begin{cases} \left(\dfrac{dv_n}{dt} + \nu A(v_n) + B(u_n,v_n) + B(v_n,u_n) - f', w_i\right) = 0 & i = 1,2,\dots,n\,. \\ (v_n(0),w_i) = (f(0) + \epsilon_n - \nu A(u_n(0)) - B(u_{0n},u_{0n}), w_i) \\ \qquad = (v_0,w_i) & i = 1,2,\dots,n\,. \end{cases}
$$

This implies $|v_n(0)| \le |v_0|$. We can then get some estimates on $v_n = \dfrac{du_n}{dt}$:

Take the inner product by g_i' and sum over i:

$$
\begin{aligned} \frac{1}{2}\frac{d}{dt}|v_n|^2 + \nu\|v_n\|^2 &\le |(B(v_n,u_n),v_n)| + \|f'\|_*\|v_n\| \\ &\le C|v_n|\,\|v_n\|\,\|u_n\| + \|f'\|_*\|v_n\| \\ &\le \frac{\nu}{2}\|v_n\|^2 + C_1\|u_n\|^2|v_n| + \frac{\nu}{4}\|v_n\|^2 + C_2\|f'\|_*^2. \end{aligned}
$$

Finally, we obtain

$$
\begin{aligned} &\frac{d}{dt}|v_n|^2 + \nu\|v_n\|^2 \le C_2\|f'\|_*^2 + C_1\|u_n\|^2|v_n|^2 \\ &|v_n(0) \le |v_0|\,. \end{aligned}
$$

Since $\|f'\|_*^2 \in L^1(0,T)$ and $C_1\|u_n\|^2$ is bounded in $L^1(0,T)$ an application of Gronwall's inequality gives $\{v_n\}$ contained in a bounded set in $L^2(0,T;V)$ and $L^\infty(0,T;H)$. Pass to the limit through a suitable sequence:

$$u_m \to u \text{ weakly in } L^2(0,T;V)$$

$$v_m \to v \text{ weakly in } L^2(0,T;V).$$

The compact injection $\Rightarrow$ $u_m \to u$ strongly in $L^2(0,T;H)$. Hence, $B(u_m,u_m) \to B(u,u)$ as in the existence theorem, and this proves u is the solution. Next we pass to the limit in (2).

$$B(u_n,v_n) \to B(u,v) \text{ weakly}$$

$$B(v_n,u_n) \to B(v,u) \text{ weakly.}$$

(This follows because of the strong convergence of $u_n \to u$ in H.
$(B(u_n,v_n),w) = -(B(u_n,w),v_n) = -\int (u_n)_j \frac{\partial w_i}{\partial x_j}(v_n)_i$ and $(u_n)_j(v_n)_i \to u_j v_i$ weakly in $L^2(0,T;L^2(\Omega))$.

This gives us $\left(\frac{dv}{dt} + \nu A(v) + B(u,v) + B(v,u) - f', w_i\right) = 0$ for all $i = 1,2,\ldots$. Since Av, ~~$B(u,v)$~~ ~~and~~ ~~$B(v,u)$ ϵ~~ $L^2(0,T;V')$ we have $\frac{dv}{dt} \in L^2(0,T;V')$.

<u>Theorem:</u> (Regularization property.) If $f, f' \in L^2(0,T;V')$, $u_0 \in H$, then $tu' \in L^2(0,T;V) \cap L^\infty(0,T;H)$, and so $u' \in L^2(\epsilon,T;V) \cap L^\infty(\epsilon,T;H)$ for any $\epsilon > 0$. Thus, even if u is not smooth at $t=0$, the solution becomes smooth after time $t = \epsilon$.

<u>Proof:</u> Same approximation with $v_n = t\frac{du_n}{dt}$. Multiply by t and take the derivative, to obtain

$$\begin{cases} \left(\frac{dv_n}{dt} + \nu A(v_n) + \nu A(u_n) + B(u_n,v_n) + B(u_n,u_n) + B(u_n,u_n) - tf' - \epsilon_n, w_i\right) \\ \qquad\qquad = 0 \quad \text{for} \quad i = 1,\ldots,n \, . \\ v_n(0) = 0 \, . \end{cases}$$

The same estimates as before give $\{v_n\}$ contained in a bounded set in $L^2(0,T;V)$ and $L^\infty(0,T;H)$, and so on .

Remark: 1) This procedure can be repeated. If $f, f', f'', \ldots, f^{(k)}$ are in $L^2(0,T;V')$, and u_0 satisfies a few natural conditions then $u, u', \ldots, u^{(k)} \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

2) If $f, f', \ldots, f^{(k)} \in L^2(0,T;V')$, $u_0 \in H$, then $u, tu', \ldots, t^k u^{(k)} \in L^2(0,T;V) \cap L^\infty(0,T;H)$. In particular, if $f \equiv 0$, $u_0 \in H$, then $u \in C^\infty(\epsilon, T;V)$ $\forall \epsilon > 0$.

Part II:

Regularity in Space: We will need some regularity result for A: What is $D(A)$?

Lemma: (Cattabriga) $w \in D(A)$ (this means $w \in V$ and $Aw \in H$) $\Leftrightarrow$ $w_i : \in H^2(\Omega) \cap H_0^1(\Omega)$ $\forall i$, div $w=0$. (This holds if Ω is bounded and smooth.)

Lemma: If $w \in D(A)$, then $B(w,w) \in H$ and $|B(w,w)| \leq C|w|^{\frac{1}{2}}\|w\| \, |Aw|^{\frac{1}{2}}$.

Proof: $|(B(w,w),\varphi)| = |\sum_{i,j} \int w_j \frac{\partial w_i}{\partial x_j} \varphi_i| \leq C(\sum_{i,j} |w_j \frac{\partial w_i}{\partial x_j}|_{L^2})|\varphi|_4$

by Cauchy-Schwartz. Thus

$$|B(w,w)|_H \leq C \sum_{i,j} |w_j \frac{\partial w_i}{\partial x_j}|_{L^2}$$

By Hölder's inequality,

$$\leq C \sum_{i,j} |w_j|_{L^4} \, |\frac{\partial w_i}{\partial x_j}|_{L^4}$$

Recall that

$$|w|_{L^4} \leq C \, |w|^{\frac{1}{2}}_{L^2} \|w\|^{\frac{1}{2}}_{H^1}$$

Hence,

$$|B(w,w)| \leq C \sum_{i,j} |w_j|^{\frac{1}{2}}_{L^2} \|w_j\|^{\frac{1}{2}}_{H^1} \|w_i\|^{\frac{1}{2}}_{H^1} \|w_i\|^{\frac{1}{2}}_{H^2}$$

$$\leq C|w|^{\frac{1}{2}}\|w\| \, |Aw|^{\frac{1}{2}}$$

by the preceding lemma.

Remark: If $w \in (H_0^1)^2 \cap (H^2)^2$, div $w = 0$, $B(w,w)$ does not have 0 divergence.

We identified H with H'. This means $B(w,w) = h + \text{grad}\, p$ where $h \in H = \{\text{div}\, h = 0\}$.

Theorem: If $f \in L^2(0,T;H)$ and $u_0 \in V$, then u' and Au are in $L^2(0,T;H)$ and $u \in L^\infty(0,T;V)$. (And so $B(u,u) \in L^2(0,T;H)$ by the equation.)

Proof: We use the special basis of eigenvectors of A. Let w_i satisfy $A w_i = \lambda w_i$. $\{w_i\}$ is an orthonormal basis for H. $u_n = \sum_{i=1}^{n} g_i(t) w_i$ satisfies

$$\left(\frac{du_n}{dt} + \nu A(u_n) + B(u_n,u_n) - f, w_i\right) = 0 \quad i = 1,2,\ldots,n\,.$$

Note: $Au_n \in \text{Span}(w_1,\ldots,w_n)$, since $Aw_i = \lambda_i w_i$. Multiply by $\lambda_i g_i$ and sum in i. Then

$$\left(\frac{du_n}{dt} + \nu A u_n + B(u_n,u_n) - f, Au_n\right) = 0 \qquad i=1,2,..,n\,.$$

Note: $$\left(\frac{du_n}{dt}, Au_n\right) = \tfrac{1}{2}\frac{d}{dt}(A u_n, u_n) = \frac{1}{2}\frac{d}{dt}\|u_n\|^2$$

$$\tfrac{1}{2}\frac{d}{dt}\|u_n\|^2 + \nu|Au_n|^2 \le |B(u_n,u_n)|\cdot|Au_n| + |f|\,|Au_n|\,.$$

Apply the general form of Young's inequality:

$$ab \le \frac{a^P}{P} + \frac{b^{P'}}{P'} \quad \text{where} \quad \frac{1}{P} + \frac{1}{P'} = 1$$

or $ab \le \epsilon\, a^P + C(\epsilon)\, b^{P'} \quad \forall \epsilon$.

Let $P = 4/3$. Then $P' = 4$. The right-hand side above is, from previous results,

$$\le \epsilon|Au_n|^2 + C(\epsilon)|u_n|^2\|u_n\|^4 + \epsilon|Au_n|^2 + C(\epsilon)|f|^2\,.$$

We obtain

$$\frac{d}{dt}\|u_n\|^2 + \eta|Au_n|^2 \le C|f|^2 + C|u_n|^2\|u_n\|^2\cdot\|u_n\|^2$$

$$\|u_n(0)\| \le C \quad \text{since } u_0 \in V\,.$$

Apply Gronwall's inequality. With $\lambda_n(t) = C|u_n|^2\|u_n\|^2$, then

$$|\lambda_n(t)|_{L^1(0,T)} \leq C$$

because $|u_n| \leq C$ and $\int_0^T \|u_n\|^2 \leq C$

Then we get $\{u_n\}$ contained in a bounded set in $L^2(0,T;H)$. Now $u \in L^\infty(0,T;V)$ and $A\,u \in L^2(0,T;H) \Longrightarrow B(u,u) \in L^4(0,T;H)$ by the lemma above, Thus $u' \in L^2(0,T;H)$.

<u>Theorem:</u> (Regularization.) If $f \in L^2(0,T;H)$, $u_0 \in H$, then $\sqrt{t}\,u \in L^\infty(0,T;V)$ and $\sqrt{t}\,u' \in L^2(0,T;H)$.

<u>Proof:</u> Multiply by $t\lambda_i g_i$ and sum in i, as before. Then

$$\tfrac{1}{2}\frac{d}{dt}(t\|u_n\|^2) - \tfrac{1}{2}\|u_n\|^2 + t|Au_n|^2 \leq t|B(u_n,u_n)|\cdot|Au_n| + t|f|\,|Au_n|$$
$$C\,t^{\frac{1}{4}}|u_n|^{\frac{1}{2}}\|u_n\|\,t^{3/4}|Au_n|^{3/2} + t|f|\,|Au_n| .$$

Apply Young's inequality again, and obtain

$$\frac{d}{dt}(t\,\|u_n\|^2) + \eta t|Au_n|^2 \leq C|t|\,|f|^2 + C\,t|u_n|^2\|u_n\|^2\cdot\|u_n\|^2 .$$

The remainder of the proof follows as usual.

<u>Remarks:</u> (On Cattabriga's results.) $D(A) = \{v \in V : v_i \in H^2(\Omega)\ \forall i\}$. Suppose $h \in H$, $Au = h$. Then each $h_i \in L^2(\Omega)$, $\operatorname{div} h = 0$, and $h\cdot n|_{\partial\Omega} = 0$. (inner product.) $Au = h$ means there exists $p \in L^2(\Omega)$ with $-\Delta u_i = h_i - \frac{\partial p}{\partial x_i}$.

If $h_i \in H^{-1}(\Omega)$, then $u \in V$, $p \in L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta u_i = h_i - \dfrac{\partial p}{\partial x_i} & (u_i \in H_0^1) \\ \operatorname{div} u = 0 . \end{cases}$$

Cattabriga's result says that if $h_i \in L^2(\Omega)$, then $u_i \in H^2(\Omega)$ and $p \in H^1(\Omega)$.

Embed this in the more general problem: Let

$$f_i \in H^{-1}(\Omega)\ ,\quad g \in L^2(\Omega) \text{ with } \int_\Omega g\,dx = 0 .$$

Then there is a unique solution (u_i,p) such that $u_i \in H_0^1(\Omega)$, $p \in L^2(\Omega)/\text{constants}$

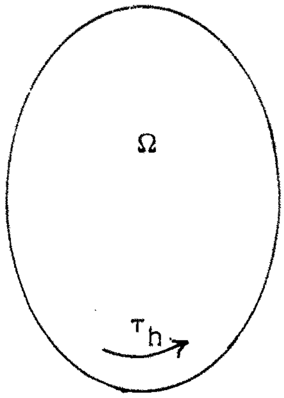
and

$$-\Delta u_i + \frac{\partial p}{\partial x_i} = f_i$$

$$\operatorname{div} u = g \qquad \text{(This is an elliptic system)}$$

Consider the regularity problem: if $f_i \in L^2(\Omega)$ and $g \in H^1$, then $u_i \in H^2$ and $p \in H^1$.

To prove this, use the Method of Translation



τ_h is a semigroup whose infinitesimal generator is Λ, a 1st order differential operator in H_0^1.

If $f_i \in D(\Lambda)$, $g \in D(\Lambda)$, then $u_i, p \in D(\Lambda)$. This gives regularity in tangential directions; the equations then give regularity in normal direction.

III. A Semilinear wave equation.

We will now consider the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = 0 \\ u(x,0) = u_0(x) \qquad\qquad (\text{in } \mathbb{R}^3.) \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \end{cases}$$

(This equation is a perturbed wave equation. It is a propagation problem; we will show that the propagation speed is 1. We will consider $\Omega = \mathbb{R}^3$.)

Formal estimates on the wave equation:

Consider

$$u'' - \Delta u = 0$$

$$u(0) = u_0 \ , \quad u'(0) = u_1$$

Multiply the equation by $\frac{du}{dt}$ and integrate: (u goes to 0 at ∞)

$$\int_{\mathbb{R}^3} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\frac{\partial u}{\partial t})^2 dx$$

$$\int_{\mathbb{R}^3} -\frac{\partial^2 u}{\partial x_1^2} \frac{\partial u}{\partial t} dx = \int_{\mathbb{R}^3} \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_i \partial_t} dx \qquad \text{(by parts)}$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\frac{\partial u}{\partial x_i})^2 dx$$

Formally, if u is smooth, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} [(\frac{\partial u}{\partial t})^2 + \sum_i (\frac{\partial u}{\partial x_i})^2] dx = 0$$

The integral is the total energy. The first term is kinetic energy, the second is potential energy.

The abstract formulation of the wave equation:

We have $V \subseteq H = H' \subseteq V'$, where $V = H^1(\mathbb{R}^3)$ has norm $\|\cdot\|$ and $H = L^2(\mathbb{R}^3)$ has norm $|\cdot|$. V is dense in H

Let $A \in \mathcal{L}(V, V')$, $A = -\Delta$. $A = A^*$, since $(Au, v) = (Av, u)$. We have the coercivity condition:

$$(Au, u) \geq \alpha \|u\|^2 - \beta |u|^2$$

$$(Au, u) = \int_{\mathbb{R}^3} \sum_i (\frac{\partial u}{\partial x_i})^2 = \|u\|^2 - |u|^2$$

Then we can express the problem as

$$u'' + Au = 0$$

$$u(0) = u_0 \in V$$

$$u'(0) = u_1 \in H$$

To derive a formal estimate, take the inner product with u':

$$(u'',u') = \tfrac{1}{2}\frac{d}{dt}|u'|^2 ,$$

$$(Au,u') = \tfrac{1}{2}\frac{d}{dt}(Au,u) \quad \text{since } A = A^*, \text{ and since}$$

$$= \tfrac{1}{2}[(Au,u') + (Au',u)] .$$

Thus we have

$$\frac{d}{dt}(|u'|^2 + (Au,u)) = 0 .$$

Integrate from 0 to T. Then

$$\|u\|_{L^\infty(0,T;V)} + |u'|_{L^\infty(0,T;H)} \le C(\|u_0\| + |u_1|) \qquad T < \infty .$$

Thus $u \in L^\infty(0,T;V)$ and $u' \in L^\infty(0,T;H)$. The u'' given by the equation is in $L^\infty(0,T;V')$.

All this is formal because (u'',u') and (Au,u') have no meaning.

A good space to look in for solutions u is $u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$, and $u'' + Au \in L^1(0,T;H)$.

<u>Existence and uniqueness for the linear problem:</u>

Hypotheses: $V \subseteq H \subseteq V'$, $A = A^*$, and $(Au,u) \ge \alpha\|u\|^2$.

<u>Theorem:</u> Let $f \in L^1(0,T;H)$, $u_0 \in V$ and $u_1 \in H$. Then there exists $u \in L^\infty(0,T;V)$ with $u' \in L^\infty(0,T;H)$, such that $u'' + Au = f$, $u(0) = u_0$, and $u'(0) = u_1$: that is, $u'' \in L^1(0,T;V')$, so $u(0)$ and $u'(0)$ have meaning.

<u>Proof:</u> Via Galerkin's method. (Note that V is separable.) Let $w_1, w_2, \ldots$ be a "basis" for V. We try to find $u_n = \sum_{i=1}^{n} g_i(t)w_i$ such that

$$\begin{cases} (\dfrac{d^2u_n}{dt^2} + Au_n - f, w_i) = 0 & i = 1,2,\ldots,n \\ u_n(0) = u_{0n} \to u_0 & \text{in } V \\ u_n'(0) = u_{1n} \to u_1 & \text{in } H . \end{cases}$$

1) Local existence of u_n .

As the w_i are linearly independent, this is a system of differential equations in g_i. We can write

$$g_i''(t) = F_i(g_1,\ldots,g_n,t) .$$

Thus solutions u_n exist and are unique on intervals $[0,t_n)$ where $t_n \leq T$.

2.) Global existence.

The formal procedure can be used now, since u, u', and u'' take values in $\text{span}(w_1,w_2,\ldots,w_n)$. Multiply the equation by $g_i'(t)$ and sum over i . Then

$$\left(\frac{d^2u_n}{dt^2} + Au_n, \frac{du_n}{dt}\right) = \left(f, \frac{du_n}{dt}\right) .$$

Using $A = A^*$, we get

$$\tfrac{1}{2}\frac{d}{dt}\left(|u_n'|^2 + (Au_n,u_n)\right) = \left(f, \frac{du_n}{dt}\right) \leq |f|\,|u_n'| .$$

Integrate from 0 to t :

$$\tfrac{1}{2}\left(|u_n'|^2 + (Au_n,u_n)\right) \leq \tfrac{1}{2}\left(|u_{1n}|^2 + (Au_{0n},u_{0n})\right) + \int_0^t |f|\cdot|u_n'|\,ds .$$

Now use Hölder's inequality with $|f|^{\frac{1}{2}}(|f|^{\frac{1}{2}}|u_n'|)$: then Young's inequality, so

$$\int_0^t |f|\,|u_n'|\,ds \leq \tfrac{1}{2}\int_0^t |f|\,ds + \tfrac{1}{2}\int_0^t |f|\,|u_n'|^2 ds .$$

Thus

$$\tfrac{1}{2}\left(|u_n'|^2 + (Au_n,u_n)\right) \leq C + \tfrac{1}{2}\int_0^t |f|\,|u_n'|^2 ds .$$

Take $\varphi_n(t) = |u_n'(t)|^2 + (Au_n,u_n)$. Then

$$\varphi_n(t) \leq C + \int_0^t \lambda(s)\varphi_n(s)ds \quad \text{with} \quad \lambda \in L^1(0,T). \quad (\lambda = |f|) .$$

Let $\psi_n(t)$ be the right-hand side of this inequality. Then

$$\psi_n'(t) = \lambda(t)\varphi_n(t) \leq \lambda(t)\psi_n(t).$$

Therefore, Gronwall's inequality gives

$$\varphi_n(t) \leq C\exp\left(\int_0^t \lambda(s)ds\right) \leq C\exp\left(\int_0^T \lambda(s)ds\right) .$$

This is a bound independent of t, and so $t_n = T$ for every n. (This gives an a priori bound on u_n.)

Hence $\{u_n\}$ is contained in a bounded set in $L^\infty(0,T;V)$ and $\{u'_n\}$ is contained in a bounded set in $L^\infty(0,T;H)$.

3.) Pass to the limit via an appropriate subsequence. Extract as subsequence $u_m \to u$ weakly* in $L^\infty(0,T;V)$ and $u'_m \to u'$ weakly* in $L^\infty(0,T;H)$. Then

$$\frac{d^2u_m}{dt^2} \to \frac{d^2u}{dt^2}$$

in the distributional sense. Take the limit in each term of the equation:

$$(Au_n, w_i)_{V',V} \to (Au, w_i) \quad \text{weakly* in } L^\infty(0,T)$$

$$\left(\frac{d^2u_n}{dt^2}, w_i\right) = \frac{d}{dt}\left(\frac{du_n}{dt}, w_i\right) \to \frac{d}{dt}\left(\frac{du}{dt}, w_i\right) \qquad \text{in } \mathcal{D}'(0,T).$$

Thus u satisfies

$$\left(\frac{d^2u}{dt^2} + Au - f, w_i\right) = 0 \quad \forall\, i.$$

(In the distributional sense.) This proves that

$$\frac{d^2u}{dt^2} \in L^1(0,T;V')$$

$u_m(0) \to u_0$ weakly (because we have estimates on u_m <u>and</u> on $\frac{du_m}{dt}$.)

For $u'_m(0)$ we have $\left(\frac{du_m}{dt}, w_i\right) \to \left(\frac{du}{dt}, w_i\right)$ weakly* in $L^\infty(0,T)$ and

$$\frac{d}{dt}\left(\frac{du_m}{dt}, w_i\right) = (f - Au_m, w_i)$$

which converges weakly in $L^1(0,T)$. Thus $\left(\frac{du_m}{dt}, w_i\right)\Big|_{t=0}$ converges to $\left(\frac{du}{dt}, w_i\right)\Big|_{t=0} = (u_1, w_i)$ for all i.

Theorem: (Regularity in time). If $f, f' \in L^1(0,T;H)$, $u_0 \in D(A)$ ($Au_0 \in H$), and $u_1 \in V$, then there exists a unique solution of

$$\begin{cases} u'' + Au = f \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases}$$

such that $u, u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;H)$, and $u' = v$ satisfies

$$\begin{cases} v'' + Av = f' \\ v(0) = u_1 \\ v'(0) = f(0) - Au_0 \end{cases}$$

Proof: (Via Galerkin's method)

First, we obtain more estimates on $\frac{du_n}{dt}$. As usual, we get u_n such that

$$\begin{cases} \left(\frac{d^2u_n}{dt^2} + Au_n - f, w_i\right) = 0 \qquad i = 1,2,\dots,n \\ u_n(0) = u_{0n} \\ u_n'(0) = u_{1n} \quad . \end{cases}$$

Put $v_n = \frac{du_n}{dt}$. Thus

$$\left(\frac{d^2v_n}{dt^2} + Av_n - f', w_i\right) = 0 \qquad i = 1,2,\dots,n \, .$$

$$v_n(0) = u_n'(0) = u_{1n} \to u_1 \text{ in } V \, .$$

Using the equation, we get also $(v_n'(0), w_i) = (u_n''(0), w_i) = (f(0) - Au_n(0), w_i)$
$i = 1,2,\dots,n.$

The trick is to choose $w_1 = u_0$ and $u_{0n} = u_0$ for all n. $v_n(0)$ is bounded in V, and $v_n'(0)$ is the projection of $f(0) - Au_0$ onto $\text{span}(w_1,\dots,w_n)$, and is bounded in H. Apply to v_n the estimates obtained in the existence proof for u.

Then we have

$\{u_n\}, \{v_n\}$ are contained in a bounded set in $L^\infty(0.T:V)$ and $\{v_n\}$ $\{\frac{dv_n}{dt}\}$ are contained in a bounded set in $L^\infty(0,T;H)$. We know that $u_m \to u$ weakly, since $\{\frac{du_n}{dt}\}$ is contained in a bounded set in $L^\infty(0,T;V)$. This gives $\frac{du}{dt} \in L^\infty(0,T;V)$.

$\{\frac{d^2u_n}{dt^2}\}$ is contained in a bounded set in $L^\infty(0,T;H)$ gives $\frac{d^2u}{dt^2} \in L^\infty(0,T;H)$. This gives the existence of a regular solution.

Uniqueness is next. Suppose u_1, u_2 are solutions. Then

$$\begin{cases} (u_1-u_2)'' + A(u_1-u_2) = 0 \\ (u_1-u_2)(0) = 0 \\ (u_1-u_2)'(0) = 0 . \end{cases}$$

Take the inner product with $(u_1-u_2)'$. Since u_1 and u_2 are regular, integration by parts is legal. Thus we obtain $\frac{d}{dt}(|u_1-u_2|^2 + (A(u_1-u_2), u_1-u_2)) = 0$. The quantity in parenthesis was 0 when t=0, so

$$|u_1-u_2|^2 + \nu\|u_1-u_2\|^2 \le |u_1-u_2|^2 + (A(u_1-u_2), u_1-u_2) \equiv 0$$

The first inequality uses the coercivity of A. Uniqueness follows.

Uniqueness holds under weaker conditions, however.

Theorem: (Uniqueness) If $f \in L^1(0,T;H)$, $u_0 \in V$ and $u_1 \in H$, then the solution u of the equation is unique in the class of functions

$$\{u \in L^\infty(0,T;V) : u' \in L^\infty(0,T;H)\} .$$

Proof: By taking differences, we need only show that $f = 0$, $u_0 = 0$, and $u_1 = 0$ imply that $u = 0$.

Take $g \in L^1(0,T;H)$ such that $g' \in L^1(0,T;H)$.

Solve

$$\begin{cases} \dfrac{d^2v}{dt^2} + Av = g \\ v(T) = 0 \\ v'(T) = 0 \end{cases}$$

Take $w(t) = v(T-t)$. Then

$$\begin{cases} w'' + Aw = g(T-t) \\ w(0) = 0 \\ w'(0) = 0 \end{cases}$$

g satisfies the hypotheses of the regularity theorem, so we can find $v, v' \in L^\infty(0,T;V)$ with $v'' \in L^\infty(0,T;H)$. Consider

$$0 = \int_0^T (u'' + Au, v)dt .$$

$$\int_0^T (u'', v)dt = -\int_0^T (u', v')dt + (u'(T), v(T)) - (u(0), v(0))$$
$$= -\int_0^T (u', v')dt.$$

We can do this since $u'' \in L^\infty(0,T;V;)$, $u' \in L^\infty(0,T;H)$, $v \in L^\infty(0,T;V)$ and $v' \in L^\infty(0,T;V)$. (v is regular)

$$= \int_0^T (u, v'')dt - (u(T), v'(T)) + (u(0), v'(0)) = \int_0^T (u, v'')dt$$

(note that the integral has meaning by our conditions on u and v.) Also,

$$\int_0^T (Au, v)dt = \int_0^T (u, Av)dt \quad \text{since } A = A^* .$$

Hence,

$$0 = \int_0^T (u'' + Au, v)dt = \int_0^T (u, v'' + Av)dt$$
$$= \int_0^T (u, g)dt,$$

for a dense class of g, so $u = 0$.

Theorem: (Identity of Energy) Suppose $f \in L^1(0,T;H)$, $u_0 \in V$, and $u_1 \in H$. Then $u \in C^0(0,T;V)$, $u' \in C^0(0,T;H)$ and $\forall\, s,t \in [0,T]$,

$$|u'(t)|^2 + (Au(t),u(t)) = |u'(s)|^2 + (Au(s),u(s)) + 2\int_s^t (f(\tau),u'(\tau))d\tau .$$

Remark: This implies that if $u \in L^\infty(0,T;V)$, $u' \in L^\infty(0,T;H)$ and $u'' + Au \in L^1(0,T;H)$, then $|u'|^2 + (Au,u)$ is absolutely continuous.

Proof: Assume f, u_0, u_1 satisfy the hypotheses of the regularity theorem. Then $u, u' \in L^\infty(0,T;V)$ and $u'' \in L^\infty(0,T;H)$, and so the conclusions are true. (We can integrate by parts.)

Next, we approach

$$\left.\begin{array}{ll} & f \text{ by } f_n \\ & u_0 \text{ by } u_{0n} \\ \text{and} & u_1 \text{ by } u_{1n} \end{array}\right\} \quad \text{satisfying the regularity conditions.}$$

$u_n - u_m$ satisfies an equation with $f_n - f_m$, $u_{0n} - u_{0m}$, and $u_{1n} - u_{1m}$.

By the original estimate,

$$\|u_n - u_m\|_{L^\infty(0,T;V)} \leq C[|f_n - f_m|_{L^1(0,T;H)} + \|u_{0n} - u_{0m}\| + |u_{1n} - u_{1m}|]$$

and $|u_n' - u_m'|_{L^\infty(0,T;H)}$ satisfies a similar inequality.

We have a Cauchy sequence $\{u_n\}$ in $L^\infty(0,T;V)$, therefore, and all the u_n are continuous; this gives $u_n \to u \in C^0(0,T;V)$ and similarly $u_n' \to u' \in C^0(0,T;H)$. Now, pass to the limit in the desired equality.

Theorem: (Regularity in space)

If $f \in L^1(0,T;V)$, $u_0 \in D(A)$, and $u_1 \in V$, then the solution u satisfies $u \in L^\infty(0,T;D(A))$, and $u' \in L^\infty(0,T;V)$.

Proof: If f, f' and $f'' \in L^1(0,T;V)$, $u_0 \in D(A^2)$ $(A\,u_0 \in D(A))$ and $u_1 \in D(A)$ then u, u' and $u'' \in L^\infty(0,T;V)$ from the theorem of regularity in time.

Obtain a new estimate by taking the inner product with Au':

$$(u'',Au') = \tfrac{1}{2}\frac{d}{dt}(Au',u') \quad \text{since } A^* = A$$
$$(Au,Au') = \tfrac{1}{2}\frac{d}{dt}|Au|^2 .$$

This gives

$$\tfrac{1}{2}\frac{d}{dt}[(Au',u') + |Au|^2] = (f,Au') \leq (Af,f)^{\frac{1}{2}}(Au',u')^{\frac{1}{2}}$$

since $A = A^*$ and $(Au,u) \geq 0$. Now, $(Au_1,u_1) + |Au_0|^2 < \infty$. By Gronwall's inequality

$$(Au',u') + |Au|^2 \leq C[\|f\|_{L^1(0,T;V)} + \|u_0\|_{D(A)} + \|u_1\|_V]^2$$

(C depends on $\|f\|$.) Now approach f by $f_n \in C^2(0,T;V)$. Also u_0 by $u_{0n} \in D(A^2)$ in $D(A)$ and u_1 by u_{1n} in $D(A)$ in V. (The approximation of f is in $L^1(0,T;V)$.) .

This gives the estimate on u. In the case $V \subsetneq H$ is a compact injection, one can use a special Galerkin basis of eigenvectors of A. (or A^{-1}). $Aw_i = \lambda_i w_i$, $\lambda_i \to \infty$.

This gives

$$\left(\frac{d^2u_n}{dt^2} + Au_n - f,\ A\frac{du_n}{dt}\right) = 0 , \quad \text{which gives the estimate.}$$

Special estimates on the wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f \\ u(0) = u_0 & \text{in } \mathbb{R}^N ,\ n > 1 . \\ \dfrac{\partial u}{\partial t}(0) = u_1 . \end{cases}$$

These are for smooth solutions.

Propagation with speed ≤ 1.

Note: $B_E(0,r_0) = \{v : \|v\|_E \leq r_0\}$.

Theorem: If the support of (u_0, u_1) is contained in $B(0,r_0)$ and $\mathrm{supp}(f(t)) \subseteq B(0,r_0+t)$, then $\mathrm{supp}(u(t)) \subseteq B(0,r_0+t)$.

Proof:

Take the inner product with $\frac{\partial u}{\partial t}\varphi(r-t)$, where φ satisfies: $\varphi(r) = 0$ on $[0,r_0]$ $\varphi(r) > 0$ on $]r_0,+\infty[$ $\varphi'(r) \geq 0$.

We will prove that

$$I(t) = \int_{\mathbb{R}^N} \left(\left|\frac{\partial u}{\partial t}(x,t)\right|^2 + \sum_i \left|\frac{\partial u}{\partial x_i}(x,t)\right|^2\right) \varphi(r-t)dx \qquad r = |x|_{\mathbb{R}^N}$$

satisfies $\frac{dI}{dt} \leq 0$.

$$I(0) = \int_{r \leq r_0} \left(|u_1|^2 + \sum_i \left|\frac{\partial u_0}{\partial x_i}\right|^2\right) \varphi(r)dx = 0 .$$

Then $I(t) \leq 0$ implies

$$\left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i \left|\frac{\partial u}{\partial x_i}\right|^2\right)(x,t)\, \varphi(r-t) = 0 \text{ a.e.}$$

so if $r - T > r_0$, $\varphi(r-T) > 0$ and so

$$\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i \left|\frac{\partial u}{\partial x_i}\right|^2 = 0 ,$$

which is what we wanted. Now, to prove the assertion above,

$$\frac{dI}{dt} = \int_{\mathbb{R}^N} 2\varphi(r-t) \left[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_i \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial t}\right]dx$$

$$-\int_{\mathbb{R}^N} \varphi'(r-t)\left[\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i \left|\frac{\partial u}{\partial x_i}\right|^2\right]dx$$

$$= \int_{\mathbb{R}^N} 2\varphi(r-t)\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}dx - \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i}\left[2\varphi(r-t)\frac{\partial u}{\partial x_i}\right]\frac{\partial u}{\partial t}dx$$

$$- \int_{\mathbb{R}^N} \varphi'(r-t)\left[\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i \left|\frac{\partial u}{\partial x_i}\right|^2\right]dx$$

$$= \int_{\mathbb{R}^N} 2\varphi(r-t)\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right) - \sum_i \int_{\mathbb{R}^N} 2\varphi'(r-t)\frac{x_i}{r}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}\, dx$$

$$- \int_{\mathbb{R}^N} \varphi'(r-t)\left(\left(\frac{\partial u}{\partial t}\right)^2 + \sum_i \left(\frac{\partial u}{\partial x_i}\right)^2\right)dx .$$

Use the fact that $\frac{\partial^2 u}{\partial t^2} - \Delta u = f$, together with the conditions on supp(f) to get the first integral 0 .

Then it suffices to show

$$(\frac{\partial u}{\partial t})^2 + \sum_i (\frac{\partial u}{\partial x_i})^2 + \sum_i 2\frac{x_i}{r}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t} \geq 0 \quad \text{a.e.}$$

Use the fact that $2|ab| \leq a^2 + b^2$ $\forall a,b \in \mathbb{R}$, together with $a = \frac{x_i}{r}\frac{\partial u}{\partial t}$ and $b = \frac{\partial u}{\partial x_i}$ to get

$$2|\frac{x_i}{r}\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x_i}| \leq (\frac{\partial u}{\partial x_i})^2 + (\frac{x_i}{r})^2(\frac{\partial u}{\partial t})^2 ,$$

and sum over i . Then

$$\sum_i (\frac{x_i}{r})^2 = \frac{|x|^2}{r^2} = 1 ,$$

so that the absolute value of

$$\sum_i 2\frac{x_i}{r}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}$$

is less than or equal to the sum of the other two terms, as desired.

In the more general case $-\Delta = -\sum \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_i}$, the speed of propagation S is less than the square root of the maximum eigenvalue of (a_{ij}).

If f=0, and supp(u_0), supp(u_1) $\subseteq \Omega$, then distance $(x,\Omega) > t \implies u(x,t) = 0$. To see this, cover Ω by small balls and apply the above result.

If $u'' + Af = 0$, then $|u'|^2 + (Au,u)$ is constant in time. That is,

$$E(t) = \int_{\mathbb{R}^N} [|\frac{\partial u}{\partial t}(x,t)|^2 + \sum_i |\frac{\partial u}{\partial x_i}(x,t)|^2]dx = \text{constant.}$$

E is the energy.

Energy cannot go to 0 in $\mathbb{R}^N$. However, if $\Omega \subseteq \mathbb{R}^N$ and Ω is bounded, the energy in Ω ,

$$\int_\Omega [|\frac{\partial u}{\partial t}|^2 + \sum_i |\frac{\partial u}{\partial x_i}|^2] dx$$

can go to 0, as t increases. (The problem is still posed in $\mathbb{R}^N$.) This corresponds to the wave passing by.

<u>Three invariants for</u> $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$.

1) $$E_1(t) = \int_{\mathbb{R}^N} [|\frac{\partial u}{\partial t}|^2 + |\text{grad}u|^2] dx$$

Recall: $\text{grad}u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$ $\quad r = |x|_{\mathbb{R}^N} = \sqrt{\sum_i |x_i|^2}$

2) $$E_2(t) = \int_{\mathbb{R}^N} [t(|\frac{\partial u}{\partial t}|^2 + |\text{grad}u|^2) + 2r \frac{\partial u}{\partial t}\frac{\partial u}{\partial r} + (N-1)u\frac{\partial u}{\partial t}] dx$$

where $\frac{\partial u}{\partial r} = \sum_i \frac{x_i}{r}\frac{\partial u}{\partial x_i}$

3) $$E_3(t) = \int_{\mathbb{R}^N} [(r^2+t^2)(|\frac{\partial u}{\partial t}|^2 + |\text{grad}u|^2) + 4tr\frac{\partial u}{\partial t}\frac{\partial u}{\partial r} + 2(N-1)tu\frac{\partial u}{\partial t} - (N-1)u^2] dx$$

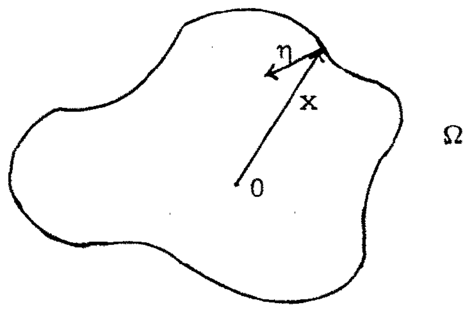
These can be obtained by multiplying the equation by something:

For E_1, $2\frac{\partial u}{\partial t}$; For E_2, $2t\frac{\partial u}{\partial t} + 2r\frac{\partial u}{\partial r} + (N-1)u$; and For E_3, $2(r^2+t^2)\frac{\partial u}{\partial t} + 4tr\frac{\partial u}{\partial r} + 2(N-1)tu$.

<u>Remarks:</u> 1) We are still in $\mathbb{R}^N$. If we work in $\Omega \neq \mathbb{R}^N$, and use $u|_{\partial\Omega} = 0$, we still have $\frac{d}{dt}E_1(t) = 0$; but

$$\frac{d}{dt}E_2(t) = \int_{\partial\Omega} |\frac{\partial u}{\partial n}|^2 (x \cdot n) d\sigma \quad \text{and} \quad \frac{d}{dt}E_3(t) = 2t\int_{\partial\Omega} |\frac{\partial u}{\partial n}|^2 (x \cdot n) d\sigma$$

(n is the normal vector to $\partial\Omega$.) If Ω is the complement of a star-shaped, bounded set in $\mathbb{R}^N$, E_2 and E_3 are decreasing, since $x \cdot \eta \leq 0$ on $\partial\Omega$:



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2) $E_3 \geq 0$. If $E_3 \leq C$, we will prove, for $N = 3$, that

$$\int_K (|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad} u|^2) dx \to 0 \text{ as } \frac{1}{t^2}$$

for a bounded set K.

Suppose u is smooth, (with compact support in $\mathbb{R}^N$) and

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in } \mathbb{R}^N. \quad N > 1 .$$

1) $$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad} u|^2) dx = \int_{\mathbb{R}^N} 2\varphi \frac{\partial u}{\partial t} f$$

$$+ \int_{\mathbb{R}^N} [\frac{\partial \varphi}{\partial t}(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad} u|^2) - 2\sum_i \frac{\partial \varphi}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}] dx$$

2) $$\frac{d}{dt}\int a_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t} dx = \int a_i \frac{\partial u}{\partial x_i} f\, dx + \int [-\tfrac{1}{2}\frac{\partial a_i}{\partial x_i}|\frac{\partial u}{\partial t}|^2$$

$$+ \tfrac{1}{2}\frac{\partial a_i}{\partial x_i}|\mathrm{grad} u|^2 - \sum_j \frac{\partial a_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial a_i}{\partial t}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial t}] dx .$$

<u>Proof:</u> Left as an exercise. Multiply the equation by $a\frac{\partial u}{\partial x_i}$ and integrate by parts. $a = a(x,t)$

3) $$\frac{d}{dt}\int \psi u \frac{\partial u}{\partial t} = \int \psi u f + \int [\psi|\frac{\partial u}{\partial t}|^2 - \psi|\mathrm{grad} u|^2 + \frac{\partial \psi}{\partial t} u \frac{\partial u}{\partial t} + \tfrac{1}{2}\Delta\psi\, |u|^2] dx.$$

If we work with functions of (r,t), using $\frac{\partial}{\partial r} = \sum \frac{x_i}{r}\frac{\partial}{\partial x_i}$, and φ depends only on r ,

(1') $$\frac{d}{dt}\int \varphi(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad} u|^2) = \int [2\varphi \frac{\partial u}{\partial t} f + \frac{\partial \varphi}{\partial t}(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad} u|^2) - 2\frac{\partial \varphi}{\partial r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial t}] dx.$$

Taking $a_i(x,t) = \frac{x_i}{r} a(r,t)$ and summing in i

(2') $$\frac{d}{dt}\int a \frac{\partial u}{\partial r}\frac{\partial u}{\partial t} = \int a \frac{\partial u}{\partial r} f + \int [-\tfrac{1}{2}(\frac{\partial a}{\partial r} + \frac{(N-1)a}{r})|\frac{\partial u}{\partial t}|^2$$

$$\frac{\partial a}{\partial t}\frac{\partial u}{\partial r}\frac{\partial u}{\partial t} + \tfrac{1}{2}(\frac{\partial a}{\partial r} + \frac{(N-3)a}{r})|\mathrm{grad} u|^2 - [\frac{\partial a}{\partial r} - \frac{a}{r}]|\frac{\partial u}{\partial r}|^2] dx.$$

As an example of the derivation, we compute the coefficient of $|\frac{\partial u}{\partial t}|^2$.

It was

$$-\tfrac{1}{2}\sum_i \frac{\partial a_i}{\partial x_i} : \sum_i \frac{\partial a_i}{\partial x_i} = \sum_i [\frac{x_i}{r}\frac{\partial a}{\partial r}\frac{x_i}{r} + a\frac{\partial}{\partial x_i}(\frac{x_i}{r})]$$

Now, $\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{1}{r} - \frac{x_i^2}{r^3}$, so $\sum_i \frac{\partial a_i}{\partial x_i} = \frac{\partial a}{\partial r} + \frac{(N-1)a}{r}$.

Next, sum the left-hand sides of (1'), (2') and (3), and suppose the result,

$$\frac{d}{dt}\int(\varphi(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad}\,u|^2) + a\,\frac{\partial u}{\partial r}\,\frac{\partial u}{\partial t} + \psi\, u\,\frac{\partial u}{\partial t})$$

has no term in $|\frac{\partial u}{\partial t}|^2$, $|\mathrm{grad}\,u|^2$, $\frac{\partial u}{\partial r}\,\frac{\partial u}{\partial t}$, and $|\frac{\partial u}{\partial r}|^2$. Then the coefficient of $|\frac{\partial u}{\partial r}|^2$ is $-[\frac{\partial u}{\partial r} - \frac{a}{r}]$, which is 0 if and only if $a(r,t) = b(t)r$; then the coefficient of $|\frac{\partial u}{\partial t}|^2$ is $\frac{\partial\varphi}{\partial t} - \frac{N}{2}\,b(t) + \psi$; that of $|\mathrm{grad}\,u|^2$ is $\frac{\partial\varphi}{\partial t} + \frac{N-2}{2}b(t) - \psi$ and that of $\frac{\partial u}{\partial r}\,\frac{\partial u}{\partial t}$ is $-2\,\frac{\partial\varphi}{\partial r} + b'(t)r$.

These are 0 if and only if $\psi = \frac{N}{2}\,b(t) - \frac{\partial\varphi}{\partial t}$ and

$$\frac{\partial\varphi}{\partial t} = \tfrac{1}{2}b(t)\ ; \quad \frac{\partial\varphi}{\partial r} = \tfrac{1}{2}b'(t)r\ .$$

This implies $b''(t) = 0$ and the general solution is

$$\varphi = \alpha + \beta t + \gamma(r^2+t^2)$$

$$b = 2(\beta + 2\gamma t)$$

$$\psi = (N-1)(\beta+2\gamma t)\,.$$

For $\alpha = 1$, $\beta = \gamma = 0$, we obtain E_1. For $\alpha = 0$, $\beta = 1$, $\gamma = 0$, we obtain E_2, and also get $\frac{\partial\psi}{\partial t}\,u\,\frac{\partial u}{\partial t} + \tfrac{1}{2}\Delta\psi|u|^2 \equiv 0$.

$\alpha = 0$, $\beta = 0$, $\gamma = 1$ implies $\frac{\partial\psi}{\partial t}\,u\,\frac{\partial u}{\partial t} + \tfrac{1}{2}\Delta\psi|u|^2 = 2(N-1)u\,\frac{\partial u}{\partial t} = \frac{d}{dt}(N-1)u^2)$.

Subtracting the right things, we get E_3.

<u>Remark:</u> $u'' - \Delta u = f$.

i) $$\frac{d}{dt}\int(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad}\,u|^2)dx = \int 2\,\frac{\partial u}{\partial t}\,f\,dx$$

ii) $$\frac{d}{dt}\int[t(|\frac{\partial u}{\partial t}|^2 + |\mathrm{grad}\,u|^2) + 2r\,\frac{\partial u}{dt}\,\frac{\partial u}{\partial r} + (N-1)u\frac{\partial u}{\partial t}]dx = \int[(2t\frac{\partial u}{\partial t} + 2r\frac{\partial u}{\partial r} + (N-1)u)f]dx.$$

<u>Question:</u> If $u'' - \Delta u = F(u)$, for what kind of functions F does this give an invariant?

Let G be a primitive of F. $\frac{d}{du}G = F$.

Then

$$\int (2t \frac{\partial u}{\partial t} + 2r \frac{\partial u}{\partial r} + (N-1)u)F(u = \frac{d}{dt} \int (2t\ G(u)) +$$
$$\sum \int \frac{\partial}{\partial x_i}(2x_i\ G(u) + \int [(N-1\ uF(u) - 2(N+1)G(u)]$$

and so,

$$\frac{d}{dt} \int [t(-) - 2t\ G(u)] = \int [(N-1)uF(u) - 2(N-1)G(u)] .$$

There is one function: $F(u) = cu^{\alpha}$ which satisfies this, where $\alpha = \frac{N+1}{N-1}$. For $N = 3$, $F(u) = cu^3$.

Recall, now, that we had the following invariants for $u'' - \Delta u = 0$ in $\mathbb{R}^N$:

1) $\int_{\mathbb{R}^N} (|\frac{\partial u}{\partial t}|^2 + |\text{grad} u|^2) dx$

2) $\int_{\mathbb{R}^N} [t(|\frac{\partial u}{\partial t}|^2 + |\text{grad} u|^2) + 2r \frac{\partial u}{\partial r} \frac{\partial u}{\partial t} + (N-1)u \frac{\partial u}{\partial t}] dx$

3) $\int_{\mathbb{R}^N} (r^2+t^2)(|\frac{\partial u}{\partial t}|^2 + |\text{grad} u|^2) + 4\ tr \frac{\partial u}{\partial t} \frac{\partial u}{\partial r} + 2(N-1)tu \frac{\partial u}{\partial t} - (N-1)u^2 [dx$.

Now we have other invariants

number	Invariant	Corresponding multiplier.
N	$\int \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t}$;	$\frac{\partial u}{\partial x_i}$
N	$\int x_i(\|\frac{\partial u}{\partial t}\|^2 + \|\text{grad} u\|^2) + 2t \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t}$;	$x_i \frac{\partial u}{\partial t} + t \frac{\partial u}{\partial x_i}$
$\frac{N(N-1)}{2}$	$\int (x_i \frac{\partial u}{\partial x_j} - x_j \frac{\partial u}{\partial x_i}) \frac{\partial u}{\partial t}$;	$x_i \frac{\partial u}{\partial x_j} - x_j \frac{\partial u}{\partial x_i}$.

<u>Remark:</u> Each multiplier satisfies $\frac{\partial^2 v}{\partial t^2} - \Delta v = 0$.

<u>Lemma:</u> If u, v satisfy $\frac{\partial^2}{\partial t^2} - \Delta = 0$, then $I = \int_{\mathbb{R}^N} (u \frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} v)$ is constant.

<u>Proof:</u> $\frac{d}{dt} I = \int_{\mathbb{R}^N} (u \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u}{\partial t^2} v) = \int (-u \Delta v + \Delta u\ v) = 0$

for u, v with compact support. (Note: we used $(\Delta u, v) = (u, \Delta v)$. The lemma holds with A in place of Δ for any $A = A^*$.)

This lemma implies the next:

Lemma: If P transforms solutions of $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ into solutions, then

$$\int \left(u \frac{\partial}{\partial t} Pu - \frac{\partial u}{\partial t} Pu\right)$$

is a constant in time.

Now we can compute all the operators $P = a_0 \frac{\partial}{\partial t} + \sum_i b_i \frac{\partial}{\partial x_i} + c$ which do this. Let $[P, \frac{d^2}{dt^2} - \Delta]$ be the commutator:

$$P\left(\frac{d^2}{dt^2} - \Delta\right) - \left(\frac{d^2}{dt^2} - \Delta\right) P$$

which is a second order differential operator. So we want

$$[P, \frac{\partial^2}{\partial t^2} - \Delta] = d(x,t)\left(\frac{\partial^2}{\partial t^2} - \Delta\right) .$$

Simple computations give

$$* \begin{cases} \forall_j \ \frac{\partial b_j}{\partial x_j} = \frac{\partial a_0}{\partial t} = -\frac{d}{2} \\ \forall_{i \neq j} \ \frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i} = 0 \\ \forall_j \ \frac{\partial b_j}{\partial t} = \frac{\partial a_0}{\partial x_j} \end{cases}$$

$$** \begin{cases} 2 \frac{\partial c}{\partial t} + \frac{\partial^2 a_0}{\partial t^2} - \Delta a_0 = 0 \\ \forall_j \ -2 \frac{\partial c}{\partial x_j} + \frac{\partial^2 b_j}{\partial t^2} - \Delta b_j = 0 \\ \frac{\partial^2 c}{\partial t^2} - \Delta c = 0 . \end{cases}$$

Using * we obtain

$$\forall_i \neq j \ \ 2 \frac{\partial^2 a_0}{\partial x_i \partial x_j} = \frac{\partial^2 b_j}{\partial x_i \partial t} + \frac{\partial^2 b_i}{\partial x_j \partial t} = 0 \quad \text{and so} \quad \frac{\partial^2 b_j}{\partial x_i \partial t} = 0 .$$

Let $\alpha = \frac{\partial^2 a_0}{\partial t^2} = \frac{\partial^2 b_j}{\partial x_j \partial t} = \frac{\partial^2 a_0}{\partial x_j^2} \; \forall_j$

$$\forall_i \; \frac{\partial \alpha}{\partial x_i} = \frac{\partial^3 a_0}{\partial x_i \partial x_j^2} = 0 \quad \text{for some } j \; (N > 1)$$

Let $\beta_i = \frac{\partial^2 a_0}{\partial x_i \partial t} = \frac{\partial^2 b_i}{\partial t^2} = \frac{\partial^2 b_j}{\partial x_i \partial x_j} \; \forall_j \quad \text{and} \quad = -\frac{\partial^2 b_i}{\partial x_k^2} \; \forall k \neq i$

$$\forall_i \; \frac{\partial \beta_i}{\partial t} = \frac{\partial \alpha}{\partial x_i} = 0 \, ; \quad \forall_i \neq j \; \frac{\partial \beta_i}{\partial x_j} = \frac{\partial^3 a_0}{\partial x_i \partial x_j \partial t} = 0$$

$$\forall_i \; \frac{\partial \beta_i}{\partial x_i} = \frac{\partial^3 a_0}{\partial x_i^2 \partial t} = \frac{\partial \alpha}{\partial t}$$

** gives then $\quad 2\frac{\partial c}{\partial t} = (N-1)\alpha$

$$2\frac{\partial c}{\partial x_j} = (N-1)\beta_j$$

and

$$0 = \frac{\partial^2 c}{\partial t^2} - \Delta c = \frac{N-1}{2}\frac{\partial \alpha}{\partial t} - \frac{N-1}{2}\sum_j \frac{\partial \beta_j}{\partial x_j}$$

$$= -\frac{(N-1)^2}{2}\frac{\partial \alpha}{\partial t} \quad .$$

So α and β_i are constants and we have the general form of a_0

$$a_0 = \frac{\alpha}{2}(t^2 + r^2) + t\sum_j \beta_j x_j + \gamma t + \sum_j \delta_j x_j + \varepsilon \; .$$

Then we deduce

$$b_i = \frac{\beta_i}{2}(t^2 - r^2) + x_i \sum_j \beta_j x_j + \alpha t x_i + \gamma x_i + \delta_i t + \bar{b}_i$$

where

$$\frac{\partial \bar{b}_i}{\partial x_i} = \frac{\partial \bar{b}_i}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \bar{b}_i}{\partial x_j} + \frac{\partial \bar{b}_j}{\partial x_i} = 0 \; \forall_{i,j}$$

$$c = \frac{N-1}{2}(\alpha t + \sum_j \beta_j x_j) + c_0 \quad .$$

Since $\frac{\partial^2 \bar{b}_i}{\partial x_j \partial x_k} = -\frac{\partial^2 \bar{b}_j}{\partial x_i \partial x_k} = \frac{\partial^2 \bar{b}_k}{\partial x_i \partial x_j} = -\frac{\partial^2 \bar{b}_i}{\partial x_j \partial x_k}$ the equal form of $\bar{b}_i$ is

$\sum_i \lambda_{ij} x_j + \mu_i$ with λ_{ij}, μ_i are constants and $\lambda_{ij} + \lambda_{ji} = 0 \ \forall_{i,j}$. To obtain the invariants we notice that

$$\int \left(\frac{\partial u}{\partial t} Pu - u \frac{\partial}{\partial t} Pu\right) dx = \int \left(a_0 \left(\frac{\partial u}{\partial t}\right)^2 + a_0 |\mathrm{grad} u|^2 + 2 \sum_j b_j \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t}\right) dx$$

$$+ \int u \frac{\partial u}{\partial t} \left(-\frac{\partial a_0}{\partial t} + \sum_j \frac{\partial b_j}{\partial x_j}\right) dx + \int \frac{|u|^2}{2} \left(-\Delta a_0 + \sum_j \frac{\partial^2 b_j}{\partial x_j \partial t} - 2 \frac{\partial c}{\partial t}\right) dx$$

so this gives

Theorem: (invariants of the equation)

$$\int \left(a_0 \left(\frac{\partial u}{\partial t}\right)^2 + a_0 |\mathrm{grad} u|^2 + 2 \sum_j b_j \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial t}\right) dx + (N-1) \int \frac{\partial a_0}{\partial t} u \frac{\partial u}{\partial t} dx$$

$-\frac{(N-1)\alpha}{2} \int |u|^2 dx$ is constant in time if u satisfies $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ (and is smooth with compact support in x) where

$$a_0 = \frac{\alpha}{2}(t^2 + r^2) + t \sum_j \beta_j x_j + \gamma t + \sum_j \delta_j x_j + \varepsilon$$

$$b_i = \frac{\beta_i}{2}(t^2 - r^2) + x_i \sum_j \beta_j x_j + \alpha t x_i + \gamma x_i + \delta_i t + \sum_j \lambda_{ij} x_j + \mu_i$$

$$c = \frac{N-1}{2}\left(\alpha t + \sum_j \beta_j x_j\right) + \text{constant}$$

with $\lambda_{ij} + \lambda_{ji} = 0$.

