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NORMS OF EXPONENTIAL SUMS

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PREFACE

The "raison d'être" of these notes is to present in a unified form that part of the theory of exponential sums which is related to the conjecture of Littlewood, i. e. the problem of finding lower bounds for $\|\exp(in_1x)+\ldots+\exp(in_Nx)\|_1$, $n_1, \ldots, n_N$ distinct integers, depending only on N. (The promissed second volume of |14| would contain a chapter treating similar matters, but it has not appeared yet).

Except for theorems (3.4), (5.2) and formula (2.17), which as far as the author knows have not appeared before in the literature, all the other results can be found in the references given in the bibliography.

After the brief introductory chapter I we devote chapters II and III to the study of the L^p, $p \geqslant 1$, and the L^1 Norms of exponential sums. Chapter IV contains some results concerning the minimum of real exponential sums and some special exponential sums are examined in the last Chapter V.

The author would like to express his thanks to the members of "L'équipe d'analyse harmonique" of the Department of Mathematics of the "Université de Paris Sud, Centre d'Orsay" and in particular to J.-P. Kahane, Y. Meyer, A. Tonge and N. Varopoulos for their helpful comments and encouragement. He would also like to thank Mme Dumas who typed (avec beaucoup de gentillesse) these notes.

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CONTENTS

NOTATIONS

Throughout these notes :

(i) We shall use the same letter C for all positive absolute constants that we encounter

(ii) Integrals without limits of integration will always be understood to be taken over $[0,2\pi]$ with respect to the normalized Lebesgue measure $\left(\int \dots = \frac{1}{2\pi} \int_0^{2\pi} \dots \right)$.

(iii) If f is an integrable 2π-periodic-function then $\hat{f}(m)$, m integer, will denote its m'th Fourier coefficient $\left(= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-imt}\, dt\right)$.

(iv) For any 2π-periodic measurable function $\|f\|_p$, $p \geq 1$ will denote its p-th norm $\left(= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f|^p \right\}^{1/p}\right)$.

(v) Unless otherwise specified, $n_1, n_2, \dots n_N$ will denote distinct positive integers.

In chapter II

(vi) All unspecified summations will be understood to be extended over the set of all suffices whose sum is zero $\left(\Sigma\, a_r\, b_s \dots = \sum_{r+s+\dots=0} a_r\, b_s \dots\right)$.

I. EXPONENTIAL SUMS IN HARMONIC ANALYSIS AND NUMBER THEORY

In this introductory chapter we give some examples of important theorems in Harmonic analysis and Number theory which are equivalent to theorems about exponential sums.

1. INTRODUCTION. "Exponential sums" are defined as finite sums f of distinct integral powers of the exponential function $\exp(ix)$:

(1.1) $$f(x) = \sum_{k=1}^{N} \exp(in_k x), \quad n_1, \ldots, n_N \text{ distinct integers}, \quad x \text{ real}.$$

There are properties of exponential sums which are shared by larger classes of trigonometric polynomials. Such classes of trigonometric polynomials are those with (i) non-negative coefficients, (ii) coefficients of absolute value 1, (iii) coefficients of absolute value not less than 1 and others. Thus for instance theorem III 3.5, which is one of our main results, holds good for the class (iii).

Our main interest in these notes will be to find estimates of the various L^p norms of exponential sums depending only on the number of summands N. This will be done in the chapters which follow (II - V).

In the following sections we cite very briefly some important application in which different estimates, e. g. local estimates of the modulus of exponential sums, play a predominant role.

2. EXPONENTIAL SUMS IN HARMONIC ANALYSIS.

The exponential sums are special trigonometric polynomials, namely those with coefficients 0 and 1 only. The last statement can be written as

(1.2) $\{\hat{f}(n)\}^2 = \hat{f}(n)$.

Here and in the sequel $\hat{g}(m)$ means the mth Fourier coefficient of g.

(1.2) implies $f * f = f$ i. e., as we usually say, f is "idempotent". The convolution" $f * g$ is defined by

(1.3) $$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, g(x-t)\, dt, \quad f,g \in L^1(0\ ,\ 2\pi).$$

Using the Riemann-Lebesgue lemma ($f \in L^1(0,2\pi)$ implies $\hat{f}(n) \to 0$) we can easily see that conversely : "Idempotent functions in $L^1(0,2\pi)$ are exponential sums".

At this point we mention an interesting theorem of Helson ([19]) characterizing the class of "infinite" exponential sums

$\sum_{k=-\infty}^{\infty} \exp(in_k x)$, $\ldots < n_{-1} < n_0 < n_1 < \ \ldots$, integers, as the class of idempotent measures m on $[0\ ,\ 2\pi]$ (i. e. $m * m = m$) if and only if the sequence $\{n_k\}$ coincides, except for at most a finite number of indices, with an arithmetic progression.

The theorem of Helson provides an example of the occurence of exponential sums in Harmonic Analysis. More important examples are related to the special exponential sum, usually referred to as the "Dirichlet kernel",

(1.4) $$D_N(x) = \sum_{k=-N}^{N} \exp(ikx).$$

The importance of D_N is due to the well-known formula

(1.5) $$S_N(x) = (D_N * f)(x)$$

where S_N is the N-th partial sum of the Fourier series of f.

One of the main problems in harmonic analysis, raised by Luzin in 1916 and solved by Carleson in 1966, was that of the almost everywhere convergence of Fourier series of square integrable functions. A few years before Carleson's affirmative answer to this question ([1]) A. Calderon proved ([40], XIII, 1.22) that the almost everywhere convergence of Fourier series in L^2 is equivalent to the uniform boundedness of the integral

$$\int_0^{2\pi} \int_0^{2\pi} D_{\min(N(x),N(y))}(x-y)\, dx\, dy \tag{1.6}$$

for all positive integral valued step functions $N(x)$, i. e. a property of the special exponential sum (1.4). Carleson's proof follows quite different lines, and a later proof by C. Fefferman ([10]) although closer to the above mentioned approach cannot be considered as a direct proof of the uniform boundedness of (1.6) (needless to say that such a direct proof would be quite wellcome.)

3. EXPONENTIAL SUMS IN NUMBER THEORY.

Exponential sums play a very important role in other branches of mathematics and in particular in number theory. In principle this is not surprising, since an exponential sum is completely determined by a set of integers $\{n_k\}$. Thus, some statements about sets of integers (in particular the "additive" properties of such sequences) can be translated into statements about exponential sums and then proved by analytical methods. In general these analytical methods belong to harmonic analysis or analytic function theory.

We give now an important theorem in number theory whose proof is based on properties of a special exponential sum.

In 1937 I. Vinogradov proved that all sufficiently large odd integers can be represented as sums of three primes (the question of whether or not all (sufficiently) large even integers can be represented as sums of two primes, the so-called Goldbach conjecture, is still open).

Let n be an odd integer and consider the exponential sum

(1.7) $$f(x) = \sum_{p<n, p \text{ prime}} \exp(ipx).$$

The coefficient $\widehat{f^3}(n)$ of e^{inx} in $f^3(x)$, as can easily be seen, equals the number of representations of n as a sum of three primes. It follows that Vinogradov's theorem is equivalent to the assertion :

(1.8) $$\widehat{f^3}(n) = \frac{1}{2\pi} \int_0^{2\pi} f^3(t)\, e^{-int}\, dt > 0,$$

n sufficiently large odd integer, that is to a property of the special exponential sum (1.7). (1.8) has been deduced from the fact (loosely speaking) that the exponential sum (1.7) is "sufficiently smaller than n" in absolute value (n being a trivial estimate) except for those x which are "sufficiently close" to rationals with "small" denominators (see [9] for the complete proof).

The theorems of Carleson and Vinogradov are two of the most difficult ones in harmonic analysis and number theory, and as we saw they are equivalent to properties of some special exponential sums. One is tempted to say : "innocent-looking" properties, but our experience up to now has shown that it was only after a good deal of ingenious work that we were able to handle them directly or indirectly.

II. L^p NORMS.

In this chapter we examine the L^p norms, $1 < p < \infty$, of exponential sums. The main result is a theorem due to Gabriel which generalizes a previous one of Hardy and Littlewood on the rearrangement of Fourier coefficients.

1. INTRODUCTION.

L^p norms are convenient measures of the average size of functions. For a 2π-periodic (measurable) function f they are defined by

$$(2.1) \qquad \|f\|_p = \left\{ \int |f(x)|^p \, dx \right\}^{1/p}, \quad p > 0$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_x |f(x)| .$$

Here and in the sequel we adopt the convention : if there are no limits of integration the symbol $\int (.)$ will mean $\frac{1}{2\pi} \int_0^{2\pi} (.) \, dx$.

Estimates of the average size of our object of investigation, i. e. exponential sums, are not only of interest in themselves but they also provide the key to the solution of some problems. Here is a classical example.

Let $f(t) = \operatorname{sgn} D_N(t)$, where D_N is the Dirichlet kernel defined in the previous chapter. If the Nth partial sum of the Fourier series of an integrable function g is denoted by $S_{N,g}$ then we have :

(2.2) $$S_{N,g}(o) = \|D_N\|_1 .$$

As we shall see in the next chapter the second member of (2.2) is of the order of $\log N$, as $N \to \infty$, and so we conclude : "The partial sums of the Fourier series of functions bounded by 1 can be as large as we please".