So there are $3 + 3N + \frac{N(N-1)}{2}$ invariants of this type, $N+2$ of them corresponding to $d(x,t) \neq 0$.

Remark: "Formally." $\frac{d^2 u}{dt^2} + Au = 0$, $A = A^*$. If $F(u, \frac{du}{dt})$ is an invariant and differentiable, as a function $F(u,v)$, then $w(t) = \frac{\partial F}{\partial v}(u(t), \frac{du}{dt}(t))$ is a solution of $\frac{d^2 w}{dt^2} + Aw = 0$.

We can prove this in $V = H = \mathbb{R}^K$.

Hence, $(\frac{du}{dt}, w) - (u, \frac{dw}{dt})$ is a constant, equal to $(u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v})$ where $v = \frac{du}{dt}$. If F is homogeneous, this is cF. All the homogeneous invariants are of the form $(u, \frac{dPu}{dt}) - (\frac{du}{dt}, Pu)$ where P is homogeneous and transforms solutions into solutions.

<u>In the case N=1 for the wave equation</u>, there are <u>nonlinear</u> transformations P taking solutions into solutions.

<u>Conjecture:</u> If $N > 1$, there is no non-affine transformation conserving smooth solutions of the wave equation.

Example for $N=1$. $u \to (\frac{\partial u}{\partial t})^2 + (\frac{\partial u}{\partial x})^2$ conserves solutions. u is a solution if and only if $u = f(x+t) + g(x-t)$.

$$w = (\frac{\partial u}{\partial t})^2 + (\frac{\partial u}{\partial x})^2 = (f'-g')^2 + (f'+g')^2 = 2(f')^2 + 2(g')^2 = f_1(x+t) + g_1(x-t)$$

The corresponding invariant is

$$\int [(\frac{\partial u}{\partial t})^2 + 3\frac{\partial u}{\partial t}(\frac{\partial u}{\partial x})^2]dx = \int (u\frac{\partial w}{\partial t} - \frac{\partial u}{\partial t} w)dx ,$$

where u is a solution.

If $N=1$, one can prove that if $F(u,v)$ satisfies $\frac{\partial^2 F}{\partial u^2} = \frac{\partial^2 F}{\partial v^2}$, then

$$\int F(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x})dx$$

is a constant. Thus there are many invariants for $N = 1$.

<u>Lemma:</u> $E_3 \geq 0$.

<u>Proof:</u> First proof: E_3 is independent of t , so take $t = 0$. Then prove

$$\int (r^2 |\text{grad} u|^2 - (N-1)u^2]dx \geq 0 \quad \text{for } u \in \mathcal{D}(\mathbb{R}^N).$$

This is a generalization of Poincare's inequality. Consider

$$\int (r^2(\frac{\partial u}{\partial x_i})^2 + 2u\frac{\partial u}{\partial x_i}\alpha x_i + \alpha^2 u^2 \frac{x_i^2}{r^2})dx = \int (r\frac{\partial u}{\partial x_i} + \alpha\frac{x_i}{r}u)^2 dx \geq 0 .$$

So by an integration by parts

$$\int r^2 (\frac{\partial u}{\partial x_i})^2 dx - \int \alpha u^2 dx + \int \alpha^2 u^2 \frac{x_i^2}{r^2} dx \geq 0 . \qquad \text{Sum in } i .$$

$$\int r^2 |\text{grad} u|^2 dx - \int N\alpha u^2 dx + \int \alpha^2 u^2 \geq 0 , \quad \text{so}$$

$$\int r^2 |\text{grad}u|^2 dx \geq (N\alpha - \alpha^2) \int |u|^2 dx \qquad \forall\alpha.$$

$\alpha=1$ gives the desired result; but $\alpha = \frac{N}{2}$ gives a better inequality:

$$\int r^2 |\text{grad}u|^2 dx \geq \frac{N^2}{4} \int |u|^2 .$$

Second Proof: $E_3 = \int \{(r^2+t^2)[|\frac{\partial u}{\partial t}|^2 + \sum_i \lambda_i^2] + 4t\frac{\partial u}{\partial t} \sum_i x_i\lambda_i\} + \frac{(N-1)(N-3)}{4} \cdot$

$$\cdot \int \frac{(r^2+t^2)}{r^2} |u|^2 dx$$

with $\lambda_i = \frac{\partial u}{\partial x_i} + \frac{N-1}{2} \frac{x_i}{r} u$.

The last term is 0 for $N = 3$, while the other portion is positive by Cauchy-Schwarz:

$$4t\, r \frac{\partial u}{\partial t} \frac{\partial u}{\partial r} + 2(N-1)t\, u \frac{\partial u}{\partial t} = 4t \frac{\partial u}{\partial t} \sum_i x_i \lambda_i$$

$$|\text{grad}u|^2 = \sum_i \lambda_i^2 - (N-1) \sum_i \frac{x_i}{r^2} u \frac{\partial u}{\partial x_i} - \frac{(N-1)^2}{4} \frac{|u|^2}{r^2} .$$

Thus, $E_3 = \int (r^2+t^2)[|\frac{\partial u}{\partial t}|^2 + \sum \lambda_i^2 + 4t \frac{\partial u}{\partial t} \sum_i x_i \lambda_i] dx - (N-1)X$

where

$$X = \int [(r^2+t^2)(\sum_i \frac{x_i}{r^2} \frac{\partial u}{\partial x_i} u + \frac{(N-1)|u|^2}{r^2} + |u|^2] dx .$$

An integration by parts gives

$$X = (3-N) \int \frac{(r^2+t^2)|u|^2}{4r^2} dx .$$

<u>Remark:</u> If $N \geq 3$ and u is a smooth solution,

$$\int_{\mathbb{R}^N} (r^2+t^2) (|\frac{\partial u}{\partial t}|^2 + \sum \lambda_i^2) + 4t \frac{\partial u}{\partial t} \sum_i x_i\lambda_i + \frac{(N-1)(N-3)}{4} \int \frac{(r^2+t^2)u^2}{r^2} dx \leq K.$$

<u>Theorem:</u> (Local Decay of Solutions) For $0 < \theta < 1$, and $N=3$,

$$\int_{r \leq \theta t} (|\frac{\partial u}{\partial t}|^2 + |\text{grad}u|^2) dx \leq C_0/t^2 \quad \text{as } t \to \infty .$$

<u>Proof:</u> If $r < \theta t$, $\theta < 1$, then $2tr \leq \frac{2\theta}{1+\theta^2} (r^2+t^2)$.

$$4t\frac{\partial u}{\partial t}\sum_i x_i\lambda_i \le 4tr\,\left|\frac{\partial u}{\partial t}\right|\left(\sum\lambda_i^2\right)^{\frac{1}{2}} \le 2\,tr\left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum\lambda_i^2\right)$$

$$\le \frac{2\theta}{1+\theta^2}(r^2+t^2)\left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i\lambda_i^2\right)$$

(The first inequality is Cauchy-Schwarz, the second is Young's.)

Now apply the remark above:

$$\left(1 - \frac{2\theta}{1+\theta^2}\right)\int_{r<\theta t}(r^2+t^2)\left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i\lambda_i^2\right)dx \le K .$$

$1 - \frac{2\theta}{1+\theta^2} = \frac{(1-\theta)^2}{1+\theta^2}$. Therefore

$$\int_{r<\theta t}\left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum_i\lambda_i^2\right)dx \le \frac{K\frac{1}{(1-\theta)^2}}{t^2} .$$

Now, by the definition of λ_i,

$$\int_{r<\theta t}\sum_i\lambda_i^2 = \int_{r<\theta t}|\operatorname{grad}u|^2 + (N-1)\sum_i\frac{x_i}{r}\frac{du}{\partial x_i} + \frac{(N-1)^2}{4r^2}u^2 .$$

An integration by parts gives

$$= \int_{r<\theta t}|\operatorname{grad}u|^2 - \int_{r<\theta t}\frac{|u|^2(N-1)(N-3)}{4r^2} + \int_{r=\theta t}\frac{(N-1)|u|^2}{2r}\,d\Sigma .$$

Thus, for $N = 3$, we have no problem. For $N > 3$,

$$\int\frac{(r^2+t^2)|u|^2}{r^2} \le K \implies \int_{r<\theta t}\frac{|u|^2}{r^2} \le \frac{K}{t^2} .$$

<u>Remark:</u> If $\eta > 1$, then

$$\int_{r\ge\eta t}\left(\left|\frac{\partial u}{\partial t}\right|^2 + |\operatorname{grad}u|^2\right) \le \frac{C_0}{t^2} .$$

<u>The nonlinear equation:</u>

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + F(u) = 0 .$$

<u>Remarks:</u>

1) Do we have the same invariants?

Answer: We keep the $1+2N+\frac{N(N-1)}{2}$ invariants for all F. We keep the $N+2$ others (the noncommuting case) only for some $F(u) = cu^{\alpha}$, where α depends on N. ($\alpha=3$ if $N=3$)

2) $F(u) = m^2 u$, $m \neq 0$, gives us the Klein-Gordon equation. It has different decay properties than the wave equation.

<u>Existence of solutions for the equation:</u>

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = 0 & \text{in } \Omega \subseteq \mathbb{R}^3 \\ u(x,0) = u_0 & \\ \dfrac{\partial u}{\partial t}(x,0) = u_1 & u|_{\partial\Omega} = 0 . \end{cases}$$

In $\mathbb{R}^N$, take $V = H_0^1(\Omega) \cap L^4(\Omega)$, $H = L^2(\Omega)$. The problem is, given $u_0 \in V$, $u_1 \in H$, to find $u \in L^\infty(0,T;V)$ such that $u' \in L^\infty(0,T;H)$ and u satisfies the equation. We also hope that, in some sense

$$\frac{d}{dt}\left(\tfrac{1}{2}|u'|_H^2 + \tfrac{1}{2}\|u\|_{H_0^1}^2 + \tfrac{1}{4}|u|_{L^4}^4 \right) = 0 .$$

Is there a unique solution which depends in a regular way on the initial data? We prove uniqueness for $N=3$. Existence works for all N.

<u>Lemma:</u> In $\mathbb{R}^3$, $H_0^1(\Omega) \subseteq L^6(\Omega)$, so $u \in H_0^1(\Omega) \Rightarrow u^3 \in L^2(\Omega)$. We have proven in $\mathbb{R}^2$ that $u \in H^1(\mathbb{R}^2) \Rightarrow u \in L^4(\mathbb{R}^2)$. We will generalize the method to have it work for $u, \frac{\partial u}{\partial x_i} \in L^P(\mathbb{R}^N)$ for $P < N$. (This Lemma is a special case of Sobolev's imbedding theorem.)

<u>Lemma:</u> Let $f_1, \ldots, f_n$ be defined on $\mathbb{R}^N$ with f_i independent of x_i, and $f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in L^{N-1}(\mathbb{R}^{N-1})$.

Then $u(x_1, \ldots, x_N) = f_1 \cdot f_2 \cdots f_N \in L^1(\mathbb{R}^N)$ and

$$\|u\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})} .$$

Proof: By induction on N. The result holds for $N=2$ by Fubini's theorem. Note that

$$\int_{x_N} f_1 \dots f_N \, dx_N = f_N \int_{x_N} f_1 \dots f_{N-1} dx_N .$$

Use Hölder's inequality, and get

$$\left|\int_{x_N} f_1 \dots f_{N-1} dx_N\right| \le |f_N| \prod_{i=1}^{N-1} \left(\int_{x_N} |f_i|^{N-1} dx_N\right)^{1/N-1} =$$

$$= |f_N(x_1, \dots, x_{N-1})| \prod_{i=1}^{N-1} \varphi_i(x_1, \dots, x_{N-1}).$$

φ_i does not depend on i, and $\varphi_i^{N-1} \in L^1(\mathbb{R}^{N-2})$. Now, $\varphi^{\frac{N-1}{N-2}} \in L^{N-2}(\mathbb{R}^{N-2})$, so, by induction

$$\prod \varphi_i^{\frac{N-1}{N-2}} \in L^1(\mathbb{R}^{N-1}) \Rightarrow \prod \varphi_i \in L^{\frac{N-1}{N-2}}(\mathbb{R}^{N-1}) .$$

$f_N \in L^{N-1}(\mathbb{R}^{N-1})$, so $f_N \prod \varphi_i \in L^1(\mathbb{R}^{N-1})$ that is $f_1 \dots f_N \in L^1(\mathbb{R}^N)$.

To prove $H_0^1(\mathbb{R}^3) \subseteq L^6(\mathbb{R}^3)$, write (for smooth u)

$$u^4 = 4 \int_{\mathbb{R}} u^3 \frac{\partial u}{\partial x_i} dx_i \le 4 \int_{\mathbb{R}} |u|^3 \left|\frac{\partial u}{\partial x_i}\right| dx_i = v_i(x) ,$$

which doesn't depend on x_i.

$$v_i \in L^1(\mathbb{R}^2), \quad \|v_i\|_{L^1} \le 4 \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2} |u^3|_{L^2} .$$

Then

$$|u|^2 \le \sqrt{v_i} \in L^2(\mathbb{R}^2).$$

By the Lemma,

$$\| |u|^6 \|_{L^1} \le \prod_{i=1}^{3} \|\sqrt{v_i}\|_{L^2(\mathbb{R}^2)} = \prod \|v_i\|^{1/2}_{L^1(\mathbb{R}^2)} \le 8 \prod \left\|\frac{\partial u}{\partial x_i}\right\|^{1/2}_{L^2} \left(\int |u|^6\right)^{3/4}$$

$$= 8 \prod \left\|\frac{\partial u}{\partial x_i}\right\|^{1/2}_{L^2} \| |u|^6 \|^{3/4}_{L^1}$$

$$\Longrightarrow \quad \| |u|^6 \|^{1/4}_{L^1} \le 8 \prod \left\|\frac{\partial u}{\partial x_i}\right\|^{1/2}_{L^2} \quad \Rightarrow \quad \|u\|_{L^6} \le 4 \prod \left\|\frac{\partial u}{\partial x_i}\right\|^{1/3} .$$

Then use the density of smooth functions to obtain the desired result.

Sololev's inequalities:

Theorem 1: Let $1 \leq P < N$ and $u \in \mathcal{D}(\mathbb{R}^N)$. Then there exists a constant C such that

$$\|u\|_{L^{P^*}} \leq C\left(\prod_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^P}\right)^{1/N}$$

where $\frac{1}{P^*} = \frac{1}{P} - \frac{1}{N}$.

Corollary: If $u \in L^P(\mathbb{R}^N)$ and $\frac{\partial u}{\partial x_i} \in L^P(\mathbb{R}^N)$, $(u \in W^{1,P}(\mathbb{R}^N))$ then $u \in L^{P^*}(\mathbb{R}^N)$.

Proof: $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,P}(\mathbb{R}^N)$, so apply that and the above theorem. ∎

Theorem 2: If $u \in \mathcal{D}(\mathbb{R}^N)$, $\lim_{s\to\infty} \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N)} \leq C\left(\prod_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^N(\mathbb{R}^N)}\right)^{1/N}$

(This corresponds to $P = N$.)

Corollary: $u \in W^{1,N}(\mathbb{R}^N)$ implies $u \in L^q(\mathbb{R}^N)$ $\forall\, q: N < q < \infty$ and $\|u\|_{L^q} < Cq$ for $q > N$.

Theorem 3: If $p > N$, and $u \in \mathcal{D}(\mathbb{R}^N)$, $\exists C$:

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|^{1-\frac{N}{P}}_{L^P(\mathbb{R}^N)}\left(\prod_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^P(\mathbb{R}^N)}\right)^{\frac{1}{P}}$$

Corollary: $u \in W^{1,P}(\mathbb{R}^N)$ implies $u \in L^\infty(\mathbb{R}^N)$ and moreover, u is Hõlderian of exponent $\theta = 1 - \frac{N}{P}$: that is

$$\max_x |u(x+h)-u(x)| \leq C|h|^\theta \left(\prod_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^P(\mathbb{R}^N)}\right)^{\frac{1}{N}}$$

Proofs: of Theorems 1, 2, 3

Consider $v = |u|^{r-1}u$

$$|v(x)| = \left|\int \frac{\partial v}{\partial x_i}\,dx_i\right| \leq \int_{\mathbb{R}} r|u|^{r-1}\left|\frac{\partial u}{\partial x_i}\right|dx_i \overset{\text{def}}{\equiv} f_i(x).$$

f_i is independent of x_i, and

$$\|f_i\|_{L^1(\mathbb{R}^{N-1})} \leq r\left\|\frac{\partial u}{\partial x_i}\right\|_{L^P}\left\||u|^{r-1}\right\|_{L^{P'}}. \quad \left(\frac{1}{P}+\frac{1}{P'}=1\right)$$

$|v|^{\frac{1}{N-1}} \leq f_i^{\frac{1}{N-1}} \in L^{N-1}(\mathbb{R}^{N-1})$. Thus by the Lemma $|v|^{\frac{N}{N-1}} \in L^1(\mathbb{R}^N)$ and

$$\int |v|^{\frac{N}{N-1}} dx \leq \prod_{i=1}^{N} \| f_i^{\frac{1}{N-1}} \|_{L^{N-1}(\mathbb{R}^{N-1})} = \prod_{i=1}^{N} \| f_i \|^{\frac{1}{N-1}}_{L^1(\mathbb{R}^{N-1})}$$

Hence

$$\int |u|^{\frac{rN}{N-1}} dx \leq r^{\frac{N}{N-1}} \Big(\prod_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|^{\frac{1}{N-1}}_{L^P(\mathbb{R}^N)} \Big) \| |u|^{r-1} \|^{\frac{N}{N-1}}_{L^{P'}(\mathbb{R}^N)} .$$

Let

$$\varphi(r) = \Big(\int |u|^{\frac{rN}{N-1}} dx \Big)^{\frac{N-1}{N}} ,$$

and let

$$\lambda = \Big(\prod_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|_{L^P(\mathbb{R}^N)} \Big)^{\frac{1}{N}} .$$

Then we have

$$\varphi(r) < \lambda r [\varphi((r-1)\alpha)]^{1/\alpha} \qquad \alpha = \frac{P'(N-1)}{N} .$$

For Theorem 1, $p < N$, so $\alpha > 1$. Then $\exists\, r_0$ such that $r_0 = (r_0 - 1)\alpha$, and so $\varphi(r_0) \leq C \lambda^{\frac{\alpha}{1-\alpha}}$; $\frac{r_0 N}{N-1} = P^*$, so $\|u\|_{L^{P^*}} \leq C\lambda$, proving Theorem 1.

For Theorem 2, $P = N$, $\alpha = 1$, so $\varphi(r) = \lambda r \varphi(r-1)$ which gives $\varphi(r) \leq \lambda^r r! \varphi(0)$. Rewriting,

$$(\varphi(r))^{\frac{1}{r}} = \|u\|_{L^{\frac{rN}{N-1}}} \leq \lambda (r!)^{\frac{1}{r}} (\varphi(0))^{\frac{1}{r}} .$$

Now, $r! \approx (\frac{r}{e})^r \sqrt{2\pi r}$ (Stirling's formula) so $(r!)^{\frac{1}{r}} \approx \frac{r}{e}$ as $r \to \infty$. Letting $r \to \infty$, we get

$$\lim_{r \to \infty} \frac{1}{r} \|u\|_{L^{\frac{rN}{(N-1)}}} \leq C\lambda .$$

<u>Remark:</u> $\exists$ u with $\frac{\partial u}{\partial x_i} \in L^N(\mathbb{R}^N)$, such that $u \notin L^P(\mathbb{R}^N)$ for every P . $u = ((\mathrm{Log}(1+r))^{\alpha}$ with $\alpha < 1 - \frac{1}{N}$ is an example. If $v = |\log r|^{\alpha}$ near 0 , $0 < \alpha < 1 - \frac{1}{N}$, $\frac{\partial v}{\partial x_i} \in L^N(\mathbb{R}^N)$. Then $v \in L^P(\mathbb{R}^N)$ $\forall\, P < \infty$, but $u \notin L^{\infty}(\mathbb{R}^N)$.

For Theorem 3; $P > N$, so $\alpha < 1$. Perform the following transformations:

1) $\varphi(r) = \lambda^r \psi(r)$, the same formula for ψ, but $\lambda = 1$.

2) $\psi(r) = \chi(r + \frac{\alpha}{1-\alpha})$. Then

$$\chi(s) \leq (s - \frac{\alpha}{1+\alpha})(\chi(\alpha s))^{\frac{1}{\alpha}} \leq s(\chi(\alpha s))^{\frac{1}{\alpha}}$$

3) $\chi(s) = s^{\frac{\alpha}{\alpha-1}} \xi(s)$. $\xi(s) \leq (\xi(\alpha s))^{\frac{1}{\alpha}}$,

and

$$\frac{\log(\xi(s))}{s} \leq \frac{\log(\xi(\alpha s))}{\alpha s}.$$

Then we obtain

$$\overline{\lim_{r\to\infty}} \frac{\log \varphi(r)}{r} \leq \log \lambda + \frac{\log \xi(s_0)}{s_0}$$

$$\forall \text{ fixed } s_0 < \infty$$

We know $u \in L^P$, that is $\varphi(r_0)$ for some special r_0.

$$\frac{\log \varphi(r)}{r} = \log(\|u\|_{L^{\frac{rN}{(N-1)}}}).$$

The bound in all r gives the bound in L^∞. This gives

$$\|u\|_{L^\infty} \leq C\lambda^{\frac{N}{P}} \|u\|_{L^P}^{1-\frac{N}{P}}.$$

To show that u is Hölderian, let $v(x) = u(x+he_i)-u(x)$. Then

$$\|\frac{\partial v}{\partial x_i}\|_{L^P} \leq 2\|\frac{\partial u}{\partial x_i}\|_{L^P}.$$

One can show

$$\|v\|_{L^P} \leq h\|\frac{\partial u}{\partial x_i}\|_{L^P}.$$

(The h is taken in the x_i direction: e_i is the unit vector in that direction.)

Hence

$$\|v(x)\|_{L^\infty} \leq C \lambda^{\frac{N}{P}} \|v\|_{L^P}^{1-\frac{N}{P}} \leq C h^{1-\frac{N}{P}} (\sum_i \|\frac{\partial u}{\partial x_i}\|_{L^P}).$$

We can do this in any direction.

Remark: If one knows $\|u\|_{L^\infty} \leq C(\|u\|_{L^P} + \sum \|\frac{\partial u}{\partial x_i}\|_{L^P})$, then by a homogeneity argument, applied to $u(\lambda x)$,

$$\|u\|_{L^\infty} \leq C\|u\|_{L^P}^{1-\frac{N}{P}} \left(\sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^P}\right)^{\frac{1}{P}} .$$

Back to the existence problem for

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = f & \text{on } \Omega \\ u(0) = u_0 & \\ & u|_{\partial\Omega} = 0 \\ u'(0) = u_1 & \end{cases}$$

$$\Omega \subseteq \mathbb{R}^N .$$

Let $V = H_0^1(\Omega) \cap L^4(\Omega)$ and $H = L^2(\Omega)$.

1) V is a reflexive, separable Banach space, under the norm

$$\|u\|_V = \|u\|_{H_0^1} + \|u\|_{L^4} .$$

What is the dual space of V ?

Remark: $\mathcal{D}(\Omega)$ is dense in V . (if Ω is bounded, open, and has smooth boundary.) The dual of V is

$$V' = (H_0^1)' + (L^4)' = H^{-1}(\Omega) + L^{4/3}(\Omega) .$$

$$f \in V' \text{ if and only if } f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + g \quad f_i \in L^2(\Omega),\ g \in L^{4/3}(\Omega) .$$

(This sum is to be considered in some larger space like the space of distributions.)

Remark: If $N \leq 4$, then $V = H_0^1(\Omega)$, since (extending by 0 outside Ω)

$$H_0^1(\Omega) \subseteq H^1(\mathbb{R}^N) \subseteq L^{P^*}(\mathbb{R}^N) \text{ where } \frac{1}{P^*} = \frac{1}{2} - \frac{1}{N} . \quad (N \neq 2) .$$

Thus

$N = 2$ $\quad H_0^1(\Omega) \subseteq L^P(\mathbb{R}^2) \quad \forall p.$

$N = 3$ $\quad H_0^1(\Omega) \subseteq L^6(\mathbb{R}^3)$

$N = 4$ $\quad H_0^1(\Omega) \subseteq L^4(\mathbb{R}^4)$.

To prove the reflexivity let $\{u_n\}$ be bounded in V then $\{u_n\}$ is bounded in H_0^1, a Hilbert space, and $\{u_n\}$ is bounded in L^4, a reflexive space, so we can choose a subsequence $\{u_m\}$ converging weakly in both spaces.

If $f \in V'$, then $f = f_1 + f_2$, where $f_1 \in H^{-1}(\Omega)$ and $f_2 \in L^{4/3}(\Omega)$. This means

$$(f,u_n) = (f_1,u_n) + (f_2,u_n) \to (f_1,u) + f_2(u) = (f,u) .$$

Thus V is reflexive.

<u>Theorem:</u> (Existence) Let $f \in L^1(0,T;H)$, $u_0 \in V$, and $u_1 \in H$. Then $\exists u$, a solution of

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = f \quad \text{in } \Omega \\ u(0) = u_0 \qquad\qquad u|_{\partial\Omega} = 0 \\ u'(0) = u_1 \end{cases}$$

such that $u \in L^\infty(0,T;V)$, and $\frac{du}{dt} \in L^\infty(0,T;H)$.

<u>Proof:</u> Via Galerkin's method.

Step 1. Take a Galerkin basis $w_1, w_2, \ldots$ of V. Find

$$u_n(t) = \sum_{i=1}^{n} g_i(t) w_i \quad \text{satisfying}$$

$$\begin{cases} \left(\dfrac{\partial^2 u_n}{\partial t^2} - \Delta u_n + u_n^3 - f, w_i\right) = 0 \qquad 1 = 1,2,\ldots,n . \\ \left.\begin{array}{l} u_n(0) = u_{0n} \\ \dfrac{\partial u_n}{\partial t}(0) = u_{1n} \end{array}\right\} \text{ in } \operatorname{Span}(w_1,\ldots,w_n) \text{ such that} \end{cases}$$

$u_{0n} \to u_0$ in V and $u_{1n} \to u_1$ in H. This is of the form $\frac{d^2 g}{dt^2} = F_i(g_1,\ldots,g_n,t)$, where F contains cubic terms, and is thus nonlinear. However, F is continuous, etc., so that we get local existence for each u_n, on an interval $[0,t_n)$.

Step 2: Find a bound on u_n independent of $t \in [0,T]$. To do this, multiply by $g_i'(t)$ and sum in i. Then use

$$\left(\frac{d^2u_n}{dt^2}, \frac{du_n}{dt}\right) = \frac{1}{2}\frac{d}{dt}\left|\frac{du_n}{dt}\right|^2 ,$$

$$\left(-\Delta u_n, \frac{du_n}{dt}\right) = \frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} = \frac{1}{2}\frac{d}{dt}\int_\Omega |\operatorname{grad} u_n|^2 dx ,$$

$$\left(u_n^3, \frac{du_n}{dt}\right) = \frac{1}{4}\frac{d}{dt}\int_\Omega |u_n|^4 dx ,$$

and

$$\left(f, \frac{du_n}{dt}\right) \le |f(t)|_H \left|\frac{du_n}{dt}\right|_H$$

to obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left|\frac{du_n}{dt}\right|^2 + \frac{1}{2}\|u_n\|^2_{H^1_0} + \frac{1}{4}|u_n|^4_{L^4}\right) \le |f|_H \left|\frac{du_n}{dt}\right|_H .$$

An application of Gronwall's inequality gives the desired bound. More precisely let $\varphi(t) = \frac{1}{2}\left|\frac{du_n}{dt}\right|^2 + \frac{1}{2}\|u_n\|^2 + \frac{1}{4}|u_n|^4_{L^4}$. Thus $\varphi'(t) \le \lambda(t)\sqrt{\varphi(t)}$ with $\lambda(t) = \sqrt{2}|f(t)|$. This proves $(\sqrt{\varphi(t)})' \le \frac{\lambda(t)}{2}$ so

$$\sqrt{\varphi(t)} \le \sqrt{\varphi(0)} + \frac{1}{\sqrt{2}}\int_c^t |f(s)|ds .$$

Then we get $\{u_n\}$ is contained in a bounded set in $L^\infty(0,T;V)$ and $\{\frac{du_n}{dt}\}$ is contained in a bounded set in $L^\infty(0,T;H)$.

Because V is reflexive, we can extract a subsequence, call it $\{u_m\}$ which converges weakly* in $L^\infty(0,T;V)$, to some u, and so that $\frac{du_m}{dt} \to \frac{du}{dt}$ weakly* in $L^\infty(0,T;H)$.

$u_m \to u$ weak* in $L^\infty(0,T;V)$ implies $\frac{\partial u_m}{\partial x} \to \frac{\partial u}{\partial x}$ weak* in $L^\infty(0,T;H)$, since $\frac{\partial}{\partial x} : V \to H$.

<u>Apply the compactness Lemma:</u> $V \subsetneq H$ is a compact injection because Ω is bounded. ($V \underset{\text{continuous}}{\subsetneq} H^1_0 \underset{\text{compact}}{\subsetneq} L^2$.)

Thus $\{u_m\}$ contained in a bounded set in $L^{\infty}(0,T;V)$ implies $\{u_m\}$ contained in a bounded set in $L^P(0,T;V)$ for any $1 < P < \infty$, and similarly $\{\frac{du_m}{dt}\}$ is contained in a bounded set in $L^P(0,T;H)$ $1 < P < \infty$, so $\{u_m\}$ is in a compact set in $L^P(0,T;H) = L^P(0,T;L^2(\Omega))$ $1 < P < \infty$. (We apply the compactness lemma with $B_0 = V$, $B_1 = B_2 = H$.). Then $u_m \to u$ strongly in $L^P(0,T;L^2(\Omega))$. $(T < \infty)$ Recall, now, that $(\frac{du_m}{dt} - \Delta u_m - u_m^3 - f, w_i) = 0$ for $1 = 1,2,\ldots,m$. With the above convergence, what can we say about u_m^3 ?

We have $u_m \to u$ strongly in $L^P(0,T;L^2(\Omega))$ and $u_m \rightharpoonup u$ weakly* in $L^P(0,T;L^4(\Omega))$ $(V \subseteq L^4(\Omega).)$

By Holder's inequality, $u_m \to u$ strongly in $L^P(0,T;L^q(\Omega))$ for $2 \leq q < 4$. (Since $\|u_m-u\|_{L^q} \leq \|u_m-u\|_{L^2}^{1-\theta}\|u_m-u\|_{L^4}^{\theta}$ where $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{4}$.)

Take $q \geq 3$ and $p \geq 3$. Then $u_m^3 \to u^3$ strongly in $L^{p/3}(0,T;L^{q/3}(\Omega))$.

Take limits:

$$(\frac{d^2u_m}{dt^2}, w_i) \to (\frac{d^2u}{dt^2}, w_i) \text{ in the sense of distributions}$$

$$(-\Delta u_m, w_i) \rightharpoonup (-\Delta u, w_i) \text{ weakly* in } L^{\infty}(0,T)$$

and

$$(u_m^3, w_i) \rightharpoonup (u^3, w_i) \text{ weakly* in } L^{\infty}(0,T).$$

This proves that u satisfies $(\frac{d^2u}{dt^2} - \Delta u + u^3 - f, w_i) = 0$ and so is a solution.

<u>Theorem:</u> (Uniqueness) For <u>$N = 3$.</u> **Let** $f \in L^1(0,T;H)$, $u_0 \in V$, and $u_1 \in H$. Then $\exists$ a unique solution $u \in L^{\infty}(0,T;V)$ with $\frac{du}{dt} \in L^{\infty}(0,T;H)$.

Proof: Suppose u and v are solutions. Let $w = u-v$. Then

$$\begin{cases} \frac{d^2w}{dt^2} - \Delta w + aw = 0 \\ w(0) = 0 \\ \frac{dw}{dt}(0) = 0 \end{cases}$$

where $a = \frac{u^3-v^3}{u-v} = u^2+uv+v^2$. Now we use $N=3$, to get, via Sobolev's inequality, $H_0^1(\Omega) \subseteq L^6(\Omega)$. So, since $u,v \in L^\infty(0,T;V)$, $a \in L^\infty(0,T;L^3(\Omega))$. We know $w \in L^\infty(0,T;H_0^1(\Omega))$ and $\frac{dw}{dt} \in L^\infty(0,T;L^2(\Omega))$. Take the inner product with $\frac{dw}{dt}$ to obtain

$$\tfrac{1}{2}\frac{d}{dt}\left(\left|\frac{dw}{dt}\right|^2 + \|w\|^2_{H_0^1}\right) = -\left(aw, \frac{dw}{dt}\right) \leq |aw|_{L^2}\left|\frac{dw}{dt}\right|_{L^2}.$$

This is valid, since $aw \in L^2$: $a \in L^3$, $w \in H_0^1 \subseteq L^6$. (Using $N=3$ again.)

Thus

$$\tfrac{1}{2}\frac{d}{dt}\left(\left|\frac{dw}{dt}\right|^2 + \|w\|^2_{H_0^1}\right) \leq \|a\|_{L^3}\|w\|_{L^6}\left|\frac{dw}{dt}\right|_{L^2}$$

$$\leq C\|a\|_{L^3}\left(\left|\frac{dw}{dt}\right|^2_{L^2} + \|w\|^2_{H_0^1}\right).$$

Since $\left(\left|\frac{dw}{dt}\right|^2 + \|w\|^2_{H_0^1}\right)\Big|_{t=0} = 0$, Gronwall's inequality implies $\left|\frac{dw}{dt}\right|^2 + \|w\|^2_{H_0^1} \equiv 0$. This proves uniqueness.

Remark: It is enough to have $a \in L^1(0,T;L^N(\Omega))$ for $N \geq 3$. Then $w \in L^\infty(0,T;L^{2N/(N-2)}(\Omega))$ by Sobolev's inequalities, so $aw \in L^1(0,T;L^2)$.

If u,v satisfy $u,v \in L^2(0,T;L^{2N}(\Omega))$, then $u \equiv v$ by the above proof.

Theorem: (Regularity). Still $N = 3$. Let $f,f' \in L^1(0,T;H)$, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, and $u_1 \in H_0^1(\Omega)$. Then $u, \frac{du}{dt} \in L^\infty(0,T;H_0^1(\Omega))$, $\frac{d^2u}{dt} \in L^\infty(0,T;L^2(\Omega))$, and hence (from the equation) $\Delta u \in L^\infty(0,T;L^2(\Omega))$. That is, $u \in L^\infty(0,T;H^2(\Omega))$.

Proof: Use Galerkin's method and obtain new estimates:

$$\left(\frac{d^2u_n}{dt^2} - \Delta u_n + u_n^3 - f, w_i\right) = 0 \qquad i = 1,2,\ldots,n \quad .$$

Take $w_1 = u_0$ (if $u_0 \neq 0$, otherwise arbitrary). Take $u_n(0) = u_0$,

$\frac{du_n}{dt}(0) = u_n \to u_1$ in H_0^1

Differentiate the equation and take the inner product with $\frac{d^2u_n}{dt^2}$.

Take $v_n = \frac{du_n}{dt}$. Then v_n satisfies

$$\left(\frac{d^2v_n}{dt^2} - \Delta v_n + 3u_n^2v_n - f', w_i\right) = 0 \qquad i = 1,2,\ldots,n.$$

$$v_n(0) = u_n \ , \qquad \{u_n\} \text{ bounded in } H_0^1 \ .$$

$$\frac{dv_n}{dt}(0) = \frac{d^2u_n}{dt^2}(0) = \text{Projection } (f(0) + \Delta u_0 - u_0^3)$$

$$\text{on span } (w_1,\ldots,w_n)$$

Remark that $f(0) + \Delta u_0 - u_0^3 \in L^3$. Then

$$\left(\frac{d^2v_n}{dt^2} - \Delta u_n + 3u_n^2v_n - f', \frac{dv_n}{dt}\right) = 0 \qquad i = 1,2,\ldots,n \ .$$

(By taking the proper multipliers and summing in i.) Hence

$$\frac{1}{2}\frac{d}{dt}\left(\left|\frac{dv_n}{dt}\right|^2 + \|v_n\|^2_{H_0^1}\right) + 3\left(u_n^2v_n, \frac{dv_n}{dt}\right) = \left(f', \frac{dv_n}{dt}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}\left|\frac{dv_n}{dt}\right|^2 + \frac{1}{2}\|v_n\|^2_{H_0^1}\right] \leq |f'|\left|\frac{dv_n}{dt}\right| + 3\left|\frac{dv_n}{dt}\right|_{L^2}\|v_n\|_{L^6}\|u_n^2\|_{L^3}$$

$$\leq |f'|\left|\frac{dv_n}{dt}\right| + C\left|\frac{dv_n}{dt}\right|_{L^2}\|v_n\|_{H_0^1}\|u_n\|^2_{L^6}$$

and $\|u_n\|^2_{L^6}$ is bounded.

By Gronwall's inequality,

$$\{\frac{dv_n}{dt}\} \text{ is contained in a bounded set in } L^\infty(0,T;L^2)\ ,$$

$$\{v_n\} \text{ is contained in a bounded set in } L^\infty(0,T;H_0^1).$$

This gives, as $n \to \infty$, the desired bounds on $\frac{\partial u}{\partial t}$.

Remark: If $f \in L^1(0,T;H_0^1)$, $u_0 \in H^2 \cap H_0^1$, and $u_1 \in H_0^1$, then $u' \in L^\infty(0,T;H_0^1)$ and $u \in L^\infty(0,T;H^2)$.

IV. Another perturbation of the wave equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + a(x,t)u^2 = f \\ u(0) = u_0 \qquad\qquad u|_{\partial\Omega} = 0\ . \\ \dfrac{du}{dt}(0) = u_1 \end{cases}$$

Theorem: (Local existence) If $\Omega \subseteq \mathbb{R}^N$ with $N \leq 4$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $|a(x,t)| \leq M \ \forall\, x \in \Omega$, $t \in [0,T]$, and $f \in L^1(0,T;L^2(\Omega))$, then $\exists\ T_0 : 0 < T_0 \leq T$ and $\exists$ a solution $u \in L^\infty(0,T_0;H_0^1(\Omega))$, with $\frac{du}{dt} \in L^\infty(0,T_0;L^2(\Omega))$.

Proof: By Sobolev's inequality, if $N \leq 4$, $\|u\|_{L^4} \leq C\|u\|_{H_0^1}$. Use the Galerkin method, and obtain estimates. Even for the finite dimensional case we may have only local existence.

We get

$$\left(\frac{d^2u_n}{dt^2} - \Delta u_n + au_n^2 - f, w_i\right) = 0 \qquad i = 1,2,\ldots,n\ .$$

Taking the appropriate linear combination, we have

$$\left(\frac{d^2u_n}{dt^2} - \Delta u_n + au_n^2 - f, \frac{du_n}{dt}\right) = 0$$

$$\Longrightarrow \quad \frac{1}{2}\frac{d}{dt}\left(\left|\frac{du_n}{dt}\right|^2 + \|u_n\|^2_{H^1_0}\right) \leq |f|\left|\frac{du_n}{dt}\right| + M\left|\frac{du_n}{dt}\right|_{L^2}|u_n^2|_{L^2}$$

$$\leq |f|\left|\frac{du_n}{dt}\right| + M\left|\frac{du_n}{dt}\right|_{L^2}|u_n|^2_{L^4},$$

$$\leq |f|\left|\frac{du_n}{dt}\right| + M\,C^2\left|\frac{du_n}{dt}\right|\|u_n\|^2_{H^1_0} .$$

If $\varphi_n(t) = \frac{1}{2}\left(\left|\frac{du_n}{dt}\right|^2 + \|u_n\|^2_{H^1_0}\right)$ we have

$$\varphi_n'(t) \leq |f|\sqrt{2\varphi_n(t)} + M C^2\sqrt{2\varphi_n(t)}\ 2\varphi_n(t) .$$

This gives no global estimate for φ_n .

We have $\varphi_n(0) \leq A$. Let T_0 be the first time where $\varphi_n(t) = A + 1$.

Then

$$\varphi_n' \leq |f|\sqrt{2(A+1)} + M\,C^2(2(A+1))^{3/2} \text{ on } [0,T_0] .$$

So

$$A+1 = \varphi_n(T_0) \leq A + \sqrt{2(A+1)}\int_0^{T_0}|f(s)|ds + M\,C^2(2(A+1))^{3/2}\,T_0$$

which gives a lower bound on T_0 depending on $|f|$ and A .

Even if $f \equiv 0$, so that $\varphi_n'(t) \leq K\varphi_n^{3/2}(t)$.

Then

$$(\varphi_n^{-\frac{1}{2}}(t))' = -\tfrac{1}{2}\varphi'\varphi^{-3/2} \geq -K/2$$

$$\varphi_n^{-\frac{1}{2}}(t) \geq \varphi_n(0)^{-\frac{1}{2}} - K/2t$$

$$\geq 0 \text{ for } K/2t \leq \varphi_n(0)^{-\frac{1}{2}}, \text{ so } \varphi_n^{\frac{1}{2}} \leq \frac{1}{\varphi_n(0)^{-\frac{1}{2}} - K/2t}$$

on $\left[0, \frac{\frac{1}{2}\varphi_n^{-\frac{1}{2}}(0)}{K}\right[$.

The bound thus depends on both A and f, as does the interval.

Now, on $[0,T_0]$ we have $\{u_n\}$ is bounded in $L^\infty(0,T_0;H_0^1)$, $\{\frac{du_n}{dt}\}$ is bounded in $L^\infty(0,T_0;L^2)$. Thus we get existence on $[0,T_0]$ exactly as before.

Suppose we have:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + a(x)u^2 = f$$

$$u(0) = u_0 \qquad u|_{\partial\Omega} = 0$$

$$u'(0) = u_1$$

where a is independent of time.

Theorem: (Global existence.) Assume $|a| \le 1$, $N \le 6$; then $|u|_{L^3}^3 \le C_0 \|u\|_{H_0^1}^3$. Assume $\|u_0\| < \frac{1}{C_0}$. If

$$\alpha = (\tfrac{1}{2}|u_1|^2 + \tfrac{1}{2}\|u_0\|^2 + \tfrac{1}{3}\int_\Omega a(x)u_0^3 dx)^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \int_0^T |f(s)|\, ds < (\frac{1}{6C_0^2})^{\frac{1}{2}},$$

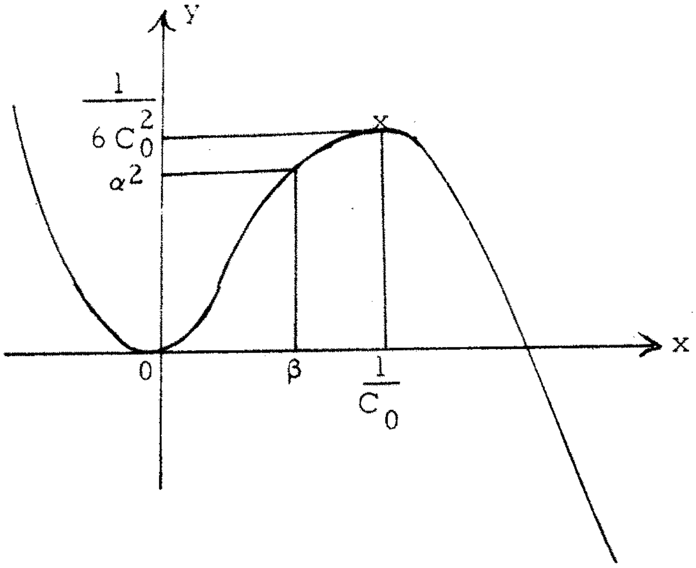
then we have existence on $[0,T]$.

Proof: Via the Galerkin method. By the usual method, obtain

$$\frac{d}{dt}[\tfrac{1}{2}|\frac{du_n}{dt}|^2 + \tfrac{1}{2}\|u_n\|_{H_0^1}^2 + \frac{1}{3}\int_\Omega a\, u_n^3\, dx] = (f, \frac{du_n}{dt}).$$

The graph of $y = \frac{x^2}{2} - \frac{C_0 x^3}{3}$ is useful for the estimate.

$0 \le \beta < \frac{1}{C_0}$ is defined by $\alpha^2 = \frac{\beta^2}{2} - \frac{C_0\beta^2}{3}$.



Suppose that on $[0,T_0[$ we have $\|u_n(s)\| \le \frac{1}{C_0}$ then

$$\tfrac{1}{2}\|u_n\|^2 + \tfrac{1}{3}\int_\Omega a\, u_n^3 dx \ \ge\ \tfrac{1}{2}\|u_n\|^2 - \frac{C_0}{3}\|u_n\|^3 \ \ge\ 0 .$$

So

$$\tfrac{1}{2}\left|\frac{du_n}{dt}\right|^2 \le \tfrac{1}{2}\left|\frac{du_n}{dt}\right|^2 + \tfrac{1}{2}\|u_n\|^2 - \frac{C_0}{3}\|u_n\|^3$$

$$\le \tfrac{1}{2}\left|\frac{du_n}{dt}\right|^2 + \tfrac{1}{2}\|u_n\|^2 + \tfrac{1}{3}\int a\, u_n^3 dx$$

$$= \tfrac{1}{2}|u_1|^2 + \tfrac{1}{2}\|u_0\|^2 + \tfrac{1}{3}\int a\, u_0^3\, dx + \int_0^t \left(f \cdot \frac{du_n}{dt}\right) ds .$$

Let $\gamma = \tfrac{1}{2}|u_1|^2 + \tfrac{1}{2}\|u_0\|^2 + \tfrac{1}{3}\int a\, u_0^3 dx$ ($\gamma \ge 0$ by hypothesis). Then

$$\tfrac{1}{2}\left|\frac{du_n}{dt}\right|^2 \le \gamma + \int_0^t |f(s)|\left|\frac{du_n}{dx}(s)\right| ds = \varphi(t) .$$

So

$$\varphi'(t) = |f(t)|\left|\frac{du_n}{dt}(t)\right| \le |f(t)|\sqrt{2\varphi(t)} .$$

So

$$\sqrt{\varphi(t)} \le \sqrt{\gamma} + \frac{1}{\sqrt{2}}\int_0^t |f(s)|\, ds \le \alpha < \left(\frac{1}{6C_0^2}\right)^{\frac{1}{2}}$$

This gives

$$\frac{1}{2}\left|\frac{du_n}{dt}\right|^2 + \frac{1}{2}\|u_n\|^2 - \frac{C_0\|u_n\|^3}{3} \leq \varphi(t) \leq \alpha^2 < \frac{1}{6C_0^2} .$$

So

$$\left|\frac{du_n}{dt}\right|^2 \leq \frac{1}{3C_0^2} \qquad \text{on } [0, T_0[$$

and

$$\frac{1}{2}\|u_n\|^2 - \frac{C_0\|u_n\|^3}{3} \leq \alpha^2 .$$

So

$$\|u_n\| \leq \beta < \frac{1}{C_0} \quad \text{on } [0, T_0[.$$

This second estimate implies that $T = T_0$ because $\|u_n\|$ is continuous. This estimate being now independent of T and n gives the existence of a solution exactly as before.

Acknowledgement

This is part of the lecture notes of a course given at the University of Wisconsin, Madison, 1974-75. I wish to thank C. Rennolet and T. Kiffe for the writing up of this part.

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J. L. Lions, Quelques méthodes de rèsolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris 1969.

This book contains a lot of examples and a good bibliography.

VARIATIONAL METHODS AND MONOTONICITY

L. Tartar

I. Introduction

In an other report [#1589] we have studied the problems

$$\frac{du}{dt} + A(u) = f, \qquad (u \in V)$$

$$\frac{d^2u}{dt^2} + A(u) = f,$$

in infinite dimensional spaces. The methods used can be divided into the following principal steps:

1) Approach by finite dimensional problems, i.e. replace V by a finite dimensional space V_n. Solve using finite dimensional theory.

2) Obtain bounds, and extract some converging sequence $u_n \to u$.

3) Prove that u is a solution to the original problem.

Step 3) can be done using two different methods:

i) Compactness methods: Use compact embeddings to get strong convergence in some space.

ii) Monotonicity methods: Use special properties of the nonlinear term, so that one can prove convergence of the nonlinear term without strong convergence of u_n.

So far the analysis has mainly relied on compactness methods, and the purpose of this report is to study the monotonicity methods.

We begin with the equation

$$A(u) = f.$$

When we reduce this to a finite dimensional problem we use:

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

Brouwer fixed point theorem: If C is a compact, convex, nonempty subset of $\mathbb{R}^N$, and T maps C continuously into C, then T has at least one fixed point in C.

We begin by studying the variational inequality:

Definition I.1. Let C be a convex set in $\mathbb{R}^N$, and let A map C into $\mathbb{R}^N$. Then u is a solution of the variational inequality in C, if $u \in C$ and $(A(u), v-u) \geq 0$ for every $v \in C$.

Remark I.1. If $u \in C^0$ (the interior of C), then $A(u) = 0$.

Theorem I.1. (Existence). If A is continuous, and C is compact and convex, then there exists a solution u to the variational inequality.

Proof. Make the counter-assumption that

$$\forall\, u \in C,\ \exists\, v \in C \ni (A(u), v-u) < 0 .$$

Defining

$$X_w = \{u \in C \mid (A(u), w-u) < 0\} ,$$

one can write that assumption as

$$\bigcup_{w \in C} X_w = C .$$

Each X_w is open in C, because the mapping $u \to (A(u), w-u)$ is continuous from C into $\mathbb{R}$. The set C is compact and covered by $\{X_w \mid w \in C\}$, so there exists a finite subcover

$$C = \bigcup_{i=1}^{p} X_{w_i} .$$

Choose a partition of unity on C, subordinate to $X_{w_1}, \ldots, X_{w_p}$, i.e. choose continuous functions $\varphi_i(x)$ $(i = 1, \ldots, p)$ such that $0 \leq \varphi_i \leq 1$, $\operatorname{supp} \varphi_i \subset X_{w_i}$, $\sum_{i=1}^{p} \varphi_i(x) = 1$ $(x \in C)$. Now consider the mapping

$$B(u) = \sum_{i=1}^{p} \varphi_i(u)\, w_i \, .$$

Then for each fixed u, $B(u)$ is a convex combination of $[w_1, \dots, w_p]$. B is continuous and maps the convex hull of $[w_1, \dots, w_p]$ into itself. By Brouwer fixed point theorem, $\exists\, u_0 \ni B(u_0) = u_0$. Hence it suffices to prove that $(A(u), B(u) - u) < 0 \;\forall\, u \in C$ in order to get a contradiction. This is done as follows:

$$u \notin X_{w_i} \implies \varphi_i(u) = 0 \implies \varphi_i(u)(A(u), w_i - u) = 0 \, .$$

On the other hand, $u \in X_{w_i} \implies \varphi_i(u) \geq 0$ and $(A(u), w_i - u) < 0 \implies \varphi_i(u)(A(u), w_i - u) \leq 0$. Now at least one $\varphi_i \neq 0$, and thus $\sum_{i=1}^{p} \varphi_i(u)(A(u), w_i - u) < 0 \iff (A(u), B(u) - u) < 0$. ■

Now look at the more general case where R^N is replaced by a locally convex (and Hausdorff) topological vector space E, with dual E'.