The same example and standard arguments of harmonic analysis show the "existence of a continuous function whose Fourier series diverges at a given point". ([40], VIII, 1.1).

In the present chapter we examine L^p norms with $p > 1$. The case $p = 1$ will be examined in chapter III.

Of particular importance are estimates depending only on the number N of summands. For the L^2 and the L^∞ norm of the exponential sum $f(x) = \exp(in_1x) + \ldots + \exp(in_Nx)$ we have

(2.3) $$\|f\|_2 = N^{1/2}, \quad \|f\|_\infty = N.$$

We do not have such simple formulae for the other norms, but in the special case of an even exponent we have the exact estimate :

(2.4) $$\Big\|\sum_{k=1}^{N} \exp(in_kx)\Big\|_{2q} \le \Big\|\sum_{k=1}^{N} \exp(ikx)\Big\|_{2q}, \quad n_1, \ldots, n_N \text{ distinct integers}, \quad q = 1, 2, \ldots$$

It is not hard to obtain explicit estimates of the second member of (2.4) and we shall do it later. Since the second member of (2.4) does not change when we replace the exponentials $\exp(ikx)$ by $\exp\{i(ak+b)x\}$, a, b integers, we can restate (2.4) as follows :

"The L^{2q}, $q = 1, 2, \ldots$, norms of exponential sums with the same number of summands is maximized when the frequencies are arranged in arithmetic progression".

2. EVEN EXPONENTS.

We state and prove now an inequality much stronger than (2.4) and postpone some comments until after the proof. We introduce first some notation.

Given a finite sequence $\{a_n\}$ of non-negative numbers we define the sequences a_n^+ and ${}^+a_n$ as the (unique) rearrangements of $\{a_n\}$ such that

(2.5) $$a_o^+ \geq a_1^+ \geq a_{-1}^+ \geq a_2^+ \geq a_{-2}^+ \geq \dots.$$
$$^+a_o \geq {}^+a_{-1} \geq {}^+a_1 \geq {}^+a_{-2} \geq {}^+a_2 \geq \dots.$$

Loosely speaking the $\{a_n^+\}$ and $\{{}^+a_n\}$ are the non-increasing "symmetric" rearrangements of $\{a_n\}$, the a_n^+ slightly overweighted to the right, the ${}^+a_n$ to the left. Note that always $a_k^+ = {}^+a_{-k}$.

When the largest value in $\{a_n\}$ occurs an odd number of times and every other value an even number of times then ${}^+a_k = a_k^+$. In this case we write $a_k^* = a_k^+ = {}^+a_k$ and call the sequence <u>symmetrical</u>. We note that symmetrical does not necessarily means $a_k = a_{-k}$. On the other hand even if $a_k = a_{-k}$ the sequence $\{a_n\}$ need not be symmetrical (it must also satisfy the condition $a_o \geq a_k$). Loosely speaking we can say that symmetrical sequences are those which admit a completely symmetric non-increasing rearrangement.

THEOREM 2.1. "<u>If</u> $\{a_r\}, \{b_s\}, \{c_t\}, \dots$ <u>are</u> k <u>finite sequences of non-negative numbers, then</u>

(2.6) $$\sum_{r_1+r_2+s_1+s_2+t_1+t_2+\dots=0} a_{r_1}a_{-r_2}b_{s_1}b_{-s_2}c_{t_1}c_{-t_2}\dots\dots \leq$$
$$\sum_{r_1+r_2+s_1+s_2+t_1+t_2+\dots=0} (a_{r_1}^+)({}^+a_{r_2})(b_{s_1}^+)({}^+b_{s_2})(c_{t_1}^+)({}^+c_{t_2})\dots$$

One can easily see that the first member of (2.6), equals

$\int |f_a|^2 |f_b|^2 |f_c|^2 \ldots.$, where $f_a(t) = \sum\limits_r a_r \exp(irt)$, $f_b(t) = \sum\limits_r b_r \exp(irt), \ldots$

Hence (2.4) is the special case of (2.6) : $a_r = b_r = c_r = \ldots = 1$ or 0 according as r belongs or not to $\{n_1, \ldots n_N\}$.

Proof. The proof consists of four steps.

In the first step we show that (2.6) is a consequence of

$$(2.7) \qquad \sum a_r b_s c_t d_\ell \ldots \leq \sum a_r^+ {}^+b_s c_t^* d_\ell^* \ldots .$$

Here $\{a_r\}$ and $\{b_s\}$ are arbitrary sequences of non negative numbers, but $\{c_t\}$, $\{d_\ell\}$, ... are assumed in addition to be symmetrical.

The summation in (2.7) is extended over the set of all suffices such that their sum is zero. We adopt this convention throughout the rest of this chapter.