Definition I. 2. Let C be a convex set in E, and let A map C into E'. Then u is a solution of the variational inequality in C, if $u \in C$ and $(A(u), v-u) \geq 0$ for every $v \in C$ (the "inner product" is between $A(u) \in E'$ and $v - u \in E$).

The same existence proof for a solution of the variational inequality goes through, if A is continuous in the right way. The only questionable statement in that proof is that the sets X_w are open. This is true if $\forall\, w \in E$, the set $\{u \in E \,|\, (A(u), w-u) \geq 0\}$ is closed, which in turn is true provided A is continuous from E into E' in the strong topology:

$u \to (A(u), w)$ is continuous (needs weak* continuity of A)

$u \to (A(u), u)$ is continuous ("requires" strong continuity of A).

Remark I. . One cannot expect a unique solution, e.g. all zeroes of A are solutions.

Remark I.3. The variational inequality arises e.g. in the following situation: Let J map $C \to \mathbb{R}$, with J differentiable, and C compact. Then J attains its minimum on C at some point; call it u_0. If $u_0 \in C^0$, then $J'(u_0) = 0$. If only $u_0 \in C$ (in general) then

$$J(u_0 + \varepsilon(v-u_0)) \geq J(u_0) \; \forall v \in C \implies (J'(u_0), v-u_0) \geq 0 \quad \forall v \in C .$$

II. Filters.

This is intended as a brief review of filters, and no proofs will be given.

Definition II.1. A filter $\mathcal{A}$ on X is a collection of subsets of X such that

i) $\phi \notin \mathcal{A}$

ii) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

iii) $A \in \mathcal{A} \;\&\; A \subset B \implies B \in \mathcal{A}$.

Definition II.2. A filterbasis $\mathcal{A}$ on X is a collection of subsets of X such that

i) $\phi \notin \mathcal{A}$

ii) $A, B \in \mathcal{A} \implies \exists\, C \in \mathcal{A},\; C \subset A \cap B$.

Remark II.1. Every filter is a filterbasis. Given a filterbasis $\mathcal{A}$ one can construct a filter $\mathcal{B}$ as follows:

$$\mathcal{B} = \{B \subset X \mid \exists\, A \in \mathcal{A}, A \subset B\} .$$

Definition II.3. A filter $\mathcal{A}$ is finer (stronger) than a filter $\mathcal{B}$, if $\mathcal{B} \subset \mathcal{A}$, i.e. every element of $\mathcal{B}$ is contained in $\mathcal{A}$.

Definition II.4. An ultrafilter is a maximal filter, i.e. if $\mathcal{A}$ is an ultrafilter and $\mathcal{B}$ is finer than $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$.

[illegible] II.1. i) Every filter is contained in an ultrafilter (i.e., given any filter $\mathcal{A}$ there exists an ultrafilter finer than $\mathcal{A}$).

ii) If $\mathcal{A}$ is an ultrafilter, then for any given $A \subset X$ either $A \in \mathcal{A}$ or $\tilde{A} \in \mathcal{A}$ ($\tilde{A}$ = the complement of $A = X \backslash A = X - A$).

Example II.1. If $A_0 \subset X_1, A_0 \neq \phi$, then $\mathcal{A} = \{A \subset X | A_0 \subset A\}$ is a filter. This is an ultrafilter if and only if A_0 has only one point.

Example II.2. If X is a topological space, and $A_0 \subset X$, $A_0 \neq \phi$, then all open sets containing A_0 form a filterbasis (neighborhood filterbasis).

Definition II.5. (A filter $\mathcal{A}$) a filterbasis $\mathcal{A}$ converges to a point x_0 in X, if it is finer than (the filter induced by) the filterbasis of neighborhoods of x_0.

Sequences can be considered as special cases of filters in the following fashion: Given a sequence $\{x_n\}$, define the filter

$$\mathcal{A} = \{A \subset X | \exists\, n_0 \in \mathbb{N} \ni (n > n_0 \Rightarrow x_n \in A)\} .$$

It is easy to see that this filter converges to some $x \in X$ if and only if the sequence converges to x.

More generally, nets can be regarded as filters. Let I be a directed set:

$$i, j \in I \Rightarrow \exists\, k \in I \ni i \leq k \ \& \ j \leq k .$$

Let $(x_i)_{i \in I}$ be a net (i.e. a mapping from I into X). Then one can define a filter $\mathcal{A}$ by

$$\mathcal{A} = \{A \subset X | \exists\, i_0 \in I \ni (i \geq i_0 \Rightarrow x_i \in A)\} .$$

It is also easy to see that the net converges if and only if the filter converges.

Remark II. 2. If $\{x_n\}$ is a sequence with corresponding filter $\mathcal{A}$, and $\{y_n\}$ is a subsequence of $\{x_n\}$ with a corresponding filter $\mathcal{B}$, then $\mathcal{B}$ is finer than $\mathcal{A}$.

Theorem II. 1. X is a compact $\Longleftrightarrow$ every ultrafilter is convergent.

Lemma II. 2. Let $A \subset X$. Then $a \in \bar{A} \Longleftrightarrow$ there exists a filterbasis on A converging to a .

Remark II. 3. If E is a topological, locally convex space, and $\mathcal{A}$ is a filter converging to $x \in E$, then it may happen that every element of $\mathcal{A}$ is unbounded.

Definition II. 6. A filter is bounded if it has at least one bounded element.

Definition II. 7. Let $\mathcal{A}$ be a filter on X , and $f: X \to Y$. Then the image of $\mathcal{A}$ under f is the filterbasis $\mathcal{B} = \{f(A) \subset Y \mid A \in \mathcal{A}\}$.

Remark II. 4. The image of an ultrafilter is a basis of an ultrafilter.

Remark II. 5. If $\mathcal{A}$ is a filter on $\mathbb{R}$ or $\bar{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, and $\mathcal{B}$ is an ultrafilter finer than $\mathcal{A}$, then $\mathcal{B}$ converges to some $x_B \in \bar{\mathbb{R}}$, because $\bar{\mathbb{R}}$ is compact.

Definition II.8 . Let $\mathcal{A}$ be a filter on $\mathbb{R}$ or $\bar{\mathbb{R}}$. Then

$$\limsup \mathcal{A} = \sup\{x_{\mathcal{B}} \in \bar{\mathbb{R}} \mid \exists\, \mathcal{B} \to x_{\mathcal{B}} \ni \mathcal{B} \text{ is finer than } \mathcal{A}\} .$$

$$\liminf \mathcal{A} = \inf\{x_{\mathcal{B}} \in \bar{\mathbb{R}} \mid \exists\, \mathcal{B} \to x_{\mathcal{B}} \ni \mathcal{B} \text{ is finer than } \mathcal{A}\}$$

III. Nonlinear Equations.

The problem is the following: We have a topological vector space E , and a mapping $A: E \to E'$. We want to solve $A(u) = 0$. The procedure will be as follows: First take finite dimensional spaces F spanned by $(w_1, \ldots, w_n)$, and solve $(A(u), w_i) = 0$ $(i = 1, \ldots, n)$ in F using finite dimensional theory. Then extract some converging filter, converging to u , and prove (with as weak

Notation: If $F \subset E$, define $A_F : F \to F^*$ (= algebraic dual of F) by

$$(A_F(u),v) = (A(u),v) \quad (u,v \in F) .$$

We shall use the following assumptions on A :

$$(1) \begin{cases} \text{i) If } F \text{ is finite dimensional, then } A_F \text{ is continuous.} \\ \text{ii) If } u_i \text{ is some filter on a compact set } K \subset E, \text{ then} \\ \qquad \left.\begin{array}{l} u_i \to u \text{ in } E \\ A(u_i) \to 0 \text{ in } E' \text{ weak}^* \\ (A(u_i),u_i) \to 0 \end{array}\right\} \Longrightarrow A(u) = 0 \end{cases}$$

The convergence of u_i in E of course depends on what topology we use on E (usually the weak top.)

Definition III.1. Let $C \subset E$ be convex. Then x_0 is an internal point of C, if

$$\forall\, y \in E ,\ \exists\, \varepsilon > 0 \ni x_0 + \varepsilon y \in C .$$

(note: this is an algebraic definition, not topological). If 0 is an internal point of C, then C is called absorbing. The set of internal points of C is denoted by $i(C)$, and its complement in C by ∂C, i.e. $\partial C = C \setminus i(C)$.

Theorem III.1. Let A satisfy (1). Let C be a compact, convex and absorbing subset of E, and let the "compatibility condition":

$$(2) \qquad (A(u),u) \geq 0 \quad (u \in \partial C)$$

hold. Then $\{u \in C \mid A(u) = 0\}$ is nonempty and compact.

Proof. Let F be a finite dimensional subspace of E. Define $C_F = C \cap F$. Then C_F is convex and absorbing in F. By (1.i), A_F is continuous on F.

Consider the variational inequality

$$(A_F(u),v-u) \geq 0 \qquad (v \in C_F) .$$

By Theorem I.1, there exists a solution $u_F \subset C_F$. Now separate two cases:

α) $\begin{cases} (A_F(u_F), u_F) \geq 0 \implies \\ (A_F(u_F), v) \geq (A_F(u_F), u_F) \geq 0 \quad (v \in C_F). \\ \text{Because } C_F \text{ is absorbing we get} \\ (A_F(u_F), v) = 0 \ (v \in F) \implies A_F(u_F) = 0. \end{cases}$

β) $\begin{cases} (A(u_F), u_F) < 0 \implies \\ \qquad u_F \in \partial C_F \ (\text{because } A_F(u_F) \neq 0). \\ \qquad u_F \in i(C) \ (\text{because of } (2)) \implies u_F \in i(C_F). \\ \text{These two statements contradict each other, and thus case } \beta \\ \text{cannot appear.} \end{cases}$

We conclude that for all F, $\exists\, u_F \ni A_F(u_F) = 0$, in particular $(A(u_F), f) = 0$ $(f \in F)$, and $(A(u_F), u_F) = 0$.

Consider the filter induced by the net $\{u_F \mid F$ a finite dimensional subspace of $E\}$. As C is compact, and $u_F \subset C$, there exists an ultrafilter u_G finer than u_F converging to some $u \in C$. Because $(A(u_F), f) = 0$ $(f \in F)$ we have $(A(u_F), f) \to 0$ $(f \in E)$, which implies $A(u_G) \to 0$ weak* in E. In the same way one gets $(A(u_G), u_G) \to 0$. By (1. ii), $A(u) = 0$.

The closedness (and hence compactness) of the zero set of A follows easily from (1. ii).

Instead of (2) one can also use a different compatibility condition:

(3) There exists a compact set $K \subset E \ni (A(u), u) = 0 \implies u \in K$.

<u>Theorem III.2.</u> Let A satisfy (1) and (3). Then $\exists\, u \in K \ni A(u) = 0$. Moreover, the set $\{v \in K \mid A(v) = 0\}$ is compact.

[illegible] Let F be a finite dimensional subspace, and choose a set C_F which is compact, convex and absorbing in F , and contains $K \cap F$ in its interior. On the boundary of C_F, $(A(u),u) \neq 0$. Without loss of generality we can assume that $(A(u),u) > 0$ for $u \in \partial C_F$ (otherwise multiply A by -1). By Theorem I.1 we can find some $u_F \subset C_F$ such that

$$(A_F(u_F),\ v - u_F) \geq 0 \qquad (v \in C_F).$$

In the same way as in the proof of Theorem 1 one gets $A_F(u_F) = 0$. In particular, $(A(u_F), u_F) = 0$, and thus by (3), $u_F \subset K$.

The last part of the proof is an exact repetition of the last part of the proof of Theorem 1 (replace the set C by the set K).

We shall now study different continuity properties more closely.

<u>Definition III. 2.</u> A is of <u>type M_0</u> if it satisfies

$$(4) \quad \begin{cases} \text{i)} & A_F \text{ is continuous for finite dimensional } F \\ \text{ii)} & \begin{cases} \text{If } u_i \text{ is some filter on a compact set } K \subset E, \text{ then} \\ \left.\begin{array}{l} u_i \to u \text{ in } E \\ A(u_i) \to f \text{ in } E' \text{ weak}^* \\ (A(u_i),u_i) \to (f,u) \end{array}\right\} \Rightarrow A(u) = f \end{cases} \end{cases}$$

<u>Definition III. 3.</u> A is of <u>type M</u> if

$$(5) \quad \begin{cases} \text{i)} & A_F \text{ is continuous for finite dimensional } F \\ \text{ii)} & \begin{cases} \text{If } u_i \text{ is some filter on a compact set } K \subset E, \text{ then} \\ \left.\begin{array}{l} u_i \to u \text{ in } E \\ A(u_i) \to f \text{ in } E' \text{ weak}^* \\ \limsup\, (A(u_i),u_i) \leq (f,u) \end{array}\right\} \Rightarrow A(u) = f \end{cases} \end{cases}$$

Definition III.4. A is monotone hemicontinuous if

i) $\forall\, u,v \in E$, the mapping $t \to (A(u+tv),v)$ is continuous (hemicontinuity).

ii) $\forall\, u,v \in E$, $(A(u) - A(v), u-v) \geq 0$ (monotonicity).

Theorem III.3. (existence): Let E be a reflexive Banach space with its weak topology. Let A be of type M_0, and satisfy the coercivity condition

$$(6) \qquad \frac{(A(u),u)}{\|u\|} \to \infty \qquad (\|u\| \to \infty) \; .$$

Then A is surjective, i.e. $A(E) = E'$.

Proof. Let $f \in E'$. Define the operator $B: E \to E'$ by $B(u) = A(u) - f$. It follows immediately from (4) that (1) holds (in particular, $u_i \to u$ means weak convergence in E). Theorem 3 then follows from Theorem 2, provided one can show that (3) is satisfied with A replaced by B. So take some $u \in E$ such that $(B(u),u) = 0$. Then $(A(u),u) = (f,u)$, and thus

$$\frac{(A(u),u)}{\|u\|} \leq \|f\|_{E'} \; .$$

By (6), $\|u\| \leq C$ for some constant C. By reflexivity, the set $K = \{u \in E \,|\, \|u\| \leq C\}$ is (weakly) compact, and thus (3) holds with A replaced by B. Applying Theorem 2 one now gets Theorem 3. ∎

Lemma III.1. If A is of type M, then A is of type M_0. If A is of type M_0, then so is $-A$.

Proof. Obvious.

Lemma III.2. If A is monotone hemicontinuous, then A is of type M (hence of type M_0).

Note: The topology on E does not matter as long as it is compatible with the duality. The interesting case is when E has its weak topology.

Proof. We begin by proving (5.ii). Suppose that

$$(7) \quad \begin{cases} u_i \to u \text{ in } E \\ A(u_i) \to f \text{ weak}^* \text{ in } E' \\ \limsup\, (A(u_i), u_i) \leq (f, u) . \end{cases}$$

Take some $v \in E$. $(A(u_i) - A(v), u_i - v) \geq 0 \iff (A(u_i), u_i) - (A(u_i), v) - (A(v), u_i) + (A(v), v) \geq 0$ (by (7)) $\Rightarrow (f,u) - (f,v) - (A(v),u) + (A(v),v) \geq 0$ $\iff (f - A(v), u-v) \geq 0$ $(v \in E)$. We want to show that $A(u) = f$. Take $w \in E$, $t > 0$, and put $v = u + tw$. Then $(f - A(u+tw), -tw) \geq 0$. Divide by t, and let $t \downarrow 0$ using the hemicontinuity of A.

This gives

$$(f - A(u), -w) \geq 0 \quad (w \in E) \Rightarrow f - A(u) = 0 .$$

This shows that condition (5.ii) holds.

We still have to verify (5.i). Take F finite dimensional. Note that A_F is monotone: $u,v,w \in F \Rightarrow (A_F(u) - A_F(v), w) = (A(u) - A(v), w)$. We shall show that $A_F : F \to F'$ is bounded, i.e. maps bounded sets of F into bounded sets of F'. This will imply continuity for the following reason: Suppose that $u_n \to u$ in F (weakly or strongly). Then $\{u_n\}$ is bounded, and hence $\{A_F(u_n)\}$ is bounded. Extract a converging subsequence: $A_F(u_{n_t}) \to f$. Then (7) holds for this subsequence, and thus by (5.ii), $A_F(u) = f$.

It only remains to show that A_F is bounded. Suppose this is not the case. Then one can find a bounded sequence u_n, which (after passing to a subsequence) converges to u, such that $\|A_F(u_n)\| \to \infty$. Extract a subsequence so that

$$\frac{A_F(u_n)}{\|A_F(u_n\|} \to \xi \in F \text{ , with } \|\xi\|_F = 1.$$

$$\frac{(A_F(u_n) - A_F(v),\ u_n - v)}{\|A_F(u_n)\|} \geq 0 \qquad (v \in F)\ .$$

But

$$\frac{A_F(u_n) - A_F(v)}{\|A_F(u_n)\|} \to \xi \qquad\qquad (n \to \infty)\ ,$$

$$u_n - v \to u - v \qquad (n \to \infty)\ ,$$

and thus

$$(\xi,\ u - v) \geq 0 \quad (v \in F) \Longrightarrow \xi = 0\ .$$

Note: In an infinite dimensional space $\xi = 0$ does not lead to a contradiction.

Example III.1. (an unbounded monotone operator). Take $E = \ell^2$, then E can be identified with E'. Consider $A: E \to E$ defined by

$$A(x_1, \ldots, x_n, \ldots) = (y_1, \ldots, y_n, \ldots)$$

with $y_n = x_n^{2n+1}$. A is well defined on all of ℓ^2, because $(x_n) \in \ell^2 \Longrightarrow x_n \to 0$ $(n \to \infty) \Longrightarrow |y_n| \leq |x_n|$ for large $n \Longrightarrow (y_n) \in \ell^2$.

Monotonicity is trivial: $(A(x_n) - A(\bar{x}_n),\ x_n - \bar{x}_n) = \sum(y_n - \bar{y}_n, x_n - \bar{x}_n) =$
$\sum(x_n^{2n+1} - \bar{x}_n^{2n+1},\ x_n - \bar{x}_n) \geq 0$.

A is not bounded: Take $u = (0, 0, \ldots, 0, 2, 0, \ldots)$. (the 2 in the n^{th} position). Then $\|u\| = 2$. $A(u) = (0, 0, \ldots, 0, 2^{2n+1}, 0, \ldots)$, $\|A(u)\| = 2^{2n+1}$.

Lemma III.3. If A is of type M_0, and bounded, then A is continuous from E strong into E' weak.

Proof. If $u_n \to u$ strongly in E, then $A(u_n)$ is bounded in E'. Extract a subsequence $A(u_n) \rightharpoonup f$ weak*. By (4), $A(u) = f$.

Lemma III.4. If A is of type M_0, and B is continuous from E weak into E' strong, then $A+B$ is of type M_0.

Proof. Let

$$\begin{cases} u_i \rightharpoonup u \text{ in } E \text{ weak} \\ A(u_i) + B(u_i) \rightharpoonup f \text{ weak}^* \text{ in } E \\ (A(u_i) + B(u_i), u_i) \to (f, u) \end{cases}$$

By the hypothesis, $B(u_i) \to (B(u)$ strongly $\Rightarrow$ $(B(u_i), u_i) \to (B(u), u)$ $\Rightarrow$

$$\begin{cases} A(u_i) \rightharpoonup f - B(u) \text{ weak}^* \\ (A(u_i), u_i) \to (f,u) - (B(u),u). \end{cases}$$

Hence $A(u) = f - B(u)$, i.e. (4.ii) holds.

Lemma III.5. Let A be of type M. Let B be bounded, weakly continuous (i.e. from E weak into E' weak*), and let $(B(u),u)$ be weakly lower semicontinuous (i.e. $u_i \rightharpoonup u \Rightarrow \liminf (B(u_i),u_i) \geq (B(u),u)$). Then $A+B$ is of type M.

For example, if B is monotone and weakly continuous, then it is of the type described above.

Proof. Let

$$(8) \qquad \begin{cases} u_i \to u \quad \text{in } E \\ A(u_i) + B(u_i) \rightharpoonup f \text{ weak}^* \text{ in } E' \end{cases}$$

Then $B(u_i) \rightharpoonup B(u)$ weak* in E', and thus

$$(9) \qquad A(u) \rightharpoonup f - B(u) \text{ weak}^* \text{ in } E' .$$

Moreover, by the weak lower semicontinuity if $(B(u),u)$, we have

$$\liminf (B(u_i),u_i) \geq (B(u),u) .$$

This, together with (8), (9) and (5.ii) gives $A(u) = f - B(u)$. Hence (5. ii) is satisfied with A replaced by $A+B$, and so $A+B$ is of type M.

We shall now change the problem slightly. Up to this point A has always mapped E into E'. Now we instead suppose that we have two topological vector spaces W and V with $W \subset V$, and W dense in V. This implies that $V' \subset W'$, and that V' is dense in W'. We next study the case when

$$\boxed{A : V \to W'}$$

<u>Theorem III.4.</u> Let K be a convex, compact and absorbing subset of V. Let A map K into W', and suppose that

$$(10) \begin{cases} \text{i)} & A_F \text{ is continuous for finite dimensional } F \\ \text{ii)} & \left.\begin{cases} u_i \in W \ \& \ u_i \to u \text{ in } V \\ A(v_i) \rightharpoonup 0 \text{ in } W' \text{ weak}^* \\ (A(u_i), u_i) \to 0 \end{cases}\right\} \implies A(u) = 0 \\ \text{iii)} & \forall u \in W \cap \partial K, \ (A(u), u) \geq 0 . \end{cases}$$

Then there exists $u \in K$ such that $A(u) = 0$.

<u>Proof.</u> Take F finite dimensional in W (note: not in V), and pick some $u_F \in K \cap F$ satisfying: $(A_F(u_F), v - u_F) \geq 0$ $(v \in K \cap F)$. This can be done because of (10.i) (cf. Theorem I.1). From (10.iii) one gets $A_F(u_F) = 0$ (see proof of Theorem 1). We get a filter in K, induced by the net u_F. Since K is compact in V (although not necessarily in W), we can find an ultrafilter u_i finer than u_F such that

$$u_i \in W, \ u_i \to u \text{ in } V ,$$

$$A(u_i) \rightharpoonup 0 \text{ in } W' \text{ weak}^* ,$$

$$(A(u_i), u_i) \to 0 .$$

By (10 ii), $A(u) = 0$. ∎

Remark III.1. We do not know in this case that the zero set of A is compact in V, as was the case before.

Theorem III.5. Let $A : V \to W'$ satisfy (10.i) and (10.ii), and suppose that

(11) $\begin{cases} \text{There exists a compact subset } K \text{ of } V \ni u \in W \ \& \ (A(u),u) = 0 \\ \implies u \in K . \end{cases}$

Then there exists $u \in K$ such that $A(u) = 0$.

Proof. The first part of the proof is exactly like the proof of Theorem 2 (take finite dimensional subspaces $F \subset W$). The second part of the proof is the same as the last part of the proof of Theorem 4. ∎

We shall now apply the theory developed above to a differential equation. Let V be a reflexive Banach space, continuously and densely imbedded in a Hilbert space H . Identify H with its dual. Then we have

$$V \subset H = H' \subset V' .$$

Let A map $V \to V'$. We want to solve

(12) $$\begin{cases} \dfrac{du}{dt} + A(u) = 0 \\ u(0) = u_0 \in H . \end{cases}$$

Theorem III. 6. Let A, V and H be as above. Let p, p' satisfy $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Define the map $\mathcal{A} : \mathcal{A}(u) = A(u(t))$ ($u \in L^p(0,T;V)$). (T is some finite number). Also suppose that

(13) $\begin{cases} \text{i) } \mathcal{A} : L^p(0,T;V) \to L^{p'}(0,T;V'), \text{ and maps bounded sets into bounded sets.} \\ \text{ii) } \mathcal{A} \text{ is of type M from } L^p(0,T;V) \text{ into } L^{p'}(0,T;V') \\ \text{iii) Coercivity condition} \\ \qquad \left.\begin{array}{l} \varphi \in L^p(0,T;V) \\ \frac{d\varphi}{dt} \in L^{p'}(0,T;V') \\ \frac{1}{2}|\varphi(t)|^2 + (\mathcal{A}\varphi,\varphi) \le \frac{1}{2}|u_0|^2 \end{array}\right\} \Longrightarrow \\ \qquad\qquad \begin{cases} \varphi \in C = \text{a bounded subset of } L^p(0,T;V) \\ \varphi(t) \in K = \text{a bounded subset of } H . \end{cases} \end{cases}$

Then there exists $u \in L^p(0,T;V)$ with $\frac{du}{dt} \in L^{p'}(0,T;V')$ such that

$$\frac{du}{dt} + \mathcal{A}u = 0 ,$$

$$u(0) = u_0 .$$

Note: $(\mathcal{A}\varphi,\varphi)$ denotes the "inner product" between $L^{p'}(0,T;V')$ and $L^p(0,t;V)$. The norm in H is denoted by $|\cdot|$, and norms in other spaces by $\|\cdot\|$, with appropriate subscripts.

<u>Remark III.2.</u> The above theorem can be used to solve more general equations than (12) when $\mathcal{A}$ is defined globally in t. In the case where $(\mathcal{A}u)(t) = A\,u(t) - f(t)$ with $f \in L^{p'}(0,T;V')$, one can show that

(13) (i)(ii) $\Longleftrightarrow$ $\begin{cases} \text{either } \alpha) \ A \text{ is affine continuous from } V \text{ into } V' \text{ (and } p \ge 2) \\ \text{or } \quad \beta) \ A \text{ is monotone hemicontinuous from } V \text{ into } V' \\ \qquad\qquad \text{satisfying } \|Au\|_{V'} \le C_0 + C_1 \|u\|_V^{p-1} . \end{cases}$

It is clear that $\alpha) \Longrightarrow \mathcal{A}$ weakly continuous bounded $\Longrightarrow$ (13) i) ii). Case $\beta)$ is very important for applications (one can give in this case direct and simpler proofs of the theorem). In this case $\mathcal{A}$ is monotone and hemicontinuity, that is continuity of $\lambda \to \int_0^T (A(u(t)+\lambda v(t)), v(t))dt$, follows from the boundedness and hemicontinuity of A.

The other implication is not obvious and we only sketch it.

Claim: If $a,b \in V$ and $(Aa - Ab, a-b) \leq 0$ then A is affine on the segment $[a,b]$.

Proof. Take

$$u_n(t) = \begin{cases} a & \text{in } \left(\frac{kT}{n}, \frac{(k+1-\theta)T}{n}\right) \\ b & \text{in } \left(\frac{(k+1-\theta)T}{n}, \frac{(k+1)T}{n}\right) \end{cases} \qquad 0 \leq k \leq n-1$$

then $\qquad u_n \to u = (1-\theta)a + \theta b \quad \text{in } L^\infty(0,T,v) \text{ weak}^*$

$\qquad Au_n \to \xi = (1-\theta)Aa + \theta b \quad \text{in } L^\infty(0,T;v') \text{ weak}^*$

and $\qquad \int_0^T (Au_n, u_n)dt \to T((1-\theta)(Aa,a) + \theta(Ab,b)) \leq \int_0^T (\xi, u)dt$

so $Au = \xi$ and the claim is proved.

Then if for some u_0, u_1 $(A u_0 - A u_1, u_0 - u_1) < 0$ one shows that A is affine on the line (u_0, u_1); then on every line through u_0; then everywhere.

Proof of Theorem III.6. Define two reflexive Banach spaces:

$$\mathscr{V} = L^p(0,T;V) \times H,$$

$$\mathscr{W} = \{(u,u(T)) \mid u \in L^p(0,T;V), \ \frac{du}{dt} \in L^{p'}(0,T;V')\}.$$

The norm on $\mathscr{V}$ is: $\|(u,\alpha)\|_{\mathscr{V}} = \|u\|_{L^p(0,T;V)} + |\alpha|_H$,

The norm in $\mathscr{W}$ is: $\|(u,u(T)\|_{\mathscr{W}} = \|u\|_{L^p(0,T;V)} + \|\frac{du}{dt}\|_{L^{p'}(0,T;V')}$.

We have seen before that $u \in L^p(0,T;V)$, $\frac{du}{dt} \in L^{p'}(0,T;V')$ implies $u \in C(0,T;H)$. Hence $\mathscr{W} \subset \mathscr{V}$.

Claim 1. $\mathscr{W}$ is dense in $\mathscr{V}$.

Proof of Claim 1. Let $(v,\gamma) \in L^{p'}(0,T;V) \times H = \mathcal{V}'$, and suppose that $((u,u(T)),(v,\gamma)) = \int_0^T (u(x),v(x))dx + (u(T),\gamma) = 0$ for every $(u,u(T)) \in \mathcal{W}$. Then in particular, $\int_0^T (u(x),v(x))dx = 0$ for every $u \in \mathcal{D}(0,T;V)$, and since $\mathcal{D}(0,T;V)$ is dense in $L^p(0,T;V)$ we get $v = 0$. If one then takes u to be a constant function in V then one gets $(u,\gamma) = 0$ $(u \in V)$, and since V is dense in H, we get $\gamma = 0$. This shows that the zero functional is the only functional in $\mathcal{V}'$ which vanishes on $\mathcal{W}$, and hence by Hahn-Banach, $\mathcal{W}$ is dense in $\mathcal{V}$.

Now define an operator $\mathcal{B} : \mathcal{V} \to \mathcal{W}'$; by $(\mathcal{B}(u,\alpha),(\varphi,\varphi(T))) = -(u, \frac{d\varphi}{dt}) + (\mathcal{A}u,\varphi) - (u_0,\varphi(0)) + (\alpha,\varphi(T))$ (the first two "inner products" are between $L^p(0,T;V)$ and $L^{p'}(0,T;V')$, and the last two are in H. We have used the fact that $\mathcal{W} \subset C(0,T;H)$).

Claim 2. u solves the original problem $\iff \mathcal{B}(u,\alpha) = 0$ and $\alpha = u(T)$.

Proof of Claim 2. $(\Longrightarrow)$: Integrate by parts. $(\Longleftarrow)$: Suppose that $(\mathcal{B}(u,\alpha),(\varphi,\varphi(T))) = 0 \ \forall(\varphi,\varphi(T)) \in \mathcal{W}$. Take $\psi \in \mathcal{D}(0,T;V)$. Then in particular, $(\psi,0) \in \mathcal{W}$, so

$$-(u, \frac{d\psi}{dt}) + (\mathcal{A}u,\psi) = 0 \quad (\psi \in \mathcal{D}(0,T;V)).$$

Thus by the definition of distribution derivatives, $\frac{du}{dt} = -\mathcal{A}u \in L^{p'}(0,T;V')$. This means that $u \in L^p(0,T;V)$, $\frac{du}{dt} \in L^{p'}(0,T;V')$, so in particular $u \in C(0,T;H)$. Integrating by parts one gets, for $\varphi \in \mathcal{W}$,

$$0 = -(u, \frac{d\varphi}{dt}) + (\frac{du}{dt},\varphi) + (u_0,\varphi(0)) + (\alpha,\varphi(T)) =$$

$$(u(0) - u_0,\varphi(0)) - (u(T) - \alpha,\varphi(T)),$$

so one must have $u(0) = u_0, u(T) = \alpha$.

We shall now use Theorem 5 to solve $\mathcal{B}(u,\alpha) = 0$,

i) (Assumption (10.i)): $\mathcal{B}$ restricted to a finite dimensional subspace is continuous, since the nonlinear term $\mathcal{A}(u)$ is continuous in finite dimensional subspaces of $L^p(0,T;V)$ (this follows from (13 ii)).

ii) (Assumption 10.ii)): Let

$$(14) \qquad \begin{cases} (u_i, u_i(T)) \rightharpoonup (u,\alpha) , \\ \mathcal{B}(u_i, u_i(T)) \rightharpoonup 0 , \\ (\mathcal{B}(u_i,u_i(T)), (u_i,u_i(T))) \to 0 \end{cases}$$

We want to show that $\mathcal{B}(u,\alpha) = 0$. $(u_i,u_i(T)) \rightharpoonup (u,\alpha)$ means

$$\begin{cases} u_i \rightharpoonup u \quad \text{weakly in } L^p(0,T;V) , \\ u_i(T) \rightharpoonup \alpha \quad \text{weakly in } H . \end{cases}$$

By (13.i), $\mathcal{A} u_i$ is bounded in $L^{p'}(0,T;V')$, so one can extract an ultrafilter converging weakly to some $\xi \in L^{p'}(0,T;V')$. Then

$$(\mathcal{B}(u_i,u_i(T)), (\varphi,\varphi(T)) = -(u_i, \frac{d\varphi}{dt}) + (\mathcal{A} u_i,\varphi) - (u_0,\varphi(0)) + (u_i(T),\varphi(T))$$
$$\to -(u, \frac{d\varphi}{dt}) + (\xi,\varphi) - (u_0, \varphi(0)) + (\alpha,\varphi(T)).$$

By (14), this is $= 0$, and so by the same argument as in the proof of Claim 2, we get $\frac{du}{dt} + \xi = 0$, $u(0) = u_0$, $u(T) = \alpha$. Thus $\frac{du}{dt} \in L^{p'}(0,T;V')$, and in particular, $(u,\alpha) \in \mathcal{W}$.

We still have to show that $\mathcal{A} u = \xi$. For this one needs the last line of (14):

$$(\mathcal{B}(u_i,u_i(T)), (u_i,u_i(T))) = \tfrac{1}{2}|u_i(T)|^2 + \tfrac{1}{2}|u_i(0)|^2$$
$$+ (\mathcal{A} u_i,u_i) - (u_0,u_i(0)) \to 0 .$$

But $\frac{1}{2}|u_i(0)|^2 - (u_0,u_i(0)) \geq -\frac{1}{2}|u_0|^2$, so

$$\liminf \tfrac{1}{2}|u_i(T)|^2 + \tfrac{1}{2}|u_i(0)|^2 - (u_0,u_i(0)) \geq$$
$$\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|u_0|^2 = \tfrac{1}{2}|u(T)|^2 - \tfrac{1}{2}|u(0)|^2$$

This then gives

$$\limsup (\mathcal{A} u_i, u_i) \leq \tfrac{1}{2}|u(T)|^2 - \tfrac{1}{2}|u(0)|^2$$
$$= -(u, \frac{du}{dt}) = (\xi, u) .$$

So we have

$$\left.\begin{array}{lll} u_i \to 0 & \text{in } L^p(0,T;V) & \text{weak} \\ \mathcal{A} u_i \to \xi & \text{in } L^p(0,T;V) & \text{weak} \\ \limsup (\mathcal{A} u_i, u_i) \leq (\xi, u) & & \end{array}\right\} \Longrightarrow \text{(by (13 ii))} \ \mathcal{A} u = \xi .$$

This then gives $\mathcal{B}(u,\alpha) = 0$, and we have verified (10 ii).

iii) (Condition (11)): $(\mathcal{B}(\varphi,\varphi(T)), (\varphi,\varphi(T))) \leq 0 \Rightarrow \frac{1}{2}|\varphi(T)|^2 + (\mathcal{A}\varphi, \varphi \leq \frac{1}{2}|u_0|^2$. It then follows from (13 iii) that $(\varphi,\varphi(T))$ is contained in a bounded subset of $\mathcal{V}$.

Now we can apply Theorem 5, and the proof is complete.

IV. Variational Inequalities.

We now return to the variational inequalities briefly treated in Section I.

We have:

E = locally convex, topological vector space,

K = a convex subset of E ,

A = $K \to E'$ (or $E \to E'$) .

Problem IV.1. (See Definition I.2.): Find $u \in K$ such that

(1) $(A(u), v-u) \geq 0 \quad \forall\, v \in K$.

By Theorem I.1, if $\dim E < \infty$, A is continuous, and K is compact and non-empty, then Problem 1 has a solution u .

We shall also consider the following generalization of Problem 1: Let φ be a convex, lower semicontinuous function in E , with values in $(-\infty,\infty]$, and $\varphi \not\equiv +\infty$.

Problem IV. 2. Find $u \in \mathcal{D}(\varphi) = \{u \in E \mid \varphi(u) < \infty\}$ such that

$$(A(u), v-u) + \varphi(v) - \varphi(u) \geq 0 \qquad \forall v \in E. \tag{2}$$

Remark IV. 1. This inequality is trivially satisfied for every $v \notin \mathcal{D}(\varphi)$.

Remark IV. 2. If the solution is an internal point of K, then $A(u) = 0$ (see Definition 4. 3. 1).

Remark IV.3. We get Problem 1 from Problem 2 by taking $\varphi(v) = 0$ $(v \in K)$, $\varphi(v) = +\infty$ $(v \notin K)$. Then $\mathcal{D}(\varphi) = K$. Moreover:

$$K \neq \emptyset \Longleftrightarrow \varphi \not\equiv +\infty$$

$$K \text{ closed} \Longleftrightarrow \varphi \text{ lower semicontinuous}$$

$$K \text{ convex} \Longleftrightarrow \varphi \text{ convex.}$$

As was mentioned in Section 4.7, the main problem is to investigate what kind of "continuity" properties of A imply existence of solutions to Problems 1-2.

Definition IV. 1. A pseudomonotone operator from E into E' is an operator A which satisfies (u_i stands for a filter):

(3)
- i) If $u_i \to u$,
  $\limsup\, (A(u_i), u_i - u) \leq 0$ then
  $\liminf\, (A(u_i), u_i - v) \geq (A(u), u-v)$ $(v \in E)$.
- ii) $\forall v \in E$, $(A(u), u-v)$ is bounded from below on bounded sets, (as a function of u).

Lemma IV. 1. If $\dim E < \infty$, then A is pseudomonotone $\Longleftrightarrow$ A is continuous.

Proof. ($\Leftarrow$). Continuity trivially implies (3 ii). Also (3 i) follows, because if $u_i \to u$, then $A(u_i) \to A(u)$, and hence $(A(u_i), u_i - v) \to (A(u), u-v)$ $(v \in E)$.

($\Rightarrow$). Let A be pseudomonotone, and let $u_i \to u$. (u_i a sequence).

Claim. Au_i is bounded.

Proof of claim. Suppose not. Then one can find a subsequence such that $\|A(u_i)\| \to \infty$, $\frac{A(u_n)}{\|A(u_n)\|} \to \xi$, with $\|\xi\| = 1$. Fix $v \in E(=\mathbb{R}^n)$. Then by (3 ii),

$$(A(u_i), u_i - v) \geq C \implies$$

$$\left(\frac{A(u_i)}{\|A(u_i)\|}, u_i - v\right) \geq \frac{C}{\|Au_i\|} \quad (\text{let } i \to \infty) \implies (\xi, u-v) \geq 0 .$$

This is true for every $v \in E$, and hence $\xi = 0$. But this contradicts the fact that $\|\xi\| = 1$. Thus we have proved the claim, i.e. Au_i is bounded.

It still remains to prove continuity of A. Let $u_i \to u$, and extract a subsequence such that $A(u_i) \to f$ for some $f \in E' = E$. Clearly $(Au_i, u_i - u) \to 0$. By (3 i), $(f, u-v) = \liminf (A(u_i), u_i - v) \geq (A(u), u-v)$ for every $v \in E$. But this implies that $f = A(u)$. Since by the same argument, every subsequence of $A(u_i)$ contains a subsequence converging to $A(u)$, we must have the complete sequence $A(u_i)$ converging to $A(u)$. This means that A is sequentially continuous, and since $E (= \mathbb{R}^n)$ is metrizable, we also have A continuous.

Remark IV. 4. If A is pseudomonotone on E, then A_F is pseudomonotone on F for every closed subspace F, so in particular A_F is continuous for every finite dimensional F.

Lemma IV. 2. A is monotone hemicontinuous $\implies$ A is pseudomonotone $\implies$ A is of type M.

Proof. Let A be monotone hemicontinuous, and let $u_i \to u$. Since $(A(u_i) - A(u), u_i - u) \geq 0$ we get $\liminf (A(u_i), u_i - u) \geq 0$. So if also $\limsup (A(u_i), u_i - u) \leq 0$, then

(4) $$\lim (A(u_i), u_i - u) = 0 .$$

Take some $v \in E$. By the monotonicity of A,

$$(A(u_i) - A(v), u_i - v) \geq 0 \iff$$

$$(A(u_i), u_i - u) + (A(u_i), u - v) - (A(v), u_i - v) \geq 0 \implies \text{(by (4))},$$

$$\liminf\, (A(u_i), u - v) \geq (A(v), u - v) .$$

Now take some $w \in E$, and put $v = u + t(w - u)$ $(t \in [0,1])$ in this inequality. Then one gets

$$\liminf\, (A(u_i), t(u-w)) \geq (A(u+t(w-u)), t(u-w)).$$

Divide by t, and let $t \to 0$. Then by the hemicontinuity of A,

$$\liminf\, (A(u_i), u-w) \geq (A(u), u-w) .$$

Using (4) once more one then gets

$$\liminf\, (A(u_i), u_i - w) \geq (A(u), u-w),$$

and we have proved (3.i).

The property (3.ii) follows trivially from the monotonicity: Fix some $v \in E$. Then $(A(u), u-v) \geq (A(v), u-v)$, and the right hand side is bounded, as a function of u, on bounded sets. This completes the proof of the first implication in Lemma 2.

Next suppose that A is pseudomonotone. By Remark 4, we have (III.5.i) satisfied. We want to show that also (III.5.ii) is satisfied. So take $u_i \rightharpoonup u$, $A(u_i) \rightharpoonup f$ weak* in E', and $\limsup\, (A(u_i), u_i - u) \leq 0$, and thus by (3.i), $\liminf\, (A(u_i), u_i - v) \geq (A(u), u-v)$ $(v \in V)$. But

$$\liminf\, (A(u_i), u_i - v) \leq \limsup\, (A(u_i), u_i - v) \leq (f, u) - (f, v),$$

and thus

$$(f, u-v) \geq (A(u), u-v) \qquad (v \in E).$$

This implies $f = A(u)$, which gives the second implication in Lemma 2, and completes the proof of Lemma 2.

Theorem IV.1. Let K be a compact, convex and nonempty subset of E. Let $A: E \to E'$ be pseudomonotone. Then there exists $u \in K$ such that (1) holds.

Proof. Without loss of generality one can assume that $0 \in K$. Let F be a finite dimensional subspace of E. By Remark 4, A_F is continuous. The set $K_F = K \cap F$ is a compact, convex and nonempty ($0 \in K_F$) subset of F. Thus by Theorem I.1, there exists $u_F \in K_F$ such that $(A(u_F), v-u_F) \geq 0$ $(v \in K_F)$. In this way we get a net $\{u_F | F = \text{ a finite dimensional subspace of } E\}$, which induces a filter. Let u_i be an ultrafilter, finer than u_F. Then by Theorem II.1, $u_i \rightharpoonup u$ for some $u \in K$. For each $v \in K$,

$$\liminf\, (A(u_i), v - u_i) \geq 0 .$$

In particular, take $v = u$. Then

$$\limsup\, (A(u_i), u_i - u) \leq 0 .$$

By pseudomonotonicity, this gives

$$0 \geq \liminf\, (A(u_i), u_i - v) \geq (A(u), u-v) \quad (v \in K),$$

and the proof of Theorem 1 is complete.

Theorem IV.2. Let K be a closed, convex, nonempty subset of E. Let $A: E \to E'$ be pseudomonotone. Moreover, suppose that there exists $v_0 \in K$, and a compact set K_1 such that

(5) $\quad u \in K, \; (A(u), v_0 - u) \geq 0 \implies u \in K_1 .$

Then there exists $u \in K \cap K_1$ such that (1) holds.

Proof. It is no loss of generality to assume that $v_0 = 0$ (shift the origin). Choose F as in the proof of Theorem 1, but replace K_F by $K_F \cap B_R$, where B_R is the ball of radius R in K. Then use Theorem I.1. to get

$u_{F,R} \in K_F \cap B_R$ satisfying

$$(A(u_{F,R}), v-u_{F,R}) \geq 0 \quad (v \in K_F \cap B_R).$$

Since $v_0 = 0 \in K_F \cap B_R$, we get from (5), $u_{F,R} \in K_F \cap K_1$. Choose some sequence $R_k \to \infty$, such that $u_{F,R} \to u_F \in K_F \to K_1$ (this can be done because $K_F \cap K_1$ is sequentially compact). Then for this u_F,

$$(A(u_F), v - u_F) \geq 0 \qquad (v \in K_F).$$

The proof is now completed just as the proof of Theorem 1, using the fact that $u_F \in K \cap K_1$, which is compact.

The following theorem generalizes Theorem 2 (cf. Remark 3):

Theorem IV.3: Let $A: E \to E'$ be pseudomonotone. Let φ be a lower semi-continuous, proper (i.e. $\varphi \not\equiv +\infty$) convex function on E. Moreover, suppose that there exists $v_0 \in \mathcal{D}(\varphi)$ such that

(6) $$(A(u), v_0-u) + \varphi(v_0) - \varphi(u) \geq 0 \Longrightarrow u \in \tilde{K},$$

where $\tilde{K}$ is a compact subset of E. Then there exists $u \in \mathcal{D}(\varphi)$ such that (2) is satisfied.

Proof. Consider the space $F = E \times \mathbb{R}$ with the product topology. Define $B: F \to F'$ by $B(u,\alpha) = (A(u),1)$ $(u \in E,\ \alpha \in \mathbb{R})$. Let $K = \{(u,\alpha) \mid \alpha \geq \varphi(u)\}$ be the "epigraph" of φ. Then K is a nonempty, closed convex set in F.

Claim. If $(u,\alpha) \in K$ satisfies

(7) $$(B(u,\alpha),\ (v,\beta) - (u,\alpha)) \geq 0 \qquad ((v,\beta) \in K),$$

then u satisfies (2), and $\alpha = \varphi(u)$.

Proof of Claim. Condition (7) means:

$$((A(u),1),\ (v-u,\beta-\alpha)) = (A(u),v-u) + \beta-\alpha \geq 0.$$

Condition (2) is trivially true if $v \notin \mathcal{D}(\varphi)$. Take $v \in \mathcal{D}(\varphi)$, and $\beta = \varphi(v)$. Then $(A(u), v-u) + \varphi(v) - \alpha \geq 0$, and since $\alpha \geq \varphi(u)$, we get $(A(u), v-u) + \varphi(v) - \varphi(u) \geq 0$. Thus (2) is satisfied. That $\alpha = \varphi(u)$ follows if one takes $v = u$. This proves the claim.

The problem (7) is of the type treated in Theorem 2, with E replaced by F. The set K satisfies the requirements of Theorem 2, and it is easy to see that $B : F \to F'$ is pseudomonotone. We still have to find the compact set K_1 used in (5). We take the special point in (5) to be $\bar{v}_0 = (v_0, \varphi(v_0))$. Then $(B(u,\alpha), \bar{v}_0 - (u,\alpha)) \geq 0 \iff (A(u), v_0 - u) + \varphi(v_0) - \alpha \geq 0 \Longrightarrow$ (since $\alpha \geq \varphi(u)$), $(A(u), v_0 - u) + \varphi(v_0) - \varphi(u) \geq 0$, and thus by (6), $u \in \tilde{K}$. This is one half of (5); we also must show that α is bounded: By the convexity and the lower semicontinuity at φ, φ is bounded from below by a continuous linear function (use e.g. the geometric form of the Hahn-Banach theorem in F). Thus $\alpha \geq \varphi(0) \geq C$, where C is some constant, independent at $u \in \tilde{K}$ ($\tilde{K}$ is weakly compact, hence bounded). To get an upper bound we use (3.ii): $(A(u), u - v_0)$ is bounded from below for $u \in \tilde{K}$, and thus $\alpha \leq \varphi(v_0) - (A(u), u - v_0)$ is bounded from above for $u \in \tilde{K}$. Thus we can take $K_1 = \tilde{K} \times I$, where I is some sufficiently large interval in formula (5). Theorem 3 now follows if one applies Theorem 2.

We shall next give a new, equivalent description of pseudomonotonicity. However, this is possible only if E is a Banach space (with its weak topology) so that one can use the Banach-Steinhaus theorem.

Definition IV. 2. $A : E \to E'$ is of type P, if

(8) $u_i \to u$, and $u_i \in$ some compact subset of $E \Longrightarrow \liminf (A(u_i), u_i - u) \geq 0$.

Example IV.1. If A is monotone, then $(A(u_i)u_i - v) \geq (A(u), u_i - u) \to 0$, so (8) holds.

Lemma IV.3. A is pseudomonotone $\Longrightarrow A$ is of type P.

Proof. Let $u_i \to 0$, and suppose to get a contradiction that $\liminf (A(u_i), u_i - u) < 0$. Then one can extract an ultrafilter u_j such that $(A(u_i), u_j - u) \to \alpha < 0$. So in particular, $\limsup (A(u_i), u_i - u) = \alpha < 0$. By pseudomonotonicity

$$\liminf (A(u_i), u_i - v) \geq (A(u), u - v) \qquad (v \in E).$$

Take $v = u$. Then

$$\alpha = \liminf (A(u_i), u_i - u) \geq (A(u), u - u) = 0 ,$$

a contradiction.

Combining this lemma with Lemma 2, we see that pseudomonotonicity implies type M_0 and type P. The converse is also true, provided one adds condition (3 ii), and also has a compactness condition on E'.

Theorem IV.4. If A is of type P and type M_0 if (3 ii) holds, and if weakly bounded sets of E' are relatively weak*-compact, then A is pseudomonotone.

Remark IV.5. The compactness condition on E' is satisfied e.g. if E is a Banach space (Banach-Alaoglu Theorem).

Proof. We must show that (3 i) holds. Note that $u_i \to u$ and $\limsup (A(u_i), u_i - u) \leq 0$ together with (8) implies $(A(u_i), u_i - u) \to 0$. Thus if in addition, $A(u_j) \rightharpoonup f$ (weak*), then we will have

$$\begin{cases} u_i \rightharpoonup u \\ A(u_i) \rightharpoonup f \\ (A(u_j),u_j) \to (f,u) \end{cases}$$

which by (III. 4. ii) implies that $A(u) = f$. Hence $(A(u_j),u_j-v) \to (f,u-v) = (A(u),u-v)$, which implies the wanted inequality $\liminf (A(u_i),u_i-v) \geq (A(u),u-v)$ $(v \in E)$. So the problem is to be able to assert that $A(u_i) \rightharpoonup f$ for some $f \in E'$.

To do this it suffices to show that (for an ultrafilter), Au_j is weak* bounded, since the weak*-compactness then gives convergence. This means that $(A(u_i),w)$ is bounded for each $w \in V$. We know that $(A(u_i), u_i-u) \to 0$. Also by (3. ii), $(A(u_i),u_j-v) \geq C(v)$ = a constant dependent on v. This implies that $(A(u_i),u-v)$ is bounded from below for each v. One can take $v = u \pm w$, then one gets $(A(u_i),w)$ bounded for each $w \in E$. This shows that $A(u_i)$ is weak* bounded, so the argument in the beginning of the proof applies.

The last part of the previous proof goes through in more generality:

Theorem IV. 5. Let $A: V \to V'$, where V is a Banach space, satisfy (3. ii). Then A is locally bounded, i. e. $\forall\, x_0 \in V\ \exists\, \varepsilon > 0 \ni \|A(x)\|_{V'}$ is bounded for $\|x-x_0\| \leq \varepsilon$.

Remark IV. 5. If A is monotone, then

$$(A(u),u-v) \geq (A(v),u-v) \quad (v \in V),$$

and the right hand side is bounded for bounded u. Thus (3. ii) holds, and so by Theorem 5, A is locally bounded.

We first prove the following lemma:

Lemma IV.4. Let $A: V \to V'$ satisfy (3.ii). Then

$$\exists\, \varepsilon, C > 0 \ni \|x\| \le \varepsilon \Rightarrow (A(x),x) \le C .$$

Proof of Lemma 4. Suppose to get a contradiction that there exists $u_n \to 0 \ni (A(u_n),u_n) \to +\infty$. Now $\|A(u_n)\|_{V'}\|u_n\|_V \ge (A(u_n),u_n)$, so

$$\left\| \frac{A(u_n)}{(A(u_n),u_n)} \right\|_{V'} \ge \frac{1}{\|u_n\|}, \quad \text{i.e.}$$

$v_n = \dfrac{A(u_n)}{(A(u_n),u_n)}$ is unbounded in V'. By the Banach-Steinhaus theorem, there exists $w \in V$ such that (v_n,w) is unbounded. Extracting a subsequence we can suppose that $(v_n,w) \to +\infty$, or that $(v_n,w) \to -\infty$. In the latter case, change $w \to -w$, so that one gets $(v_n,w) \to +\infty$. Now

$$(A(u_n),u_n-w) = (A(u_n),u_n)\left(1 - \frac{(A(u_n),w)}{(A(u_n),u_n)}\right) = (Au_n,u_n)(1-(v_n,w)) \to -\infty ,$$

because the first factor $\to +\infty$, and the second factor $\to -\infty$. But this contradicts (3.ii), and the proof is complete.

Proof of Theorem 5. By Lemma 4, there exist ε, $C > 0$ such that

$$\|x\| \le \varepsilon \Rightarrow (A(x),x) \le C .$$

Thus by (3.ii), $(A(x),w) = (A(x),x) - (A(x),x-w)$ is bounded from above for each $w \in V$, and for $\|x\| \le \varepsilon$. Replace w by $-w$, then one gets $|(A(x),w)|$ bounded for $\|x\| \le \varepsilon$, and each fixed $w \in V$. This means that the image in V' of the ball $\|x\| \le \varepsilon$ is weak*-bounded. By the Banach-Steinhaus theorem, weak* boundedness implies norm-boundedness, i.e. $\|A(x)\|_{V'} \le C$ for $\|x\| \le \varepsilon$, and the proof is complete.

In the last theorem for this subsection we specialize to a monotone hemicontinuous operator:

Theorem IV. 6. Let $A : E \to E$ be monotone hemicontinuous. Let K be a nonempty, compact and convex subset of E. Then the set of $u \in K$ satisfying (1) is nonempty, closed and convex.

Proof. Nonemptyness follows from Theorem 1, combined with Lemma 2.

Claim. u satisfies (1) $\Longleftrightarrow (A(v), v-u) \geq 0 \quad (v \in K)$. (note: $A(u)$ replaced by $A(v)$).

Proof of Claim. $(\Longrightarrow)$: $(A(v), v-u) \geq (A(u), v-u)$, so this decision is trivial. $(\Longleftarrow)$: Suppose that $(A(v), v-u) \geq 0 \quad (v \in K)$. Let $v = u + \theta(w-u)$, $\theta \in (0,1]$, and $w \in K$. Then $(A(u + \theta(w-u)), \theta(w-u)) \geq 0$. Divide by θ, and let $\theta \to 0$; using the hemicontinuity of A one then gets $(A(u)\, w-u) \geq 0 \quad (w \in K)$. This proves the claim.

Now, for each v, $C_v = \{u \in K \,|\, (A(v), v-u) \geq 0\}$ is a closed and convex subset of E. Thus the set of solutions to the variational inequality, which is given by $\bigcap_{v \in E} C_v$ is closed and convex.

Remark IV. 6. If A is strictly monotone, i.e. $u \neq v \Longrightarrow (A(u) - A(v), u-v) > 0$, then the variational inequality (1) has a unique solution: Let u_1, u_2 be solutions. Then

$$\left.\begin{array}{l} (A(u_1), v-u_1) \geq 0 \quad (v \in K), \text{ take } v = u_2 \\ (A(u_2), v-u_2) \geq 0 \quad (v \in K), \text{ take } v = u_1 \end{array}\right\} \overset{\text{(add)}}{\Longrightarrow}$$

$(A(u_1) - A(u_2), u_1-u_2) \leq 0$. By monotonicity, $(A(u_1) - A(u_2), u_1-u_2) \geq 0$, so $(A(u_1) - A(u_2), u_1-u_2) = 0$, which by strict monotonicity implies $u_1 = u_2$.

V. Minimizing Convex Functions.

Here we shall study minima of a function φ, which is supposed to be lower semicontinuous, convex and everywhere defined function on E, which is a topological vector space.

Definition V.1. $f = E \to F$ (E and F topological vector spaces) is weakly differentiable at v_0 if there exists a continuous linear operator $A: E \to F$ such that $\forall e \in E$, $\forall a \in F'$,

$$\frac{d}{dt}(f(x_0 + te), a)\Big|_{t=0} = (Ae, a)_{F,F'} .$$

We then write $f'(x_0) = A$.

Remark V.1. If E and F are Banach spaces, and f is Fréchet differentiable, i.e. if

$$\frac{1}{t\|e\|_E} \|f(x_0+te) - f(x_0) - tAe\|_{F'} \to 0 ,$$

uniformly for $\|e\|_E \leq 1$, then f is weakly differentiable.

Remark V.2. If $F = \mathbb{R}$, then $A: E \to \mathbb{R}$, i.e. $A \in E'$.

Lemma V.1. If φ is a convex, everywhere defined, and weakly differentiable function E, then φ' is monotone hemicontinuous on E.

Proof. Restrict φ to the segment $[u,v]$. Then $t \to \varphi(u + t(v-u))$ is convex in $[0,1]$. The derivative of this function at zero is by definition $(\varphi'(u), v-u)$. The derivative at 1 is $(\varphi'(v), v-u)$. By convexity, the derivative increases $(\varphi'(u), v-u) \leq (\varphi\ (v), v-u)$, so we get monotonicity.

The hemicontinuity follows from the fact that the function $\psi(t) = \varphi(u+ t(v-u))$ is everywhere differentiable, and hence $\psi'(t)$ is continuous, i.e.

$$\psi'(t) = (\varphi'(u+t(v-u)), v-u) \to (\varphi'(u), v-u) \quad (t \to 0) .$$

This completes the proof of Lemma 1.

We now get a new proof of Theorem IV.1 in this special case. We begin with a lemma:

Lemma V.2. Let φ be convex, everywhere defined and weakly differentiable on a Banach space E. Then φ is continuous.

Proof. By Lemma 1 and Remark 4.4.5, $\varphi' : E \to E'$ is locally bounded, i.e. $\forall\, x_0 \in E\ \exists\, \varepsilon, C > 0 \ni \|\varphi'(x)\|_{E'} \leq C$, for $\|x-x_0\| \leq \varepsilon$. Thus for $\|x-x_0\| \leq \varepsilon$,

$$|\varphi(x) - \varphi(x_0)| = \left|\int_0^1 (\varphi'(x_0 + s(x-x_0)), x-x_0)ds\right| \leq C\|x-x_0\|_E ,$$

and we have continuity.

Theorem V.1. Let $A = \varphi'$, where φ is a lower semicontinuous, convex and weakly differentiable function on a topological vector space E. Let K be a compact, nonempty and convex subset of E. Then there exists $u \in K$ such that $(A(u), v-u) \geq 0$ $(v \in K)$. Moreover, a point $u \in K$ is a solution if and only if φ attains its minimum on K at u.

Remark V.3. The lower semicontinuity is automatically satisfied if E is a Banach space with some topology compatible with the duality (see Lemma 2).

Proof of Theorem 1. Since φ is lower semicontinuous and K is compact, φ attains its minimum at some point $u \in K$. Take some $v \in K$, and define $\psi(t) = \varphi(u+t(v-u))$, $t \in [0,1]$. The function ψ attains the minimum in $[0,1]$ at $t = 0$, so $\psi'(0) \geq 0$. But $\psi'(0) = (\varphi'(u), v-u) = (A(u), v-u)$, and thus u is a solution to the variational inequality.

We have now proved existence of a solution, and that points where φ attains its minimum are solutions to the variational inequality. The converse direction remains. So suppose that for some $u \in K$, $(\varphi'(u), v-u) \geq 0$

$(v \in K)$. Take some $v \in K$, and define $\psi(t) = \varphi(u+t(v-u))$. Then $\psi'(0) = (\varphi'(u), v-u) \geq 0$. The convexity of ψ on $[0,1]$ then implies that $\psi(1) \geq \psi(0)$, i.e. $\varphi(v) \geq \varphi(u)$. This shows that u is a minimizing point for φ.

We shall next study a special case of the previous one, where A is linear, i.e. φ is quadratic. Here we work in a Banach space.

Theorem V.2. Let V be a reflexive Banach space, and let $A \in L(V,V')$ be coercive:

(1) there exists $\alpha > 0$ such that

$$(Au,u) \geq \alpha\|u\|_V^2 \quad (u \in V).$$

Let K be a closed, convex, nonempty subset of V, and let $f \in V'$. Then there exists $u \in K$ such that

(2) $$(Au, v-u) \geq (f, v-u) \qquad (v \in K).$$

Proof. Define a new operator $B:V \to V'$ by $Bu = Au - f$. Then by (1), B is strictly monotone. Hemicontinuity follows from the linearity of A. One can then use Theorem IV.2 to get existence, provided the set $\{u \in V | (Bu,u) \leq 0\}$ is compact (take $v_0 = 0$ in IV.5. But $(Bu,u) \leq 0 \Leftrightarrow (Au,u) \leq (f,u) \leq \|f\|_{V'}\|u\|_V \Rightarrow$ (by (1)) $\alpha\|u\|_V^2 \leq \|f\|_{V'}\|u\|_V \Rightarrow \alpha\|u\|_V \leq \|f\|_{V'}$, i.e. the set is bounded. Putting the weak topology on V, this set is then relatively compact (note: the hemicontinuity of B is independent of the topology on V). Theorem 2 now follows from Theorem IV.2.

In the case when V is a Hilbert space, we can give a different, more direct proof, which only uses Theorem 1, and not the harder Theorem IV.2. We begin with some lemmas:

Lemma V. 3. Let $A \in L(V,V')$. Then (Au,u) is convex $\iff$ $(Au,u) \geq 0$ $(u \in V)$.

Proof. (Au,u) is convex $\iff$ (Au,u) restricted to an arbitrary line in V is convex $\iff$ $\psi(t) = (A(u+tv), u+tv)$ is convex $\forall\, u,v \in V$). $\iff$ $\psi'(t)$ is nondecreasing in t $\forall\, u,v \in V$. But $\psi'(t) = (A(u+tv),v) + (Av, u+tv) =$ $= (Au,v) + 2t(Av,v) + (Av,u)$ is nondecreasing in t $\forall\, v \in V \iff (Av,v) \geq 0$ $\forall\, v \in V$. ∎

Alternative proof of Theorem 1. (When V is a Hilbert space): We begin with the special case when A is self-adjoint, i.e. $A = A^*$. By (1) and Lemma 3, the function $\varphi(0) = \frac{1}{2}(Au,u) - (f,u)$ is convex. By the computation in the proof of Lemma 3 it is also weakly differentiable (in fact even Fréchet differentiable), and $(\varphi'(u),v) = \frac{1}{2}[(Au,v) + (Av,u)] - (f,v) =$ $= \frac{1}{2}((A+A^*)u,v) - (f,v) = (Au-f,v)$, $\varphi'(u) = Au-f$. Theorem 2 is then a consequence at Theorem 1.

Before we treat the case $A \neq A^*$, we need another lemma.

Lemma V. 4. Let $S f = u$ be the solution of $(Au-f, v-u) \geq 0$ $(v \in K)$, whenever a solution exists (u is unique because of (1)). Then S is Lipschitz continuous with Lipschitz constant $1/\alpha$, on its domain.

Proof of Lemma 4. Let $u_1 = S f_1$, $u_2 = S f_2$. Then

$$(Au_1 - f_1, u_2 - u_1) \geq 0 ,$$

$$(Au_2 - f_2, u_1 - u_2) \geq 0 , \qquad \text{and}$$

$(A(u_1-u_2),(u_1-u_2)) \leq (f_1-f_2, u_1-u_2) \leq \|f_1-f_2\|_{V'} \|u_1-u_2\|_V$. By (1), $(A(u_1-u_2), u_1-u_2) \geq \alpha\|u_1-u_2\|_V^2$ so one gets $\alpha\|u_1-u_2\|_V^2 \leq \|f_1-f_2\|_{V'}\|u_1-u_2\|$, i.e. either $u_1=u_2$, or $\alpha\|u_1-u_2\|_V \leq \|f_1-f_2\|_{V'}$. This proves the lemma; taking $f_1 = f_2$ one also gets uniqueness of u).

Alternative proof of Theorem 1. (continued). We shall solve the case A $A \neq A^*$ by gradually deforming the symmetric operator $\frac{1}{2}(A+A^*)$ into the operator A. Define

$$A_\theta = (1-\theta)\,\tfrac{1}{2}(A+A^*) + \theta A \qquad (\theta \in [0,1]) .$$

Then A_θ is coercive:

$$(A_\theta u,u) = \tfrac{1}{2}(1+\theta)(Au,u) + \tfrac{1}{2}(1-\theta)(A^* u,u) = (Au,u) \geq \alpha \|u\|_V^2 .$$

Thus by the first part of the proof, the problem

(3) $$(A_\theta u-f, v-u) \geq 0 \qquad (v \in K)$$

has a solution $u \in K$ when $\theta = 0$. Now for arbitrary $\theta \in [0,1]$, define $S_\theta f = u$ as the solution of (3), whenever a solution exists. Lemma 4 can be applied with A replaced by A_θ (since A_θ is coercive), and we find that S_θ is Lipschitz continuous on its domain, with Lispchitz constant $1/\alpha$.

Claim: If $\mathcal{D} S_\theta = V'$, then $\mathcal{D} S_\eta = V'$ for $|\eta-\theta| \leq \gamma$, where $\gamma < 2\alpha/M$, $M = \|A-A^*\|_{L(V,V')}$.

Assume for the moment that this claim is true. Since $\mathcal{D} S_0 = V'$, one then gets to $A_1 = A$ in a finite number of steps, and has solved the original problem.

It remains to prove the claim. We want to solve $(A_\eta u-f, v-u) \geq 0$ $(v \in K)$. Consider this as

$$(A_\theta u - (f + A_\theta u - A_\eta u), v-u) \geq 0 ,$$

i.e. $u = S_\theta(f + A_\theta u - A_\eta u)$, if it exists. $A_\theta - A_\eta = \frac{1}{2}(\theta-\eta)(A-A^*)$, so $\|A_\theta - A_\eta\|_{L(V,V')} \leq \frac{1}{2}|\theta-\eta| M \leq \frac{1}{2}\gamma M$. Thus the function $u \to S_\theta(f + A_\theta u - A_\eta u)$ has a Lipschitz constant $\frac{1}{2}\gamma M/\alpha < 1$. One can now use the contraction mapping principle to get a solution u (note that $u \in K$, since the range of S_θ is contained in K). This completes the proof.

VI. Operators in $H_0^1(\Omega)$.

Let Ω be a bounded open set in $\mathbb{R}^N$. We define

$$V = H_0^1(\Omega) = \{u \in L^2(\Omega) \mid \frac{\partial u}{\partial x_i} \in L^2(\Omega), u\Big|_{\partial\Omega} = 0\},$$

$$H = L^2(\Omega).$$

We have seen before that the boundedness of Ω implies compactness of the injection $V \hookrightarrow H$.

$$V' = H^{-1}(\Omega) = \{f = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \,\Big|\, f_i \in L^2(\Omega)\}.$$

We study the following type of operators:

$$\text{(1)} \qquad A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} [a_i(x,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N})] + a_0(x,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N}).$$

The basic assumptions on A will be:

(2)

i) $a_j(x,u,p_1,\dots,p_N)$ is measurable in x and continuous in $u,p_1,\dots,p_N$ $(j = 1,\dots,N)$,

ii) $|a_j(x,u,p_1,\dots,p_N)| \le \lambda(x) + C(|u| + \sum_{i=1}^{N} |p_i|)$ where $\lambda \in L^2(\Omega)$, and C is some constant $(j = 1,\dots,N)$,

iii) the mapping $(p_1,\dots,p_N) \to (a_1,\dots,a_N)$, $\mathbb{R}^N \to \mathbb{R}^N$ is monotone for each fixed (x,u) or equivalently

$$\sum_{i=1}^{N} [a_i(x,u,p_1,\dots,p_N) - a_i(x,u,g_1,\dots,g_N)](p_i - g_i) \ge 0 \quad (x,u \in \mathbb{R}).$$

Lemma VI.1. Let A satisfy (1) and (2.i) and (2.ii). Then A is continuous from

Proof. It is enough to show that $u \to a_j(x,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N})$ is continuous from V strong into $L^2(\Omega)$ strong. This follows from:

Lemma VI.2. Let A satisfy (1) and (2.i) -(2.ii). Then the mapping $(u,p_1,\dots,p_N) \to a_j(x,u,p_1,\dots,p_N)$ is continuous from $(L^2(\Omega))^{N+1}$ strong $\to$ $L^2(\Omega)$ strong.

Before we can prove Lemma 2 we first need the following variant of:

Lebesgue's dominated convergence theorem. Let

$$\psi_n(x) \le v_n(x) \le \varphi_n(x) \quad (x \in \Omega); \; \psi_n(x) \to \psi(x), \quad v_n(x) \to v(x) \text{ and}$$

$\varphi_n(x) \to \varphi(x)$ a.e. in Ω. Moreover, let

$$\int_\Omega \psi_n(x)\,dx \to \int_\Omega \psi(x)sx \; ; \; \int_\Omega \varphi_n(x)dx \to \int_\Omega \varphi(x)dx \, .$$

Then $\int_\Omega v_n(x)dx \to \int_\Omega v(x)dx$.

Proof. Define $w_n(x) = v_n(x) - \psi_n(x)$, $w(x) = v(x) - \psi(x)$. Then $w_n(x) \ge 0$, and $w_n(x) \to w(x)$ a.e. in Ω. Thus by Fatou's lemma,

$$\int_\Omega v - \int_\Omega \psi = \int_\Omega w \le \liminf \int_\Omega w_n = \liminf \left[\int_\Omega v_n - \int_\Omega \psi_n\right] =$$

$$\liminf \int_\Omega v_n - \int_\Omega \psi \, . \quad \text{Hence } \liminf \int_\Omega v_n \ge \int_\Omega v \, .$$

In the same way (replace v_n by $-v_n$, ψ_n by $-\varphi_n$) one gets

$$\limsup \int_\Omega v_n \le \int v \, .$$

Thus $\lim \int_\Omega v_n = \int_\Omega v$, as claimed. ∎

Proof of Lemma 2. By (1 ii), $|a_j(x,u,p_1,\dots,p_N))| \le \lambda(x) + c(|u| + \sum |p_i|) \in L^2(\Omega)$, so we only have to prove that $a_j(x,u,p_i)$ (short for $a_j(x,u,p_1,\dots,p_N)$) is measurable to show that $a_j(x,u,p_i) \in L^2(\Omega)$. Since u and $p_1,\dots,p_N$ are measurable there exist $N+1$ sequences of simple functions $u^k,p_1^k,\dots,p_N^k \to u,p_1,\dots,p_N$ a.e. in Ω. As a is continuous in u $u,p_1,\dots,p_N$, we have $a_j(x,u^k,p_i^k) \to a_j(x,u,p_i)$ a.e. in Ω, so a is a limit a.e. of such functions. Measurability of these functions will therefore imply measurability of a. Fix k. There exists a partition of Ω into measurable sets such that on each set the functions $u^k, p_1^k, \ldots, p_N^k$ are constants. Thus by (2i) $a_j(x, u^k, p_i^k)$ is measurable on each set. Adding the pieces together one finds that $a_j(x, u^k, p_i^k)$ is measurable. Thus, as we noticed above, we get $a_j(x, u, p_i)$ measurable.

We still have to show continuity of a. Let $(u^n, p_1^n, \ldots, p_N^n) \to (u, p_1, \ldots, p_N)$ in $(L^2(\Omega))^{N+1}$ strong. Passing to a subsequence we can get pointwise convergence in all $N+1$ components a.e. in Ω. Then for this subsequence, $a_j(x, u^n, p_i^n) \to a_j(x, u, p_i)$ a.e. in Ω. Use (2.ii) and apply Lebesgue's dominated convergence theorem to $|a_j(x, u^n, p_i^n) - a_j(x, u, p_i)|^2$ to conclude that $a_j(x, u^n, p_i^n) \to a_j(x, u, p_i)$ in $L^2(\Omega)$. ∎

Now condition (2.iii) enters the picture:

<u>Lemma VI.3.</u> Let A satisfy (1)-(2), and give V its weak topology. Then A is of type P (see Definition 4.4.2).

<u>Proof.</u> Let $u_n \rightharpoonup u$ weakly in V. Then, since the imbedding $V \hookrightarrow H$ is compact, $u_n \to u$ strongly in H. By (1) and an integration by parts (denote $a_i(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N})$ by $a_i(x, u, \frac{\partial u}{\partial x_j})$)

$$(A(u_n), u_n - u) = \sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \frac{\partial u_n}{\partial x_j})(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx + \int_\Omega a_0(x, u_n, \frac{\partial u_n}{\partial x_j})(u_n - u)dx.$$

We have to show that $\liminf (A(u_n), u_n - u) \geq 0$. By the strong convergence in $H = L^2(\Omega)$, the last term goes to 0 as $n \to \infty$. By (2 iii),

$$\int_\Omega \sum_{i=1}^{N} [(a_i(x, u_n, \frac{\partial u_n}{\partial x_j})) - a_i(x, u_n, \frac{\partial u}{\partial x_j})] (\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx \geq 0 .$$

Thus

$$\liminf\,(A(u_n), u_n - u) \geq \liminf \int_\Omega \sum_{i=1}^{N} a_i(x, u_n, \frac{\partial u}{\partial x_j})(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx.$$

However, this is zero because $\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \rightharpoonup 0$ weakly in $L^2(\Omega)$ and $a_i(x, u_n, \frac{\partial u}{\partial x_j}) \to a_i(x, u, \frac{\partial u}{\partial x_j})$ strongly in $L^2(\Omega)$ by Lemma 2. ■

We eventually want to know that A is pseudomonotone. We have already got type P, but we still need type M_0, to get pseudomonotonicity (see Theorem IV.4). Type M_0 does not follow automatically; one needs something more, e.g.

(3) $\begin{cases} \text{There exists } \alpha > 0 \text{ such that} \\ \sum_{i=1}^{N} [(a_i(x,u,p_1,\ldots,p_N) - a_i(x,u,g_1,\ldots,g_n)](p_i - g_i) \geq \alpha \sum_{i=1}^{N} |p_i - g_i|^2. \end{cases}$

<u>Lemma VI.4.</u> Let A satisfy (1)-(3). Let $u_n \rightharpoonup u$ weakly in V, and $(A(u_n), u_n - u) \to 0$. Then $u_n \to u$ in V (strongly).

This auxiliary lemma will be used in the proof of Lemma 5 below.

<u>Proof of Lemma 4.</u> Using the same arguments as in the proof of Lemma 3, and then adding the condition $(A(u_n), u_n - u) \to 0$ and the assumption (3), one gets

$$0 = \lim_{n\to\infty} (A(u_n), u_n - u) = \lim_{n\to\infty} \{ \int_\Omega a_0(x, u_n, \frac{\partial u_n}{\partial x_j})(u_n - u)dx$$

$$+ \int_\Omega \sum_{i=1}^{N} a_i(x, u_n, \frac{\partial u}{\partial x_j})(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx$$

$$+ \int_\Omega \sum_{i=1}^{N} [a_i(x, u_n, \frac{\partial u_n}{\partial x_j}) - a_i(x, u_n, \frac{\partial u}{\partial x_j})](\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx \}$$

$$= \lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} [a_i(x, u_n, \frac{\partial u_n}{\partial x_j}) \quad a_i(x, u_n, \frac{\partial u}{\partial x_j})](\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i})dx$$

$$\geq \alpha \limsup \int_\Omega \sum_{i=1} |\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}|^2 dx .$$

Thus we get strong convergence in the derivatives. We already had strong convergence of u in $L^2(\Omega)$, so we get strong convergence in V.

Lemma VI.5. Let A satisfy (1)-(3). Then A is of type M (and hence of type M_0) from V weak into V'.

Proof. Take

$$(4)\quad \begin{cases} u_n \to u \text{ weak in } V, \\ A(u_n) \to f \text{ weak}^* \text{ in } V' \ (= \text{weak in } V') \\ \limsup\, (A(u_n), u_n) \le (f,u). \end{cases}$$

We want to show that $A(u) = f$. By Lemma 3, $\liminf\, (A(u_n), u_n - u) \ge 0$. This together with (4) gives $\lim\, (A(u_n), u_n - u) = 0$. Thus by Lemma 4, $v_n \to v$ strongly in V. Hence by Lemma 1, $A(u_n) \to A(u)$ strongly in V'. This together with (4) gives $A(u) = f$, and completes the proof. ■

The condition (3) is not the only way to get type M_0. One can do without it if $a_0(x,u,p_1,\ldots,p_N)$ is nice:

$$(5)\quad \begin{cases} \text{The mapping } u \to a_0(x,u,\frac{\partial u}{\partial x_j}) \text{ is continuous from } V \text{ weak into} \\ L^2(\Omega) \text{ weak.} \end{cases}$$

Lemma VI.6. Let A satisfy (1), (2) and (5). Then A is of type M from V weak into V'.

Proof. Let (4) hold. Then by (5), $a_0(x,u_n,\frac{\partial u_n}{\partial x_j}) \to a_0(x,u,\frac{\partial u}{\partial x_j})$ weakly in $L^2(\Omega)$. Since $u_n \to u$ strongly in $L^2(\Omega)$, we get as before

$$\int_\Omega a_0(x,u_n,\frac{\partial u_n}{\partial x_j})(u_n - u)dx \to 0 .$$

Thus if we define a new operator $\tilde{A}$ by

$$\tilde{A}(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}[a_i(x,u,\frac{\partial u}{\partial x_j})] = A(u) - a_0(x,u,\frac{\partial u}{\partial x_j}),$$

then (4) holds with A replaced by $\tilde{A}$, and f replaced by $f-a_0(x,u,\frac{\partial u}{\partial x})$. This shows that there is no loss of generality in assuming that $a_0 \equiv 0$.

As before we get $(A(u_n),u_n-u) \to 0$. By (2 ii), $a_i(u_n)$ is bounded in $L^2(\Omega)$. Passing to an ultrafilter, again denoted by u_n, we get $a_i(u_n) \rightharpoonup h_i$ weakly in $L^2(\Omega)$, for some functions h_i . This means that

$$0 = \lim \int_\Omega \sum_{i=1}^N a_i(x,u_n,\frac{\partial u_n}{\partial x_j})(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i})dx$$

$$= \lim \int_\Omega \sum_{i=1}^N a_i(x,u_n,\frac{\partial u_n}{\partial x_j})\frac{\partial u_n}{\partial x_i}dx - \int_\Omega \sum_{i=1}^N h_i\frac{\partial u}{\partial x_i}dx ,$$

i.e. $\int_\Omega \sum_{i=1}^N a_i(x,u_n,\frac{\partial u_n}{\partial x_j})\frac{\partial u_n}{\partial x_i}dx \to \int_\Omega \sum_{i=1}^N h_i\frac{\partial u}{\partial x_i}dx$. By (2 iii) , $\int_\Omega \sum_{i=1}^N (a_i(x,u_n,\frac{\partial u_n}{\partial x_j}) - a_i(x,u_n,p_j))(\frac{\partial u_n}{\partial x_i}-p_i)dx \geq 0$, for arbitrary $p_1,\ldots,p_N$ $p_1,\ldots,p_N \in L^2(\Omega)$. Passing to the limit (note that $u_n \to u$ strongly in $L^2(\Omega)$, and using Lemma 2 to take care of the $a_i(x,u_n,p_j)\frac{\partial u_n}{\partial x_i}$ term) one gets

$$\int_\Omega \sum_{i=1}^N (h_i - a_i(x,u,p_j))(\frac{\partial u}{\partial x_i} - p_i) \geq 0 .$$

Take $p_i = \frac{\partial u}{\partial x_i} + tg_i$; then the above inequality becomes

$$t \int_\Omega \sum_{i=1}^N (h_i - a_i(x,u,\frac{\partial u}{\partial x_j} + tg_j))g_i \geq 0 \qquad (g_i \in L^2) .$$

Divide by $t > 0$, let $t \to 0+$, and use Lemma 2. This gives

$$\int_\Omega \sum_{i=1}^N (h_i - a_i(x,u,\frac{\partial u}{\partial x_j}))g_i \geq 0 \qquad (g_i \in L^2) .$$

Replace g_i by $-g_i$, then one gets the opposite inequality. This shows that $a_i(x,u,\frac{\partial u}{\partial x_j}) = h_i$, and completes the proof. ∎

To apply the existence theorems for solutions of $(A(u),v-u) \geq 0$ $(v \in K)$ we still need one of the two following conditions: Either K is convex and compact, or K is closed and convex, and

$$(6)\quad \begin{cases} \text{there exists } v_0 \in K \text{ and a compact set } K_0 \text{ such that} \\ (A(u), v_0 - u) \geq 0 \Rightarrow u \in K_0 \end{cases}$$

(see Theorems IV.1-IV.2). The condition (6) is e.g. implied by:

$$(7)\quad \begin{cases} \text{there exists } \beta > 0 \text{ and } f \in L^1(\Omega) \text{ such that} \\ a_0(x,u,p_j)u + \sum_{i=1}^{N} a_i(x,u,p_j)p_i \geq f(x) + \beta \sum_{i=1}^{N} |p_i|^2 . \end{cases}$$

Lemma VI.7. Let A satisfy (1), (2) and (7). Give V its weak topology. Then (6) holds.

Proof. It suffices to show that $(A(u), v_0 - u) \geq 0$ implies boundedness of u in V. Write

$$(A(u), v_0 - u) = \int_\Omega a_0(x,u,\frac{\partial u}{\partial x_j})(v_0 - u)$$
$$+ \int_\Omega \sum_{i=1}^{N} a_i(x,u,\frac{\partial u}{\partial x_j})(\frac{\partial v_0}{\partial x_i} - \frac{\partial u}{\partial x_i}) \geq 0 .$$

By (7),

$$\beta \int_\Omega \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|^2 \leq \int_\Omega \sum_{i=1}^{N} a_i(x,u,\frac{\partial u}{\partial x_j})\frac{\partial u}{\partial x_i} + \int_\Omega a_0(x,u,\frac{\partial u}{\partial x_j})u - \int_\Omega f(x) .$$

Thus $\leq \int_\Omega \sum_{i=1}^{N} a_i(x,u,\frac{\partial u}{\partial x_j})\frac{\partial v_0}{\partial x_i} + a_0(x,u,\frac{\partial u}{\partial x_j})\, v_0 - f$

$$\beta \|u\|_V^2 \leq \sum_{i=1}^{N} |\frac{\partial v_0}{\partial x_i}|_H |a_i(x,u,\frac{\partial u}{\partial x_j})|_H$$
$$+ |v_0|_H\, |a_0(x,u,\frac{\partial u}{\partial x_j})|_H + C .$$

Using also (2 ii) one gets

$$\beta \|u\|_V^2 \leq c_1 + c_2 \|u\|_V ,$$

where c_1, c_2 are constants. This gives boundedness of $\|u\|_V$, and completes the proof.

We shall now look closer at the particular operator $-\Delta u + u^3$ which was studied in connection with the "nonlinear wave equation". Here we consider the stationary problem

$$(3) \qquad \begin{cases} -\Delta u + u^3 = f , \\ u\big|_{\partial\Omega} = 0 . \end{cases}$$

The term $-\Delta$ is good, but u^3 causes problems, because it does not satisfy $|u|^3 \leq \lambda(x) + C(|u| + \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|)$. We must change the space V :

$$V = H_0^1(\Omega) \cap L^4(\Omega)$$
$$V' = H^{-1}(\Omega) + L^{4/3}(\Omega) .$$

Then $A(u) = -\Delta u + u^3$ maps $V \to V'$.

Lemma VI.8. A is monotone hemicontinuous on V .

Proof. Monotonicity follows from the fact that

$$(A(u_1) - A(u_2), u_1 - u_2) = \int_\Omega \sum_{i=1}^{N} (\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i})^2 dx$$
$$+ \int_\Omega (u_1^3 - u_2^3)(u_1 - u_2) dx \geq 0 .$$

A is in fact more than hemicontinuous: It is continuous from V strong into V' strong: Let $u_n \to u$ strongly in V, i.e. $u_n \to u$ in $H_0^1(\Omega)$ and $u_n \to$ in $L^4(\Omega)$. Then $-\Delta u_n \to -\Delta u$ in $H^{-1}(\Omega)$ and $u_n^3 \to u^3$ in $L^{4/3}(\Omega)$, which gives strong convergence in V'.

Lemma VI.9. A is coercive:

$$(A(u_1) - A(u_2), u_1 - u_2) \geq \|u_1 - u_2\|^2_{H_0^1(\Omega)} + \tfrac{1}{4}|u_1 - u_2|^4_{L^4(\Omega)} .$$

Proof. It suffices to check that $(a^3 - b^3)(a-b) \geq \frac{1}{4}|a-b|^4$ for $\forall\, a, b \in \mathbb{R}$. This is easy: $(a^3-b^3)(a-b) = (a-b)^2(a^2 + ab + b^2) \geq \frac{1}{2}(a-b)^2(a^2+b^2)$ $\geq \frac{1}{4}(a-b)^2(a^2+b^2 - 2ab) = \frac{1}{4}(a-b)^4$.

This is enough to imply (6). Take any $v_0 \in K$, and let $(A(u), v_0 - u) \geq 0$.

Then

$$\|u-v_0\|^2_{H^1_0(\Omega)} + \tfrac{1}{4}|u-v_0|^4_{L^4(\Omega)} \leq$$

$$(A(u) - A(v_0), u-v_0) \leq -A(v_0), u-v_0)$$

$$\leq C(\|u-v_0\|_{H^1_0(\Omega)} + |u-v_0|_{L^4(\Omega)}),$$

so one gets boundedness of $\|u\|_V$.

Theorem VI.1. (Existence and uniqueness): If K is closed, convex and nonempty in $V = H^1_0(\Omega) \cap L^4(\Omega)$ and $f' \in V'$, then there exists a unique $u \in K$ such that

$$(-\Delta u + u^3 - f, v-u) \geq 0 \qquad (v \in K).$$

Existence follows from Theorem IV.2 combined with Lemma 8 and the fact that (6) holds. Uniqueness is a consequence of Lemma 9 and Remark IV.6.

VII. Continuous Dependence of Solutions.

We begin with a concrete example, i.e. the operator $A(u) = -\Delta u + u^3$ from the previous section.

Example VII.1. The mapping $f \rightarrow u$ given by Theorem VI.1 is locally Hölder continuous.

Proof. Take two functions f_1 and f_2 with corresponding solutions u_1 and u_2. Then

$$(A(u_1) - f_1, u_2 - u_1) \geq 0,$$

$$(A(u_2) - f_2, u_1 - u_2) \geq 0,$$

and thus

$$(A(u_1) - A(u_2), u_1-u_2) \leq (f_1-f_2, u_1-u_2) .$$

This gives (use also Lemma VI.9)

$$\|u_1-u_2\|^2_{H^1_0} + \tfrac{1}{4}|u_1-u_2|^4_{L^4(\Omega)} \leq \|f_1-f_2\|_{V'} \|u_1-u_2\|_V .$$

Write $\|u_1-u_2\|_{H^1_0} = x$, $|u_1-u_2|_{L^4(\Omega)} = y$, $\|f_1-f_2\|_{V'} = a$, then this becomes ($\|\cdot\|_V = \|\cdot\|_{H^1_0} + |\ |_{L^4(\Omega)}$)

$$x^2 + \tfrac{1}{4}y^4 \leq a(x+y) \Longrightarrow$$

$$(x^2-ax) + (\tfrac{1}{4}y^4 - ay) \leq 0 \Longrightarrow$$

$$(x - a/2)^2 + (\tfrac{1}{4}y^4 - ay + \tfrac{3}{4}a^{4/3}) \leq \tfrac{1}{4}a^2 + \tfrac{3}{4}a^{4/3} .$$

The function $\frac{1}{4}y^4 - ay + \frac{3}{4}a^{4/3}$ is nonnegative (its minimum is obtained at $y = a^{1/3}$) and thus

$$x \leq a/2 + (\tfrac{1}{4}a^2 + \tfrac{3}{4}a^{4/3})^{\frac{1}{2}} ,$$

which gives a local Hölder bound for x in terms of a (not global because of the different exponents on the right hand side). A similar expression can be obtained for y. One gets

$$x = O(a^{2/3}) \qquad (a \to 0) ,$$

$$y = O(a^{1/3}) \qquad (a \to 0).$$

This proves the statement of Example 1.

We go on to more general cases:

<u>Theorem VII.1.</u> Let V be reflexive and let $A: V \to V'$ be of type M_0 and satisfy

$$\lim_{\|u\|_V \to \infty} \frac{(Au, u)}{\|u\|_V} = \infty .$$

Also suppose that A is one-to-one. Then the mapping $A^{-1}: f \to u$ is continuous from V' strong into V weak.

Proof. Let $f_n \to f$ in V', and let $A(u_n) = f_n$. Then by the coercivity condition, u_n belongs to a bounded set of V. One can extract a subsequence $u_k \rightharpoonup u$ in V weak. Then

$$\begin{cases} u_k \rightharpoonup u \text{ in } V \text{ weak}, \\ A(u_k) = f_k \to f \text{ in } V' \text{ strong}, \\ (A(u_k, u) \to (f,u), \end{cases}$$

which implies $A(u) = f$ because A is of type M_0. The uniqueness of the solution $A(u) = f$ then implies that the complete sequence u_n tends to u in V weak.

A similar theorem also holds for the variational inequality:

Theorem VII. 2. Let A be pseudomonotone. Let K be compact, convex and nonempty, and let the variational inequality

$$(A(u) - f, v-u) \geq 0 \qquad (v \in K)$$

have a unique solution $u \in K$. Then the mapping $f \to u$ is continuous from V' strong into V weak.

Proof. Let $f_n \to f$ strongly in V', and define the corresponding $u_n \in K$ as above. Extract a subsequence $u_k \rightharpoonup u$ weakly in V. We have

$$(A(u_k) - f_k, v-u_k) \geq 0 \ (v \in K),$$

so in particular

$$(A(u_k) - f_k, u - u_k) \geq 0 .$$

This gives (since $f_k \to f$ in V')

$$\limsup (A(u_k), u_k-u) \leq 0 .$$

The pseudomonotonicity then gives

$$\liminf\,(A(u_k), u_k - v) \geq (A(u), u-v) \qquad (v \in K).$$

But $\quad (f, u-v) = \lim\,(f_k, v_k - v) \geq \liminf\,(A(u_k), u_k - v) \geq (A(u), u-v)$,

so u is the solution corresponding to f . The uniqueness of the solution again gives convergence of the complete sequence.

VIII. The Method of Penalization.

Here we shall give a different proof of the existence of a solution of the variational inequality. The idea is to approach the given problem by a sequence of problems, which can be solved easily.

The requirement

$$(A(u), v-u) \geq 0 \qquad (v \in K) ,$$

where $K \subset V$ is some closed convex set, can be formulated as

(1) $$(A(u), v-u) + \varphi(v) - \varphi(u) \geq 0 \qquad (v \in V) ,$$

where

$$\varphi(u) = \begin{cases} 0 , & u \in K \\ \infty , & u \notin K . \end{cases}$$

Similarily, $(A(u), v-u) \geq (f, v-u)$ becomes

(2) $$(A(u), v-u) + (\varphi(v) + f, v)) - (\varphi(u) + (f, u)) \geq 0 .$$

By Theorem VII.2, if $f_n \to f$ strongly in V' , and the other hypothesis of that theorem hold, then the solutions $u_n \rightharpoonup u$ weakly in V .

Taking a sequence f_n in (2) means that one replaces the convex function $\varphi(v) + (f, v)$ by a sequence of functions $\varphi(v) + (f_n, v)$. More generally, one can replace the function φ in (1) by some possibly different sequence φ_n of convex, lower semicontinuous functions, and solve

(3) $$(A(u_n), v-u_n) + \varphi_n(v) - \varphi_n(u) \geq 0 \qquad (v \in V).$$

A question arises: What conditions on φ_n will make the solutions u_n converge to the solution u of (1)? (Note that (2) can be considered as a special case of (1), by changing the function φ.)

Theorem VIII.1. Let

i) $\varphi_n(v) \to \varphi(v) \qquad (v \in \mathcal{D}\varphi)$,

ii) $u_n \rightharpoonup u \Rightarrow \varphi(u) \leq \liminf \varphi_n(u_n)$,

iii) The solutions u_n of (3) remain bounded,

iv) A be pseudomonotone and bounded.

Then there exists a subsequence $u_k \rightharpoonup u$ weakly in V, where u is a solution of (2).

Remark VIII.1. If $\varphi_n(v) = \psi(v) + (f_n, v)$, then i) holds if $f_n \rightharpoonup f$ weakly, and ii) holds if $f_n \to f$ strongly.

Proof of Theorem 1. Since the sequence u_n is bounded we can extract a weakly convergent subsequence $u_k \rightharpoonup u$. By the boundedness of A, $A(u_k)$ is bounded in V'.

Take some $v \in \mathcal{D}\varphi$. Then $v \in \mathcal{D}\varphi_k$ for sufficiently large k.

$$\varphi_k(u_k) \leq \varphi_k(v) + (A(u_k), v-u_k),$$

so $\varphi_k(u_k)$ is bounded from above. By (ii), $u \in \mathcal{D}\varphi$, and

$$\varphi(u) \leq \liminf [\varphi_k(v) + (A(u_k), v-u_k)] \qquad (v \in \mathcal{D}\varphi).$$

Take $v = u$. Then one gets, using (i)

$$\limsup (A(u_k), u_k - u) \leq 0.$$

By pseudomonotonicity,

$$\liminf (A(u_k), u_k - v) \geq (A(u), u-v) \qquad (v \in V).$$

Thus for $v \in \text{Dom}\,\varphi$,

$$\begin{aligned}\varphi(u) &\leq \varphi(v) + \liminf\,(A(u_k),\ v-u_k) \\ &\leq \varphi(v) + \limsup\,(A(u_k),\ v-u_k) \\ &= \varphi(v) - \liminf\,(A(u_k), u_k - v) \\ &\leq \varphi(v) - (A(u), u-v).\end{aligned}$$

This completes the proof.

Example VIII.1. Take K_n decreasing, convex, closed. Let $\bigcap K_n = K \neq \phi$.
Define

$$\varphi_n(v) = \begin{cases} (f,v), & \text{if } v \in K_n \\ \infty, & \text{if } v \notin K_n \end{cases}$$

Solve (3). This is equivalent to solving

$$(A(u_n), v-u_n) \geq (f, v-u_n) \qquad (v \in K_n)$$

on K_n. Clearly Condition i) is true.

That also ii) holds is not too hard to see either: Let $u_n \rightharpoonup u$, and $\liminf \varphi_n(u_n) < \infty$. Then $u_n \in K_n$ for all sufficiently large n, and therefore $u \in \bigcap K_n = K$. This gives $\varphi(u) = (f,u) = \lim \varphi_n(u_n)$, so ii) holds.

We now come to the method of penalization.

Theorem VIII.2. Let V be reflexive. Let $K \subset V$ be closed, convex and nonempty. Let $A: V \to V'$ be pseudomonotone, bounded and coercive: $\exists\, v_0 \in K \ni (A(u), v_0 - u) \geq 0 \Longrightarrow u$ belongs to a bounded set. Moreover, suppose that there exists $B: V \to V'$ monotone hemicontinuous such that $K = \{u \mid B(u) = 0\}$. For $\varepsilon > 0$, define u_ε as the solution of

$$A(u_\varepsilon) + \frac{1}{\varepsilon} B(u_\varepsilon) = 0$$

(which does exist). Then there exists a sequence $u_\eta \rightharpoonup u$ $(\eta \to 0)$ such that

$u \in K$ is a solution of

$$(A(u), v-u) \geq 0 \quad (v \in K) .$$

Note that we already know from Theorem IV. 2 that a solution u of the variational inequality does exist. However, this gives a new existence proof, under the new, somewhat more restrictive conditions.

The proof begins with a lemma:

Lemma VIII. 1. Let V be reflexive. Let $A, B: V \to V'$ be pseudomonotone, and let A be bounded. Then $A+B$ is pseudomonotone.

Proof of Lemma 1. By Lemmas IV. 2 and IV. 3, A and B are of type M and type P. This trivially implies that $A+B$ is of type P (see Definition IV. 2).

Claim. $A+B$ is of type M .

Proof of Claim. Let

$$\begin{cases} u_i \rightharpoonup u \\ A(u_i) + B(u_i) \rightharpoonup f \\ \limsup\, (A(u_i) + B(u_i), u_i) \leq (f, 0) . \end{cases}$$

We want to show that $A(u) + B(u) = f$. By the boundedness of A , we can get a subsequence u_k such that $A(u_k) \rightharpoonup g$. This gives $B(u_k) \rightharpoonup f-g$. Since B is of type P we get

$$\liminf\, (B(u_k), u_k) \geq (f-g, u) \Longrightarrow$$

$$\limsup\, (A(u_k), u_k)$$

$$\leq \limsup\, (A(u_k) + B(u_k), u_k) - \liminf\, (B(u_k), u_k)$$

$$\leq (f, u) - (f-g, u) = (g, u) .$$

Thus $A(u) = g$.

In the same way one shows that $B(u) = f-g$. This proves the claim. The Lemma now follows from Theorem IV. 4.

<u>Proof of Theorem 2</u>. By Lemma 1, $A + \frac{1}{\varepsilon}B$ is pseudomonotone. Thus in particular, $A + \frac{1}{\varepsilon}B$ is of type M_0. Take v_0 as in the statement of Theorem 2. Suppose that $(A(u) + \frac{1}{\varepsilon}B(u), u-v_0) \leq 0$. Since $v_0 \in K$ we get $B(v_0) = 0$, and so by monotonicity,

$$(B(u), u-v_0) \geq (B(v_0), u-v_0) \geq 0 .$$

Hence

$$(A(u), u-v_0) \leq 0$$

which implies that u is contained in a bounded set. Take v_0 to be the origin in V, and apply Theorem III. 2. This yields the existence of $u_\varepsilon \in V$ satisfying

$$A(u_\varepsilon) + \frac{1}{\varepsilon}B(u_\varepsilon) = 0 \implies$$
$$(A(u_\varepsilon) + \frac{1}{\varepsilon}B(u_\varepsilon), u_\varepsilon - v_0) = 0 \implies \text{(as above)}$$
$$(A(u_\varepsilon), u_\varepsilon - v_0) \leq 0 ,$$

and hence $u_\varepsilon \in$ bounded set. Extract $u_\eta \to u$ $(\eta \to 0)$. Since $A(u_\eta)$ is bounded we get $B(u_\eta) = -\eta A(u_\eta) \to 0$ (strongly). We thus have

$$\left.\begin{cases} u_\eta \to u \\ B(u_\eta) \to 0 \\ (B(u_\eta), u_\eta) \to 0 \end{cases}\right\} \implies B(u) = 0 \implies u \in K .$$

Take any $v \in K$, and note that $B(v) = 0$.

$$(A(u_\eta), v-u_\eta) = -\frac{1}{\eta}(B(u_\eta), v - u_\eta)$$
$$= \frac{1}{\eta}(B(u_\eta) - B(v), u_\eta - v) \geq 0 \implies$$
$$\liminf (A(u_\eta), v-u_\eta) \geq 0 .$$

Substitute v for u. Then

$$u_\eta \to u ,$$

$$\liminf\, (A(u_\eta), u - u_\eta) \geq 0$$

and so by pseudomonotonicity,

$$0 \geq \liminf -\frac{1}{\eta}(B(v) - B(u_\eta), v - u_\eta)$$

$$= \liminf\, (A(u_\eta), u_r - v) \geq (A(u), u-v) .$$

This gives $(A(u),u-v) \leq 0$, and completes the proof.

IX. Variational inequalities related to the stationary heat equation.

We begin with the general equation on dynamics:

$\rho(x,t)$ = density (1 unknown)

$u(x,t)$ = velocity (3 unknowns in $\mathbb{R}^3$)

$\sigma_{ij}(x,t)$ = stress tensor (9, or rather 6 unknowns)

$e(x,t)$ = internal energy (≈ temperature) (1 unknown)

$\bar{g}(x,t)$ = energy flow (3 unknowns)

Equations

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial}{\partial x_i}(\rho u_i) = 0 .$$

Conservation of momentum:

$$\rho\left(\frac{\partial u_i}{\partial t} + \sum_j \frac{\partial u_i}{\partial x_j} u_j\right) = f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i = 1,2,3),$$

where $f = (f_1, f_2, f_3)$ is an external force.

Conservation of angular momentum:

$$\sigma_{ij} = \sigma_{ji} , \qquad 1 \leq i,j \leq 3 .$$

Conservation of energy (w = potential energy):

$$\sigma\left(\frac{\partial e}{\partial t} + \sum_j \frac{\partial e}{\partial x_j} u_j\right) = \rho w - \sum_i \frac{\partial g_i}{\partial x_i} + \sum_{ij} \sigma_{ij} \frac{\partial u_i}{\partial x_j} .$$

Apply these equations to a solid: put $u = 0$, ρ = constant. The second equation gives the tension, which we do not care about at the present. The last equation becomes

$$(1) \qquad \rho \frac{\partial e}{\partial t} = \rho w - \sum_i \frac{\partial g_i}{\partial x_i} .$$

The classical hypothesis on the material is that

$$(2) \qquad g_i = - \sum_j a_{ij} \frac{\partial e}{\partial x_j} ,$$

where (a_{ij}) is a positive definite matrix i.e. $\sum a_{ij} \xi_i \xi_j \geq 0$ $(\xi \in \mathbb{R}^3)$. Roughly this means that the heat flow tends to even out the temperature, but it doesn't have to flow in the direction of the negative gradient, because the material can have different properties in different directions. Let $a_{ij} = a_{ij}(x)$. Then the positive definitions of (a_{ij}) means

$$(g, \nabla e) \leq 0 .$$

More generally, one can suppose that

$$g_i = -a_i(x,e, \frac{\partial e}{\partial x_j}) ,$$

with

$$\sum_i a_i(x,e,\xi_1,\xi_2,\xi_3)\xi_i \geq 0 .$$

The needed mathematical assumption is

$$\sum_i [a_i(x,e,\xi) - a_i(x,e,\eta)](\xi_i - \eta_i) \geq 0$$

which gives monotonicity. Note that this is automatically satisfied in the matrix case (a_{ij}). For simplicity we only use below the matrix form.

Substituting (2) into (1) we get the heat equation

$$\frac{\partial e}{\partial t} - \sum_{ij} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial e}{\partial x_j}) = w .$$

If the solid is isotropic, then

$$a_{ij} = a(e)\,\delta_{ij}$$

and one gets

$$\frac{\partial e}{\partial t} - \sum \frac{\partial}{\partial x_i}\left(a(e)\frac{\partial e}{\partial x_i}\right).$$

Frequently one even takes $a(e) =$ constant > 0, independent of e.

We also need some boundary conditions. There are three different types of reasonable conditions:

1) give temperature on the boundary

2) give the flux of heat at the boundary:

$\sum_j a_{ij} \frac{\partial e}{\partial x_j} \cos(n,x) = -(g \cdot n)$ given, where n is the outward normal. We define $\frac{\partial e}{\partial \nu_A} = -(g \cdot n)$. $\frac{\partial e}{\partial \nu_A} > 0$ means transfer of heat into our set Ω.

For an isotropic material $\frac{\partial e}{\partial \nu_A}$ is proportional to $\frac{\partial e}{\partial n}$ (normal derivative).

(3) give some relation between $u|_{\partial\Omega}$ and $\frac{\partial e}{\partial \nu_A}$, e.g.

$$e(x) \leq e_0 \implies \frac{\partial e}{\partial \nu_A} = \text{function of } e(x) \quad \text{(heating)}$$

$$e(x) > e_0 \implies \frac{\partial e}{\partial \nu_A} = 0 \quad \text{(no flux of heat).}$$

We begin by studying the stationary problem for an isotropic solid in case 3):

$$\begin{cases} -\sum \frac{\partial}{\partial x_i}\left(a(e)\frac{\partial e}{\partial x_i}\right) = w & (x \in \Omega) \\ \frac{\partial e}{\partial n} + F(x,e) = 0 & (x \in \partial\Omega) \end{cases}$$

We shall see that this problem has a solution, provided $a(e) \geq \alpha > 0$, and $F(x,e)$ is increasing in e. We assume $\partial\Omega$ to be sufficiently smooth.

We solve the problem by transferring it into a problem exclusively on $\partial\Omega$. For simplicity, take the case

$$\begin{cases} -\Delta u - w & \text{in } \Omega \\ \frac{\partial u}{\partial n} + F(x,u) = 0 & \text{on } \partial\Omega \end{cases}$$

The procedure is roughly the following:

First solve the problem

$$(3) \qquad \begin{cases} -\Delta u = w \,, \\ u\big|_{\partial\Omega} = v \,, \end{cases}$$

where v is a given function on $\partial\Omega$. If $w \in L^2(\Omega)$, $v \in H^{\frac{1}{2}}(\partial\Omega)$ (= restriction of $H^1(\Omega)$ to $\partial\Omega$), and Ω is a bounded, open, regular set, then there exists a unique $u \in H^1(\Omega)$ solving (3). One can then define an affine mapping $T : H^{\frac{1}{2}}(\partial\Omega) \to (H^{\frac{1}{2}}(\partial\Omega))'$. This mapping is monotone. The original problem is equivalent to

$$Tu + F(x,u) = 0 \,.$$

We now look at the details of the transformation.

<u>Step 1.</u> The equation (3).

For $v \in H^{\frac{1}{2}}(\partial\Omega) = \{\text{restr. to } \partial\Omega \text{ of } H^1(\Omega)\}$, we define

$$\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{u|_{\partial\Omega}=v} \|u\|_{H^1(\Omega)} \,.$$

Take any $u_0 \in H^1(\Omega) \ni u_0|_{\partial\Omega} = v$. Then $u - u_0$ satisfies

$$\begin{cases} -\Delta(u-u_0) = w + \Delta u_0 \in H^{-1}(\Omega) = (H_0^1(\Omega))' \\ u - u_0 \in H_0^1(\Omega) \,. \end{cases}$$

The operator $-\Delta$ is an isomorphism between H_0^1 and H^{-1}, it is in fact a canonical isomorphism because $(-\Delta u, v)$ = the scalar product of u and v in H_0^1 (use the Lax-Milgram lemma; the same is also true for other more general monotone hemicontinuous and coercive operators). This means that the equation

$$-\Delta\tilde{u} = w + \Delta u_0$$

has a solution $\tilde{u} \in H_0^1(\Omega)$. Defining $u = u_0 + \tilde{u}$ we get a solution of (3).

Step 2. Define $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}$. This would be easy if $u \in H^2(\Omega)$, but that is not the case.

Lemma IX.1. Define $X = \{u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega)\}$, and give X the (Hilbert space) norm

$$\|u\|_X^2 = \|u\|_{H^1(\Omega)}^2 + |\Delta u|_{L^2(\Omega)}^2 .$$

Then $D(\bar{\Omega})$ is dense in X.

Remark IX.1. The solution u of (3) satisfies $\Delta u = w \in L^2(\Omega)$, so $u \in X$.

Proof of Lemma 1. (Outline, cf. the chapter on Navier-Stokes): Use a partition of unity and a change of coordinates to reduce the proof to the two cases $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{R}_+^n$, where the proof is straightforward.

Lemma IX.2. The map $u \to \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} : X \to (H^{\frac{1}{2}}(\Omega))'$, define by extending the corresponding map defined on $D(\bar{\Omega})$, is linear and continuous.

Proof. Let $v \in H^{\frac{1}{2}}(\Omega)$, and choose $v_0 \in H^1(\Omega) \ni u_0|_{\partial\Omega} = V$. This can be done in a continuous way: the mapping $v \to u_0$ is continuous from $H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$.

Now let $u \in D(\bar{\Omega})$. Then by an integration by parts

$$\int_\Omega (-\Delta u, u_0) = \int_\Omega \sum_{i=1}^3 \frac{\partial u_0}{\partial x_i} \frac{\partial u}{\partial x_i} - \int_{\partial\Omega} \frac{\partial u}{\partial n} u_0$$

(note that $\frac{\partial u_0}{\partial x_i} \in L^2(\Omega)$, and $u_0 \in H^{\frac{1}{2}}(\partial\Omega) \in L^2(\partial\Omega)$). This gives

$$|\int_{\partial\Omega} \frac{\partial u}{\partial n} u_0| \leq |\Delta u|_{L^2(\Omega)} |u_0|_{L^2(\Omega)}$$

$$+ \sum | \frac{\partial u_0}{\partial x_i}|_{L^2(\Omega)} | \frac{\partial u}{\partial x_i}|_{L^2(\Omega)}$$

$$\leq C\|u_0\|_{H^1} \|u\|_X \implies$$

$$|\int_{\partial\Omega} \frac{\partial u}{\partial n} v | \leq C\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \|u\|_X \implies$$

$$\|\frac{\partial u}{\partial n}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_X \qquad (u \in D(\bar{\Omega})) .$$

Since by Lemma 1, $D(\bar{\Omega})$ is dense in X, we get the same inequality for all $u \in X$.

Remark IX. 2. If one only knows that $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$, then one can show that $\frac{\partial u}{\partial n} \in (H^{3/2}(\partial\Omega))^1 = H^{-3/2}(\partial\Omega)$. This result, as well as the one of Lemma 2, is best possible.

Putting everything together we find that we can construct a continuous affine mapping $T: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ defined as $T(v) = \frac{\partial u}{\partial n}$, where u is the solution of (3).

Remark IX. 3. Using Lemma 1 one can prove that

$$\int_\Omega -\Delta u_1 u_2 = \int_\Omega \sum_{i=1} \frac{\partial u_1}{\partial x_i} \frac{\partial u_2}{\partial x_i} - \int_{\partial\Omega} \frac{\partial u_1}{\partial n} u_2$$

for $u_1 \in X$, $u_2 \in H^1(\Omega)$, where the last integral is considered as an "inner-product" between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Applying this twice one gets

$$\int_\Omega -(\Delta u_1)u_2) = \int_\Omega -(\Delta u_2)u_1 - \int_{\partial\Omega} \frac{\partial u_1}{\partial n} u_2 + \int_{\partial\Omega} \frac{\partial u_2}{\partial n} u_1 .$$

for $u_1, u_2 \in X$.

<u>Step 3.</u> T is monotone.

Take

$$\begin{cases} -\Delta u_1 = w \\ u_1\big|_{\partial\Omega} = v_1 \end{cases} \qquad \begin{cases} -\Delta u_2 = w \\ u_2\big|_{\partial\Omega} = v_2 \end{cases}$$

$$\Longrightarrow \begin{cases} -\Delta(u_1 - u_2) = 0 \\ (u_1 - u_2\big|_{\partial\Omega} = v_1 - v_2 \end{cases}$$

Substitute $u_1 = u_2 = u_1 - u_2$ in the last formula in Remark 3. This gives

$$\int_{\partial\Omega} \frac{\partial}{\partial n}(u_1 - u_2)(u_1 - u_2) = \int_{\Omega} \sum \left| \frac{\partial}{\partial x_i}(u_1 - u_2) \right|^2$$

$$= \|u_1 - u_2\|^2_{H^1(\Omega)} - |u_1 - u_2|^2_{L^2(\Omega)} \geq 0 .$$

Thus T is not only monotone, but satisfies

$$(Tv_1 - Tv_2, v_1 - v_2) \geq C\|v_1 - v_2\|^2_{H^{\frac{1}{2}}(\partial\Omega)} - |u_1 - u_2|^2_{L^2(\Omega)}$$

In fact one can show that for $\varepsilon > 0$, $v \to \varepsilon v + Tv$ is coercive (go back to the integral formula).

Combining Steps 1-3 with Theorem III.3 and Lemma III.2 we get

<u>Corollary IX.1.</u> Let $F: H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ be monotone hemicontinuous, and satisfy

$$(F(v_1) - F(v_2), v_1 - v_2) \geq \varepsilon\|v_1 - v_2\|^2_{L^2(\partial\Omega)} .$$

Let Ω be a bounded, regular, open set, and let $w \in L^2(\Omega)$. Then there exists a unique solution $u \in H^1(\Omega)$ of

$$\begin{cases} -\Delta u = w \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} + F(u) = 0 . \end{cases}$$

If F is defined pointwise: $u(x) \to F(x,u(x))$ with $F(x,r)$ measurable in x for each r, continuous increasing in r for almost any x, and satisfies

$$|F(x,r)| \leq \lambda(x) + c\,|r|, \quad \text{where } \lambda \in L^2,$$

then F is monotone hemicontinuous from $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$. To find $\varepsilon > 0$ we have to assume $\dfrac{F(x,r_1) - F(x,r_2)}{r_1 - r_2} \geq \varepsilon$. a.e.

Remark IX.4. The corollary does not cover the case $F \equiv 0$ because of the coercivity condition. In this case the solution does not exist unless the compatibility condition

$$\int_\Omega w = 0$$

holds. This is true because for every $u \in X$, $\int_\Omega -\Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n}$, and this is zero if $F \equiv 0$. If $\int_\Omega w = 0$, then infinitely many solutions exist (i.e. the solution contains an undetermined constant).

So far we have only discussed the stationary heat equation with non-linear boundary conditions. Now we want to discuss a variational inequality

$$V = H_0^1(\Omega)$$

$A: V \to V'$ given by $A = -\Delta$

A is linear, monotone: $(Au-Av,u-v)$

$$= \int_\Omega \sum_i \left|\frac{\partial(u-v)}{\partial x_i}\right|^2 = \|u-v\|_V^2 .$$

We can conclude from Theorem V.2 that if K is a closed, convex set in V, and if $f \in V' = H^{-1}(\Omega)$, then there exists a unique $u \in K$ such that $(Au - f, v-u) \geq 0 \; (v \in K)$.

Example IX.1. Take $K = \{v \in H_0^1(\Omega) \mid v \geq \varphi$ a.e. in $\Omega\}$, where $\varphi \in L^2(\Omega)$ Suppose that $K \neq \phi$. If $\varphi \in H^1(\Omega)$, then $K \neq \phi \Longleftrightarrow \varphi|_{\partial\Omega} \leq 0$. The set K is closed: Let $v_n \in K \to v$ in V, then for a subsequence $v_k \to u$ a.e. $\Longrightarrow$ $v \geq \varphi$ a.e. in Ω. It is also convex, so it is also weakly closed.

Thus the inequality

$$(4) \qquad \begin{cases} (Au - f, v-u) \geq 0 & (v \in K) \\ u \geq \varphi \quad \text{a.e.} & \text{in } \Omega \end{cases}$$

has a solution $u \in H^1(\Omega)$. One can give a closer description of this solution. Take $w \in D(\Omega)$, $w \geq 0$. Then $u + w \in K$. Put $v = u + w \Longrightarrow (Au - f, w) \geq 0$ $(w \in D(\Omega), w \geq 0)$. $\Longrightarrow$ $Au - f$ is a positive distribution $\Longrightarrow$ $Au - f$ is induced by a positive measure, so we get

$$\begin{cases} Au - f \geq 0 & \text{(in the sense of measures)} \\ u \geq \varphi & \text{(a.e.)} \end{cases}$$

Formally one can further improve this characterization. For the following discussion, suppose that u and φ are smooth functions. Also suppose that in some open set ω, $u > \varphi$. Then for $w \in D(\omega)$, and for $|\varepsilon|$ sufficiently small, $u + \varepsilon w \geq \varphi$ on ω. Take $v = u + \varepsilon w$, $\varepsilon > 0$ and small $\Longrightarrow$ $(Au - f, \varepsilon w) \geq 0 \Longrightarrow (Au - f, w) \geq 0$. Here w is anything in $D(\omega)$, so $Au - f = 0$. So formally we get

$$(5) \qquad \left\{ \begin{matrix} -\Delta u - f \geq 0 \\ u - \varphi \geq 0 \end{matrix} \right\}$$ and for each $x \in \Omega$ one of these must be an equality,

i.e.

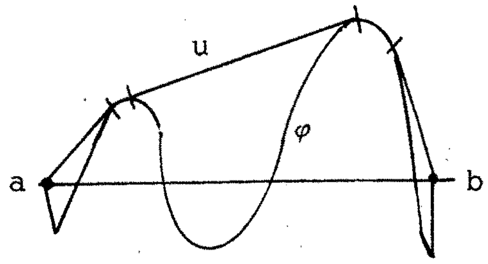
$$\begin{cases} u > \varphi \Longrightarrow -\Delta u - f = 0 \\ u = \varphi \Longrightarrow -\Delta u - f \geq 0 \end{cases}$$

Conversely, suppose that u is smooth, and satisfies (5). Then it solves (4):

$$\int_\Omega (-\Delta u - f, v-u) = \int_{u=\varphi} (-\Delta u - f, v - \varphi) \geq 0 .$$

The problem is to prove that a solution of (5) is smooth enough for this to make sense.

Geometrical solution for $N = 1$, $\Omega = (a,b)$



Solution: u = how a tense string looks fastened at a and b, pulled over the function φ (take $f \equiv 0$)

either $u = \varphi$ and $-u'' \geq 0$,

or $u > \varphi$ and $u'' = 0$.

1st case: u concave

2nd case: u linear.

Note that even if $\varphi \in C^\infty$, we have in general $u \in C^1(a,b)$, $u \notin C^2(a,b)$. However, if $\varphi'' \in L^p(a,b)$, then either $u''(x) = 0$, or $u''(x) = \varphi''(x)$, so $u'' \in L^p$, and $\|u''\|_{L^p(a,b)} \leq \|\varphi''\|_{L^p(a,b)}$. Here we have a restriction on the smoothness of the solution which is built into the variational inequality: smooth data does not necessarily give smooth solutions, as the heat equation does. For the heat equation, $f \in H^s(\Omega) \Rightarrow u \in H^{s+2}(\Omega)$ $(s \geq -1, s \neq -\frac{1}{2})$. Here one can prove much less (let $\varphi \in C^\infty$): $f \in H^s \Rightarrow u \in H^{s+2}$ if $s < \frac{1}{2}$, but not for bigger s.

Theorem IX.1. (regularity for Example 1 with $\varphi = 0$). Let Ω be a bounded, regular open set, and let $f \in L^p(\Omega)$ for some p, $2 \leq p < \infty$. Define

$K = \{v \in H_0^1(\Omega) | \; v \geq 0$ a.e. in $\Omega\}$. Then the solution $u \in K$ of

$$(-\Delta u - f, v - u) \geq 0 \qquad (v \in K)$$

satisfies $|\Delta u|_{L^p(\Omega)} \leq 2|f|_{L^p(\Omega)}$

Remark IX. 5. This will imply $u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$. By Sobolev's embedding theorem, $u, \frac{\partial u}{\partial x_i} \in L^g(\Omega)$, $\frac{1}{g} = \frac{1}{p} - \frac{1}{N}$. Also $u \in L^r(\Omega)$, $\frac{1}{r} = \frac{1}{p} - \frac{2}{N}$, provided $p < \frac{N}{2}$. If $p > \frac{N}{2}$, then $u \in C^0(\bar{\Omega})$.

Proof of Theorem 1. Define

$$\beta(u(x)) = \begin{cases} 0, & \text{if} \quad u(x) \geq 0 \\ u, & \text{if} \quad u(x) < 0. \end{cases}$$

Clearly $\beta : L^2(\Omega) \to L^2(\Omega)$, and β is monotone, hemicontinuous and bounded: $|\beta(u) - \beta(v)|_{L^2} \leq |u-v|_{L^2}$. Also $\beta(u) = 0 \iff u \geq 0$ a.e. Consider the problem

$$-\Delta u_\varepsilon + \frac{1}{\varepsilon}\beta(u_\varepsilon) = f$$

in $V = H_0^1(\Omega)$. We can use Theorem VIII. 2 to conclude that we find a sequence $u_\varepsilon \rightharpoonup u$ weakly in V, where $u \in K$ is the solution of

$$(-\Delta u - f, v-u) \geq 0 \qquad (v \in K).$$

To get the inequality $|-\Delta u|_{L^p(\Omega)} \leq 2|f|_{L^p(\Omega)}$, it suffices to prove that

$$|-\Delta u_\varepsilon|_{L^p(\Omega)} \leq 2|f|_{L^p(\Omega)},$$

because the mapping $u \to |u|$ is convex and continuous, hence weakly lower semicontinuous in V.

Case $p = 2$.

$$u_\varepsilon \in V \implies \beta(u_\varepsilon) \in L^2(\Omega), \; -\Delta u_\varepsilon \in L^2(\Omega).$$

Lemma IX. 3. If γ is Lipschitz continuous, monotone on $\mathbb{R}$, $\gamma(0) = 0$, and $u \in H_0^1(\Omega)$, then $(-\Delta u, \gamma(u)) \geq 0$.

Apply this lemma to $\gamma = \beta$, $u = u_\varepsilon$. This yields $(-\Delta u_\varepsilon, \beta(u_\varepsilon)) \geq 0$.

Multiply the equation by $-\Delta u_\varepsilon \implies$

$$|-\Delta u_\varepsilon|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}(-\Delta u_\varepsilon, \beta(u_\varepsilon)) = (f, -\Delta u_\varepsilon) \implies$$

$$|-\Delta u_\varepsilon|^2_{L^2(\Omega)} \leq |f|_{L^2(\Omega)} |-\Delta u_\varepsilon|_{L^2(\Omega)}$$

This gives the desired inequality

$$|-\Delta u_\varepsilon|_{L^2(\Omega)} \leq |f|_{L^2(\Omega)}$$

(note: we do not even need the factor 2 here).

Proof of Lemma 3. Suppose first that $\gamma \in C^1(\mathbb{R})$. Then one can integrate by parts and get

$$(-\Delta u, \gamma(u)) = \sum_i \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} \gamma(u)$$

$$= \sum_i \int_\Omega \gamma'(u) \left(\frac{\partial u}{\partial x_i}\right)^2 \geq 0 .$$

In the general case, take a sequence $\gamma_n \in C^1(\mathbb{R})$, γ_n monotone with $\gamma_n(0) = 0$ such that $\gamma_n(t) \to \gamma(t)$ for all t, and $|\gamma_n'(t)|$ is uniformly bounded. Then $\gamma_n(u) \to \gamma(u)$ in $L^2(\Omega)$, so

$$(-\Delta u, \gamma(u)) = \lim (-\Delta u, \gamma_n(u)) \geq 0 .$$

This completes the proof of Lemma 3, and finishes the Case $p = 2$.

Case $p > 2$. Apply Lemma 3 (or rather a straightforward generalization of Lemma 3) to $\gamma(t) = |\beta(t)|^{p-2} \beta(t)$ (which is only locally Lipschitz) to get

$(-\Delta u_\varepsilon, \gamma(u_\varepsilon)) \geq 0 \implies$

$$\frac{1}{\varepsilon} |\beta(u_\varepsilon)|^p_{L^p(\Omega)} = \frac{1}{\varepsilon} \; \beta(u_\varepsilon), \gamma(u_\varepsilon))$$

$$\leq (f, \gamma(u_\varepsilon)) \leq |f|_{L^p(\Omega)} |\gamma(u_\varepsilon)|_{L^{p'}(\Omega)} \implies$$

$$\frac{1}{\varepsilon} |\beta(u_\varepsilon)|_{L^p(\Omega)} \leq |f|_{L^p(\Omega)}$$

Substituting this into the equation for u_ε one gets $|-\Delta u_\varepsilon|_{{}_\epsilon p_{(\Omega)}} \leq 2|f|_{L^p(\Omega)}$.

This completes the proof of Theorem 1.

X. A Hammerstein Equation.

The classical Hammerstein equation is of the form

$$u(x) + \int_\Omega K(x,y)\, f(y, u(y))dy = g(x) .$$

It is of the abstract form

$$u + K(A(u)) = g ,$$

where

$$(Au)(x) = f(x, u(x))$$

$$(Ku)(x) = \int_\Omega K(x,y) u(y) dy .$$

We let X be a reflexive Banach space ($\implies X'$ is reflexive), and suppose that

$$A: X \to X'$$

$$K: X' \to X .$$

Remark X.1. It is no loss of generality to assume that $K(0) = 0$, $A(0) = 0$; otherwise put

$$\tilde{K}(v) = K(v + A(0)) - K(A(0)) ,$$

$$\tilde{A}(u) = A(u) - A(0) \qquad \Longrightarrow$$

$$u + \tilde{K}(\tilde{A}(u)) = g - K(A(0)) .$$

Another possible normalization is to take $K(0) = 0$, $g = 0$ obtained by putting $u = g+v$, $B(v) = A(g+v)$, $u+K(A(u)) = g \Longrightarrow v + K(B(v)) = 0$.

Lemma X.1. $\exists\, u \in X : u + KAu = 0 \iff$

$$\exists u \in X, v \in X' : \begin{cases} -v + Au = 0 \\ u + Kv = 0 \end{cases} .$$

Proof. Obvious.

Now consider the map $\mathcal{A} : X \times X' \to X' \times X$, i.e. $V \to V'$ where $V = X \times X'$, defined by

$$(u,v) \to (-v + Au, u + Kv) .$$

Lemma X.2. $\mathcal{A}$ is monotone $\iff$ K and A are monotone.

Proof. Take $(u_1,v_1),(u_2,v_2) \in V$. Then

$$(\mathcal{A}(u_1,v_1) - \mathcal{A}(u_2,v_2), (u_1,v_1) - (u_2,v_2))$$

$$= ((-v_1 + A(u_1)) - (-v_2 + A(u_2)), u_1-u_2)$$

$$+ ((u_1 + K(v_1)) - (u_2 + K(v_2)), v_1 - v_2)$$

$$= (A(u_1) - A(u_2), u_1 - u_2) + (K(v_1) - K(v_2), v_1 - v_2) .$$

This clearly gives the direction $(\Leftarrow)$. To get the other one, take first $v_1 = v_2$ and then $u_1 = u_2$.

Lemma X.3. $\mathcal{A}$ is hemicontinuous $\iff$ K and A are hemicontinuous.

Proof. The same type of calculation gives

$$(\mathcal{A}(u_1 + tu_2, v_1 + tv_2), (u_2,v_2))$$

$$= (-v_1,u_2) + (u_1,v_2) + A(u_1 + tu_2,u_2) + K(v_1 + tv_2,v_2) ,$$

from which the statement follows.

Lemma X.4. $\mathcal{Q}$ is pseudomonotone $\Leftrightarrow$ A and K are pseudomonotone.

Proof. Essentially the same calculation.

Remark X.2. The operator A and K seem to play different roles in the equation $u + K(A(u)) = 0$. However, $u + K(A(u)) \Leftrightarrow$

$$\begin{cases} -v + A(u) = 0 \\ u + K(v) = 0 . \end{cases}$$

Put $\tilde{K}(v) = -K(-v)$, $\tilde{v} = -v \implies$

$$\begin{cases} u + \tilde{K}(\tilde{v}) = 0 \\ \tilde{v} + A(u) = 0 \end{cases} \implies$$

$$\tilde{v} + A(\tilde{K}(\tilde{v})) = 0 ,$$

where A and $\tilde{K}$ have "changed places". Also notice that $\tilde{K}$ is monotone if K is .

We begin with the question of uniqueness. Suppose that A and K are monotone hemicontinuous:

Case 1. If A is strictly monotone, then we have uniqueness, because let

$$u + K(A(u)) = 0$$

$$v + K(A(v)) = 0 ,$$

take the difference, and multiply by $A(u) - A(v)$: $0 \geq (u-v, A(u)-A(v)) + (K(A(u)) - K(A(v)), A(u) - A(v)) \geq (u-v, A(u) - A(v))$. So $u = v$.

Case 2. If K is strictly monotone, then we have uniqueness: Same computation gives $Au = Av$, and thus $u = -K(A(u)) = -K(A(v)) = v$.

Case 3. If $(A(a) - A(b), a-b) = 0 \implies A(a) = A(b)$, then we also have uniqueness (use the reasoning in Case 1 to get $A(u) = A(v)$, and then continue as in Case 2). Clearly the same is true if one requires the same thing of K .

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Definition X.1. $M : V \to V'$ is n-monotone, if $\forall\, u_1, \ldots, u_n$,

$$\sum_{i=1}^{n} (M u_i, u_i - u_{i+1}) \geq 0 \quad (\text{define } u_{n+1} = u_1) .$$

M is cyclically monotone, if it is n-monotone for all n.

Lemma X.5. Let A be tri-monotone and hemicontinuous. Then

$$(A(a) - A(b), a-b) = 0 \implies A(a) = A(b) .$$

Proof. Apply trimonotonicity with the triple $a, u, b \implies$

$$(A(a), a-u) + (A(u), u-b) + (A(b), b-a) \geq 0 .$$

Thus using also $(A(a) - A(b), a-b) = 0$ we get

$$(A(a), b-u) + (A(u), u-b) \geq 0 .$$

Put $u = b + tc$ and use hemicontinuity:

$$(A(b+tc) - A(a), tc) \geq 0 \quad (c \in V) \implies$$

$$(A(b) - A(a), c) \geq 0 \quad (c \in V) \implies$$

$$A(a) = A(b) .$$

In the particular case when

$$K\, u(x) = \int_{\Omega} K(x,y)\, u(y)\, dy ,$$

where K is monotone and $K(x,y) = K(y,x)$ we have K cyclically monotone (the derivative of $\varphi(u) = \frac{1}{2}(Ku,u)$ is $\frac{1}{2}(K + K^*) = K$). Thus in this case we have uniqueness.

One can also look at the problem of uniqueness in the following way: If u_1 and u_2 solve $u + K(A(u)) = 0$, then (u_1, v_1) and $(u_2 . v_2)$ solve $\mathcal{A}(u,v) = 0$, where $v_1 = A(u_1)$ and $v_2 = A(u_2)$, $\mathcal{A}(u,v)$ is monotone, so the set of solutions of $\mathcal{A}(u,v) = 0$ is closed and convex. Hence also

$$((1 - \theta)u_1 + \theta\, u_2, (1 - \theta)v_1 + \theta\, v_2)$$

$$= ((1 - \theta\, u_1 + \theta\, u_2, A((1 - \theta)u_1 + \theta\, u_2))$$

must be a solution of $\mathcal{Q}(u,v) = 0$. This implies that

A is affine on the segment $[u_1, u_2]$.

Similarily one gets

K is affine on the segment $[v_1, v_2]$.

There may be some cases where these informations lead to either $u_1 = u_2$ or $v_1 = v_2$.

In the following existence theorem we assume monotonicity of A and K . One could modify the argument to work when A and K are pseudo-monotone, and $(K(v),v) \geq 0$ $(v \in V)$. The same remark applies to Theorem 2 below.

<u>Theorem X.1.</u> (existence). Let A and K be monotone, hemicontinuous; $A : X \to X'$ and $K : X' \to X$, where X is a reflexive Banach space. Also let $K(0) = 0$ (cf. Remark 1). Moreover, suppose that $\exists\, R > 0 \ni (Au,u) < 0 \Longrightarrow \|u\| \leq R$. Then there exists $u \in X$ such that $u + K(A(u)) = 0$. (Actually one can find u satisfying $\|u\| \leq R$) . We first need:

<u>Lemma X.6.</u> Let X be a Banach space, and suppose that $A: X \to X'$ satisfies

$$\forall\, v \in X \quad \inf_{\|u\| \leq R} \; (A(u), u-v) > -\infty$$

Then for every constant K , the image under A of the set

$$S = \{u \in X \mid \|u\| \leq R \, , \; (A(u),u) \leq K\}$$

is bounded in X'.

<u>Proof.</u> Take any $w \in X$. Then

$$(A(u),w) = (Au,u) - (A(u), u-w)$$

so

$$\sup_{u \in S} (A(u),w) \leq K - \inf_{u \in S} (A(u), u-w) < +\infty$$

Replacing w by -w one gets

$$\sup_{u \in S} |(A(u), w)| < \infty .$$

Hence by the Banach-Steinhaus theorem,

$$\sup_{u \in S} \|A(u)\|_{X'} < \infty ,$$

which proves the lemma.

Proot of Theorem 1. By Lemma 6 A maps the set

$$\{u \in X \mid \|u\| \leq R , (A(u),u) \leq 0\}$$

into a bounded set of X' , say inside the ball $\|y\| \leq S$. On $X \times X'$, consider

$$C = \{(u,v) \mid \|u\| \leq R+1, \|v\| \leq S + 1\}.$$

By Theorem IV.1. combined with Lemmas 2 and 3, there exists $(\bar{u},\bar{v}) \in C$ such that

$$(\mathcal{Q}(\bar{u},\bar{v}), (u,v) - (\bar{u},\bar{v})) \geq 0 \qquad \forall (u,v) \in C .$$

This is equivalent to

$$(-\bar{v} + A(\bar{u}), u-\bar{u}) + (\bar{u} + K(\bar{v}), v-\bar{v}) \geq 0 .$$

Take $u = v = 0 \Rightarrow (A(\bar{u}),\bar{u}) \leq (K(\bar{v}),\bar{v}) \leq 0$ by monotonicity of K, and the fact that $K(0) = 0$ thus $(A(u),u) \leq 0$.

Case 1. $(A(\bar{u}),\bar{u}) = 0 \Rightarrow K(\dot{\bar{v}},\bar{v}) = 0 \Rightarrow (-\bar{v} + A(\bar{u}),u) + (\bar{u} + K(\bar{v}),v) \geq 0$ $\forall (u,v) \in C \Rightarrow -\bar{v} + A(\bar{u}) = 0$; $\bar{u} + K(\bar{v}) = 0$, which is exactly what we want to show.

Case 2. $(A(\bar{u}),\bar{u}) < 0 \Rightarrow \|\bar{u}\| \leq R$. Take $v = \bar{v}$, and u in a neighborhood of $\bar{u} \Rightarrow -\bar{v} + A(\bar{u}) = 0$. However, this implies $|\bar{v}| \leq S$ by the way we chose S .

Now choose $u = \bar{u}$, and v in a neighborhood ov $\bar{v} \Longrightarrow \bar{u} + K(\bar{v}) = 0$.
This completes the proof. ∎

Application to the Hammerstein Equation. Take $X = L^p(\Omega)$ $(1 < p < \infty$, and suppose that $|f(y,r)| \leq g(y) + c|r|^{p-1}$ where $g \in L^{p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. Also let f satisfy the Carathéodory Condition: measurability in y and continuity in r. Then $A(u) = f(x, u(x))$ maps $L^p(\Omega)$ continuously in $L^{p'}(\Omega)$. To get monotonicity one must require f to be nondecreasing in r. With these assumptions we have A monotone hemicontinuous. However, to get "$(A(u),u) \geq 0$" for large u we still need coercivity:

$$r\, f(x,r) \geq \varepsilon |r|^p - h(y) ,$$

where $h \in L^1$; for the following reason:

Let $K u = \int K(x,y)u(y)dy$ map $L^{p'}(\Omega)$ into $L^p(\Omega)$ be monotone hemicontinuous, and try to solve

$$u + K(A(u)) = \varphi .$$

Using Remark 1 one can reduce this problem to Theorem 1, However, this procedure replaces the operator $A(v)$ by $\tilde{A}(v) = A(\varphi + v)$, and the coercivity is needed to get $(A(\varphi + v), v) \geq 0$ for large v.

Theorem X. 2. Let A and K be monotone hemicontinuous; $A: X \to X'$ and $K: X' \to X$, where X is a reflexive Banach space. Moreover, suppose that $\exists\, R' > 0 \ni (Au,u) < 0 \Longrightarrow \|A(u)\| \leq R'$ and that K is bounded, with $K(0) = 0$. Then there exists $u \in X$ such that $u + K(A(u)) = 0$. (Actually one can find u satisfying $\|Au\| \leq R'$.

Remark X. 3. We replaced $(Au,u) < 0 \Longrightarrow \|u\| \leq R$ by the weaker condition $(Au,u) < 0 \Longrightarrow \|Au\| \leq R'$ but we suppose now that K is bounded.

Proof. Let K map $\{v \in X' \mid \|v\| \leq R' + 1\}$ into $\{u \in X \mid \|u\| \leq S\}$. Like in the proof of Theorem 1 we can find $(\bar{u},\bar{v}) \in$

$$C = \{(u,v) \in X \times X' \mid \|u\| \leq S + 1 ,\ \|v\| \leq R' + 1\} ,\quad \ni$$

$$(-\bar{v} + A(\bar{u}),\ u - \bar{u}) + (\bar{u} + K(\bar{v}),\ v - \bar{v}) \geq 0 .$$

Take $u = v = 0 \Rightarrow A(\bar{u},\bar{u}) \leq -K(\bar{v},\bar{v}) \leq 0$. Thus $(A(\bar{u}),\bar{u}) \leq 0$.

Case 1. $(A(\bar{u}),\bar{u}) = 0 \Rightarrow (K(\bar{v}),\bar{v}) = 0$, and the conclusion follows as in Theorem 1.

Case 2: $(A(\bar{u}),\bar{v}) < 0 \Rightarrow \|A(\bar{u})\| \leq R'$.

$$(-\bar{v} + A(\bar{u}),u) + (\bar{u} + K(\bar{v}),v) \geq (A(\bar{u}),\bar{u}) + (K(\bar{v}),\bar{v})$$

$$= (A(\bar{u}),\ \bar{u} + K(\bar{v})) + (K(\bar{v}),\ \bar{v} - A(\bar{u})) .$$

Minimizing over $u,v \in C$ one gets

$$-(S+1)\|-\bar{v} + A(\bar{u})\|_{X'} - (R'+1)\|\bar{u} + K(\bar{v})\|_{X'} \geq$$

$$\|A(\bar{u})\|_{X'}\|\bar{u} + K(\bar{v})\|_{X} - \|K(\bar{v})\|_{X}\|\bar{v} - A(\bar{u})\|_{X'}$$

$$\geq -R'\|\bar{u} + K(\bar{v})\|_{X} - S\|\bar{v} - A(\bar{u})\|_{X'}$$

so $\bar{v} - A(\bar{u}) = 0$ and $\bar{u} + K(\bar{v}) = 0$ which completes the proof.

Application. Let f be as in the application following Theorem 1, except for the coercivity. Consider the equation

$$u + Kf(x,u) = \varphi .$$

Without loss of generality we can take $f(x,0) = 0$ (subtract $f(x,0)$ from $f(x,u)$, and shift the error to the right hand side). As before, define $A(v) = f(x,\varphi+v)$. This time to verify that

$(A(v),v) \leq 0 \Rightarrow A(v)$ is bounded in $L^{p'}(\Omega)$ we do not need coercivity: $(A(v),v) \leq 0 \Leftrightarrow \int_\Omega u\, f(x,u) \leq \int_\Omega \varphi\, f(x,u)$, where $u = v + \varphi$, so we want to prove that $\int_\Omega u\, f(x,u) \leq \int_\Omega \varphi\, f(x,u) \Rightarrow |f(x,u)|_{L^{p'}(\Omega)}$ is bounded.

Define $\beta(x) = \alpha|f(x,u(x))|^{p'-1}$, where α is a constant, to be specified below. Then for $|u(x)| \leq \beta(x)$,

$$\begin{aligned}|f(x,u)| &\leq g(x) + C|u(x)|^{p-1} \\ &\leq g(x) + C\,\alpha^{p-1}\,|f(x,u(x))|^{(p-1)(p'-1)} \\ &= g(x) + C\,\alpha^{p-1}\,|f(x,u(x))| .\end{aligned}$$

Choose α so that $C\,\alpha^{p-1} = \frac{1}{2}$. Then one gets

$$|f(x,u)| \leq \tfrac{1}{2}g(x) . \qquad (|u(x)| \leq \beta(x)) .$$

Thus in particular,

$$(1) \qquad \int_{|u| \leq \beta} |f(x,u(x))|^{p'} dx \leq 2^{p'} \, |g|^{p'}_{L^{p'}(\Omega)}$$

Returning to the assumption

$$\int_\Omega u\, f(x,u)\, dx \leq \int_\Omega \varphi\, f(x,u) dx$$

we get (note that $u f(x,u) \geq 0$ by monotonicity and the fact that $f(x,u) = 0$)

$$\begin{aligned}\int_{|u| \geq \beta} |f(x,u)|^{p'} dx &= \frac{1}{\alpha} \int_{|u| \geq \beta} \beta\,|f(x,u)|\,dx \\ \leq \frac{1}{\alpha} \int_{|u| \geq \beta} |u\, f(x,u)|\,dx &\leq \frac{1}{\alpha} \int_\Omega |u\, f(x,u)|\,ds \\ = \frac{1}{\alpha} \int_\Omega u\, f(x,u) dx &\leq \frac{1}{\alpha} \int_\Omega \varphi\, f(x,u) dx \\ &\leq \frac{1}{\alpha}\, |\varphi|_{L^p(\Omega)}\, |f(x,u)|_{L^{p'}(\Omega)}\end{aligned}$$

Combining this with (1) we get

$$|f(x,u)|^{p'}_{L^{p'}(\Omega)} \leq C_1 + C_2\, |f(x,u)|_{L^{p'}(\Omega)} ,$$

which yields the desired boundedness, since $p' > 1$.

Remark X.4. The same type of proof goes through for $p = \infty$. One must replace the condition on f by : f is nondecreasing in r, and for every fixed r, $f(x,r)$ is integrable. The reflexivity of the Banach space is destroyed, but here A maps bounded sets in $L^\infty(\Omega)$ into sets of the form

$$\{v \in L^1(\Omega) \,|\, |v(x)| \leq u(x) \in L^1(\Omega)\},$$

which happen to be weakly compact in $L^1(\Omega)$ and this ensures existence by a suitable generalization of Theorem 2.

Acknowledgment

This is part of the lecture notes of a course given at the University of Wisconsin-Madison in 1974-75. I wish to thank O. Staffans for the redaction of this part.

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LINEAR CONTROL THEORY AND RICCATI EQUATIONS

L. Tartar

1. Preliminaries

Definition 1.1. If X is a topological space and f is a real function on X, f is called lower semicontinuous (l.s.c.) at $x_0 \in X$ if for every $\epsilon > 0$, there exists $V(x_0)$, a neighborhood of x_0, such that

$$f(x) \geq f(x_0) - \epsilon \quad \text{for every } x \in V(x_0).$$

Remark 1.2. If X is metrizable, f is l.s.c. at x_0 if and only if

$$f(x_0) \leq \liminf_{n \to \infty} f(y_n) \quad \text{whenever } y_n \to x_0 \text{ as } n \to \infty.$$

Definition 1.3. A real function f is called l.s.c. on X if f is l.s.c. at x_0 for every $x_0 \in X$.

Theorem 1.4. Let X be a compact topological space, and f a l.s.c. real function on X. Then f attains its minimum on X.

The proof of Theorem 1.4 makes use of the following lemma, which follows easily from Definition 1.1:

Lemma 1.5. A real function f on X is l.s.c. on X if and only if the set

$$\{x \in X: f(x) \leq \ell\}$$

is closed in X for every $\lambda \in \mathbb{R}$.

Proof of Theorem 1.4. Let $\alpha = \inf_{x \in X} f(x)$.

Of course, at this point α may be $-\infty$. Choose a sequence $\{\alpha_n\}$, such that $\alpha_n \searrow \alpha$ as $n \to \infty$. Let $\Omega_n = \{x \in X: f(x) \leq \alpha_n\}$, $n = 1,2,3,\ldots$. $\{\Omega_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty, closed sets by the definition of α and Lemma 1.5. So, since X is compact, we may choose

$$x_0 \in \bigcap_{n=1}^{\infty} \Omega_n$$

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

Then $f(x_0) \leq \alpha_0$ for every n, so $f(x_0) \leq \alpha \leq f(x_0)$. Hence f attains its minimum at x_0. In the case when X is metrizable a somewhat shorter proof is available. Let $\{x_n\}$ be a minimizing sequence, i.e.

$$f(x_n) \to \alpha = \inf_{x \in X} f(x) \qquad \text{as } n \to \infty .$$

Since X is compact, there exists a subsequence $\{x_m\}$ and a point $x_\infty \in X$ such that $x_m \to x_\infty$ as $m \to \infty$. Hence $f(x_\infty) \leq \liminf_{m \to \infty} f(x_n) = \alpha \leq f(x_\infty)$. Therefore f attains its minimum at x_∞.

Definition 1.6. Let f be a real function on a set E. Then

$$\{(x,\lambda) \in E \times \mathbb{R} : f(x) \leq \lambda\}$$

is called the epigraph of f.

Application 1.7. Let E be a topological vector space and f a convex function on E. Suppose on E we have two topologies τ_1 and τ_2 which are locally convex and have the same dual space. Then f is l.s.c. with respect to τ_1 if and only if f is l.s.c. with respect to τ_2 if and only if epigraph f is closed in either $E_{\tau_1} \times \mathbb{R}$ or $E_{\tau_2} \times \mathbb{R}$. This is a result of the following lemmas and remark.

Lemma 1.8. Let f be a real function on E. Then f is convex on E if and only if epigraph f is a convex subset of $E \times \mathbb{R}$.

Proof: Suppose f is convex on E. Choose $(x_1,\lambda_1), (x_2,\lambda_2) \in$ epigraph f, and $\theta \in [0,1]$. Then $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2) \leq \theta\lambda_1 + (1-\theta)\lambda_2$. So $(\theta x_1 + (1-\theta)x_2, \theta\lambda + (1-\theta)\lambda_2) = \theta(x_1,\lambda_1) + (1-\theta)(x_2,\lambda_2) \in$ epigraph f. Conversely, if epigraph f is convex, $x_1, x_2 \in E$, $\theta \in [0,1]$, $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$ since $(x_1,f(x_1))$ and $(x_2,f(x_2))$ and hence $(\theta x_1 + (1-\theta)x_2, \theta f(x_1) + (1-\theta)f(x_2))$ is in epigraph f.

Lemma 1.9. Let f be a real function on E. Then f is l.s.c. on E if and only if epigraph f is closed in $E \times \mathbb{R}$.

Proof: We give the proof for the case where E is metrizable. Suppose epigraph f is closed. Choose $x_n \to x$, we wish to show that

$$f(x) \leq \liminf_{n \to \infty} f(x_n) = \alpha .$$

If $\alpha = +\infty$, there is nothing to prove. If $-\infty < \alpha < \infty$, choose a subsequence $\{x_m\}$ such that $f(x_m) \to \alpha$ as $m \to \infty$. Then $(x_m, f(x_m)) \in$ epigraph f for every m and

$$(x_m, f(x_m)) \to (x, \alpha) \in \text{epigraph } f .$$

So $f(x) \leq \alpha = \liminf_{n \to \infty} f(x_n)$ as desired. If $\alpha = -\infty$, choose a subsequence $\{x_m\}$ such that

$$f(x_m) \searrow -\infty \quad \text{as} \quad m \to \infty .$$

Then $(x_m, f(x_{m_0})) \in$ epigraph f for $m \geq m_0$ since $f(x_m) \leq f(x_{m_0})$ for $m \geq m_0$ And $(x_m, f(x_{m_0})) \to (x, f(x_{m_0}))$ as $m \to \infty$. Hence $(x, f(x_{m_0}) \in$ epigraph f or $f(x) \leq f(x_{m_0})$ for every m_0. But then $f(x) \leq \alpha = -\infty$, contradiction.

Conversely, suppose f is l.s.c. on E, and $(x_n, \lambda_n) \to (x, \lambda)$ as $n \to \infty$ with $(x_n, \lambda_n) \in$ epigraph f or $f(x_n) \leq \lambda_n$ for $n = 1, 2, 3, \ldots$. Then $f(x) \leq \liminf_{n \to \infty} f(x_n) \leq \liminf_{n \to \infty} \lambda_n = \lambda$.

So $(x, \lambda) \in$ epigraph f, and epigraph f is closed in $E \times \mathbb{R}$.

Remark 1.10. If E_{τ_1}. E_{τ_2} are as in application 1.7, the theorem of Hahn-Banach implies that they have the same closed, convex sets. This, together with the preceeding lemmas, proves the assertion of application 1.7.

Remark 1.11. If f is a strongly continuous, convex real function on a Banach space E, then f is weakly l.s.c. on E.

Proof: Apply remark 1.10 with τ_1 the strong topology, and τ_2 the weak topology on E. Since E_{τ_1}, E_{τ_2} have the same dual space, a convex set is weakly closed if and only if it is strongly closed by remark 1.10

Now apply the above with E replaced by $E \times \mathbb{R}$. Since f is convex, strongly continuous, epigraph f is convex, strongly closed in $E \times \mathbb{R}$, so it is weakly closed and f is weakly l.s.c.

Application 1.12. Let E be a reflexive Banach space, f a strongly continuous, convex function defined on a closed, bounded, convex subset C of E. Then, f attains its minimum on C.

Proof: Since C is strongly closed and convex, C is weakly closed. (See remark 1.12). Therefore since C is bounded and E is reflexive, C is weakly compact. The function f is weakly l.s.c. by remark 1.12, so the result follows directly from Theorem 1.4.

Application 1.13. In a reflexive Banach space E, each point has a projection on a closed, convex set C.

Proof: Fix $x \in E$. We wish to show that $f(y) = \|x-y\|$ attains its minimum on C. It is easy to see that f is strongly continuous, convex. Let $\alpha = \inf_{y \in C} \|x-y\|$, $\tilde{C} = C \cap \{y : \|x-y\| \leq \alpha + 1\}$. Then the minimum for f on $\tilde{C}$ and C are the same and f attains its minimum on $\tilde{C}$, since $\tilde{C}$ is closed, bounded, and convex, by application 1.12.

Example 1.14. An example of a nonreflexive space E with a closed subspace C with the property that if $x \notin C$, then x has no projection on C.

Let $E = L^1(0,1)$

$$C = \{f \in L^1(0,1) : \int_0^1 tf(t)dt = 0\}$$

C is clearly closed subspace of E .

For $f \in C$, $g \in E$

$$\|f-g\|_1 = \int_0^1 |f(t)-g(t)|dt \geq \int_0^1 t|f(t)-g(t)|dt$$

$$\geq \left|\int_0^1 t(f(t)-g(t))dt\right| = \left|\int_0^1 tg(t)dt\right| .$$

And equality can hold if and only if

$$|f(t)-g(t)| = t|f(t)-g(t)| \quad \text{a.e.}$$

which implies that $g = f \in C$. Therefore $g \notin C$ implies that $\|f-g\|_1 > \left|\int_0^t tg(t)dt\right|$ for every $f \in C$. However, $\text{dist}(g,C) = \left|\int_0^1 tg(t)dt\right|$.

Define

$$f_n(t) = \begin{cases} g(t) & \text{for } 0 < t < 1 - \frac{1}{n} \\ a_n \int_0^{1-\frac{1}{n}} s\,g(s)ds & \text{for } 1 - \frac{1}{n} \leq t \leq 1 \end{cases}$$

Then $\int_0^1 tf_n(t)dt = \int_0^{1-\frac{1}{n}} tg(t)dt + a_n \int_0^{1-\frac{1}{n}} sg(s)ds \int_{1-\frac{1}{n}}^1 t\,dt$

$$= \left[\int_0^{1-\frac{1}{n}} tg(t)dt\right]\left(1+a_n\left(\frac{1}{n} - \frac{1}{2n^2}\right)\right) .$$

So, taking $a_n = -\frac{2n^2}{-2n}$, $f_n \in C$.

$$\|g-f_n\|_1 = \int_{1-\frac{1}{n}}^1 |g(t) - f_n(t)|dt \leq \int_{1-\frac{1}{n}}^1 |g(t)|dt + \int_{1-\frac{1}{n}}^1 |f_n(t)|dt$$

$$= \int_{1-\frac{1}{n}}^1 |g(t)|dt + \frac{1}{n}\left(\frac{2n^2}{2n-1}\right)\left|\int_0^{1-\frac{1}{n}} tg(t)dt\right|$$

$$= \int_{1-\frac{1}{n}}^1 |g(t)|dt + \frac{1}{1-\frac{1}{2n}}\left|\int_0^{1-\frac{1}{n}} tg(t)dt\right| \to \left|\int_0^1 t\,g(t)dt\right|$$

as $n \to \infty$.

This proves that $\mathrm{dist}(g,C) = |\int_0^1 tg(t)dt|$, and, since

$$\|f-g\|_1 = |\int_0^1 tg(t)dt| \quad \text{implies } f = g ,$$

every $g \notin C$ has no projection on C .

2. Control of linear systems with quadratic (convex) cost function.

Introduction.

We suppose the system state $y(t)$ satisfies a linear evolution equation of the form

$$(2.1) \qquad \begin{cases} \frac{dy}{dt} + Ay(t) = f(t) + Bu(t) & \text{for } 0 \le t \le T \\ y(0) = y_0 \end{cases}$$

where y_0 and f are given data and $T < \infty$. The function $u(t)$ is called the control and is to be chosen (perhaps with some constraints) to minimize a cost function $J(u)$ of the form

$$(2.2) \qquad J(u) = \int_0^T Q_1(y,u)dt + Q_2(y(T))$$

where Q_1, Q_2 are quadratic.

A control minimizing $J(u)$, when it exists, is called optimal.

Example 2.1. Consider a simple case of (2.1),

$$(2.3) \qquad \begin{cases} \frac{dy}{dt} = u(t) & 0 \le t \le T \\ y(0) = 0 \end{cases}$$

where the control $u(t)$ has the constraint

$$(2.4) \qquad u(t) \in [-1,+1] \qquad \text{a.e.}$$

and the cost function is given by

$$(2.5) \quad J(u) = \int_0^T (|y(x)|^2 - |u(t)|^2 dt = \int_0^T (|\int_0^t u(s)ds|^2 - |u(t)|^2)dt .$$

Claim: $J(u) > -T$ for all u satisfying (2.4).

Proof: $\int_0^T |y(t)|^2 dt \geq 0$ and $\int_0^T |u(t)|^2 dt \leq T$ by (2.4). So

$$J(u) = \int_0^T (|y(t)|^2 - |u(t)|^2)dt \geq -T .$$

If equality holds, $u^2 = 1$ a.e. and $y^2 = 0$ a.e. But, if $y = 0$, the $u = \frac{dy}{dt} = 0$ in the distribution sense. This contradicts $u^2 = 1$ a.e. and proves the claim.

Claim: $\inf J(u) = -T$, the inf taken over all u satisfying (2.4).

Define

$$u_n(t) = \begin{cases} +1 & \text{for } t \varepsilon [\frac{2K}{N}, \frac{2K+1}{N}) \cap [0,T] \quad K = 0,1,2,\ldots \\ -1 & \text{for } t \varepsilon [\frac{2K+1}{N}, \frac{2K+2}{N}) \cap [0,T) \quad K = 0,1,2,\ldots \end{cases}$$

Then $u_n^2(t) = 1$ for $t \varepsilon [0,T]$ and since $y_n(t) = \int_0^t u_n(s)ds$ $|y_n(t)| \leq \frac{1}{n}$ for $t \varepsilon [0,T]$. So

$$J(u_n) = \int_0^T (|y_n(t)|^2 - |u_n(t)|^2)dt = \int_0^T |y_n(t)|^2 dt - T$$

$$\leq -T + \frac{T}{n^2} \rightarrow -T \text{ as } n \rightarrow \infty .$$

This proves the claim. In this example no optimal control exists.

Remark 2.2. In example 2.1, $u_n \rightarrow 0$ weakly-star in $L^\infty(0,T)$ as $n \rightarrow \infty$, i.e. $\int_0^T f(t)u_n(t)dt \rightarrow 0$ as $n \rightarrow \infty$ for every $f \in L^1(0,T)$, even though $u_n^2 = 1$ for every n.

Proof: For simplicity of notation, take $T = 1$. Since $C[0,1]$ is dense in $L^1(0,1)$ it is enough to show that

$$\int_0^1 f(t)u_n(t)dt \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for } f \in C[0,1].$$

Since such f are uniformly continuous on $[0,1]$ for each n, there is an $\epsilon(\frac{1}{n}) > 0$ such that

$$|f(t+\frac{1}{n}) - f(t)| < \epsilon(\frac{1}{n}) \qquad \text{for } t \in [0,1]$$

and $\epsilon(\frac{1}{n}) \to 0$ as $n \to \infty$.

Now $$\left|\int_{\frac{2K}{n}}^{\frac{2K+2}{n}} f(t)u_n(t)dt\right| = \left|\int_{\frac{2K}{n}}^{\frac{2K+1}{n}} f(t)dt - \int_{\frac{2K+1}{n}}^{\frac{2K+2}{n}} f(t)dt\right|$$

$$= \left|\int_{\frac{2K}{n}}^{\frac{2K+1}{n}} (f(t) - f(t+\frac{1}{n}))dt\right| \leq \frac{1}{n}\epsilon(\frac{1}{n}) .$$

Therefore, for even n,

$$\left|\int_0^1 f(t)u_n(t)dt\right| \leq \sum_{K=0}^{\frac{n}{2}-1} \left|\int_{\frac{2K}{n}}^{\frac{2K+2}{n}} f(t)u_n dt\right| \leq \frac{1}{2}\epsilon(\frac{1}{n}) \to 0 \quad \text{as } n \to \infty$$

and, for odd n,

$$\left|\int_0^1 f(t)u_n(t)dt\right| \leq \sum_{K=0}^{\frac{n-1}{2}-1} \left|\int_{\frac{2K}{n}}^{\frac{2K+2}{n}} f(t)u_n(t)dt\right| + \left|\int_{\frac{n-1}{n}}^{n} f(t)u_n(t)dt\right| \leq \frac{1}{2}\epsilon(\frac{1}{n}) + \frac{1}{n}\|f\|_\infty$$

which also $\to 0$ as $n \to \infty$ and completes the proof. This remark motivates the following

<u>Example 2.3:</u>

Let $C = \{(u,v) \in L^\infty(0,T) \times L^\infty(0,T)$: there exists u_n, with $\|u_n\|_\infty \leq 1$, satisfying $u_n \rightharpoonup u$ and $u_n^2 \rightharpoonup v$ weakly star in $L^\infty(0,T)\}$

i.e. C is the weak-star closure in $L^\infty(0,T) \times L^\infty(0,T)$ of the set $\{(u,u^2)$: $\{(u,u^2) : \|u\|_\infty \leq 1\}$. For $(u,v) \in C$ define y by (2.3) and let

$$\tilde{J}(u,v) = \int_0^T (|y(t)|^2 - v(t))dt .$$

<u>Claim:</u> If $(u_n, u_n^2) \rightharpoonup (u,v)$ weakly-star as $n \to \infty$, then $J(u_n) \to \tilde{J}(u,v)$ as $n \to \infty$.

Proof: Since $u_n \rightharpoonup u$ weakly-star,

$$y_n(t) = \int_0^t u_n(s)ds \to \int_0^t u(s)ds = y(t) \quad \text{for each } t \in [0,T] .$$

By considering $f \equiv 1$ in $L^1(0,T)$. And

$$|y_n(t)|^2 = (\int_0^t u_n(s)ds)^2 \le (\int_0^T |u_n(s)|ds)^2 \le T^2 .$$

The dominated convergence theorem then implies that

$$\int_0^T |y_n(t)|^2 dt \to \int_0^T |y(t)|^2 dt \quad \text{as} \quad n \to \infty$$

and

$$\int_0^T u_n^2(t)dt \to \int_0^T v(t)dt \quad \text{since } u_n^2 \rightharpoonup v \text{ weakly star as } n \to \infty .$$

This proves the claim.

It is easy to see, as in example 2.1, that

$$\tilde{J}(u,v) \ge -T \quad \text{for } (u,v) \in C$$

and if u_n is chosen as in example 2.1,

$$(u_n, u_n^2) \rightharpoonup (0,1) \text{ by remark 2.2,}$$

so $(0,1) \in C$ and $\tilde{J}(0,1) = -T$. Hence the "generalized problem"

$$\inf_{(u,v) \in C} \tilde{J}(u,v)$$

has the optimal control $(0,1)$, which may be thought of as a "generalized solution" to Example 2.1.

We now consider equation (2.1) in an abstract setting. Let V be a real Hilbert space continuously and densely imbedded in a real Hilbert space H . H is identified with its dual so that $V \subset H \subset V'$. Let $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ denote the norms in V, H, and V' respectively. Suppose $A \in \mathcal{L}(V,V')$ satisfying $(Au,u) \ge \alpha\|u\|^2$, for some $\alpha > 0$. (or, more generally, $(Au,u) \ge \alpha\|u\|^2 - \beta|u|^2$) .

Theorem 2.4. Under the above hypothesis, if $y_0 \in H$ and $f \in L^2(0,T;V')$, there is a unique y satisfying

$$(2.6)\quad \begin{cases} \frac{dy}{dt} + Ay(t) = f(t) & 0 < t \leq T \\ y(0) = y_0 \end{cases}$$

having the properties:

$$y \in L^2(0,T;V) \cap C(0,T;H)$$
$$\frac{dy}{dt} \in L^2(0,T,V')$$

and y is a continuous function of f and y_0.

Proof: follows from previous results.

Remark 2.5.

Let E be a real reflexive Banach space. The set of admissible controls for equation (2.1) is denoted by U_{ad}. U_{ad} will always be a closed, convex subset of $L^2(0,T;E)$.

If $B \in \mathcal{L}(E;V')$, then by Theorem 2.4, (2.1) has a unique solution y for each $u \in U_{ad}$, and y is a continuous affine function of u.

Remark 2.6. We will consider a cost function of the form

$$(2.7)\quad J(u) = \int_0^T (\tfrac{1}{2}(Cy,y)+(d,y) + \tfrac{1}{2}(Nu,u) + (v,u))dt + \tfrac{1}{2}(Ky(T),y(T)) + (k, y(T))$$

where

$$(2.8)\quad \begin{cases} C \in \mathcal{L}(V,V'), \ C = C^*, \ C \geq 0, \ d \in L^2(0,T;V') \\ N \in \mathcal{L}(E,E'), \ N = N^*, \ N \geq 0, \ v \in L^2(0,T;E) \\ K \in \mathcal{L}(H,H), \ K = K^*, \ K \geq 0, \ k \in H \end{cases}$$

Lemma 2.7. $u \to J(u)$ is a quadratic strongly continuous function on $L^2(0,T;E)$.

Proof: Follows immediately from (2.7) and the fact that y depends continuously on u.

Lemma 2.8. If $C \geq 0$, (i.e. $(C\omega,\omega) \geq 0$ for every $\omega \in V$), $N \geq 0$, and $K \geq 0$, then $J(u)$ is a convex function of u.

Proof: Follows immediately from the fact that y is an affine function of u and the the general result that

$$\omega \to (R\omega,\omega) \text{ is convex if } R \geq 0 .$$

This result is a consequence of the identity

$$(2.8)\quad (R(\theta\omega_1 + (1-\theta)\omega_2), \theta\omega_1 + (1-\theta)\omega_2) = \theta(R\omega_1,\omega_1) + (1-\theta)(R\omega_2,\omega_2) - \theta(1-\theta)(R(\omega_1-\omega_2),\omega_1-\omega_2) \qquad \text{for } 0 \leq \theta \leq 1 .$$

Remark 2.9. We will assume the coercivity condition that either

i) u_{ad} is bounded

or ii) there exist $\nu > 0$ such that $(Nu,u) \geq \nu \|u\|_E^2$ for all $u \in E$.

We will need the following lemmas which are a consequence of previous results

Lemma 2.10. Let $W = \{u \in L^2(0,T;V) : \frac{du}{dt} \in L^2(0,T;V')\}$ with $\|u\|_W^2 = \int_0^T (\|u(t)\|^2 + \|\frac{du}{dt}\|_*^2)dt$.

Then

i) W is a Hilbert space

ii) $\mathcal{D}([0,T];V)$ is dense in W

iii) $W \subset C([0,T];H)$

iv) for $u,v \in W$

$$\int_s^t ((u(\sigma), \frac{dv}{dt}(\sigma)) + (\frac{du}{dt}(\sigma),v(\sigma))\, d\sigma = (u(t),v(t)) - (u(s),v(s)) .$$

So, in particular, taking $u = v$,

$$\frac{d}{dt}|u(t)|^2 = 2(\frac{du}{dt}(t),u(t)) \in L^1(0,T).$$

Lemma 2.11. Suppose $A \in \mathcal{L}(V,V')$, $(Au,u) \geq \alpha\|u\|^2$, $\alpha > 0$, $u \in V$. Then $y \in W \to (\frac{dy}{dt} + Ay, y(0))$ is an isomorphism of W onto $L^2(0,T;V') \times H$. Or, equivalently, for every $f \in L^2(0,T;V')$ and $y_0 \in H$ there exists a unique $y \in W$ satisfying

$$(2.8) \qquad \begin{cases} \frac{dy}{dt} + Ay = f \\ y(0) = y_0 . \end{cases}$$

(Note this is a direct consequence of Theorem 2.4). And taking the scalar product of (2.9) with y and applying Young's inequality we obtain the estimate

$$(2.10) \quad \tfrac{1}{2}\frac{d}{dt}|y(t)|^2 + \alpha\|y(t)\|^2 \leq \|f(t)\|_*\|y(t)\| \leq \epsilon\|y(t)\|^2 + \frac{1}{4\epsilon}\|f(t)\|_*^2, \ \epsilon > 0 .$$

Taking $\epsilon = \frac{\alpha}{2}$ we obtain

$$(2.11) \qquad \|y\|_{L^2(0,T;V)} \leq (\text{constant})(|y_0| + \|f\|_{L^2(0,T;V')})$$

and a similar inequality is true with the left side replaced by $|y|_{C([0,T];H)}$ or $\|\frac{dy}{dt}\|_{L^2(0,T;V')}$

Let us summarize the situation to this point. By replacing f in Lemma 2.11 by $f + Bu$ where $B \in \mathcal{L}(E,V')$ and $u \in L^2(0,T;E)$, we have that each control u in a given nonempty, closed, convex subset U_{ad} uniquely determines a state $y(t)$ determined by the equation

$$\begin{cases} \frac{dy}{dt} + Ay(t) = f(t) + Bu(t) & 0 \leq t \leq T \\ y(0) = y_0 & \text{where } y_0 \in H \text{ is given.} \end{cases}$$

We seek to minimize over $u \in U_{ad}$ the cost function $J(u)$ given by (2.7). We have seen that if C, K, N satisfy (2.8) then $J(u)$ is a continuous convex function of u. The existence of an optimal control, or a $u \in U_{ad}$ minimizing $J(u)$ is guaranteed by the following

Theorem 2.12. Assume the coercivity condition that either

i) U_{ad} is bounded

or ii) there exists $\nu > 0$ such that $(Nu,u) \geq \nu \|u\|_E^2$ for all $u \in E$.

Then, under the above hypotheses, an optimal control exists. If $(Nu,u) > 0$ for $u \neq 0$, this optimal control is unique.

Proof: Since $J(u)$ is convex, strongly continuous, it is weakly l.s.c. by Remark 1.11. So, if U_{ad} is bounded, it is weakly compact in $L^2(0,T;E)$, and $J(u)$ attains its minimum on U_{ad} by Theorem 1.4. This completes the proof of Case i).

Now assume ii) holds. Write

$$J(u) = \frac{1}{2}\int_0^T (Nu,u)dt + J_1(u)$$

where $J_1(u)$ is convex, continuous. Since J_1 is convex, continuous, it is bounded below by affine function, so there exists $a,b \geq 0$ such that

(2.12) $$J_1(u) \geq -a\|u\|_{L^2(0,T;E)} - b .$$

So, by (2.12) and ii),

(2.13) $$J(u) \geq \frac{\nu}{2}\|u\|^2_{L^2(0,T;E)} - a\|u\|_{L^2(0,T;E)} - b .$$

Fix $u_0 \in U_{ad}$. Let $\mathcal{B} = \{u \in L^2(0,T;E) : \frac{\nu}{2}\|u\|^2 - a\|u\| - b \leq J(u_0)\}$. By completing the square it is easy to see that $\mathcal{B}$ is a bounded subset of $L^2(0,T;E)$.

By (2.13),

(2.14) $$\inf_{u \in U_{ad}} J(u) = \inf_{u \in U_{ad} \cap \mathcal{B}} J(u)$$

and $U_{ad} \cap \mathcal{B}$ is a bounded set to which case i) applies.

To prove uniqueness, note that if $(Nu,u) > 0$ for $u \neq 0$, then

$u \to (Nu,u)$ is strictly convex (see (2.8)).

Hence $u \to J(u)$ is strictly convex, that is

(2.15) $$J(\theta u_1 + (1-\theta)u_2) < \theta J(u_1) + (1-\theta)J(u_2) \text{ for } 0 < \theta < 1, \quad u_1 \neq u_2 .$$

If J attains its minimum on U_{ad} at both u_1 and u_2 then $J(\theta u_1 + (1-\theta)u_2) < \min J$ by (2.15). But $\theta u_1 + (1-\theta)u_2 \in U_{ad}$ since U_{ad} is convex. Contradiction.

Lemma 2.13. $u \to J(u)$ is differentiable in the sense that

$$\frac{d}{d\lambda} J(u_1+\lambda u_2) \text{ exists at } \lambda = 0$$

for all $u_1, u_2 \in L^2(0,T;E)$.

Proof: Fix $u_1, u_2 \in L^2(0,T;E)$. Let $u = u_1 + \lambda u_2$. Then the solution y of

$$\begin{cases} \frac{dy}{dt} + Ay = f + B(u_1 + \lambda u_2) \\ y(0) = y_0 \end{cases}$$

is given by

(2.16) $$y = y_1 + \lambda z$$

where

$$\begin{cases} \frac{dy}{dt} + Ay_1 = f + Bu_1 \\ y_1(0) = y_0 \end{cases}$$

and

$$\begin{cases} \frac{dz}{dt} + Az = Bu_2 \\ z(0) = 0 \end{cases}$$

Clearly $\lambda \to y$ is differentiable with

$$\frac{dy}{d\lambda} = z \qquad \text{by (2.16).}$$

And

$$J(u_1 + \lambda u_2) = J(u) = \int_0^T (\tfrac{1}{2}(C(y_1 + \lambda z), y_1 + \lambda z) + (d, y_1 + \lambda z)$$

$$+ \tfrac{1}{2}(N(u_1 + \lambda u_2), u_1 + \lambda u_2) + (v, u_1 + \lambda u_2))\, dt$$

$$+ (K(y_1(T) + \lambda z(T), y_1(T) + \lambda z(T)) + (k, y_1(T) + \lambda z(T)) .$$

Hence

$$\frac{d}{d\lambda} J(u_1 + \lambda u_2) = \int_0^T [\tfrac{1}{2}(Cz, y_1 + \lambda z) + \tfrac{1}{2}(C(y_1 + \lambda z), z)$$

$$+ (d,z) + \tfrac{1}{2}(Nu_2, u_1 + \lambda u_2)$$

$$+ \tfrac{1}{2}(N(u_1 + \lambda u_2), u_2) + (v, u_2)]dt$$

$$+ (K(z(T)), y_1(T) + \lambda z(T)) + (K(y_1(T) + \lambda z(T)), z(T))$$

$$+ (k, z(T)) .$$

Using the hypothesis that $C = C^*$, $N = N^*$, $K = K^*$ we obtain, by putting $\lambda = 0$, that

$$\frac{d}{d\lambda} J(u_1 + \lambda u_2)\Big|_{\lambda=0} = \int_0^T [(Cy_1, z) + (d,z) + (Nu_1, u_2) + (v, u_2]dt \tag{2.17}$$

$$+ (K(y,(T))), z(T)) + (k, z(T))$$

<u>Lemma 2.14.</u> $u_0 \in U_{ad}$ is an optimal control if and only if

$$\frac{d}{d\lambda} J(u_0 + \lambda(\omega - u_0))\Big|_{\lambda=0} \geq 0 \quad \text{for every } \omega \in U_{ad} . \tag{2.18}$$

<u>Proof:</u> Suppose u_0 is an optimal control. Since U_{ad} is convex,

$$u_0 + \lambda(\omega - u_0) \in U_{ad} \quad \text{for } \lambda \varepsilon [0,1] .$$

(2.19) So $\quad J(u_0 + \lambda(\omega - u_0)) \geq J(u_0)$ for $\lambda \varepsilon [0,1]$.

By Lemma 2.3, $J(u_0 + \lambda(\omega - u_0))$ has a two-sided derivative at $\lambda = 0$, and by (2.19)

$$\frac{d}{d\lambda} J(u_0 + \lambda(\omega - u_0)) = \frac{d^+}{d\lambda} J(u_0 + \lambda(\omega - u_0)) = \lim_{\lambda \downarrow 0} \frac{J(u_0 + \lambda(\omega - u_0)) - J(u_0)}{\lambda} \geq 0 .$$

Conversely, suppose (2.18) holds. Let $R(\lambda) = J(u_0 + \lambda(\omega - u_0))$, then R is convex, $R(0) = J(u_0)$, and $R'(0) \geq 0$. Since R is convex, $R(\lambda) \geq R(0) + \lambda R'(0)$ for $\lambda \geq 0$. Therefore $J(\omega) = R(1) \geq R(0) = J(u_0)$ for all $\omega \in U_{ad}$.

Definition 2.15. The adjoint state P is given by the equation

$$(2.20) \qquad \begin{cases} -P'(t) + A^*P(t) = Cy(t) + d(t) & 0 \leq t \leq T \\ P(T) = K\,y(T) + k \end{cases}$$

Lemma 2.16. For every $u_1, u_2 \in L^2(0,T:E)$

$$\frac{d}{d\lambda} J(u_1 + \lambda u_2)\Big|_{\lambda=0} = \int_0^T (B^*P_1 + Nu_1 + V, u_2)dt$$

where P_1 satisfies (2.20) with $y = y_1$, y_1 satisfying (2.1) with $u = u_1$.

Proof: By (2.20), with z as in the proof of Lemma 2.13,

$$(2.21) \qquad \int_0^T (Cy_1 + d, z)dt + (Ky_1(T) + k, z(T)) = \int_0^T (-p_1' + A^*P_1, z)dt + (P_1(T), z(T)).$$

Using integration by parts, and the fact that z satisfies

$$\begin{cases} z' + Az = Bu_2 \\ z(0) = 0 \end{cases}$$

we obtain

$$\int_0^T (-P_1' + A^*P_1, z)dt + (p_1(T), z(T) = -\int_0^T (P_1', z)dt + \int_0^T (P_1, Az)dt + (P_1(T), z(T))$$

$$= \int_0^T (P_1, z')dt + \int_0^T (P_1, Az)dt$$

$$= \int_0^T (P_1, z' + Az)dt = \int_0^T (P_1, Bu_2)dt$$

or

$$(2.22) \qquad \int_0^T (-P_1' + A^*P_1, z)dt + (P_1(T), z(T)) = \int_0^T (B^*P_1, u_2)dt.$$

Combining (2.17), (2.21) and (2.22) we obtain the desired result.

Combining Lemmas 2.14 and 2.16 we have

<u>Theorem 2.17.</u> (Condition of optimality) u_1 is an optimal control if and only if

(2.23) $$\int_0^T (B^* P_1 + Nu_1 + v, w-u_1)dt \geq 0 \text{ for every } w \in U_{ad}.$$

<u>Remark 2.18.</u> If $U_{ad} = L^2(0,T;E)$, then (2.23) can hold if and only if

$$B^* P_1 + Nu_1 + v = 0 .$$

Hence u_1 is an optimal control if and only if

(2.24) $$\begin{cases} y_1' + Ay_1 = f + Bu_1 \\ -P_1' + A^* P_1 = Cy_1 + d \\ B^* P_1 + Nu_1 + v = 0 \\ y_1(0) = y_0 \\ P_1(T) = Ky_1(T) + k . \end{cases}$$

Or, equivalently,

(2.25) $$\begin{cases} y_1' + Ay_1 = f - BN^{-1}(B^* P_1 + v) \\ -P_1' + A^* P_1 = Cy_1 + d \\ y_1(0) = y_0 \\ P_1(T) = Ky_1(T) + k \end{cases}$$

<u>Note:</u> The hypothesis $(Nu,u) \geq \nu \|u\|_E^2$, $\nu > 0$ of Remark 2.9 implies directly that $N^{-1} \in \mathcal{L}(E',E)$ with $\|N^{-1}\|_{\mathcal{L}(E',E)} \leq \frac{1}{\nu}$.

3. The Riccati Equation

We will consider (2.25) beginning at some intermediate time $s \in [0,T]$ writing (2.25) as

$$(3.1) \qquad \begin{cases} y' + Ay + DP = g \\ -P' + A^*P - Cy = d \\ P(T) = Ky(T) + k \\ y(s) = h \end{cases}$$

where

$$(3.2) \qquad g = f - BN^{-1}v$$

$$(3.3) \qquad D = BN^{-1}B^*$$

and $h \in H$ is given.

If p,y satisfy (3.1) then $P = P_1 + P_2$ and $y = y_1 + y_2$ where P_1, y_1 satisfy

$$(3.4) \qquad \begin{cases} y_1' + Ay_1 + DP_1 = 0 \\ -P_1' + A^*P_1 - Cy_1 = 0 \\ P_1(T) = Ky_1(T) \\ y_1(S) = h \end{cases}$$

and P_2, y_2 satisfy

$$(3.5) \qquad \begin{cases} y_2' + Ay_2 + DP_2 = g \\ -P_2' + A^*P_2 - Cy_2 = d \\ P_2(T) = Ky_2(T) + k \\ y_2(S) = 0 \end{cases}$$

Since $P_1(S)$ is a linear function of h by (3.4) we write $p_1(S) = \mathbb{P}(S)h$, and denoting $P_2(S)$ by $r(S)$, independent of h, we obtain the form

(3.6) $$P(S) = \mathbb{P}(S)h + r(S) \quad \text{for } S \in [0,T]$$

(3.7) where $\mathbb{P}(S) \in \mathcal{L}(H,H)$, $r(S) \in H$ for $S \in [0,T]$.

If $\mathbb{P}$ and r are known, then the optimal control u at an intermediate time $S \in [0,T]$ may be calculated from the present state $y(S)$ of the system, independently of the initial state y_0. For, setting $h = y(S)$, we have

$$P(S) = \mathbb{P}(S)y(S) + r(S)$$

and from 2.24 $u(S) = -N^{-1}(B^*P(S) + v(S))$. So

(3.8) $$u(S) = -N^{-1}(B^*(\mathbb{P}(S)y(S) + r(S)) + v(S))$$

In this section we will show that, in a sense to be made precise later, $\mathbb{P}$ and r satisfy the equations

(3.9) $$\begin{cases} -\mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \mathbb{P}D\mathbb{P} = C \\ \mathbb{P}(T) = K \end{cases}$$

(3.10) $$\begin{cases} -r' + A^*r + \mathbb{P}Dr = d + \mathbb{P}(f - BN^{-1}v) \\ r(T) = k \end{cases}$$

We begin by developing some basic properties of $\mathbb{P}$.

<u>Lemma 3.1.</u> $\mathbb{P}^*(S) = \mathbb{P}(S)$ for every $S \in [0,T]$.

<u>Proof:</u> Suppose P, y satisfy (3.4) with $y(S) = h$ and $\bar{p}, \bar{y}$ satisfy (3.4) with $\bar{y}(S) = \bar{h}_1$ so that $P(S) = \mathbb{P}(S)h$ and $\bar{p}(S) = \mathbb{P}(S)\bar{h}$. Then

$$(\mathbb{P}(S)\ h, \bar{h}) = (P(S), \bar{y}(S) = -\int_S^T \frac{d}{dt}(p(t), \bar{y}(t)dt + (P(T), \bar{y}(T))$$

$$= \int_S^T [(A\bar{y}, P) + (D\bar{P}, P) - (A^*P, \bar{y}) + (Cy, \bar{y})]dt + (Ky(T), \bar{y}(T))$$

by (3.4). Now C, K are self-adjoint by hypothesis and $D = B^*N^{-1}B$ is self-adjoint since N is self-adjoint. So we obtain

$$(\mathbb{P}(S)h,\bar{h}) = \int_0^T [(\bar{y},A^*P) + (\bar{P},DP) - (P,A\bar{y}) + (y,C\bar{y})]dt + (y(T),K\bar{y}(T))$$

$$= \int_S^T [(Ay,\bar{P}) + (DP,\bar{P}) - (A^*\bar{P},y) + C\bar{y},y)]dt + (K\bar{y}(T),y(T))$$

since $(\bar{y},A^*P) - (P,A\bar{y}) = 0 = (Ay,\bar{P}) - (A^*\bar{P},y)$. So

$$(\mathbb{P}(S)h,\bar{h}) = -\int_S^T \frac{d}{dt}(\bar{P}(t), y(t))dt + (\bar{P}(T),y(T)$$

$$= (\bar{P}(S),y(S)) = (\mathbb{P}(S)\bar{h},h) .$$

<u>Lemma 3.2.</u> $\mathbb{P}(S) \geq 0$, i.e. $(\mathbb{P}(S)h,h) \geq 0$ for every $h \in H$.

<u>Proof:</u> As in the proof of Lemma 3.1, with $h = \bar{h}$, $P = \bar{P}$,

(3.11) $$(\mathbb{P}(S)h,h) = \int_S^T [(DP,P) + (Cy,y)]dt + (Ky(T),y(T)) \geq 0$$

since $D \geq 0$, $C \geq 0$, and $K \geq 0$.

<u>Remark 3.3.</u> $N \geq 0$ implies $D \geq 0$ since

$$(Dh,h) = (BN^{-1}B^*h,h) = (N^{-1}B^*h,B^*h)$$

$$= (\bar{h},N\bar{h}) \geq 0 \text{ where } \bar{h} = N^{-1}B^*h .$$

<u>Remark 3.4.</u> $\frac{1}{2}(\mathbb{P}(S)h,h)$ is the optimal cost for the control problem on $[S,T]$ starting from $y(S) = h$ with $f = v = d = k = 0$.

<u>Proof:</u> By 2.24 , since $v = 0$, the optimal control is $u = -N^{-1}B^*P$. Therefore

$$(DP,P) = (BN^{-1}B^*P,P) = (N^{-1}B^*P,B^*P)$$

$$= (-N^{-1}B^*P, - NN^{-1}B^*P) = (Nu,u).$$

So that

(3.12) $$J(u) = \frac{1}{2} \int_S^T [(Cy,y) + (Nu,u)]dt + (Ky(T),y(T)) = \frac{1}{2}(\mathbb{P}(S)h,h) \quad \text{by (3.11).}$$

<u>Remark 3.5.</u> If $\bar{C} \geq C$, $\bar{N} \geq N$, $\bar{K} \geq K$ then clearly from (3.12)

$$\bar{J}(u) \geq J(u)$$

and so by Remark 3.4.

$$(\bar{\mathbb{P}}(S)h,h) \geq (\mathbb{P}(S)h,h) \quad \text{or } \bar{\mathbb{P}} \geq \mathbb{P} .$$

This is an important monotonicity property which will be discussed in greater detail elsewhere.

In all of the following considerations we take $f = v = d = k = 0$.

Theorem 3.6. There is a constant M, independent of S , such that $\|P(S)\|_{\mathcal{L}(H,H)} \leq M$ for $0 \leq S \leq T$.

Proof: We first show there exists M such that $|(P(S)h,h)| \leq M|h|^2$ for $h \in H$, and $0 \leq S \leq T$. By Remark 3.4,

$$\tfrac{1}{2}(\mathbb{P}(S)h,h) = \inf_{u \in U_{ad}} J(u) \leq J(0) \tag{3.13}$$

J(0) is the optimal cost for the control problem with optimal control $u = 0$, that is when $B = 0$ which gives $D = 0$.

The proof then follows from (3.13) and the following claim. $J(0) \leq M|h|^2$ for every $h \in H$.

By definition

$$J(0) = \tfrac{1}{2}\int_S^T (Cy,y)dt + \tfrac{1}{2}(Ky(T),y(T)) \tag{3.14}$$

where y satisfies

$$\begin{cases} y' + Ay = 0 \\ y(S) = h \end{cases} \tag{3.15}$$

Multiplying both sides of (3.15) by y and using the coercivity condition

$$(A\omega,\omega) \geq \alpha\|\omega\|^2 \qquad \text{for all } \omega \in V \tag{3.16}$$

we obtain

$$\tfrac{1}{2}\frac{d}{dt}|y(t)|^2 + \alpha\|y(t)\|^2 \leq 0 \qquad \text{for } t \in [S,T] .$$

Integrate both sides from S to T and use $y(S) = h$,

$$\tfrac{1}{2}|y(T)|^2 + \alpha\int_S^T \|y(t)\|^2 dt \leq \tfrac{1}{2}|h|^2 . \tag{3.17}$$

Therefore, from (3.14),

$$J(0) \le \tfrac{1}{2}\|C\|_{\mathfrak{L}(V,V')} \int_S^T \|y(t)\|^2 dt + \tfrac{1}{2}\|K\|_{\mathfrak{L}(H,H)} |y(T)|^2$$
$$\le \tfrac{1}{2}M|h|^2 \qquad \text{by (3.17)}$$

with

$$M = 2\max\{\frac{\|C\|_{\mathfrak{L}(V,V')}}{2\alpha}, \ \|K\|_{\mathfrak{L}(H,H)}\},$$

Hence, by (3.13),

(3.18) $$|(\mathbb{P}(S)h,h)| \le M|h|^2 \quad \text{for } h \in H, \ 0 \le S \le T.$$

Now since $\mathbb{P}(S) = \mathbb{P}^*(S)$, $0 \le S \le T$,

(3.19) $$(\mathbb{P}(S)h,k) = \frac{1}{4\lambda}[(\mathbb{P}(S)(h+\lambda k),h+\lambda k) - (\mathbb{P}(S)(h-\lambda k),h-\lambda k)]$$

for $\lambda > 0$, $h,k \in H$. So, by (3.18)

(3.20) $$|(\mathbb{P}(S)h,k)| \le \frac{M}{4\lambda}(|h+\lambda k|^2 + |h-\lambda k|^2) = \frac{M}{4\lambda}(2|h|^2 + 2\lambda^2|k|^2)$$
$$= \frac{M}{2}(\frac{|h|^2}{\lambda} + \lambda|k|^2).$$

Taking $k = \mathbb{P}(S)h$ and $\lambda = \frac{|h|}{|\mathbb{P}(S)h|}$ in (3.20) gives

(3.21) $$|\mathbb{P}(S)h|^2 \le M|\mathbb{P}(S)h||h|$$

which completes the proof.

Theorem 3.7. If $D = 0$, A is independent of t and $C \in \mathfrak{L}(H,H)$, then

(3.22) $$\mathbb{P}(S)h = e^{-(T-S)A^*} K e^{-(T-S)A}h + \int_S^T e^{-(t-s)A^*} C e^{-(t-s)A} h dt.$$

Proof: A generates a semigroup of bounded operators e^{-tA} where $z(t) = e^{-tA}z_0$ satisfies

(3.23) $$\begin{cases} z' + Az = 0 \\ z(0) = z_0 \end{cases}$$

The backwards, nonhomogeneous problem

$$(3.24)\qquad \begin{cases} z' + A^* z = q \\ z(T) = zT \end{cases}$$

has the solution

$$z(S) = e^{-(T-S)A^*} z_T + \int_S^T e^{-(t-s)A^*} q(t)dt$$

we can explicitly solve

$$(3.25)\qquad \begin{cases} y' + Ay = 0 \\ y(s) = h \\ -P + A^* P = (y) \\ P(T) = Ky(T) \end{cases}$$

by (3.24),

$$(3.26)\qquad \mathbb{P}(S)h = P(S) = e^{-(T-S)A^*} P(T) + \int_S^T e^{-(t-s)A^*} Cy(t)dt$$

and by (3.25), $y(t) = e^{-(t-s)A}h$ for $s \leq t \leq T$.

So substituting $y(t) = e^{-(t-s)A}h$ and $P(T) = Ky(T) = Ke^{-(T-S)A}h$ into (3.26) gives (3.22).

Definition 3.8. $D{:}A) = \{h \in V : Ah \in H\}$.

The proof of Theorem 3.10 will require the following result of general semigroup theory.

Lemma 3.9. If $h \in D(A)$, then $e^{-\sigma A} h \in D(A)$ for $\sigma \geq 0$, and

$$(3.27)\qquad \frac{d}{d\sigma} e^{-\sigma A}h = -Ae^{-\sigma A}h = -e^{-\sigma A} Ah, \quad \sigma \geq 0.$$

Theorem 3.10. If $D = 0$, A is independent of t, $C \in \mathcal{L}(H,H)$, and $h,k \in D(A)$ then

$$(3.28)\qquad \frac{d}{ds}(\mathbb{P}(S)h,k) = (\mathbb{P}(S)Ah,k) + (\mathbb{P}(S)h,Ak) - (Ch,k).$$

Proof: By (3.22) and the fact that $(e^{-\sigma A})^* = e^{-\sigma A^*}$,

$$(\mathbb{P}(S)h,k) = (K\,e^{-(T-S)A}h,\ e^{-(T-S)A}k) + \int_S^T (Ce^{-(t-s)A}h, e^{-(t-s)A}k)dt .$$

Hence, by (3.27),

$$\begin{aligned}\frac{d}{ds}(\mathbb{P}(S)h,k) &= (Ke^{-(T-S)A}h, e^{-(T-S)A}Ak) \\ &+ (Ke^{-(T-S)A}Ah,\ e^{-(T-S)A}k) \\ &+ \int_S^T (Ce^{-(t-s)A}h,\ e^{-(t-s)A}\,Ak)dt \\ &+ \int_S^T (Ce^{-(t-s)A}Ah,\ e^{-(t-s)A}k)dt - (Ch,k) .\end{aligned}$$

By rearranging terms one easily obtains (3.29).

Remark 3.11. Formally speaking, (3.28) implies that

$$((-\mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} - C)h,k) = 0 \quad \text{for } h,k \in D(A)$$

so that in this sense $\mathbb{P}$ satisfies the Riccati equation

$$(3.29) \qquad \begin{cases} -\mathbb{P} + \mathbb{P}A + A^*\mathbb{P} = C \\ \mathbb{P}(T) = K \end{cases}$$

which is (3.9) with $D = 0$.

Remark 3.12. Theorems 3.7 and 3.10 hold, with the same proofs, if $C = C(t) \in L^\infty(0,T\,;\mathfrak{L}(H,H))$.

We will nowshow that $\mathbb{P}$ satisfies (3.25) in a stronger sense.

Theorem 3.13. Suppose

i) A is independent of t

ii) $C \in L^\infty(0,T;\mathfrak{L}(H,H))$.

Then $\mathbb{P}(S)$, defined by (3.4) has the following property:

If $z \in L^2(0,T;V)$, $z' \in L^2(0,T;V')$, and $z' + Az \in L^2(0,T;H)$, then $Pz \in L^2(0,T;V)$, $(\mathbb{P}z)' \in L^2(0,T;V')$ and

(3.30) $$(\mathbb{P}z)' \quad A^*\mathbb{P}z = Cz - \mathbb{P}(z' + Az)$$

$$\mathbb{P} \in L^{\infty}(0,T; \mathcal{L}(H,H)) \quad \text{and} \quad \mathbb{P}(T) = K .$$

Proof: Suppose, for the moment, that $z \in C^1(0,T;D(A))$ and choose $k \in D(A)$. Then

$$\frac{d}{ds}(\mathbb{P}(S)z(S),k) = (\mathbb{P}'(S)z(S),k) + (\mathbb{P}(S)z'(S),k)$$

$$= (\mathbb{P}(S)Az(S),k) + (\mathbb{P}(S)z(S),Ak) - (Cz(S),k) + (\mathbb{P}(S)z'(S),k)$$

by (3.28)

Or, in another form,

(3.31) $$-\frac{d}{ds}(\mathbb{P}(S)z(S),k) + (A^*\mathbb{P}(S)z(S),k) + (\mathbb{P}(S)(z'(S) + Az(S)),k) = (Cz(S),k) ,$$

for all $k \in D(A)$. Hence $q(S) = \mathbb{P}(S)z(S)$ satisfies

(3.32) $$\begin{cases} -q' + A^*q = Cz - \mathbb{P}(z' + Az) \\ q(T) = Kz(T) \end{cases}$$

Note that (3.32) has a solution q satisfying $q \in L^2(0,T:V)$, $q' \in L^2(0,T;V')$ if z satisfies only the hypotheses of this theorem. For such z choose a sequence $z_n \in C^1(0,T;D(A))$ satisfying $z_n \to z$ in $L^2(0,T;V)$, $z_n' \to z'$ in $L^2(0,T;V')$, $z_n' + Az_n \to z' + Az$ in $L^2(0,T;H)$, and $z_n(T) \to z(T)$ in H. Putting $z = z_n$ and $q = q_n = \mathbb{P}(S)z_n(S)$ in (3.32) and letting $n \to \infty$ completes the proof. Note that $\mathbb{P} \in L^{\infty}(0,T;\mathcal{L}(H,H))$ by Theorem 3.6.

Corollary 3.14. If $h \in D(A)$, $\mathbb{P}(S)h \in L^2(0,T;V)$ and $\mathbb{P}'(S)h \in L^2(0,T;V')$.

Proof: Take $z(S) \equiv h$ in Theorem 3.13.

Remark 3.15. The nonlinear case $D_1 \in L^{\infty}(0,T;\mathcal{L}(H,H))$, D_1 not necessarily zero can be reduced to the case $D_1 = 0$ if we allow A and C to be functions of t.

Proof: Put $\tilde{A}(t) = A + D_1(t)\mathbb{P}(t)$. Then if A satisfies $(Au,u) \geq \alpha\|u\|^2 - \beta|u|^2$ and $\mathbb{P}, D_1 \in L^\infty(0,T;\mathcal{L}(H,H))$, we have

$$(\tilde{A}(t)u,u) = (Au,u) + (D_1(t)\mathbb{P}(t)u,u)$$
$$\geq \alpha\|u\|^2 - \beta|u|^2 - [\|D_1\mathbb{P}\|_{L^\infty(0,T;\mathcal{L}(H,H))}]|u|^2$$
$$\geq \alpha\|u\|^2 - \gamma|u|^2$$

which is an estimate of the same type. Now if $\mathbb{P}(S)h = P(S)$ where P satisfies

$$(3.33)\qquad \begin{cases} y' + Ay + D_1P = 0 \\ -P' + A^*P - Cy = 0 \\ P(T) = Ky(T) \\ y(S) = h \end{cases}$$

Then

$$y' + Ay + D_1\mathbb{P}y = 0$$

and

$$-P' + A^*P + \mathbb{P}D_1P = Cy + \mathbb{P}D_1P = (C + \mathbb{P}D_1\mathbb{P})y \ .$$

Using $\tilde{A}^* = A^* + \mathbb{P}D_1$, since $\mathbb{P}, D_1$ are symmetric, and putting $\tilde{C} = C + \mathbb{P}D_1\mathbb{P}$ we have

$$(3.34)\qquad \begin{cases} y' + \tilde{A}y = 0 \\ -P' + \tilde{A}^*P - \tilde{C}y = 0 \\ P(T) = Ky(T) \\ y(S) = h \end{cases}$$

which is the same as (3.33) except that $D_1 = 0$ and $\tilde{A}, \tilde{C}$ necessarily depend on t. Therefore it only remains to investigate (3.34) in the case where A depends on t.

We begin with a regularity result.

Theorem 3.16. Suppose $A(t) \in \mathcal{L}(V,V')$ with $(A(t)u,u) \geq \alpha\|u\|^2 - \beta|u|^2$ for $t \in [0,T]$, and there is a constant M independent of t such that $\|A(t)\|_{\mathcal{L}(V,V')} \leq M$. In addition, assume $A'(t) \in \mathcal{L}(V,V')$ and $\|A'(t)\|_{\mathcal{L}(V,V')} \leq M$ for all $t \in [0,T]$. Then the solution of

(3.35)
$$\begin{cases} y + Ay = f \\ y(0) = y_0 \end{cases}$$

has the following regularity property:

If $f, f' \in L^2(0,T;V')$, $y_0 \in V$, and $f(0) - A(0)y_0 \in H$, then
$$y, y' \in L^2(0,T;V) \cap L^\infty(0,T;H)$$
and
$$y'' \in L^2(0,T;V') .$$

Proof: The direct proof is suggested by the following formal procedure:

Formally differentiate (3.35) to obtain

(3.36)
$$y'' + Ay' + A;y = f' .$$

Putting $t = 0$ in (3.35) we have

(3.37)
$$y'(0) = f(0) - A(0)y_0 .$$

So y' formally satisfies

(3.38)
$$\begin{cases} (y')' + Ay' = f' - A'y \\ y'(0) = f(0) - A(0)y_0 \end{cases} .$$

Consider the equation

(3.39)
$$\begin{cases} z' + Az = f' - A'y \\ z(0) = f(0) - A(0)y_0 \end{cases} .$$

Since $\|A'(t)\| \leq M$, and $y \in L^2(0,T;V)$ and $f' \in L^2(0,T;V')$, $f' - A'y \in L^2(0,T;V')$.

Therefore, since $f(0) - A(0)y_0 \in H$ by hypothesis, by previous results we know that (3.39) has a solution z satisfying

$$z \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{3.40}$$
$$z \in L^2(0,T;V').$$

By our formal work, we would hope to show that $y' = z$, or more precisely, defining

$$\bar{y}(t) = y_0 + \int_0^t z(S)ds \tag{3.41}$$

we wish to show that $\bar{y} = y$. Note that $\bar{y} \in L^2(0,T;V)$, $\bar{y}' = z \in L^2(0,T;V)$.

Now

$$\frac{d}{dt}(A\bar{y}) = Az + A'\bar{y}$$

and

$$(A\bar{y})(0) = A(0)\bar{y}(0) = A(0)y_0 .$$

Hence

$$A(t)\bar{y}(t) = A(0)y_0 + \int_0^t A(S)z(S)ds + \int_0^t A'(S)\bar{y}(S)ds$$

and

$$\bar{y}'(t) = z(t) = z_0 + \int_0^t z'(S)ds .$$

Adding, and using (3.39) we obtain

$$\begin{aligned} \bar{y}' + A\bar{y} &= f(0) + \int_0^t (f'(S) - A'(S)y(S))ds + \int_0^t A'(0)\bar{y}(S)ds \\ &= f(t) - \int_0^t A'(S)y(S)ds + \int_0^t A'(S)\bar{y}(S)ds . \end{aligned} \tag{3.42}$$

Therefore, $r = \bar{y} - y$ satisfies

$$\begin{cases} r' + Ar = \int_0^t A'(S)r(S)ds \\ r(0) = 0 \end{cases} \tag{3.43}$$

with $r \in L^2(0,T;V)$, $r' \in L^2(0,T;V)$.

Multiply both sides of (3.43) by r to obtain

$$(3.44)\quad \tfrac{1}{2}\frac{d}{dt}|r(t)|^2 + \alpha\|r(t)\|^2 \le M\|r(t)\|\int_0^t\|r(s)\|ds = \frac{M}{2}\frac{d}{dt}\left(\int_0^t\|r(s)\|ds\right)^2 .$$

Integrating both sides of (3.44) from 0 to t we have,

$$(3.45)\qquad \tfrac{1}{2}|r(t)|^2 + \alpha\int_0^t\|r(s)\|^2 ds \le \frac{M}{2}\left(\int_0^t\|r(s)\|ds\right)^2 .$$

Apply Schwarz's inequality and rearrange terms to obtain

$$(3.46)\qquad |r(t)|^2 \le (Mt - 2\alpha)\int_0^t\|r(S)\|^2 ds$$

Hence $r(t) = 0$ for $t \le \frac{2\alpha}{M}$.

Now $\tilde{r}(t) = r(t + \frac{2\alpha}{M})$ satisfies (3.43) with $A(t)$ replaced by $\tilde{A}(t) = A(t + \frac{2\alpha}{M})$. So $r(t) = 0$ for $t \le 2(\frac{2\alpha}{M})$ Continuing this process shows that $r \equiv 0$ and $y = \bar{y}$, completing the proof of regularity.

Our next goal is to prove the validity of Theorem 3.13 when A depends on t, and $(A(t)u,u) \ge \alpha\|u\|^2 - \beta|u|^2$,

Remark 3.17. Let $z(t) = e^{-\beta t}y(t)$, then $y' + Ay = 0$ implies $z' + (A + \beta I)z = 0$ and $((A + \beta I)u,u) \ge (Au,u) + \beta(u,u) \ge \alpha\|u\|^2$. So we may assume that $\beta = 0$.

Lemma 3.18. Suppose $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, and A is constant on $]t_{i-1},t_i[$ for $i = 1,2,\ldots,n$. Then the conclusion of Theorem 3.13 holds.

Proof: Choose $i \in \{1,2,\ldots,n\}$ and on the interval $[t_{i-1},t_i]$ consider the problem

$$(3.47)\qquad \begin{cases} y' + Ay = 0 \\ -P' + A^*P = Cy \\ P(t_i) = \mathbb{P}(t_i)y(t_i) \end{cases}$$

here, as usual, $\mathbb{P}$ is defined by (3.4), however this $\mathbb{P}$ might also be thought of as defined by (3.47) on $[t_{i-1},t_i[$, since the solutions z,q on $[t_{i-1},t_i]$ of

$$(3.48)\qquad \begin{cases} z' + Az = 0 \\ -q' + A^* q = Cz \end{cases}$$

are determined uniquely by the data $z(t_{i-1})$ and $q(t_i)$. Now, if $z \in L^2(0,T;V)$ $z' \in L^2(0,T;V')$, $z' + Az \in L^2(0,T;H)$, then z has the same properties in $]t_{i-1},t_i[$, and since A is constant on $]t_{i-1},t_i[$, Theorem 3.13 implies that

$$(3.49)\qquad \mathbb{P} z \in L^2(t_{i-1},t_i;V),\ (\mathbb{P}z)' \in L^2(t_{i-1},t_i;V') \quad \text{and}$$

$$(3.50)\qquad -(\mathbb{P}z)' + A^*\mathbb{P}z = (z - \mathbb{P}(z' + Az) \quad \text{on} \quad [t_{i-1},t_i] .$$

Since this is true for every $i = 1,2,\ldots,n$, $\mathbb{P}z \in L^2(0,T;V)$, $(\mathbb{P}z)' \in L^2(0,T;V)$ and (3.49) holds on $[0,T]$. This completes the proof.

<u>Theorem 3.19.</u> Suppose $A \in \mathcal{L}^\infty(0,T;\mathcal{L}(V,V'))$, and there exist positive constants α,β such that

$$(3.51)\qquad (A(t)u,u) \geq \alpha\|u\|^2 - \beta|u|^2 \quad \text{for } u \in V,\ 0 \leq t \leq T .$$

Then the conclusion of Theorem 3.13 holds.

<u>Proof:</u> By Remark 3.17 we may assume $\beta = 0$.

For $N = 1,2,3,\ldots$ define

$$A_N(t) = \frac{N}{T}\int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} A(S)ds \quad \text{for} \quad t \in]\frac{iT}{N}, \frac{(i+1)T}{N}] ,\ i = 0,1,\ldots,N-1 .$$

Each A_n is piecewise constant and

$$(3.52)\qquad \|A_n\|_{L^P(0,T;\mathcal{L}(V,V'))} \leq \|A\|_{L^P(0,T;\mathcal{L}(V,V')}$$

for $N = 1,2,3,\ldots$, $1 \leq P \leq +\infty$, since

$$(3.53)\qquad \|A_N(t)\| \leq \frac{N}{T}\int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} \|A(S)\|ds \leq \frac{N}{T}(\int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} \|A(S)\|^P ds)^{1/P} (\frac{T}{N})^{1/P'}$$

$$= (\frac{N}{T})^{1/P} (\int_{\frac{iT}{N}}^{\frac{(1+1)T}{N}} \|A(S)\|^P ds)^{1/P}$$

so that

(3.54) $$\int_0^T \|A_N(t)\|^P dt = \sum_{i=0}^{N-1} \int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} \|A_N(t)\|^P dt \leq \frac{T}{N}\left[\frac{N}{T}\int_{\frac{iT}{N}}^{\frac{(i+1)T}{N}} \|A(S)\|^P ds\right]$$

$$= \int_0^T \|A(t)\|^P dt .$$

This is the argument for $1 \leq P < \infty$, and the case $P = \infty$ is clear. (3.52) implies that $L_N : A \to A_N$ is a uniformly bounded family of linear maps on $L^P(0,T;\mathcal{L}(V,V''))$. If $A \in C([0,T];\mathcal{L}(V,V''))$, it is easy to see that $L_N A(t) \to A(t)$ as $N \to \infty$ in $\mathcal{L}(V,V')$ uniformly for $t \in [0,T]$. Hence $A_N = L_N \to A$ in $L^P(0,T;\mathcal{L}(V,V'))$, and by a density argument we obtain

(3.55) $$A_N \to A \text{ as } N \to \infty \text{ in } L^P(0,T,\mathcal{L}(V,V')) \text{ for } 1 \leq P < +\infty .$$

(Note that $P = +\infty$ must now be excluded.)

We will also need the following result:

(3.56) $$A_N V \to AV \text{ in } L^2(0,T;V') \text{ strongly for every } V \in L^2(0,T;V).$$

This is true by a similar agrument: $M_N : V \to A_N V$ is a bounded linear map from $L^2(0,T;V)$ to $L^2(0,T;V')$ with norm dominated by $\|A\|_{L^\infty(0,T;\mathcal{L}(V,V')}$ If $V \in C(0,T;V)$, $A_N V \to AV$ is $L^2(0,T;V')$ strongly, by (3.55) with $P = 2$, so a density argument proves (3.56).

Next, we prove the following

<u>Claim:</u> Suppose $f_N \to f$ in $L^2(0,T;V')$ strongly and $u_{0N} \to u_0$ in H strongly. Let u_N, u be the unique solution of

(3.57) $$\begin{cases} u_N' + A_N u_N = f_N \\ u_N(0) = u_{0N} \end{cases} \quad \text{and} \quad \begin{cases} u' + Au = f \\ u(0) = u_0 \end{cases}$$

Then $u_N \to u$ in $L^2(0,T;V)$ strongly and $C(0,T,H)$ strongly.

Proof of claim: Let $w = u_N - u$, then w satisfies

$$(3.58)\qquad \begin{cases} w_N' + A_N w_N = g_N = f_N - f - (A_N u - Au) \\ w_N(0) = w_{0N} = u_{0N} - u_0 \end{cases}$$

Our hypotheses, together with (3.56) imply that $g_N \to 0$ in $L^2(0,T;V')$ strongly, and $w_{0N} \to 0$ in H strongly. Hence, since all the A_N are coercive with the same α, standard arguments complete the proof of the claim.

Now $\mathbb{P}_N(S)h$ is defined as $P_N(S)$ where

$$\begin{cases} y_N' + A_N y_N = 0 \\ y_N(S) = h \end{cases} \qquad \begin{cases} -P_N' + A_N^* P_N = Cy_N \\ P_N(T) = Ky_N(T) . \end{cases}$$

An application of the claim, shown that $y_N \to y$ in $L^2(S,T;V) \cap C^0(S,T;H)$ strongly where y is the solution of

$$\begin{cases} y' + Ay = 0 \\ y(S) = h . \end{cases}$$

Hence $Cy_N \to Cy$ in $L^2(S,T;V')$ strongly and $Ky_N(T) \to Ky(T)$ in H strongly. Now, applying the claim to P_N we have that $P_N \to P$ in $L^2(S,T;V) \cap C(S,T;H)$ strongly where P is the solution of

$$\begin{cases} -P' + A^* P = Cy \\ P(T) = Ky(T) \end{cases}$$

That is, $\mathbb{P}_N(S)h \to \mathbb{P}(S)h$ in H strongly for every $h \in H$, and $S \in [0,T]$.

Let z be given satisfying $z \in L^2(0,T;V)$, $z' \in L^2(0,T;V;)$, $z' + Az \in L^2(0,T;H)$. Let $f = z' + Az$, and let z_N be the solution of

$$\begin{cases} z_N' + A_N z_N = f \\ z_N(0) = z_0 . \end{cases}$$

By applying the claim, we have that $z_N \to z$ in $L^2(0,T;V) \cap C(0,T;H)$ strongly, so $\mathbb{P}_N(t)z_N(t) \to \mathbb{P}(t)z(t)$ in H strongly for every $t \in [0,T]$, since $\mathbb{P}_N(t)h \to \mathbb{P}(t)h$ for every $h \in H$, and $\|\mathbb{P}_N(t)\|_{\mathcal{L}(H,H)}$ is uniformly bounded by Theorem 3.6 and the fact that the A_N are coercive with the same α. Since A_N is piecewise constant and z_N satisfies $z_N \in L^2(0,T;V)$, $z_N' \in L^2(0,T;V')$, $z_N' + Az_N = f \in L^2(0,T;H)$, Lemma 3.18 implies that $\mathbb{P}_N z_N$ satisfies

$$(3.59)\qquad \begin{cases} -(\mathbb{P}_N z_N)' + A_N^*(\mathbb{P}_N z_N) = Cz_N - \mathbb{P}_N f \\ (\mathbb{P}_N z_N)(T) = Kz_N(T) \quad . \end{cases}$$

An application of Lebesgue's dominated convergence theorem shows that $\mathbb{P}_N f \to \mathbb{P}f$ in $L^2(0,T;H)$ strongly, so that $Cz_N - \mathbb{P}_N f \to Cz - \mathbb{P}f$ $L^2(0,T;H)$ strongly and $Kz_N(T) \to Kz(T)$ in H strongly since $z_N \to z$ in $C(0,T;H)$ strongly. Therefore, by yet another application of the claim, $\mathbb{P}_N z_N \to \zeta$ in $L^2(0,T;V) \cap C(0,T;H)$ strongly where ζ is the solution of

$$(3.60)\qquad \begin{cases} -\zeta' + A^*\zeta = Cz - \mathbb{P}f \\ \zeta(T) = Kz(T) \end{cases}$$

By above, $\mathbb{P}_N z_N \to \mathbb{P}_z$ pointwise in H strongly, so $\mathbb{P}z = \zeta$ and the proof is complete.

We now summarize the above results in the following:

Theorem 3.20. Suppose

i) $A \in L^\infty(0,T;\mathcal{L}(V,V'))$ and there exist $\alpha,\beta > 0$ such that $(A(t)u,u) \geq \alpha\|u\|^2 - \beta|u|^2$ for $u \in V$, $0 \leq t \leq T$.

ii) $C \in L^\infty(0,T;\mathcal{L}(H,H))$, $C(t) = C^*(t) \geq 0$.

iii) $D_1 \in L^\infty(0,T;\mathcal{L}(H,H))$, $D_1(t) = D_1^*(t)$.

iv) $K \in \mathcal{L}(H,H)$, $K = K^* \geq 0$.

Then for $0 \leq S \leq T$, the mapping $\mathbb{P}(S)$ from H to H defined by $\mathbb{P}(S)h = P(S)$ where P satisfies

$$(3.61)\qquad \begin{cases} y' + Ay + D_1 P = 0 \\ -P' + A^* P - Cy = 0 \\ P(T) = Ky(T) \\ y(S) = h \end{cases}$$

has the property that $\mathbb{P} \in \mathcal{L}^{\infty}(0,T;\mathcal{L}(H,H))$ and satisfies the Riccati equation

$$(3.62)\qquad \begin{cases} -\mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \mathbb{P}D_1\mathbb{P} = C \\ \mathbb{P}(T) = K \end{cases}$$

in the sense that if

$$(3.63)\qquad z \in L^2(0,T;V), \quad z' \in L^2(0,T;V'), \quad z' + Az \in L^2(0,T;H),$$

then

$$(3.64)\qquad \begin{cases} -(\mathbb{P}z)' + \mathbb{P}(z' + Az) + A^*\mathbb{P}z + \mathbb{P}D_1\mathbb{P}z = Cz \\ (\mathbb{P}z)(T) = Kz(T). \end{cases}$$

Proof: Combine Theorem 3.19 with Remark 3.15 and note that

$$\tilde{A} = A + C_1\mathbb{P} \in L^{\infty}(0,T;\mathcal{L}(H,H))$$

since $\mathbb{P}, D_1 \in L^{\infty}(0,T;\mathcal{L}(H,H))$ and $\tilde{A}^* = A^* + \mathbb{P}D_1$, since $\mathbb{P} = \mathbb{P}^*$ and $D_1 = D_1^*$. Furthermore, $\tilde{C} = C + \mathbb{P}D_1\mathbb{P} \in L^{\infty}(0,T;H)$ and $\tilde{C}^* = \tilde{C}$, and if $z' + Az \in L^2(0,T;H)$, then $z' + \tilde{A}z = z' + Az + \mathbb{P}D_1 z \in L^2(0,T;H)$ since $\mathbb{P}D_1 \in L^{\infty}(0,T;\mathcal{L}(H,H))$, $z \in C(0,T;H)$.

Now we return to the complete system

$$(3.65)\qquad \begin{cases} y' + Ay + D_1 P = g = f - BN^{-1}v \\ -P' + A^* P - Cy = d \\ P(T) = Ky(T) + k \\ y(S) = h \end{cases}$$

where $D_1 = BN^{-1}B^*$. Recall that the solution $P(S)$ is of the form $P(S) = \mathbb{P}(S)h + r(S)$. We wish to find an equation satisfied by r.

Theorem 3.21. Suppose the hypotheses of Theorem 3.20 hold and $f - BN^{-1}V = g \in L^2(0,T;H)$, then r satisfies

$$(3.66) \qquad \begin{cases} r' + A^*r + \mathbb{P}D_1 r = d + \mathbb{P}(f - BN^{-1}V) \\ r(T) = k \end{cases}$$

Proof. $r(S) = P(S) - \mathbb{P}(S)y(S)$ where $y \in L^2(0,T;V)$, $y' \in L^2(0,T,V')$, $y' + Ay = g - D_1P \in L^2(0,T:H)$ by hypothesis. Hence by Theorem 3.20,

$$(3.67) \qquad -(\mathbb{P}y)' + A^*\mathbb{P}y = Cy - \mathbb{P}D_1\mathbb{P}y - \mathbb{P}(y' + Ay).$$

Since

$$(3.68) \qquad -P' + A^*P = Cy + d,$$

we subtract (3.67) from (3.68) to obtain

$$(3.69 \qquad -r' + A^*r = d + \mathbb{P}D_1\mathbb{P}y + \mathbb{P}(g-D_1P) = d + \mathbb{P}g - \mathbb{P}D_1 r,$$

which yields (3.66).

Remark 3.22. Equations (3.62) and (3.66) may be used to determine the optimal control u_1 in the case $U_{ad} = L^2(0,T;E)$ (i.e. without constraints), by the following steps:

(1) Solve (3.62) to determine $\mathbb{P}$ (e.g. if $V = H = \mathbb{R}^N$, $\mathbb{P}$ satisfies a system of $\frac{N(N+1)}{2}$ ordinary differential equations).

(2) Substitute the result of (1) in (3.66) and calculate r. (e.g. if $V = H = \mathbb{R}^N$, r satisfies a system of N ordinary differential equations.)

(3) Find the state y by solving

$$(3.70) \qquad \begin{cases} y' + Ay = f - BN^{-1}B^*(\mathbb{P}(t)y(t) + r(t)) . \\ y(S) = h . \end{cases}$$

(4) The optimal control u is then given by

(3.71) $$u_1(t) = -N^{-1}B^*(\mathbb{P}(t)y(t) + r(t)) .$$

Remark 3.23. The Riccati equation (3.62) for $\mathbb{P}$ may be motivated by the following formal derivation:

Consider the control problem

(3.72) $$\begin{cases} y' + Ay = Bu, & u \in U_{ad} \subset L^2(0,T;E) \\ y(S) = h \end{cases}$$

with optimal cost starting from the state h at times given by

(3.73) $$\varphi(S,h) = \min_{u \in U_{ad}} \int_S^T (\tfrac{1}{2}(Cy,y) + \tfrac{1}{2}(Nu,u))dt + \tfrac{1}{2}(Ky(T),y(T)) .$$

Recall from Remark 3.4 that $\varphi(S,h) = \frac{1}{2}(\mathbb{P}(S)h,h)$, we wish to find an equation formally satisfied by φ. We first obtain an approximation of $\varphi(S-\Delta S,h)$, the optimal cost starting from h at time $S-\Delta S$. Assume there is a convex set $E_{ad} \subset E$ such that $U_{ad} = \{u : u(t) \in E_{ad} \text{ a.e.}\}$. During the time $[S-\Delta S,S]$ apply the constant control $u_0 \in E_{ad}$, then at time S the state y(S) is given by

(3.74) $$y(S) \approx h + \Delta S(Bu_0 - Ah) ,$$

using (3.72). Furthermore,

(3.75) $$\int_{S-\Delta S}^S \tfrac{1}{2}[(Cy,y) + (Nu_0,u_0)]dt \approx \tfrac{1}{2}\Delta S[(Ch,h) + (Nu_0,u_0)] .$$

Now during the time (S,T] apply the optimal control, then, combining the above we obtain

(3.76) $$\varphi(S-\Delta S,h) \approx \min_{u_0 \in E_{ad}} \{\tfrac{1}{2}\Delta S[(Ch,h) + (Nu_0,u_0)] + \varphi(S,h+\Delta S(Bu_0 - Ah))\}$$

$$\approx \min_{u_0 \in E_{ad}} \tfrac{1}{2}\{ \Delta S[(Ch,h) + (Nu_0,u_0)] + \varphi(S,h) + (\tfrac{\partial\varphi}{\partial h}(S,h), \Delta S(Bu_0-Ah))\}.$$

We also have the approximation

(3.77) $$\varphi(S - \Delta S, h) \approx \varphi(S,h) - \Delta S \frac{\partial\varphi}{\partial S}(S,h) .$$

Combining (3.76) and (3.77) we obtain

(3.78) $$-\frac{\partial\varphi}{\partial S}(S,h) \approx \min_{u_0 \in E_{ad}} \{\tfrac{1}{2}(Ch,h) + \tfrac{1}{2}(Nu_0, u_0) + (\frac{\partial\varphi}{\partial h}(S,h), Bu_0 - Ah)\}.$$

Since $\varphi(S,h) = \frac{1}{2}(\mathbb{P}(S)h,h)$,

(3.79) $$\frac{\partial\varphi}{\partial h} = \mathbb{P}(S)h , \qquad \frac{\partial\varphi}{\partial S} = \tfrac{1}{2}(\mathbb{P}'(S)h,h)$$

and, if $E_{ad} = E$, by (2.24) the optimal u_0 is given by

(3.80) $$Nu_0 + B^*\mathbb{P}(S)h = 0 .$$

Substituting (3.79) and (3.80) into (3.78) and using $D_1 = BN^{-1}B^*$ we obtain

(3.81) $$\begin{aligned} -\tfrac{1}{2}(\mathbb{P}'(S)h,h) &= \tfrac{1}{2}(Ch,h) + \tfrac{1}{2}(\mathbb{P}(S)h, D_1\mathbb{P}(S)h) - (\mathbb{P}(S)h, D_1\mathbb{P}(S)h) - (\mathbb{P}(S)h, Ah) \\ &= \tfrac{1}{2}(Ch,h) - \tfrac{1}{2}(\mathbb{P}(S)D_1\mathbb{P}(S)h,h) - \tfrac{1}{2}(A^*\mathbb{P}(S)h,h) - \tfrac{1}{2}(\mathbb{P}(S)Ah,h), \end{aligned}$$

since $\mathbb{P} = \mathbb{P}^*$. Hence

(3.82) $$-(\mathbb{P}'(S)h,h) + (\mathbb{P}(S)Ah,h) + (A^*\mathbb{P}(S)h,h) + (\mathbb{P}(S)D_1\mathbb{P}(S)h,h) = (Ch,h)$$

which is formally equivalent to (3.62). The cost function is a function defined by a minimum. The preceeding formal argument may be made mathematically accurate by means of the following theorem. We will investigate the continuity and differentiability properties of functions of the form

(3.83) $$g(x) = \min_{\lambda \in \Lambda} f(x,\lambda) , \qquad x \in X$$

where X is a metric space, Λ is a compact topological space, and $f: X \times \Lambda \to R$.

Theorem 3.24. Suppose

i) $x \to f(x,\lambda)$ is continuous from X to $\mathbb{R}$ for every $\lambda \in \Lambda$,

ii) $(x,\lambda) \to f(x,\lambda)$ is l.s.c. from $X \times \Lambda$ to $\mathbb{R}$.

Then g is continuous from X to $\mathbb{R}$ and for every $x \in X$, (3.84) $$g(x) = f(x,\lambda) \quad \text{for } \lambda \in \Lambda_x$$

where Λ_x is a (nonempty) compact subset of Λ.

Proof: By ii), $\lambda \to f(x,\lambda)$ is l.s.c. from Λ to $\mathbb{R}$ for each fixed $x \in X$. Hence, by Theorem 1.4, this function attains its minimum $g(x)$ for $\lambda \in \Lambda_x \subset \Lambda$, Λ_x compact. By i) $f(x,\lambda)$ is u.s.c. as a function of x for every $\lambda \in \Lambda$, hence the minimum $g(x)$ is u.s.c. It remains to show that g is l.s.c. Choose $x \in X$ and $x_n \to x$, with $\lambda_n \in \Lambda x_n$. So that $g(x_n) = f(x_n,\lambda_n)$. Since Λ is compact, there exists $\lambda \in \Lambda$ such that $\lambda_m \to \lambda$ for some subsequence $\{\lambda_m\}$ of $\{\lambda_n\}$. Then, by ii),

(3.85) $$g(x) \le f(x,\lambda) \le \liminf_{m\to\infty} f(x_m,\lambda_m) = \liminf_{m\to\infty} g(x_m) .$$

Since such a subsequence exists for every $x_n \to x$, g is l.s.c., and the proof is complete.

Theorem 3.25. Suppose $X = [0,\infty)$, the hypotheses of Theorem 3.24 hold, and, in addition,

iii) for every $\lambda \in \Lambda$, $f(x,\lambda)$ has a right derivative

(3.86) $$h(x,\lambda) = \frac{\partial f}{\partial x_+}(x,\lambda), \qquad x \ge 0 ,$$

and $(x,\lambda) \to h(x,\lambda)$ is l.s.c. from $X \times \Lambda$ to $\mathbb{R}$.

Then g has a right derivative at 0 given by

(3.87) $$\frac{\partial g}{\partial x_+}(0) = \min_{\lambda \in \Lambda_0} h(0,\lambda),$$

where Λ_0 is our former Λ_x with $x = 0$.

Proof: For $x > 0$, since $g(x) \le f(x,\lambda)$, $\lambda \in \Lambda$, and $g(0) = f(0,\lambda)$, $\lambda \in \Lambda_0$,

(3.88) $$\frac{g(x) - g(0)}{x} \le \frac{f(x,\lambda) - f(0,\lambda)}{x} \quad \text{for } \lambda \in \Lambda_0$$

Hence

(3.89) $$\limsup_{x\to 0} \frac{g(x) - g(0)}{x} \le \min_{\lambda \in \Lambda_0} h(0,\lambda) .$$

The theorem will be proved if we show that the opposite inequality holds with lim sup replaced by lim inf. Choose a sequence $\{x_n\} \subset (0,\infty)$ with $x_n \to 0$. Choose $\lambda_n \in \Lambda x_n$, and $\mu \in \Lambda$ such a subsequence $\lambda_m \to \mu$. Note that $\mu \in \Lambda_0$, since by ii) and the continuity of g,

(3.90) $$f(0,\mu) \le \liminf f(x_m,\lambda_m) = \liminf g(x_m) = g(0) .$$

Let $\epsilon > 0$ be given. By iii), there exists $\alpha > 0$ and $N(\mu)$ a neighborhood of μ in Λ such that

(3.91) $$h(x,\lambda) \ge h(0,\mu) - \epsilon \quad \text{for } 0 < x < \alpha ,\ \lambda \in N(\mu) .$$

Integrating (3.91) from 0 to x we obtain

(3.92) $$f(x,\lambda) - f(0,\lambda) \ge x(h(0,\mu) - \epsilon) ,\quad 0 < x < \alpha,\quad \lambda \in N(\mu) .$$

For large m, $0 < x_m < \infty$, and $\lambda_m \in N(\mu)$, so by (3.92),

(3.93) $$\begin{aligned} g(x_m) = f(x_m,\lambda_m) &\ge f(0,\lambda_m) + x_m(h(0,\mu) - \epsilon \\ &\ge g(0) + x_m(h(0,\mu) - \epsilon). \end{aligned}$$

Hence, for every $\epsilon > 0$,

(3.94) $$\liminf_{m\to\infty} \frac{g(x_m) - g(0)}{x_m} \ge h(0,\mu) - \epsilon$$

and, therefore, since $\mu \in \Lambda_0$,

(3.95) $$\liminf_{m\to\infty} \frac{g(x_m) - g(0)}{x_m} \ge \min_{\lambda \in \Lambda_0} h(0,\lambda) .$$

Since for every sequence $x_n \to 0$, such a subsequence $\{x_n\}$ exists, this completes the proof.

Remark 3.26. Suppose $X = \mathbb{R}$, the hypotheses of Theorem 3.25 hold, and $f(x,\lambda)$ has a two-sided derivative $h(x,\lambda)$ with $(x,\lambda) \to h(x,\lambda)$ continuous. Then $\bar{f}(x,\lambda) = f(-x,\lambda)$ has a right derivative $-h(-x,\lambda)$ which is l.s.c., so by Theorem 3.25, $\bar{g}(x) = g(-x)$ has a right derivative at 0.

So $g(x)$ has a left derivative at 0 given by

$$\frac{\partial g}{\partial x_-}(0) = -\min_{\lambda \in \Lambda_0} (-h(0,\lambda)) = \max_{\lambda \in \Lambda_0} h(0,\lambda) . \tag{3.96}$$

Note that the left and right derivative are equal if Λ_0 is a singleton.

Acknowledgement.

This is part of the lecture notes of a course given at the University of Wisconsin, Madison in 1974-75.

I wish to thank D. Brewer for the redaction of this part.

REFERENCE

Most of the material comes from

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EQUATIONS WITH ORDER PRESERVING PROPERTIES

L. Tartar

1. Introduction: Linear Results

We have seen in [4] that the solution $\mathbb{P}(t)$ of

$$(1.1) \qquad \begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \mathbb{P}D_1\mathbb{P} = D_2 \\ \mathbb{P}(0) = \mathbb{P}_0 \end{cases}$$

is increasing in $\mathbb{P}_0$, D_2 with respect to the order relation on $\mathcal{L}(H, H)$ given by

$$(1.2) \qquad Q \geq 0 \quad \text{if and only if} \quad (Qh, h) \geq 0 \quad \text{for every} \quad h \in H \ .$$

In this report we will investigate the class of operators which preserve this monotonicity property when substituted for the nonlinear term $\mathbb{P}D_1\mathbb{P}$ in (1.1).

We will first investigate monotonicity properties of second order equations of the type

$$(1.3) \qquad \begin{cases} -\Delta u = f \\ u(x) = 0, \quad x \in \partial\Omega \end{cases}$$

with respect to the order on $L^2(\Omega)$ given by

$$(1.4) \qquad u \geq v \quad \text{if and only if} \quad u(x) \geq v(x) \quad \text{for a.e.} \quad x \in \Omega \ .$$

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

Lemma 1.1. If $u \in H^1(\Omega)$, then $u_+ \in H^1(\Omega)$, and

$$\frac{\partial}{\partial x_i}(u_+) = \begin{cases} 0 & \text{if } u(x) \le 0 \\ \frac{\partial u}{\partial x_i} & \text{if } u(x) > 0 \end{cases} \qquad \text{a.e.,}$$

where $u_+(x) = \max(u(x), 0)$.

Proof. The proof will follow from a more general result.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz continuous, i.e. $|\varphi'(x)| \le K$ a.e. for some constant K, and suppose $\varphi(0) = 0$. We will show that if $u \in H^1(\Omega)$, then $\varphi(u) \in H^1(\Omega)$.

Case 1: Suppose $\varphi \in C^1(\mathbb{R})$. If u is smooth then

$$\frac{\partial}{\partial x_i}\varphi(u) = \varphi'(u)\frac{\partial u}{\partial x_i}$$

so that

(1.5) $$\left|\frac{\partial}{\partial x_i}\varphi(u)\right| \le K\left|\frac{\partial u}{\partial x_i}\right| .$$

If $u \in H^1(\Omega)$, for any ω such that $\bar{\omega} \subset \Omega$, u may be approximated in $H^1(\omega)$ by smooth functions u_n, so that since (1.5) holds in ω for each u_n, it holds in ω for u. Hence, since $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, (1.5) implies that $\frac{\partial}{\partial x_i}\varphi(u) \in L^2(\Omega)$, and $|\varphi(u)| \le K|u|$ shows that $\varphi(u) \in L^2(\Omega)$, hence $\varphi(u) \in H^1(\Omega)$.

Case 2: Using the result of case 1 we may drop the assumption $\varphi \in C^1$ by an approximation argument. If φ is Lipschitz, there exists $\varphi_n \in C^1(\mathbb{R})$, $|\varphi_n'| \le K$, such that $\varphi_n(t) \to \varphi(t)$ uniformly for $t \in \mathbb{R}$.

Then $\varphi_n(u) \to \varphi(u)$ uniformly, so that $\varphi(u) \in L^2(\Omega)$, and since

$$\left|\frac{\partial}{\partial x_i} \varphi_n(u)\right| \leq K\left|\frac{\partial u}{\partial x_i}\right| ,$$

$\{\frac{\partial}{\partial x_i} \varphi_n(u)\}$ is a bounded sequence in $L^2(\Omega)$, and so has a weak limit which must equal $\frac{\partial}{\partial x_i} \varphi(u)$ in the sense of distributions. Therefore

$$\left|\frac{\partial}{\partial x_i} \varphi(u)\right|_{L^2(\Omega)} \leq K\left|\frac{\partial u}{\partial x_i}\right|_{L^2(\Omega)} \quad \text{and} \quad \varphi(u) \in H^1(\Omega) .$$

Case 3. Suppose the sequence $\varphi_n \in C^1(\mathbb{R})$ of case 2 has the additional property that $\varphi_n'(t) \to \psi(t)$ pointwise, where $\varphi(t) = \int_0^t \psi(s)ds$.

In this case,

$$\frac{\partial}{\partial x_i} \varphi(u) = \psi(u) \frac{\partial u}{\partial x_i} \quad \text{a.e. for all } u \in H^1(\Omega) . \tag{1.6}$$

To see this, note that by case 1,

$$\frac{\partial}{\partial x_i} \varphi_n(u) = \varphi_n'(u) \frac{\partial u}{\partial x_i}, \quad n = 1, 2, 3, \ldots$$

and letting $n \to \infty$ proves (1.6). Returning to the proof of the lemma, we take $\varphi(t) = \max(t, 0)$. It is easy to see that Case 3 holds for this φ with, for example

$$\psi(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 . \end{cases}$$

Applying (1.6) with this ψ completes the proof.

Remark 1.2. Lemma 1.1 shows in particular that if $u \in H^1(\Omega)$, $A = \{x \in \Omega : u(x) = 0\}$, then $\frac{\partial u}{\partial x_i}(x) = 0$ for a.e. $x \in A$.

Theorem 1.3. Consider the bilinear form on $H^1(\Omega)$ given by

$$a(u, v) = \int_\Omega \Big(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_i a_i \frac{\partial u}{\partial x_i} v + a_0 uv\Big)dx \tag{1.7}$$

where $a_{ij} \in L^\infty(\Omega)$, a_i, $a_0 \in L^\infty(\Omega)$, $i, j = 1, \ldots, N$. Suppose V is a closed subspace of $H^1(\Omega)$ such that

i) a is coercive on V, i.e. there exists $\alpha > 0$ such that $a(u, u) \geq \alpha \|u\|^2$ for every $u \in V$, and

ii) for all $u \in V$, $u_+ \in V$.

Let L be a continuous linear form on V such that

$$L(v) \geq 0 \quad \text{whenever} \quad v \in V,\ v \geq 0 . \tag{1.8}$$

Then the unique solution u of

$$\begin{cases} a(u, v) = L(v) \quad \text{for all} \quad v \in V \\ u \in V \end{cases} \tag{1.9}$$

satisfies $u \geq 0$.

Proof. Note that the existence of a unique solution is guaranteed by Lax-Milgram's theorem. Let $u_- = -\min(u, 0)$, so that $u = u_+ - u_-$. By ii) since $u \in V$, $u_+ \in V$, and so $u_- \in V$. Putting $v = u_-$ in (1.9) we obtain

$$a(u_+ - u_-, u_-) = L(u_-) \geq 0 \tag{1.10}$$

since $u_- \geq 0$. Now $a(u_+, u_-) = 0$ for every $u \in H^1(\Omega)$ because $(\frac{\partial}{\partial x_i} u_+)(\frac{\partial}{\partial x_j} u_-) = 0$ a.e., by Lemma 1.1 (at a.e. $x \in \Omega$, at least one factor is zero). Therefore, by (1.10) and i)

(1.11) $$0 \leq L(u_-) = a(u_+ - u_-, u_-) = -a(u_-, u_-) \leq -\alpha \|u_-\|^2 .$$

Hence $u_- = 0$, so $u = u_+ \geq 0$.

Example 1.4. Let $V = H^1(\Omega)$ and

$$L(v) = \int_\Omega fvdx + \int_{\partial\Omega} gvd\sigma$$

where $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, (or since $v \in H^1(\Omega)$ implies $v \in H^{\frac{1}{2}}(\partial\Omega)$, one could take $g \in H^{-\frac{1}{2}}(\partial\Omega)$). Then $a(u,v) = L(v)$ for all $v \in H^1(\Omega)$ means that

$$-\sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + \sum_i a_i \frac{\partial u}{\partial x_i} + a_0 u = f \quad \text{in} \quad \Omega$$

and, at least formally,

$$\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \cos(\vec{n}, x_i) = g \quad \text{on} \quad \partial\Omega .$$

Theorem 1.3 say that if $f, g \geq 0$ and a is coercive on $H^1(\Omega)$, then $u \geq 0$. A particular case is if $f, g \geq 0$, $a_0 > 0$ then the solution of

$$\begin{cases} -\Delta u + a_0 u = f & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = g & \text{on} \quad \partial\Omega \end{cases}$$

satisfies $u \geq 0$.

Example 1.5. Let $V = H_0^1(\Omega)$, $L(v) = \int_\Omega fvdx$, $f \in L^2(\Omega)$. Then $a(u,v) = L(v)$ for every $v \in H_0^1(\Omega)$ means that u satisfies

$$\begin{cases} -\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum a_i \frac{\partial u}{\partial x_i} + a_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and Theorem 1.3 says that $f \geq 0$ implies $u \geq 0$.

Example 1.6. $\Omega = [0, \ell_1] \times [0, \ell_2]$, a square, with a periodicity condition

$$V = \{u \in H^1(\Omega) : u(0,y) = u(\ell_1, y) \text{ for } y \in [0, \ell_2]$$
$$\text{and/or } u(x,0) = u(x, \ell_2) \text{ for } x \in [0, \ell_1]\} .$$

Example 1.7. Let $V = H^1(\Omega)$, consider for $\epsilon > 0$ the bilinear form

$$a_\epsilon(u,v) = a(u,v) + \frac{1}{\epsilon}\int_{\partial\Omega} uvd\sigma .$$

Then a_ϵ is coercive whenever a is and

$$a_\epsilon(u_+, u_-) = 0 \quad \text{for every } u \in V .$$

Taking $L(v) = \int_\Omega fv + \int_{\partial\Omega} gv$ as in Example 1.4, $a(u_\epsilon, v) = L(v)$ for every $v \in V$ means that

$$(1.12) \qquad \begin{cases} -\sum \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_\epsilon}{\partial x_j}\right) + \sum a_i \frac{\partial u_\epsilon}{\partial x_i} + a_0 u_\epsilon = f \quad \text{in } \Omega \\ \sum_{i,j} a_{ij}\frac{\partial u_\epsilon}{\partial x_j}\cos(\vec{n}, x_i) + \frac{1}{\epsilon} u_\epsilon = g \end{cases}$$

and Theorem 1.3 says that $f \geq 0$, $g \geq 0$ implies $u_\epsilon \geq 0$.

Example 1.8. In Example 1.7, replace $L(v)$ by $L_\epsilon(v) = \int_\Omega fv + \frac{1}{\epsilon}\int_{\partial\Omega} gv$.

Then $a_\epsilon(u_\epsilon, v) = L_\epsilon(v)$ for every $v \in V$ means that u_ϵ satisfies (1.12) with g replaced by $\frac{1}{\epsilon} g$. We will show that as $\epsilon \to 0$, $u_\epsilon \to u$ weakly in $H^1(\Omega)$ where u satisfies

$$(1.13) \qquad \begin{cases} Au = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $Au = -\sum \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum a_i \frac{\partial u}{\partial x_i} + a_0 u$. In other words, a non-homogeneous Dirichlet problem may be obtained as a limit of variational problems. Given the truth of the above remark, if $f, g \geq 0$ then $u_\epsilon \geq 0$ for every $\epsilon > 0$, so $u \geq 0$. We will now prove that $u_\epsilon \to u$ satisfying (1.13). Choose $w \in H^1(\Omega)$ such that $w = g$ on $\partial\Omega$. (i.e. we require that $g \in H^{\frac{1}{2}}(\partial\Omega)$). Let $v_\epsilon = u_\epsilon - w$. Since $a_\epsilon(u_\epsilon, v) = L_\epsilon(v)$,

$$a(w + v_\epsilon, v) + \frac{1}{\epsilon}\int_{\partial\Omega}(w + v_\epsilon)v = \int_\Omega fv + \frac{1}{\epsilon}\int_{\partial\Omega} gv .$$

So, since $w = g$ on $\partial\Omega$,

$$(1.14) \qquad a(v_\epsilon, v) + \frac{1}{\epsilon}\int_{\partial\Omega} v_\epsilon v = \int_\Omega fv - a(w, v) \quad \text{for } v \in V .$$

Putting $v = v_\epsilon$ in (1.14) we obtain, for some constant C,

$$(1.15) \qquad \alpha\|v_\epsilon\|^2 + \frac{1}{\epsilon}\int_{\partial\Omega}|v_\epsilon|^2 \leq \int_\Omega fv_\epsilon - a(w, v_\epsilon) \leq C\|v_\epsilon\| ,$$

where $\|\cdot\|$ is taken in $H^1(\Omega)$. By (1.15), $\|v_\epsilon\| \leq C/\alpha$ for all $\epsilon > 0$, so that, taking a subsequence, there exists $v_\infty \in H^1(\Omega)$ such that $v_\epsilon \to v_\infty$ weakly in $H^1(\Omega)$ as $\epsilon \to 0$. Also, by (1.15),

$$\frac{1}{\epsilon}\int_{\partial\Omega} |v_\epsilon|^2 \leq C/\alpha, \quad \text{so that} \quad v_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{in} \quad L^2(\partial\Omega) .$$

Hence $v_\infty \in H_0^1(\Omega)$. By (1.14), for $v \in \mathcal{D}(\Omega)$ we have

$$a(v_\epsilon, v) = \int_\Omega fv - a(w, v)$$

so letting $\epsilon \to 0$ we obtain

$$a(v_\infty + w, v) = \int_\Omega fv \quad \text{for} \quad v \in \mathcal{D}(\Omega) . \tag{1.16}$$

Finally, (1.16) implies that $u = v_\infty + w$ satisfies (1.13).

<u>Example 1.9.</u> Let $a(u,v)$, defined as before, be coercive on $V = H^1(\Omega)$ with Ω bounded and $\beta \geq a_0(x) \geq \alpha > 0$ for every $x \in \Omega$. Consider the problem

$$\begin{cases} Au = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases}$$

with $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$. We will show that

$$u(x) \geq \min\left(\inf_{x\in\Omega} \frac{1}{\alpha} f(x), \ \inf_{x\in\Omega} \frac{1}{\beta} f(x), \ \inf_{y\in\partial\Omega} g(y)\right) \tag{1.17}$$

$$u(x) \leq \max\left(\sup_{x\in\Omega} \frac{1}{\alpha} f(x), \ \sup_{x\in\Omega} \frac{1}{\beta} f(x), \ \sup_{y\in\partial\Omega} g(y)\right) . \tag{1.18}$$

Let k denote the right-hand side of (1.17). Then, if $k \leq 0$, $f(x) - a_0(x)k \geq f(x) - \alpha k \geq 0$, and, if $k \geq 0$, $f(x) - a_0(x)k \geq f(x) - \beta k \geq 0$.

Also $g(y) - k \geq 0$ for $y \in \partial\Omega$. Hence, by Example 1.18, the solution $u - k$ of

$$\begin{cases} A(u - k) = f - a_0 k & \text{in } \Omega \\ u - k = g - k & \text{on } \partial\Omega \end{cases}$$

satisfies $u - k \geq 0$, or $u \geq k$ which is (1.17). Similarly, let k denote the right-hand side of (1.18). Then if $k \leq 0$, $f(x) - a_0(x)k \leq f(x) - \beta k \leq 0$ and, if $k \geq 0$, $f(x) - a_0(x)k \leq f(x) - \alpha k \leq 0$. So, by the same reasoning as before, $u - k \leq 0$ and (1.18) follows.

In general an ordering on a real Hilbert space H is defined by a convex cone P in H where we write $u \geq 0$ if and only if $u \in P$. We will, in addition, require that P has the following

<u>Property 1.10</u>. $P = \{u \in H : (u, v) \geq 0 \text{ for every } v \in P\}$.

This property implies that P is closed as the intersection of closed sets, and that if $u, -u \in P$, then $(u, -u) \geq 0$ so that $u = 0$.

<u>Lemma 1.11</u>. Every $u \in H$ has a unique decomposition $u = u_+ - u_-$ satisfying $u_+, u_- \geq 0$ and $(u_+, u_-) = 0$.

<u>Proof</u>. Let u_+ be the projection of u on the closed, convex set P. The projection is characterized by

(1.19) $$(u - u_+, v - u_+) \leq 0 \quad \text{for every } v \in P .$$

Taking $v = 0$ and $v = 2u_+$ in (1.19) we obtain

(1.20) $$(u - u_+, u_+) = 0 ,$$

which, implies when combined with (1.19) that

$$(u - u_+, v) \leq 0 \quad \text{for every} \quad v \in P .$$

Hence, by Property 1.10, $u_+ - u \in P$, setting $u_- = u_+ - u$ we have $u = u_+ - u_-$, $u_+, u_- \geq 0$, and $(u_+, u_-) = 0$ by (1.20).

To prove uniqueness, suppose $u = a - b$ with $a, b \geq 0$, $(a, b) = 0$. Then

$$(u - a, v - a) = (-b, v - a) = (-b, v) \leq 0$$

for every $v \in P$, since $b, v \in P$. Hence a is the projection of u on P or $a = u_+$, and $b = u_+ - u = u_-$. This completes the proof. Note that since $-u = u_- - u_+$, the uniqueness shows that u_- is the projection of $-u$ on P.

Example 1.12. Let $H = L^2(\Omega)$, $P = \{u : u(x) \geq 0 \ \text{a.e.} \ x \in \Omega\}$. Property 1.10 is satisfied since

$$\int_\Omega u(x)v(x)dx \geq 0 \quad \text{for every} \quad v \geq 0 \quad \text{implies} \quad u \geq 0 ,$$

and, as before, $u_+(x) = \max(u(x), 0)$.

Example 1.13. Let $H = (L^2(\Omega))^N$, with

$$(u, v) = \sum_{i=1}^{N} \int_\Omega u_i(x)v_i(x)dx, \ u = (u_1, \ldots, u_N), \ v = (v_1, \ldots, v_N) .$$

Take $P = \{u : u_i(x) \geq 0 \ \text{a.e.} \ x \in \Omega, \ i = 1, \ldots, N\}$.

Example 1.14. Let $H = \mathcal{L}_S(\mathbb{R}^N, \mathbb{R}^N)$, the set of symmetric N by N matrices. If $M = (m_{ij})$, $N = (n_{ij})$, $m_{ij} = m_{ji}$, $n_{ij} = n_{ji}$, the inner product is given by

(1.21) $$\langle M, N\rangle = \sum_{i,j} m_{ij}n_{ij} = \text{trace}(MN) .$$

We take $P = \{M : (Mx, x) \geq 0$ for every $x \in \mathbb{R}^N\}$. We wish to show that P satisfies Property 1.10. Suppose $M, N \geq 0$, then in the orthogonal basis of M, $M = \text{diag}\{m_1, \ldots, m_N\}$ with $m_i \geq 0$, and $N = (n_{ij})$ with $n_{ii} \geq 0$. So $\langle M, N\rangle = \text{trace}(MN) = \sum_i m_i n_{ii} \geq 0$.

Conversely, suppose $\langle M, N\rangle \geq 0$ for every $N \geq 0$. Let $N = (n_{ij})$ with $n_{ij} = \xi_i \xi_j$, $\xi_i \in \mathbb{R}^N$. Then $(Nx, x) = \sum_{i,j} \xi_i\xi_j x_i x_j = (\sum_i \xi_i x_i)^2 \geq 0$ for every $x \in \mathbb{R}^N$, so $N \geq 0$, for any choice of ξ_i. Now $\langle M, N\rangle = \sum_{i,j} m_{ij}\xi_i\xi_j \geq 0$ for every $\xi_i \in \mathbb{R}^N$, implies by definition that $M \geq 0$.

Note to find M^+ for $M \in \mathcal{L}_S(\mathbb{R}^N, \mathbb{R}^N)$, convert to the orthogonal basis of M so that $M = \text{diag}(m_1, \ldots, m_N)$, then $M^+ = \text{diag}\{m_1^+, \ldots, m_N^+\}$ and transform back to the original basis.

<u>Remark 1.15</u>. The orderings of Examples 1.11 and 1.12 are lattice orders, that is given $u_1, u_2 \in H$ there exists $v = \sup(u_1, u_2) \geq u_1, u_2$ satisfying $w \geq u_1$ and $w \geq u_2$ implies $w \geq v$ for every $w \in H$. The order of Example 1.14 is not a lattice order; for example if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\sup(A, B)$ does not exist. To see this, note that an easy computation shows that $W = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_4 \end{pmatrix} \geq A$ if and only if

(1.22) $$w_1 \geq 1, \; w_4 \geq 0, \quad \text{and} \quad w_2 \leq (w_1 - 1)w_4 \, .$$

Furthermore $W \geq B$ if and only if

(1.23) $$w_1 \geq 0, \; w_4 \geq 0, \quad \text{and} \quad (w_2 - 1)^2 \leq w_1 w_4 \, .$$

It is impossible for both (1.22) and (1.23) to hold if $w_4 = 0$, however for every $\delta > 0$, (1.22) and (1.23) hold if $w_4 = \delta$, $w_1 = \frac{1}{4\delta} + 1$, and $w_2 = \frac{1}{2}$. Therefore $\sup(A, B)$ does not exist.

To avoid this difficulty we will henceforth employ a <u>different definition of sup</u> which coincides with the usual concept for lattice orderings, but is also defined for nonlattice orderings.

<u>Definition 1.16</u>. If $u_1, u_2 \in H$, $\sup(u_1, u_2) = u_1 + (u_2 - u_1)_+ = u_2 + (u_1 - u_2)_+$.

<u>Definition 1.17</u>. If $u_1, u_2 \in H$, $\inf(u_1, u_2) = u_1 - (u_1 - u_2)_+ = u_2 - (u_2 - u_1)_+$.

Note that $\sup(u, 0) = u_+$, $\inf(u, 0) = -(-u)_+ = -u_-$.

2. Variational Inequalities

Let V, H be real Hilbert spaces with $V \subset H \subset V'$, where V' denotes the dual space of V and H is identified to its dual. As usual, $\|\cdot\|$ will denote the norm in V, $|\cdot|$ the norm in H and $\|\cdot\|_*$ the norm in V'. Let H be ordered by a convex cone $P \subset H$. If $f \in V'$, $f \geq 0$ will mean that $(f,v) \geq 0$ for every $v \in V \cap P$. This is the induced ordering on V'. We make the following assumptions:

(2.1) a is a bilinear form on $V \times V$ satisfying $a(u,u) \geq \alpha \|u\|^2$, $\alpha > 0$.

(2.2) If $u \in V$, then $u_+ \in V$.

(2.3) If $u \in V$, $a(u_+, u_-) \leq 0$.

Theorem 2.1. Assume (2.1) - (2.3) hold. If $f \in V'$, $f \geq 0$, then the solution u of

$$\text{(2.4)} \qquad \begin{cases} a(u,v) = (f,v) & \text{for every } v \in V \\ u \in V \end{cases}$$

satisfies $u \geq 0$.

Proof. Since $u \in V$, $u_+ \in V$, so $u_- \in V$. Taking $v = u_-$ in (2.4) we obtain

$$a(u_+ - u_-, u_-) = (f, u_-) \geq 0$$

since $f \geq 0$ and $u_- \in V \cap P$. Hence, by (2.3),

$$a(u_-, u_-) \leq a(u_+, u_-) \leq 0 .$$

So, by (2.1), $u_- = 0$, and $u = u_+ \geq 0$.

Example 2.2. Let $H = L^2(\Omega)$, $V = H^1(\Omega)$,

$$(2.5)\qquad a(u,v) = \int_\Omega \left(\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_i \left(a_i \frac{\partial u}{\partial x_i} v + b_i u \frac{\partial v}{\partial x_i} \right) + a_0 uv \right) dx$$

$$+ \int_{\partial\Omega} buv d\sigma - \int_\Omega \int_\Omega m(x,y)u(x)v(y)dxdy$$

$$\int_{\partial\Omega} \int_{\partial\Omega} n(\sigma,\tau)u(\sigma)v(\tau)d\sigma d\tau ,$$

where $m \geq 0$, $n \geq 0$, $a_{ij}, a_i, b_i, a_0, b \in L^\infty(\Omega)$, $m, n \in L^\infty(\Omega \times \Omega)$. If there is $C > 0$ such that $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq C \sum_i \xi_i^2$ for $\xi_i \in \mathbb{R}^N$, then there exist $\alpha, \beta > 0$ such that $a(u,u) \geq \alpha \|u\|^2 - \beta |u|^2$ An example of this type is

$$(2.6)\qquad \begin{cases} -\Delta u + \lambda u - (\int_\Omega u(x)dx)\varphi = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Theorem 2.1 may be used to deduce that if $f \geq 0$, $\varphi \geq 0$, then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, (2.6) has a solution u satisfying $u \geq 0$.

<u>Example 2.3</u>. Let $H = (L^2(\Omega))^N$, $V = (H_0^1(\Omega))^N$,

$$(2.7)\qquad a(u,v) = \sum_K a_K(u_K, v_K) + \sum_{K,\ell} \int_\Omega m_{K\ell}(x)u_K(x)v_\ell(x)dx$$

where the bilinear forms a_K are of the type given in Example 2.2, and, in order that the compatibility condition (2.3) holds, we assume that $m_{K\ell} \leq 0$ if $K \neq \ell$. In this case, the variational inequality (2.4) gives rise to coupled equations of the form

$$(2.8)\qquad \begin{cases} A_K u_K + \sum_{\ell} m_{K\ell} u_\ell = f_K & \text{in } \Omega \\ u_K = 0 & \text{on } \partial\Omega . \end{cases}$$

A particular example of this type is

$$(2.9)\qquad \begin{cases} -\Delta u_1 + au_1 - bu_2 = f_1 \\ -\Delta u_2 - Cu_1 + du_2 = f_2 \end{cases}$$

where $b, C \geq 0$. If the coercivity condition holds, then Theorem 2.1 says that if $f_1, f_2 \geq 0$, then $u_1, u_2 \geq 0$. Note that if a Green's function is used to solve for u_2 in terms of u_1, then (2.9) becomes an integral equation.

We will now prove a monotonicity result for the evolution equation

$$(2.10)\qquad \begin{cases} (\frac{du}{dt}, v) + a(u,v) = (f,v) & \text{for all } v \in V, \quad 0 \leq t \leq T \\ u(0) = u_0 . \end{cases}$$

Theorem 2.4. Suppose that each of the following hold:

(2.11) There is a constant $\alpha > 0$ such that $a(u,u) \geq \alpha \|u\|^2$.

(2.12) If $u \in V$, then $u_+ \in V$ and there is a constant $C > 0$ such that $\|u_+\| \leq C\|u\|$.

(2.13) $a(u_+, u_-) \leq 0$ for every $u \in V$.

If $f \in L^2(0,T;V')$, $u_0 \in H$, $f \geq 0$, $u_0 \geq 0$, then the solution $u \in L^2(0,T;V) \cap C(0,T;H)$ satisfies $u \geq 0$. The proof will require the aid of the following lemma.

Lemma 2.5. Suppose V satisfies (2.12), $u \in L^2(0,T;V) \cap C(0,T;H)$, and $\frac{du}{dt} \in L^2(0,T;V')$, then

$$(2.14) \qquad \int_0^T \left(\frac{du}{dt}, u_-(t)\right)dt = \frac{1}{2}|u_-(0)|^2 - \frac{1}{2}|u_-(T)|^2 .$$

Proof. We first prove (2.14) for $u \in C^1(0,T;V)$. Suppose $u \in C^1(0,T;V)$ and extend so that $u \in C^1(\mathbb{R};V)$.

$$(2.15) \quad \int_0^T \left(\frac{du}{dt}, u_-(t)\right)dt = \lim_{h\to 0} \int_0^T \left(\frac{u(t+h)-u(t)}{h}, u_-(t)\right)dt$$

$$= \lim_{h\to 0}\left[\int_0^T \left(\frac{u_+(t+h)-u_+(t)}{h}, u_-(t)\right)dt - \int_0^T \left(\frac{u_-(t+h)-u_-(t)}{h}, u_-(t)\right)dt\right].$$

Suppose $h > 0$. Then since $(u_+(t+h), u_-(t)) \geq 0$ and $(u_+(t), u_-(t)) = 0$, (2.15) implies that

$$(2.16) \quad \int_0^T \left(\frac{du}{dt}, u_-(t)\right)dt \geq \lim_{h\to 0^+}\left(-\int_0^T \left(\frac{u_-(t+h)-u_-(t)}{h}, u_-(t)\right)dt\right) .$$

Since $|u_-(t+h) - u_-(t)| \leq |u(t+h) - u(t)|$ and $u \in C^1(0,T;V)$, u_- has at least a weak derivative and satisfies

$$\lim_{h\to 0} -\int_0^T \left(\frac{u_-(t+h)-u_-(t)}{h}, u_-(t)\right)dt = -\int_0^T \left(\frac{du_-}{dt}, u_-(t)\right)dt$$

$$= -\frac{1}{2}\int_0^T \frac{d}{dt}|u_-(t)|^2 dt = \frac{1}{2}|u_-(0)|^2 - \frac{1}{2}|u_-(T)|^2 .$$

This, combined with (2.16), yields

$$\int_0^T \left(\frac{du}{dt}, u_-(t)\right)dt \geq \frac{1}{2}|u_-(0)|^2 - \frac{1}{2}|u_-(T)|^2 .$$

The opposite inequality is obtained by taking $h < 0$. This proves (2.14) for $u \in C^1(0,T;V)$. For u satisfying $u \in L^2(0,T;V) \cap C(0,T;H)$, $\frac{du}{dt} \in L^2(0,T;V')$, choose a sequence $u_n \in C^1(0,T;V)$ such that $u_n \to u$ strongly in $L^2(0,T;V) \cap C(0,T;H)$ and $\frac{du_n}{dt} \to \frac{du}{dt}$ strongly in $L^2(0,T;V')$. By (2.12), the sequence $\{(u_n)_-\}$ lies in a bounded subset of $L^2(0,T;V)$. Since $v \to v_-$ is a contraction on H and $u_n \to u$ strongly in H and uniformly in t, $(u_n)_- \to u_-$ strongly in H and uniformly in t. Therefore, taking a subsequence if necessary we have $(u_n)_- \to u_-$ weakly in $L^2(0,T:V)$. Passing to the limit in (2.14) with $u = u_n$ completes the proof of Lemma 2.5.

Proof of Theorem 2.4. We know that (2.10) has a solution u satisfying the hypotheses of Lemma 2.5. By (2.12) we may take $v = u_-$ in (2.10) and integrate to obtain

$$\int_0^T (\frac{du}{dt}, u_-(t))dt + \int_0^T a(u_+(t), u_-(t))dt - \int_0^T a(u_-(t), u_-(t))dt$$

$$= \int_0^T (f(t), u_-(t))dt \geq 0$$

since $f \geq 0$, $u_- \geq 0$. By Lemma 2.5 and (2.13) we have

$$|u_-(0)|^2 - |u_-(t)|^2 - \int_0^T a(u_-(t), u_-(t))dt \geq 0 \ . \tag{2.17}$$

Since $u_0 \geq 0$, $u\ (0) = 0$, so by (2.11) and (2.17),

$$\alpha \int_0^T \|u_-(t)\|^2 dt \leq 0 \ .$$

Hence $u \geq 0$ as element of $L^2(0,T;V)$. This completes the proof of Theorem 2.4.

Remark 2.6. Theorem 2.4 remains true if (2.11) is replaced by $a(u,u) \geq \alpha \|u\|^2 - \beta |u|^2$, by use of the transformation $u(t) = w(t)e^{\beta t}$ which yields the same equation for w with f replaced by $e^{-\beta t}f$ and $a(u,v)$ replaced by $\bar{a}(u,v) = a(u,v) + \beta(u,v)$. Hence $\bar{a}$ satisfies (2.11).

We now consider a variational inequality

$$(2.18) \qquad \begin{cases} a(u, v-u) \geq (f, v-u) & \text{for every } v \in K \\ u \in K \end{cases}$$

where $a : V \times V \to \mathbb{R}$, satisfies the coercivity condition $a(u,u) \geq \alpha \|u\|^2$, V satisfies the compatibility condition: $u \in V$ implies $u_+ \in V$, and $a(u_+, u_-) \leq 0$ for every $u \in V$. K is a nonempty, closed convex subset of V. This problem is a special case of

$$(2.19) \qquad \begin{cases} a(u, v-u) + j(v) - j(u) \geq (f, v-u) & \text{for every } v \in V \\ u \in D(j) \end{cases}$$

where $j : V \to (-\infty, +\infty]$ is lower semicontinuous, convex, and proper. Here $D(j) = \{v \in V : j(v) < +\infty\}$. The problem (2.19) becomes (2.18) if

$$(2.20) \qquad j(v) = \begin{cases} 0 & \text{for } v \in K \\ +\infty & \text{for } v \notin K . \end{cases}$$

Theorem 2.7. Suppose a is a coercive bilinear form on $V \times V$, $V \subset H \subset V'$, and V is compatible with the ordering in H in the sense described above. Suppose j is a lower semicontinuous, proper, convex function on V satisfying the compatibility condition

$$j(\inf(u,v)) + j(\sup(u,v)) \leq j(u) + j(v) \quad \text{for every} \quad u,v \in V . \tag{2.21}$$

Let u_i be the solution of (2.19) with $f = f_i$, $i = 1, 2$. Then $f_1 \leq f_2$ in V' implies $u_1 \leq u_2$ in V.

<u>Proof</u>. Recall that $\inf$ and $\sup$ are defined by

$$\inf(u,v) = u - (u-v)_+ = v - (v-u)_+, \quad \sup(u,v) = u + (v-u)_+ = v + (u-v)_+ .$$

The condition (2.21) implies that if $u, v \in D(j)$ then $\inf(u,v)$, $\sup(u,v) \in D(j)$.

Taking $v = \inf(u_1, u_2)$ in the inequality satisfied by u_1 we obtain

$$a(u_1, -(u_1 - u_2)_+) + j(\inf(u_1,u_2)) - j(u_1) \geq (f_1, -(u_1 - u_2)_+) . \tag{2.22}$$

Taking $v = \sup(u_1, u_2)$ in the inequality satisfied by u_2 we obtain

$$a(u_2, (u_1 - u_2)_+) + j(\sup(u_1,u_2)) - j(u_2) \geq (f_2, (u_1 - u_2)_+) . \tag{2.23}$$

Adding (2.22) to (2.23) and applying (2.21) we have

$$a(u_2 - u_1, (u_1 - u_2)_+) \geq (f_2 - f_1, (u_1 - u_2)_+) \geq 0 \tag{2.24}$$

since $f_2 - f_1 \geq 0$. Therefore

$$a((u_1 - u_2)_+, (u_1 - u_2)_+) = a(u_1 - u_2, (u_1 - u_2)_+) + a((u_1 - u_2)_-, (u_1 - u_2)_+) \leq 0$$

since the terms on the right are ≤ 0 by (2.24) and hypothesis. By the coercivity hypothesis, this implies that $(u_1 - u_2)_+ = 0$, hence $u_1 \leq u_2$. This completes the proof.

Remark 2.8. The above proof shows that the same results hold for the problem

$$(2.25)\qquad \begin{cases} a(u, v - u) \geq L(v - u) & \text{for every } v \in K \\ u \in K \end{cases}$$

where L is a linear functional on V and K satisfies the compatibility condition

$$(2.26)\qquad u_1, u_2 \in K \text{ implies } \inf(u_1, u_2) \text{ and } \sup(u_1, u_2) \in K .$$

Under the above hypothesis, if u_i is the solution of (2.25) with $L = L_i$, $i = 1, 2$, then $L_1(v) \leq L_2(v)$ for every $v \geq 0$ implies $u_1 \leq u_2$.

Example 2.9. Let $V = H^1(\Omega)$, $K = \{u \in V : u \geq 0$ a.e. in $\Omega\}$. Note that the compatibility condition (2.26) is satisfied. If one chooses $f, g \in V'$ and defines

$$a(u, v) = \int_\Omega \text{grad } u \text{ grad } v + uv, \quad L(v) = \int_\Omega fv + \int_{\partial\Omega} gv$$

then the variational inequality (2.25) gives rise to the problem

$$\begin{cases} -\Delta u + u - f \geq 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} - g \geq 0 & \text{on } \partial\Omega \\ u \geq 0 \end{cases}$$

where equality holds in the first two inequalities on the set where $u > 0$. Theorem 2.7 guarantees that if $f_1 \leq f_2$ and $g_1 \leq g_2$, then the corresponding solutions satisfy $u_1 \leq u_2$.

If one takes $K = \{u \in V : u \geq 0$ a.e. on $\partial\Omega\}$, then the same problem arises except that now $-\Delta u + u - f = 0$ in Ω.

Example 2.10. Let $V = H^1(\Omega)$, $H = L^2(\Omega)$, with $j(u) = \int_\Omega \varphi(x, u(x))dx$ where $\varphi(x, v)$ is measurable in x, convex in v, and $\varphi(x, 0) = 0$ for every $x \in \Omega$. The compatibility condition becomes

$$(2.27) \quad \int_\Omega (\varphi(x, \inf(u_1, u_2)) + \varphi(x, \sup(u_1, u_2)))dx \leq \int_\Omega (\varphi(x, u_1) + \varphi(x, u_2))dx .$$

Since at each point $x \in \Omega$ either $\inf(u_1(x), u_2(x)) = u_1(x)$ and $\sup(u_1(x), u_2(x)) = u_2(x)$ or vice-versa, we in fact have equality in (2.27). This setting gives rise to a problem of the type

$$(2.28) \quad \begin{cases} -\Delta u + u + \dfrac{\partial\varphi}{\partial v}(x, u) = f \\ \dfrac{\partial u}{\partial n} = g . \end{cases}$$

Example 2.11. We now consider a situation of the type considered in Example 2.10 for a system. Let $V = (H^1(\Omega))^N$, $H = (L^2(\Omega))^N$, and

$$j(u) = \int_\Omega \varphi(x, u_1(x), \ldots, u_N(x))dx$$

where $u = (u_1, \ldots, u_N)$. We assume that φ is measurable in x and that the mapping $(u_1, \ldots, u_N) \to \varphi(x, u_1, \ldots, u_N)$ is convex for each fixed $x \in \Omega$. For smooth φ this means that $\left(\dfrac{\partial^2\varphi}{\partial u_i \partial u_j}\right)_{i,j}$ is a positive matrix. The compatibility condition requires that

$$(2.29) \quad \begin{cases} \varphi(x, \inf(u_1, v_1), \ldots, \inf(u_N, v_N)) + \varphi(x, \sup(u_1, v_1), \ldots, \sup(u_N, v_N)) \\ \qquad \leq \varphi(x, u_1, \ldots, u_N) + \varphi(x, v_1, \ldots, v_N) \quad \text{for a.e. } x \in \Omega,\ u, v \in V . \end{cases}$$

To see what this means take $u_i = v_i$ for $i = 3, \ldots, N$, $v_1 = u_1 + a$, $v_2 = u_2 + b$, $a, b \geq 0$. Define $\psi(u, v) = \varphi(x, u, v, u_3, \ldots, u_N)$ where $x \in \Omega$, and $u_3, \ldots, u_N \in H^1(\Omega)$ are fixed. Then the compatibility condition requires that

$$\psi(u, v) + \psi(u + a, v + b) \leq \varphi(u, v + b) + \psi(u + a, v) \tag{2.30}$$

for every $a, b \geq 0$, $u, v \in H^1(\Omega)$. Comparing the Taylor expansions about (u, v) of both sides of (2.30) we see that if (2.30) holds then $\frac{\partial^2 \psi}{\partial u \partial v} \leq 0$.

Conversely, if $\frac{\partial^2 \psi}{\partial u \partial v} \leq 0$ then

$$\psi(u, v) + \psi(u + a, v + b) - \psi(u, v + b) - \psi(u + a, v) = \int_{\tilde{v}=v}^{v+b} \int_{\tilde{u}=u}^{u+a} \frac{\partial^2 \psi}{\partial \tilde{u} \partial \tilde{v}} \, d\tilde{u} d\tilde{v} \leq 0$$

so (2.30) holds.

In general, for smooth φ, the compatibility condition (2.29) holds if and only if

$$\frac{\partial^2 \varphi}{\partial u_i \partial u_j} \leq 0 \quad \text{for} \quad i \neq j . \tag{2.31}$$

This setting gives rise to problems of the type

$$A_i u_i + \frac{\partial \varphi}{\partial u_i}(x, u_1, \ldots, u_N) = f_i, \; i = 1, \ldots, N, \quad \text{boundary conditions .} \tag{2.32}$$

If $F_i = \frac{\partial \varphi}{\partial u_i}$ then $(u_1, \ldots, u_N) \to (F_i(u), \ldots, F_N(u))$ is monotone and $\frac{\partial F_i}{\partial u_j} \leq 0$ for $i \neq j$.

This situation will be generalized to problems of this type which do not arise from variational inequalities in the next section.

3. An Abstract Fixed Point Theorem Applied to a Nonvariational Equation

In this section we will consider problems of the type

$$(3.1) \qquad \begin{cases} -\Delta u + F(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where F is not necessarily of the form given in (2.32).

Theorem 3.1. Let Ω be a bounded open subset of $\mathbb{R}^N$ with $f \in L^2(\Omega)$. Suppose there is a constant $K \geq 0$ such that

$$(3.2) \qquad \left|\frac{\partial F}{\partial u}(u)\right| \leq K \quad \text{for every } u \in \mathbb{R} .$$

Suppose there exists two functions $u_-, u_+ \in H^1(\Omega) \cap L^\infty(\Omega)$ such that

$$(3.3) \qquad \begin{cases} -\Delta u_- + F(u_-) \leq f \leq -\Delta u_+ + F(u_+) & \text{in } \Omega \\ u_- \leq 0 \leq u_+ & \text{on } \partial\Omega . \end{cases}$$

Then there exists a solution u of (3.1) satisfying $u_- \leq u \leq u_+$ a.e. in Ω.

More precisely, there exists solutions u_{min}, u_{max} of (3.1) such that $u_- \leq u_{min} \leq u_{max} \leq u_+$ and whenever u is a solution of (3.1) satisfying $u_- \leq u \leq u_+$, then $u_{min} \leq u \leq u_{max}$. (Note that here the notation u_-, u_+ does not denote $-\inf(u,0)$ and $\sup(u,0)$ as in previous sections.)

Proof. Choose $\lambda \geq K$, then by (3.2) $\lambda I - F$ is increasing on $[u_-, u_+]$. Consider the problem

$$(3.4) \qquad \begin{cases} -\Delta \bar{u} + \lambda \bar{u} = \lambda u - F(u) + f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega , \end{cases}$$

where we have chosen u such that $u_- \leq u \leq u_+$. By previous results (3.4) has a unique solution $\bar{u}$ and hence defines a mapping S given by $\bar{u} = S(u)$. It is not difficult to see that S has the following properties:

i) S is increasing, i.e. if $u_1 \leq u_2$ then $S(u_1) \leq S(u_2)$. This follows from the fact that if $u_1 \leq u_2$, then $\lambda u_1 - F(u_1) + f \leq \lambda u_2 - F(u_2) +$ by above.

ii) If $u_- \leq u \leq u_+$, then $u_- \leq S(u) \leq u_+$. This follows from $-\Delta\bar{u} + \lambda\bar{u} = \lambda u - F(u) + f \geq \lambda u_- - F(u_-) + f \geq -\Delta u_- + \lambda u_-$, $\bar{u} = 0 \geq u_-$ on $\partial\Omega$, and opposite inequalities involving u_+.

iii) S is continuous from $L^2(\Omega)$ to $L^2(\Omega)$ since $\lambda u - F(u)$ is Lipschitz continuous.

To obtain the minimal fixed point u_{min}, define $u_0 = u_-$ and $u_{n+1} = S(u_n)$ for $n = 0, 1, 2, \ldots$. By i) $u_{n+1} \geq u_n$ for $n = 0, 1, 2, \ldots$ and by ii) $u_n \leq u_+$ for $n = 0, 1, 2, \ldots$. Since $\{u_n\}$ is an increasing sequence which is bounded above it converges strongly in $L^2(\Omega)$ to some $u_{min} \in L^2(\Omega)$. By iii) $u_{n+1} = S(u_n) \to S(u_{min})$ strongly in $L^2(\Omega)$, so that u_{min} is a fixed point of S and clearly $u_- \leq u_{min} \leq u_+$. The fixed point u_{max} is obtained in the same way, only taking $v_0 = u_+$ and taking the limit of a decreasing sequence $\{v_n\}$ which is bounded below. Now suppose w is a solution of (3.1), i.e. $Sw = w$, and $u_- \leq w \leq u_+$. Then by definition $u_0 \leq w \leq v_0$ and if $u_n \leq w \leq v_n$, then

$u_{n+1} = Su_n \le Sw = w \le Sv_n \le v_{n+1}$. Hence, by induction, $u_n \le w \le v_n$ for $n = 0, 1, 2, \ldots$ and passing to the limit we have $u_{min} \le w \le u_{max}$. This completes the proof.

Our next objective is to prove an abstract fixed point theorem which can be applied to mappings S which are increasing, but not necessarily continuous.

<u>Theorem 3.2</u>. Let E be an ordered space, S an increasing mapping from E to itself. Assume the ordering on E is such that every bounded nonempty subset of E has a supremum and an infimum. Suppose there exist $u_-, u_+ \in E$ such that

$$u_- \le Su_- \le Su_+ \le u_+ . \tag{3.5}$$

Then S has minimal and maximal fixed points u_{min}, u_{max} satisfying $u_- \le u_{min} \le u_{max} \le u_+$. (Once u_-, u_+ are known to exist, the remaining hypotheses need only hold on the interval $[u_-, u_+]$.

<u>Proof</u>. Define $X_- = \{u \in [u_-, u_+] \mid u \le Su\}$ and $X_+ = \{u \in [u_-, u_+] \mid u \ge Su\}$. By (3.5), $u_- \in X_-$ and $u_+ \in X_+$, hence X_- has a supremum v^* and X_+ has an infimum u^*. For every $u \in X_-$, $u \le Su$ and $u \le v^*$, so that $u \le Su \le Sv^*$. Since this holds for every $u \in X_-$, we have $v^* \le Sv^*$. Since S is increasing, $Sv^* \le S(Sv^*)$ so that $Sv^* \in X_-$ and hence $Sv^* \le v^*$. These results show that $v^* = Sv^*$. Similar arguments show that $Su^* = u^*$. Finally if $u \in [u_-, u_+]$ and $S(u) = u$, then $u \in X_- \cap X_+$ so that $u^* \le u \le v^*$. This completes the proof.

Remark 3.3. An increasing mapping $S : E \to E$ still has a minimal fixed point in $[u_-, u_+]$ if the partial ordering on E is assumed to have the weaker property that bounded nonempty subsets of E are inductive, i.e. every totally ordered subset of a bounded nonempty set in E has a maximal element. If moreover, $\sup(a, b)$ exists for every $a, b \in [u_-, u_+]$, then S also has a maximal fixed point in $[u_-, u_+]$.

Proof. It is an easy consequence of the Hausdorf Maximality Principle that under the above conditions every nonempty, bounded subset B of E has a maximal element m in the sense that if $b \in B$ and $b \geq m$, then $b = m$. Define $Y_- = \{u \in X_- : u \leq v \text{ for every } v \in X_+\}$. Y_- is nonempty since $u_- \in Y_-$ so let u^* be a maximal element of Y_-. Now $S : Y_- \to Y_-$ since if $u \leq v$ for every $v \in X_+$ then $Su \leq Sv$ and $Sv \leq v$ for every $v \in X_+$ so $Su \leq v$ for every $v \in X_+$. Furthermore, $Su \in X_-$ whenever $u \in X_-$, hence $Su \in Y_-$. Since $u^* \in X_-$, $u^* \leq Su^*$, so the maximality of u^* and the fact that $Su^* \in Y_-$, imply that $u^* = Su^*$. The fixed point u^* is minimal since if w is another fixed point then $w \in X_+$. Therefore, since $u^* \in Y_-$, $u^* \leq w$.

Now suppose $\sup(a, b)$ exists for $a, b \in [u_-, u_+]$. Let v^* be a maximal element of X_-. Then $v^* \leq Sv^*$ since $v^* \in X_-$, and $S : X_- \to X_-$ imply that $v^* = Sv^*$. We will show that v^* is a maximal fixed point in $[u_-, u_+]$. Suppose $w \in [u_-, u_+]$ and $Sw = w$. Let $u = \sup(v^*, w)$, then $Su \geq Sv^* = v^*$ and $Su \geq Sw = w$. Hence

$Su \geq \sup(v^*, w) = u$, so $u \in X_-$. Now $u \geq v^*$, so $u = v^*$ and therefore $v^* \geq w$. This completes the proof.

Example 3.4. We will show that the space $L^2(\Omega)$ with the usual ordering possesses the property required by Theorem 3.2. That is, if $u_-, u_+ \in L^2(\Omega)$ and $\{u_\alpha : \alpha \in A\}$ is a collection of elements of $L^2(\Omega)$ in the interval $[u_-, u_+]$, then $\inf_\alpha u_\alpha$ and $\sup_\alpha u_\alpha$ exist as elements of $L^2(\Omega)$. Before proceeding to the proof, note that the try $(\inf_\alpha u_\alpha)(x) = \inf_\alpha u_\alpha(x)$ fails since the function on the left is undefined on the union of uncountably many sets of measure zero.

Proof. Instead we employ a proof using measures. Without loss of generality, suppose $u_- = 0$. For $v \in L^2(\Omega)$, $v \geq 0$ define

$$(3.6)\quad L(v) = \sup\{\sum_{i=1}^{N} \int_\Omega u_{\alpha_i} v_i dx : \alpha_i \in A, v_i \geq 0, \sum_{i=1}^{N} v_i \leq v, \ N = 1, 2, 3, \ldots\} .$$

We will first show that if $v \geq 0$, $w \geq 0$, then $L(v + w) = L(v) + L(w)$. Let $\epsilon > 0$ be given, then there exist $N, P > 0$, $\{\alpha_i\}$, $\{\beta_j\} \subset A$, and $\{v_i\}$, $\{w_j\}$ with $\sum_{i=1}^{N} v_i \leq v$, $\sum_{j=1}^{P} w_j \leq w$ such that

$$L(v) \leq \sum_{i=1}^{N} \int_\Omega u_{\alpha_i} v_i dx + \epsilon$$

and

$$L(w) \leq \sum_{j=1}^{P} \int_\Omega u_{\beta_j} w_j dx + \epsilon .$$

Hence,

$$L(v) + L(w) \leq L(v + w) + 2\epsilon \quad \text{for every} \quad \epsilon > 0,$$

or
$$L(v) + L(w) \leq L(v + w) .$$

Conversely,

$$L(v + w) \leq \sum_{i=1}^{N} \int_\Omega u_{\alpha_i} \varphi_i dx + \epsilon \tag{3.7}$$

where $\sum \varphi_i \leq v + w$ and some appropriate choice of N and $\{\alpha_i\}$. Let

$$v_i(x) = \begin{cases} \dfrac{\varphi_i v(x)}{v(x) + w(x)}, & v(x) + w(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and define w_i similarly with v replaced by w. Then $\varphi_i = v_i + w_i$ and $\sum v_i \leq v$, $\sum w_i \leq w$. Substituting in (3.7) it is easy to see that $L(v + w) \leq L(v) + L(w) + \epsilon$ for every $\epsilon > 0$, or $L(v + w) \leq L(v) + L(w)$.

Next we extend L to all of $L^2(\Omega)$ by defining $L(v) = L(\sup(v, 0)) - L(-\inf(v, 0))$. By, above, L is linear, and since

$$|L(v)| \leq \int_\Omega |u_+ v| dx \leq \|u_+\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} ,$$

L is a bounded linear functional on $L^2(\Omega)$. Hence there exists $u^* \in L^2(\Omega)$ such that

$$L(v) = \int_\Omega u^* v \, dx \quad \text{for every} \quad v \in L^2(\Omega) .$$

Since $\int_\Omega u^* v \, dx = L(v) \leq \int_\Omega u_+ v \, dx$ for every $u \geq 0$, $u^* \leq u_+$. Furthermore, $\int_\Omega u^* v \, dx = L(v) \geq \int_\Omega u_\alpha v \, dx$ for every $\alpha \in A$, $v \geq 0$, so $u^* \geq u_\alpha$ for every $\alpha \in A$. Finally, if $w \geq u_\alpha$ for every $\alpha \in A$, then

$$\int_\Omega wv\,dx \geq L(v) = \int_\Omega u^* v\,dx \quad \text{for every} \quad v \geq 0 ,$$

so $u^* \leq w$. Therefore $u^* = \sup_\alpha u_\alpha$, the $\inf_\alpha u_\alpha$ may be found similarly. This completes the proof.

Remark 3.5. At the beginning of this section we proved the existence of maximal and minimal solutions for the problem (3.1) under the assumption that $|F'| \leq K$. By use of the preceding fixed point Tneorem 3.2 one may weaken this assumption to considering a function F of the form $F = F_0 - F_1$ where F_0 is maximal monotone and F_1 is monotone. One applies Theorem 3.2 to the mapping $Su = \bar{u}$ given by

$$(3.8) \qquad \begin{cases} -\Delta\bar{u} + F_0(\bar{u}) = F_1(u) & \text{in} \quad \Omega \\ \bar{u} = 0 & \text{on} \quad \partial\Omega . \end{cases}$$

The existence of this decomposition implies that F is locally of bounded variation. If F is known only to be continuous, the same results still hold, however now the mapping S arises from a decomposition of F which depends on u. This situation is related to quasi-variational inequalities which will be studied in the following section.

4. Quasi-variational Inequalities

In this section we will consider a quasi-variational inequality of the form

$$(4.1)\qquad \begin{cases} a(u, v-u) \geq (f, v-u) & \text{for every } v \in K(u) \\ u \in K(u) \end{cases}$$

where now $K(u)$ is a family of closed, convex, nonempty sets depending on u. This, as before, is a special case of the more general quasi-variational inequality

$$(4.2)\qquad \begin{cases} a(u, v-u) + j_u(v) - j_u(u) \geq 0 & \text{for every } v \in V \\ u \in \text{Dom } j_u , \end{cases}$$

where j_u is a family of convex, lower semicontinuous, proper functions from V to $(-\infty, +\infty]$.

The problem (4.1) is equivalent to finding a fixed point for the mapping $T(u) = \bar{u}$ given by

$$(4.3)\qquad \begin{cases} a(\bar{u}, v-\bar{u}) \geq (f, v-\bar{u}) & \text{for every } v \in K(u) \\ \bar{u} \in K(u) \end{cases}$$

and (4.2) is equivalent to finding a fixed point for the mapping $T(u) = \bar{u}$ given by

$$(4.4)\qquad \begin{cases} a(\bar{u}, v-\bar{u}) + j_u(v) - j_u(\bar{u}) \geq 0 & \text{for every } v \in V \\ \bar{u} \in \text{Dom } j_u \end{cases}$$

where (4.3) and (4.4) are variational inequalities for each $u \in V$.

Definition 4.1. Let j, k be l.s.c., proper, convex functions on an ordered space V. We write $j \propto k$ if

$$j(\inf(u,v)) + k(\sup(u,v)) \leq j(u) + k(v)$$

for every $u, v \in V$. Recall that $\inf(u,v) = u - (u-v)_+ = v - (v-u)_+$ and $\sup(u,v) = v + (u-v)_+ = u + (v-u)_+$.

The proof of the main theorem of this section will require the following lemma.

Lemma 4.2. As usual let $V \subset H \subset V'$, with a coercive on $V \times V$, $u_+ \in V$ for every $u \in V$ and $a(u_+, u_-) \leq 0$ for every $u \in V$. Consider the variational inequality

$$(4.5) \qquad \begin{cases} a(u, v-u) + j(v) - j(u) \geq (f, v-u) & \text{for every } v \in V \\ u \in V . \end{cases}$$

Suppose u_i is the solution of (4.5) with $j = j_i$, $i = 1, 2$. Then $j_1 \propto j_2$ implies $u_1 \leq u_2$.

Proof. Take $v = \inf(u_1, u_2)$ in the inequality for u_1 and $v = \sup(u_1, u_2)$ in the inequality for u_2 to obtain

$$a(u_1, -(u_1 - u_2)_+) + a(u_2, (u_1 - u_2)_+) \geq 0$$

or

$$a(u_1 - u_2, (u_1 - u_2)_+) \leq 0 .$$

Therefore, $a((u_1 - u_2)_+, (u_1 - u_2)_+) = a(u_1 - u_2, (u_1 - u_2)_+) + a((u_1 - u_2)_-, (u_1 - u_2)_+) \leq 0$, and since a is coercive we obtain $(u_1 - u_2)_+ = 0$ or $u_1 \leq u_2$.

Theorem 4.3. Let the hypotheses of Lemma 4.2 hold with a coercive on $V \times V$ in the sense that

$a(u,u) \geq \alpha \|u\|^2 - \beta |u|^2$ for every $u \in V$ and fixed $\alpha > 0, \beta \geq 0$.

Suppose that $\{j_u : u \in V\}$ is a family of l.s.c., proper, convex functions on V which satisfies the compatibility condition

$$(4.6) \qquad j_{u_1} \propto j_{u_2} \quad \text{whenever} \quad u_1 \leq u_2 .$$

For $\mu \geq \beta$ we can solve the variational inequality

$$(4.7) \quad \begin{cases} a(\bar{u}, v - \bar{u}) + \mu(\bar{u} - u, v - \bar{u}) + j_u(v) - j_u(\bar{u}) \geq 0 & \text{for every } v \in V \\ \bar{u} \in \operatorname{Dom} j_u . \end{cases}$$

Let the mapping T_μ be given by $T_\mu(u) = \bar{u}$. Then T_μ is increasing. (Note that a fixed point of T_μ is a solution of the quasi-variational inequality, (4.2).)

Proof. Suppose $u_1 \leq u_2$, $\bar{u}_1 = T_\mu u_1$, $\bar{u}_2 = T_\mu u_2$. By hypothesis, $a(u,v) + \mu(u,v)$ is coercive on V for $\mu \geq \beta$ and satisfies the compatibility condition $a(u_+, u_-) + \mu(u_+, u_-) \leq 0$. It is easy to see that $\bar{u}_i$ satisfies the variational inequality (4.5) with $j_i(v) = j_{u_i}(v) - \mu(u_i, v)$, $i = 1, 2$. Since $j_{u_1} \propto j_{u_2}$, and $-\mu(u_1, v) \propto -\mu(u_2, v)$ we have $j_1 \propto j_2$. Therefore, by Lemma 4.2, $\bar{u}_1 \leq \bar{u}_2$, or $T_\mu u_1 \leq T_\mu u_2$.

Remark 4.4. If

$$j_u(v) = \begin{cases} (f, v) & v \in K(u) \\ +\infty & v \notin K(u) \end{cases}$$

as in (4.1), then the compatibility condition (4.6) means that if $u_1 \leq u_2$ and $v_i \in K(u_i)$, $i = 1, 2$, then $\inf(v_1, v_2) \in K(u_1)$ and $\sup(v_1, v_2) \in K(u_2)$.

Proof. The above condition implies that if the right-hand side of Def. 4.1 is finite, then so is the left, and, in fact, equality holds by direct substitution.

Example 4.5. Let $V = H^1(\Omega)$, $H = L^2(\Omega)$, and

$$K(u) = \{v \in V : M_1 u \le v \le M_2 u\} \tag{4.8}$$

where M_1, M_2 are increasing functions from H to V. The compatibility condition amounts to proving that if $u_1 \le u_2$, $M_1u_1 \le v_1 \le M_2u_1$, and $M_1u_2 \le v_2 \le M_2u_2$, then $M_1u_1 \le \inf(v_1, v_2)$, and $\sup(v_1, v_2) \le M_2u_2$. Since M_1 is increasing, $v_2 \ge M_1u_2 \ge M_1u_1$, so, since $v_1 \ge M_1u_1$, $\inf(v_1, v_2) \ge M_1u_1$. The remaining inequality is proved similarly.

Note that if M_iu is not always in $H^1(\Omega)$ for $u \in L^2(\Omega)$, ($i = 1$ or 2), then we must add the condition that $K(u) \ne \emptyset$ for $u \in L^2(\Omega)$. For example if $u(x) = \begin{cases} 2 & \text{for } 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$ then

$$K(u) = \{v \in H^1(\Omega) : u \le v \le u + 1\} = \emptyset .$$

To apply the fixed point theorem of Section 3 to the increasing mapping T_μ we require two elements u_-, u_+ of V satisfying $u_- \le u_+$, and $u_- \le T_\mu u_- \le T_\mu u_+ \le u_+$. The following theorem gives sufficient conditions for the existence of u_-, similar conditions may be formulated for u_+. We will work in the setting of the quasi-variational inequality (4.1).

Theorem 4.6. Suppose the hypotheses of Theorem 4.3 hold and there exists $u_- \in V$ such that $a(u_-, v) \le (f, v)$ for every $v \in V$, $v \ge 0$.

Suppose $\sup(u_-, v) \in K(u_-)$ for every $v \in K(u_-)$. Then $u_- \leq T_\mu u_-$ where $T_\mu u_- = \bar{u}$ is given by

$$(4.9) \quad \begin{cases} a(\bar{u}, v - \bar{u}) + \mu(\bar{u} - u_-, v - \bar{u}) \geq (f, v - \bar{u}) \quad \text{for every } v \in K(u_-) \\ \bar{u} \in K(u_-) . \end{cases}$$

Proof. By hypothesis $\sup(u_-, \bar{u}) \in K(u_-)$. Putting $v = \sup(u_-, \bar{u})$ in (4.9) and letting $q = (u_- - \bar{u})_+$ we obtain

$$a(\bar{u}, q) + \mu(\bar{u} - u_-, q) \geq (f, q) .$$

Now, by hypothesis

$$a(-u_-, q) \geq (-f, q) \quad \text{since} \quad q \in V, \ q \geq 0 ,$$

adding these results we obtain

$$a(\bar{u} - u_-, q) - \mu |q|^2 \geq 0 ,$$

so by the compatibility condition on a,

$$a(q, q) + \mu |q|^2 \leq 0$$

which by coercivity gives $q = 0$ or $u_- = \bar{u} = T_\mu u_-$.

Remark 4.7. By a similar argument, a sufficient condition for the existence of $u_+ \in V$ such that $T_\mu u_+ \leq u_+$ is that $u_+ \in V$ satisfy $a(u_+, v) \geq (f, v)$ for every $v \in V$, $v \geq 0$, and $\inf(u_+, v) \in K(u_+)$ for every $v \in K(u_+)$.

Example 4.8. Let $V = H^1(\Omega)$, $H - L^2(\Omega)$, and $a(u, v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v + \int_\Omega uv$.

Suppose $f \geq 0$ and consider the variational inequality

$$\begin{cases} a(u, v) \geq (f, v) \quad \text{for every } v \in K \\ u \in K \end{cases}$$

where $K = \{v \in V : \varphi \le v \le \psi\}$, $\varphi, \psi \in H$. Suppose $\varphi \le 0 \le \psi$ so that $K \ne \phi$. In this case we may take $u_- = 0$, since $a(0, v) = 0 \le (f, v)$ for every $v \ge 0$ and $\sup(0, v) \in K$ for every $v \in K$. Let w be the solution of

$$(4.10) \qquad \begin{cases} a(w, v) = (f, v) & \text{for every } v \in V \\ w \in V . \end{cases}$$

Since $f \ge 0$, we know that $w \ge 0$. Since, in particular, $a(w, v) \ge (f, v)$ for every $v \ge 0$ and $\inf(w, v) \in K$ for every $v \in K$, we may take $u_+ = w$, by Remark 4.7.

Example 4.9. Consider Example 4.8 in the quasi-variational case

$$K(u) = \{u \in V : M_1 u \le v \le M_2 u\}$$

as in Example 4.5. Suppose the increasing mappings M_1, M_2 satisfy $M_1 w \le 0 \le M_2 0$ where $w \ge 0$ is the solution of (4.10) as before. Then, as in the preceding example, it is easy to see that we may take $u_- = 0$ and $u_+ = w$.

Example 4.10. Let V, H, a be as in Example 4.8 but consider the more general quasi-variational inequality (4.2) with

$$j_u(v) = \int_\Omega \varphi(x, u(x), v(x))dx ,$$

where φ is smooth. We require that $\frac{\partial^2 \varphi}{\partial v^2} \ge 0$ so that j_u is convex. To satisfy the compatibility condition we need that $u_1 \le u_2$ implies

(4.11) $\varphi(x, u_1, \inf(v_1, v_2)) + \varphi(x, u_2, \sup(v_1, v_2)) \leq \varphi(x, u_1, v_1) + \varphi(x, u_2, v_2)$

for every $v_1, v_2 \in V$, a.e. $x \in \Omega$. By integrating over a rectangle as in Section 2 it is not difficult to see that (4.11) will hold if $\frac{\partial^2 \varphi}{\partial u \partial v} \leq 0$.

This setting gives rise to a problem of the type

$$\begin{cases} -\Delta \bar{u} + \frac{\partial \varphi}{\partial v}(x, u, \bar{u}) = 0 \\ \text{boundary conditions.} \end{cases}$$

Example 4.11. Consider the case of a system, with $V = (H_0^1(\Omega))^N$, $H = (L^2(\Omega))^N$,

$$a(u, v) = \sum_{i=1}^{N} \int_\Omega \operatorname{grad} u_i \operatorname{grad} v_i dx + \sum_{i,j=1}^{N} \int_\Omega \lambda_{i,j} u_i v_j dx$$

where $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$. The bilinear functional a satisfies a coercivity condition of the usual type and since

$$a(u_+, u_-) = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \int_\Omega \lambda_{ij} u_{i_+} u_{j_-} dx$$

we require $\lambda_{ij} \leq 0$ for $i \neq j$ so that the compatibility condition $a(u_+, u_-) \leq 0$ holds.

In this setting, consider the quasi-variational inequality (4.2) with

$$j_u(v) = \int_\Omega \varphi(x, u_1, \ldots, u_N, v_1, \ldots, v_N) dx$$

where φ is a convex function of $v_1, \ldots, v_N$. We assume that $\frac{\partial^2 \varphi}{\partial v_i \partial v_j} \leq 0$ for $i \neq j$ and, in order to satisfy the compatibility condition, we assume that $\frac{\partial^2 \varphi}{\partial u_i \partial v_j} \leq 0$ for $i, j = 1, \ldots, N$. (See Example 4.10

the case $N = 1$ is discussed.) This setting formally solves a system of the form

$$
(4.12) \qquad \begin{cases} -\Delta u_i + F_i(u_1, \ldots, u_N) = f_i & \text{in } \Omega \\ u_i = 0 \text{ on } \partial\Omega, & i = 1, \ldots, N , \end{cases}
$$

where $F_i(u_1, \ldots, u_N) = \sum_{j=1}^{N} \lambda_{ij} u_j + \frac{\partial \varphi}{\partial v_i}(x, u_1, \ldots, u_N, u_1, \ldots, u_N)$. By the above assumptions on φ, F_i satisfies the compatibility condition $\frac{\partial F_i}{\partial u_j} \leq 0$ if $i \neq j$ since

$$
\frac{\partial F_i}{\partial u_j} = \frac{\partial^2 \varphi}{\partial u_j \partial v_i}(x, u_1, \ldots, u_N, u_1, \ldots, u_N) + \frac{\partial^2 \varphi}{\partial v_j \partial v_i}(x, u_1, \ldots, u_N, u_1, \ldots, u_N) \leq 0 .
$$

Conversely, if

$$
(4.13) \qquad F_i \in C^1, \quad \text{satisfying } \frac{\partial F_i}{\partial u_j} \leq 0,\ i \neq j,\ \frac{\partial F_i}{\partial u_i} \leq K,\ K \geq 0 .
$$

Then we may take

$$
\varphi(u_1, \ldots, u_N, v_1, \ldots, v_N) = \sum_{i=1}^{N} [\tfrac{1}{2} K v_i^2 - (K u_i - F_i(u_1, \ldots, u_N)) v_i]
$$

so that $\frac{\partial^2 \varphi}{\partial v_i \partial u_j} = \frac{\partial F_i}{\partial u_j} \leq 0$ if $i \neq j$ and $\frac{\partial^2 \varphi}{\partial v_i \partial u_i} = \frac{\partial F_i}{\partial u_i} - K \leq 0$.

Under the conditions (4.13) is also possible to consider (4.12) by a more direct approach. Assume that there exist $u_- = (u_{1_-}, \ldots, u_{N_-})$, $u_+ = (u_{1_+}, \ldots, u_{N_+})$, $u_-, u_+ \in V$, satisfying

$$\begin{cases} -\Delta u_{i_-} + F_i(u_-) \leq f_i \leq -\Delta u_{i_+} + F_i(u_+) & \text{in } \Omega \\ u_{i_-} \leq 0 \leq u_{i_+} & \text{on } \partial\Omega, \quad i = 1, \ldots, N . \end{cases}$$

Then there exist solutions u_{min}, u_{max} of (4.12) (possibly equal) such that $u_- \leq u_{min} \leq u_{max} \leq u_+$ and any solution u of (4.12) such that $u_- \leq u \leq u_+$ satisfies $u_{min} \leq u \leq u_{max}$.

This is proved by considering a mapping $T(u) = \bar{u}$ defined by

$$(4.14) \qquad \begin{cases} -\Delta \bar{u}_i + K\bar{u}_i = Ku_i - F_i(u) + f_i & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega . \end{cases}$$

One shows that T is increasing and continuous and that $T^n u_-, T^n u_+$ converge strongly to the desired solutions u_{min}, u_{max} as $n \to \infty$. Note that this problem is not genuinely quasi-variational since (4.14) is of variational type.

We conclude this section with an example which is genuinely quasi-variational.

<u>Example 4.12</u>. We will formally discuss the problem of a plate heated by a support, the shape of the plate being allowed to change with temperature. Consider the problem (formally)

$$
(4.15) \quad \begin{cases} -\Delta u - \theta = 0 \quad \text{in} \quad \Omega \\ -\Delta\theta = g \geq 0 \quad \text{in} \quad \Omega \\ \text{with, on} \quad \partial\Omega, \ u \geq 0, \ \frac{\partial u}{\partial n} \geq 0, \ u\frac{\partial u}{\partial n} = 0 \\ \text{and} \quad \theta \geq 0, \ \theta = 0 \quad \text{where} \quad u = 0, \ \frac{\partial\theta}{\partial n} = 0 \quad \text{where} \quad u > 0 \, . \end{cases}
$$

Here θ represents temperature and u the displacement of the plate. These boundary conditions give rise to the sets

$K(u,\theta) = \{(v,\eta) : v \geq 0, \ \eta \geq 0, \ \text{on} \ \partial\Omega, \ \text{and} \ \eta = 0 \ \text{where} \ u = 0 \ \text{on} \ \partial\Omega\}$.

We will show that K satisfies the compatibility condition. Suppose $(u_1, \theta_1) \leq (u_2, \theta_2)$, and choose $(v_i, \eta_i) \in K(u_i, \theta_i)$, $i = 1, 2$. We wish to show that $\inf((v_1, \eta_1), (v_2, \eta_2)) \in K(u_1, \theta_1)$ and $\sup((v_1, \eta_1), (v_2, \eta_2)) \in K(u_2, \theta_2)$. The first amounts to showing that $\inf(\eta_1, \eta_2) = 0$ where $u_1 = 0$, which is true since $\eta_1 = 0$ where $u_1 = 0$ and $\eta_2 \geq 0$. The second amounts to showing that $\sup(\eta_1, \eta_2) = 0$ where $u_2 = 0$ which is true since $\eta_2 = 0$ where $u_2 = 0$, $\eta_1 = 0$ where $u_1 = 0$, and $u_1 \leq u_2$.

The methods used above give that (4.15) has a solution (u, θ) such that $(0,0) \leq (u,\theta) \leq (\bar{u}, \bar{\theta})$ where $(\bar{u}, \bar{\theta})$ satisfies

$$
\begin{cases} -\Delta\bar{\theta} = g \quad \text{in} \quad \Omega\, , \ \bar{\theta} = 0 \quad \text{on} \quad \partial\Omega \\ -\Delta\bar{u} = \bar{\theta} \quad \text{in} \quad \Omega\, , \ \bar{u} = 0 \quad \text{on} \quad \partial\Omega\, . \end{cases}
$$

5. Order Relations on Symmetric Matrices

In this section we will consider the space $\mathcal{L}_S(H, H)$, with $H = \mathbb{R}^N$, of symmetric matrices ordered by $\mathbb{P} \geq Q$ if $(\mathbb{P}x, x) \geq (Qx, x)$ for every $x \in \mathbb{R}^N$. We will apply the theorems of this chapter on increasing maps to generalize the results obtained in [4] on the Riccati equation.

Consider the linear problem

$$(5.1) \qquad \begin{cases} \mathbb{P}' + \mathbb{P}A + B\mathbb{P} = F \\ \mathbb{P}(0) = \mathbb{P}_0 . \end{cases}$$

Recall that if $B = A^*$, $\mathbb{P}_0 = \mathbb{P}_0^*$, $F = F^*$, then $\mathbb{P}(t) = \mathbb{P}^*(t)$, and moreover, if $\mathbb{P}_0 \geq 0$, $F \geq 0$, then $\mathbb{P} \geq 0$.

We wish to generalize (5.1) to the nonlinear case

$$(5.2) \qquad \begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} = F(\mathbb{P}) \\ \mathbb{P}(0) = \mathbb{P}_0 \end{cases}$$

where F is a function from $\mathcal{L}_S(H, H)$ to itself satisfying $F(\mathbb{P}) = F(\mathbb{P})^*$ whenever $\mathbb{P} = \mathbb{P}^*$. Consider the iterative process $\mathbb{P} \to \overline{\mathbb{P}}$ where $\mathbb{P} \in L^\infty(0, T; \mathcal{L}_S(H, H))$ and $\overline{\mathbb{P}}$ is the solution of

$$\begin{cases} \overline{\mathbb{P}}' + \overline{\mathbb{P}}A + A^*\overline{\mathbb{P}} = F(\mathbb{P}) \\ \overline{\mathbb{P}}(0) = \mathbb{P}_0 \end{cases}$$

or, more generally, introduce $M \in \mathcal{L}(H, H)$ and consider the equation

$$(5.3) \qquad \begin{cases} \overline{\mathbb{P}}' + \overline{\mathbb{P}}(A + M) + (A^* + M^*)\overline{\mathbb{P}} = F(\mathbb{P}) + \mathbb{P}M + M^*\mathbb{P} \\ \overline{\mathbb{P}}(0) = \mathbb{P}_0 . \end{cases}$$

We seek a fixed point of the mapping $\mathbb{P} \to \overline{\mathbb{P}}$ defined by (5.3). To this end we would wish to know whether it is reasonable to require that $F(\mathbb{P}) + \mathbb{P}M + M^*\mathbb{P}$ is increasing. In the case of the Riccati equation $F(\mathbb{P}) = -\mathbb{P}D_1\mathbb{P} + D_2$, which is not increasing since $0 \leq Q \leq \mathbb{P}$ does not necessarily imply that $Q^2 \leq \mathbb{P}^2$. For this reason we will consider a quasi-variational setting

$$(5.4) \quad \begin{cases} \mathbb{P}'_{n+1} + \mathbb{P}_{n+1}(A + M_n) + (A^* + M_n)\mathbb{P}_{n+1} = \mathbb{P}_n M_n + M_n^*\mathbb{P}_n + F(\mathbb{P}_n) \\ \mathbb{P}_{n+1}(0) = \mathbb{P}_0 . \end{cases}$$

Consider the more general problem

$$(5.5) \quad \begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \varphi(t, \mathbb{P}) = 0 \\ \mathbb{P}(0) = \mathbb{P}_0 . \end{cases}$$

We will prove the following existence theorem:

<u>Theorem 5.1</u>. Suppose $\varphi(t, \mathbb{P}) = a(t)\mathbb{P} + \varphi_1(t, \mathbb{P}) + \mathbb{P}\varphi_2(t, \mathbb{P})\mathbb{P}$ where $\mathbb{P} \to \varphi_i(t, \mathbb{P})$ is decreasing for $i = 1, 2$. Suppose there exist $\mathbb{P}_-, \mathbb{P}_+ \in L^\infty(0, T; \mathcal{L}_S(H, H))$, $\mathbb{P}_-(t) \leq \mathbb{P}_+(t)$ for $t \in [0, T]$ satisfying

$$(5.6) \quad \begin{cases} \mathbb{P}'_- + \mathbb{P}_- A + A^*\mathbb{P}_- + \varphi(t, \mathbb{P}_-) \leq 0 \leq \mathbb{P}'_+ + \mathbb{P}_+ A + A^*\mathbb{P}_+ + \varphi(t, \mathbb{P}_+) \\ \mathbb{P}_-(0) \leq \mathbb{P}_0 \leq \mathbb{P}_+(0) . \end{cases}$$

Moreover, assume that there is a constant K such that

$$(5.7) \quad \|\varphi_i(t, \mathbb{P}) - \varphi_i(t, Q)\| \leq K\|P - Q\|$$

for $\mathbb{P}_- \leq \mathbb{P}$, $Q \leq \mathbb{P}_+$, $0 \leq t \leq T$, $i = 1, 2$.

Then (5.5) has a solution $\mathbb{P}$ such that $\mathbb{P}_- \leq \mathbb{P} \leq \mathbb{P}_+$ and if $\mathbb{P}_0$ increases in the interval $[\mathbb{P}_-(0), \mathbb{P}_+(0)]$, $\mathbb{P}(t)$ increases for every $t \in [0, T]$.

<u>Remark 5.2</u>. If $a(t) \leq 0$, $a(t)\mathbb{P}$ is decreasing in $\mathbb{P}$ so can be put in the term ϕ_1. If $a(t) \geq 0$, $a(t)\mathbb{P} = \mathbb{P}(a(t)\mathbb{P}^{-1})\mathbb{P}$, and since $a(t)\mathbb{P}^{-1}$ is decreasing in $\mathbb{P}$ it can be put in the term φ_2, except that a false singularity is introduced.

<u>Proof of Theorem 5.1</u>. As suggested earlier, we will employ an iterative method on the problems

$$(5.8)\quad \begin{cases} \mathbb{P}'_{n+1} + \mathbb{P}_{n+1}(A + M_n) + (A^* + M_n^*)\mathbb{P}_{n+1} - \mathbb{P}_n M_n - M_n^* \mathbb{P}_n + \varphi(t, \mathbb{P}_n) = 0 \\ \mathbb{P}_{n+1}(0) = \mathbb{P}_0 \end{cases}$$

where $M_n = \varphi_2(t, \mathbb{P}_n)\mathbb{P}_n + \lambda I$, and λ satisfies

$$(5.9)\quad 2\lambda \geq \sup_{0 \leq t \leq T} \{a(t) + \|\mathbb{P}_+(t) - \mathbb{P}_-(t)\| \max(\|\varphi_2(t, \mathbb{P}_+(t))\|, \|\varphi_2(t, \mathbb{P}_-(t)\|)\} .$$

We will say that Q is a lower solution of (5.5) if Q satisfies the same inequalities as $\mathbb{P}_-$ in (5.6), and an upper solution if it satisfies the same inequalities as $\mathbb{P}_+$ in (5.6).

The proof is accomplished in three steps:

<u>Step 1</u>. If $\mathbb{P}_- \leq \mathbb{P}_n \leq \mathbb{P}_+$, then $\mathbb{P}_- \leq \mathbb{P}_{n+1} \leq \mathbb{P}_+$.

<u>Step 2</u>. If $\mathbb{P}_n$ is a lower solution, then $\mathbb{P}_n \leq \mathbb{P}_{n+1}$ and $\mathbb{P}_{n+1}$ is a lower solution. If $\mathbb{P}_n$ is an upper solution, then $\mathbb{P}_n \geq \mathbb{P}_{n+1}$ and $\mathbb{P}_{n+1}$ is an upper solution.

Step 3. If $\mathbb{P}_1 = \mathbb{P}_-$, then $\{\mathbb{P}_n\}$ is monotone increasing and converges to the solution $\mathbb{P}$. If $\mathbb{P}_1 = \mathbb{P}_+$, then $\{\mathbb{P}_n\}$ is monotone decreasing and converges to the solution $\mathbb{P}$.

Note that uniqueness is guaranteed by the Lipschitz hypothesis. (See Remark 5.3.) We will only prove Step 1; the proofs of Step 2 is similar and Step 3 is easy. We will show that if $\mathbb{P}_n \geq \mathbb{P}_-$, then $\mathbb{P}_{n+1} \geq \mathbb{P}_-$. The analogous statement with $\mathbb{P}_+$ may be proven similarly. Since $\mathbb{P}_{n+1}$ satisfies (5.8) and $\mathbb{P}_-(0) \leq \mathbb{P}_0 = \mathbb{P}_{n+1}(0)$ it suffices to show that $\mathbb{P}_-$ satisfies

$$\mathbb{P}'_- + \mathbb{P}_-(A + M_n) + (A^* + M_n^*)\mathbb{P}_- - \mathbb{P}_n M_n - M_n^* \mathbb{P}_n + \varphi(t, \mathbb{P}_n) \leq 0 .$$

By (5.6), this will be true if we can show that

$$\mathbb{P}_- M_n + M_n^* \mathbb{P}_- - \mathbb{P}_n M_n - M_n^* \mathbb{P}_n + \varphi(t, \mathbb{P}_n) - \varphi(t, \mathbb{P}_-) \leq 0 . \tag{5.10}$$

Using the definition M_n and the decomposition of φ a simple computation shows that

$$\begin{aligned} \mathbb{P}_- M_n + M_n^* \mathbb{P}_- - \mathbb{P}_n M_n - M_n^* \mathbb{P}_n + \varphi(t, \mathbb{P}_n) - \varphi(t, \mathbb{P}_-) = \\ (2\lambda - a)(\mathbb{P}_- - \mathbb{P}_n) - (\mathbb{P}_n - \mathbb{P}_-)\varphi_2(t, \mathbb{P}_n)(\mathbb{P}_n - \mathbb{P}_-) \\ + (\varphi_1(\mathbb{P}_n) - \varphi_1(\mathbb{P}_-)) + \mathbb{P}_-(\varphi_2(\mathbb{P}_n) - \varphi_2(\mathbb{P}_-))\mathbb{P}_- . \end{aligned} \tag{5.11}$$

Since $\mathbb{P}_n \geq \mathbb{P}_-$, φ_1 and φ_2 decreasing, the last two terms of (5.11) are nonpositive. To show that the remainder of (5.11) is nonpositive we will make use of the following

Claim. If $A, B \in \mathcal{L}_S(\mathbb{R}^N, \mathbb{R}^N)$, $B \geq 0$, then $BAB \leq \|A\| \, \|B\| B$.

Using the claim with $A = \varphi_2(t, \mathbb{P}_-)$, $B = \mathbb{P}_n - \mathbb{P}_-$ we have that

$$(\mathbb{P}_n - \mathbb{P}_-)\varphi_2(t, \mathbb{P}_n)(\mathbb{P}_n - \mathbb{P}_-) \le (\mathbb{P}_n - \mathbb{P}_-)\varphi_2(t, \mathbb{P}_-)(\mathbb{P}_n - \mathbb{P}_-)$$
$$\le \|\varphi_2(t, \mathbb{P}_-)\| \, \|\mathbb{P}_n - \mathbb{P}_-\|(\mathbb{P}_n - \mathbb{P}_-)$$
$$\le \|\varphi_2(t, \mathbb{P}_-)\| \, \|\mathbb{P}_+ - \mathbb{P}_-\|(\mathbb{P}_n - \mathbb{P}_-) \ .$$

Therefore, by (5.9),

$$(2\lambda - a)(\mathbb{P}_- - \mathbb{P}_n) - (\mathbb{P}_n - \mathbb{P}_-)\varphi_2(t, \mathbb{P}_n)(\mathbb{P}_n - \mathbb{P}_-)$$
$$\le (2\lambda - a(t) - \|\varphi_2(t, \mathbb{P}_-)\| \, \|\mathbb{P}_+ - \mathbb{P}_-\|)(\mathbb{P}_- - \mathbb{P}_n) \le 0 \ .$$

Given the truth of the claim, this completes the proof of Step 1.

To prove the claim, note that we wish to show that

$$(BABx, x) \le \|A\| \, \|B\|(Bx, x) \quad \text{for every} \quad x \in \mathbb{R}^N \ .$$

Now, $(BABx, x) = (ABx, Bx) \le \|A\| \, |Bx|^2$, so it will suffice to show that

(5.12) $$|Bx|^2 \le \|B\|(Bx, x) \quad \text{for every} \quad x \in \mathbb{R}^N \ .$$

Defining a new scalar product, $(x, y)_* = (Bx, y)$ with corresponding norm $\|x\|_*^2 = (x, x)_* = (Bx, x)$ we have by Cauchy-Schwartz that

$$|(Bx, x)_*|^2 \le \|Bx\|_*^2 \|x\|_*^2$$

which implies that

$$|Bx|^4 = |(Bx, Bx)|^2 \le (B^2x, Bx)(Bx, x) \le \|B\| \, |Bx|^2(Bx, x)$$

and proves (5.12).

The fact that the solution of (5.5) is an increasing function of the initial data $\mathbb{P}_0$ is an application of the above proof, since if

$\mathbb{P}_0 \le \tilde{\mathbb{P}}_0$, the solution $\mathbb{P}$ of (5.5) with $\mathbb{P}(0) = \mathbb{P}_0$ is a lower solution for the problem (5.5) with solution $\tilde{\mathbb{P}}$ satisfying $\tilde{\mathbb{P}}(0) = \tilde{\mathbb{P}}_0$. Hence $\mathbb{P} \le \tilde{\mathbb{P}}$. This completes the proof of Theorem 5.1.

<u>Remark 5.3</u>. Consider the problem

$$\text{(5.13)} \qquad \begin{cases} \mathbb{P}' = \mathbb{P}^{\frac{1}{2}} \\ \mathbb{P}(0) = 0 \, . \end{cases}$$

The mapping $\mathbb{P} \to \mathbb{P}^{\frac{1}{2}}$ is increasing, but not Lipschitz continuous. Uniqueness does not hold since $\mathbb{P}(t) = \frac{t^2}{4} Q$ is a solution for any $Q \in \mathcal{L}_S(\mathbb{R}^N, \mathbb{R}^N)$ satisfying $Q^2 = Q$.

<u>Example 5.4</u>. Let $\varphi(t, \mathbb{P}) = \mathbb{P}D\mathbb{P} - F$, independent of t, so that $_1(\mathbb{P}) \equiv -F$, $\varphi_2(\mathbb{P}) \equiv D$, D, F fixed. Then (5.5) becomes the Riccati equation

$$\begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \mathbb{P}D\mathbb{P} = F \\ \mathbb{P}(0) = \mathbb{P}_0 \, . \end{cases}$$

Suppose $\mathbb{P}_0, D, F \ge 0$. Then one may take $\mathbb{P} = 0$ and $\mathbb{P}_+$ to be the solution of the linear problem

$$\text{(5.14)} \qquad \begin{cases} Q' + QA + A^*Q = F \\ Q(0) = \mathbb{P}_0 \end{cases}$$

which is known to be positive. Then

$$\mathbb{P}'_+ + \mathbb{P}_+A + A^*\mathbb{P}_+ + \mathbb{P}_+D\mathbb{P}_+ - F = \mathbb{P}_+D\mathbb{P}_+ \ge 0$$

so that $\mathbb{P}_+$ satisfies (5.6).

<u>Example 5.5</u>. Let $\varphi(\mathbb{P}) = a\mathbb{P} + \mathbb{P}D\mathbb{P} - F$. If $a > 0$, $D \geq 0$, one can find a lower solution $\mathbb{P}_-$ even for small negative values of $\mathbb{P}_0$ and F. However, for large negative values of $\mathbb{P}_0$ and F the solution $\mathbb{P}(t)$ may tend to $-\infty I$. If one can find an upper solution $\mathbb{P}_+$ such that $\mathbb{P}_+(t) \to -\infty I$ as $t \to t_0$, then, of course, the solution $\mathbb{P}$ must tend to $-\infty I$ as $t \to \tilde{t}$ for some $\tilde{t} \leq t_0$.

<u>Example 5.6</u>. Let $\varphi(\mathbb{P}) = -\sum_{i=1}^{n} B_i^* \mathbb{P} B_i + \mathbb{P}D\mathbb{P} - F$, then $\varphi_1(\mathbb{P}) = -\sum_{i=1}^{n} B_i^* \mathbb{P} B_i - F$, $\varphi_2(\mathbb{P}) = D$ are decreasing and we are led to the problem

$$\begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} - \sum_{i=1}^{n} B_i^* \mathbb{P} B_i + \mathbb{P}D\mathbb{P} = F \\ \mathbb{P}(0) = \mathbb{P}_0 . \end{cases}$$

<u>Example 5.7</u>. Let $\varphi_1(\mathbb{P}) \equiv -F$, $\varphi_2(\mathbb{P}) = (M + \sum_{i=1}^{n} C_i^* \mathbb{P} C_i)^{-1}$ where $M \geq 0$ is invertible. (This comes from a stochastic control problem)

$$\begin{cases} \mathbb{P}' + A\mathbb{P} + \mathbb{P}A^* + \mathbb{P}(M + \sum_{i=1}^{n} D_i^* \mathbb{P} D_i)^{-1}\mathbb{P} = F \\ \mathbb{P}(0) = \mathbb{P}_0 . \end{cases}$$

If $\mathbb{P}_0 \geq 0$, $F \geq 0$, we may take $\mathbb{P}_- = 0$ and $\mathbb{P}_+$ to be the solution of the linear problem (5.14).

Example 5.8. Let $\varphi_2(\mathbb{P}) = (M + \sum_{i=1}^N D_i^* \mathbb{P} D_i)^{-1}$ as in the preceding example and let

$$\varphi_1(\mathbb{P}) = - \sum_{i=1}^N B_i^* \mathbb{P} B_i + (\sum_{i=1}^N B_i^* \mathbb{P} D_i)\varphi_2(\mathbb{P})(\sum_{i=1}^N D_i^* \mathbb{P} B_i) .$$

To show that φ_1 is decreasing suffices to show that if $\mathbb{P}, Q \in \mathcal{L}_S$, $Q \geq 0$, then

$$\frac{d}{dt} \varphi_1(\mathbb{P} + tQ)\Big|_{t=0} \leq 0 . \tag{5.15}$$

For brevity, let $R = \sum_{i=1}^N D_i^* \mathbb{P} B_i$, $S_i = D_i^* Q B_i$, $L = \varphi_2(\mathbb{P})$. Computation shows that

$$\frac{d}{dt} \varphi_1(\mathbb{P} + tQ)\Big|_{t=0} = - \sum_{i=1}^N B_i^* Q B_i + (\sum_{i=1}^N S_i^*)LR$$

$$+ R^* L(\sum_{i=1}^N S_i) - R^* L(\sum_{i=1}^N D_i^* Q D_i)LR = \sum_{i=1}^N X_i$$

where $X_i = - B_i^* Q B_i + S_i^* LR + R^* L S_i - R^* L D_i^* Q D_i LR$. Since $Q \geq 0$, $T = Q^{\frac{1}{2}}$ exists. Choose $x \in \mathbb{R}^N$ and let $a = TB_i x$, $b = TD_i LRx$. Then

$$(X_i x, x) = - (B_i^* Q B_i x, x) + (B_i^* Q D_i LRx, x) + (R^* L D_i^* Q B_i x, x)$$
$$- (R^* L D_i^* Q D_i LRx, x)$$
$$= - (TB_i x, TBix) + (TD_i LRx, TB_i x) + (TB_i x, TD_i LRx)$$
$$- (TD_i LRx, TD_i LRx) = - (a, a) + (b, a) + (a, b) - (b, b)$$
$$= - (a - p, a - b) \leq 0 .$$

This proves (5.15) and shows that φ_1 is decreasing.

Remark 5.9. Consider again the Ricatti equation

$$\begin{cases} \mathbb{P}' + \mathbb{P}A + A^*\mathbb{P} + \mathbb{P}D\mathbb{P} = F \\ \mathbb{P}(0) = \mathbb{P}_0 \end{cases}$$

where we take $F \geq 0$ to be independent of t. Suppose $\mathbb{P}_-$ and $\mathbb{P}_+$ may be found, also independent of t. If $\mathbb{P}_0 = \mathbb{P}_-$, then $\mathbb{P}(t)$ is increasing as $t \to \infty$ tends to $\mathbb{P}_\infty^{min}$, a solution of the corresponding stationary problem. If $\mathbb{P}_0 = \mathbb{P}_+$, then $\mathbb{P}(t)$ decreases and converges to $\mathbb{P}_\infty^{max}$, a solution of the stationary problem. Let us consider an example in which the stationary problem has more than one solution.

$$\begin{cases} \mathbb{P}' - \mathbb{P} + \mathbb{P}^2 = 0 \\ \mathbb{P}(0) = \mathbb{P}_0 \ . \end{cases}$$

If $\mathbb{P}_- = 0$, $\mathbb{P}_+ = kI$, $k \geq 1$, every orthogonal projection is a stationary solution and there are infinitely many stationary solutions between $\mathbb{P}_-$ and $\mathbb{P}_+$. If $\mathbb{P}_- = \varepsilon I$, $0 < \varepsilon < 1$, the I is the only stationary solution between $\mathbb{P}_-$ and $\mathbb{P}_+$.

If $\mathbb{P}_\infty^{min} = \mathbb{P}_\infty^{max}$, any solution $\mathbb{P}$ with $\mathbb{P}_- \leq \mathbb{P}_0 \leq \mathbb{P}_+$ must converge to $\mathbb{P}_\infty^{min} = \mathbb{P}_\infty^{max}$ as $t \to \infty$. In the above example, $\mathbb{P}_0 \geq \varepsilon I$, implies that $\mathbb{P}(t) \to I$ as $t \to \infty$.

Acknowledgment

This is part of the lecture notes of a course given at the University of Wisconsin-Madison in 1974-75.

I wish to thank D. Brewer for the redaction of this part.

References

For quasi-variational inequalities see [1]; for classical ideas using positivity see [2] and for applications to Riccati equations see [3].

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