In the second step it is shown that (2.7) is a consequence of its special case corresponding to three sequences only ($\{a_r\}$ and $\{b_s\}$ are arbitrary and $\{c_t\}$ symmetrical).

In the third step we show that we may even assume that the a_r,s, b_s's and c_t's are 0 or 1.

This special case of (2.7) (three sequences only consisting of zero's and one's) is proved in the fourth and last step.

Step 1. We recall our convention on summation and write :

$$B_m = \sum b_{s_1} b_{-s_2} \quad , \quad C_n = \sum c_{t_1} c_{-t_2} , \ldots$$

Obviously $B_m = B_{-m}$, $C_n = C_{-n}$, ... and (by Schwarz's inequality) $B_m \leq B_o$, $C_n \leq C_o$, ... It follows that the sequences $\{B_m\}$, $\{C_n\}$, ... are symmetrical.

On observing that for any permutation φ of the indices we have (trivial proof)

(2.8) $$a_r^+ = a_{\varphi(r)}^+ , \qquad {}^+a_r = {}^+a_{\varphi(r)}$$

and using (2.7) we obtain

(2.9) $$\Sigma\, a_{r_1}\, a_{-r_2}\, b_{s_1}\, b_{-s_2}\, c_{t_1}\, c_{-t_2} = \Sigma\, a_{r_1}\, a_{-r_2}\, B_m\, C_n \cdots$$
$$\leq \Sigma\, a_{r_1}^+\, {}^+a_{r_2}\, B_m^*\, C_n^* \cdots \quad \text{(by (2.7))}$$
$$= \sum_m X_{-m}\, B_m^*$$

where $$X_m = \sum_{r_1+r_2+n+\ldots=m} a_{r_1}^+\, {}^+a_{r_2}\, C_n^* \cdots$$
$$= \sum_{r+n+\ldots=m} \bar{a}_r\, C_n^* \cdots$$

and $$\bar{a}_r = \sum_{r_1+r_2=m} a_{r_1}^+\, {}^+a_{r_2}.$$

We observe that $\{\bar{a}_r\}$ is a symmetrical sequence. Indeed

$$\bar{a}_{-r} = \sum_{-r_1-r_2=r} a_{r_1}^+\, {}^+a_{r_2}$$
$$= \sum_{-r_1-r_2=r} {}^+a_{-r_1}\, a_{-r_2}^+ \qquad \text{(recall that } a_r^+ = {}^+a_{-r})$$
$$= \sum_{r_1+r_2=r} {}^+a_{r_1}\, a_{r_2}^+$$
$$= \sum_{r_1+r_2=r} a_{r_1}^+\, {}^+a_{r_2} = \bar{a}_r$$

and by Schwarz's inequality $\bar{a}_r \leq a_o$.

Moreover $\bar{a}_r$ is non-increasing for $r \geq 0$. To see this we write for simplicity $a_r = a_r^+ = {}^+a_r$ and argue as follows :

If $n > 0$ then

$$\bar{a}_n - \bar{a}_{n+1} = \sum_r a_r\, a_{r-n} - \sum_r a_r\, a_{r-n-1}$$
$$= a_o(a_{-n} - a_{-n-1}) + \left\{a_1(a_{1-n} - a_{-n}) + a_{-1}(a_{-1-n} - a_{-2-n})\right\} + \ldots$$

$$\left\{a_k(a_{k-n} - a_{k-n-1}) + a_{-k}(a_{-k-n} - a_{-k-n-1})\right\} + \ldots$$

Since $a_{-k-n} - a_{-k-n-1} \geq 0$ (recall that $n > 0$, $k \geq 0$) and $a_k \geq a_{-k}$ the last sum is not less than

$$a_o(a_{-n}-a_{-n-1}) + a_1(a_{1-n} - a_{-n} + a_{-1-n} - a_{-2-n}) + \ldots$$
$$+ a_k(a_{k-n} - a_{k-n-1} + a_{-k-n} - a_{-k-n-1}) + \ldots$$
$$= (a_o-a_1)(a_{-n}-a_{-n-1}) + (a_1-a_2)(a_{1-n} - a_{-2-n}) + \ldots + (a_k - a_{k+1})(a_{k-n} - a_{-k-n-1}) + \ldots$$

which is obviously non-negative.

Trivial modifications of the last argument show that for any symmetrical sequences $\{a_r\}$, $\{b_s\}$ the sequence $\sum_{r+s=m} a_r^* b_s^*$ is symmetrical and non-increasing for $m \geq 0$. Using this fact and a trivial induction we obtain the following

LEMMA 2.1. "Given any finite number of symmetrical sequences $\{a_r\}$, $\{b_s\}$, $\{c_t\}$, ..., the sequence $\left\{\sum_{r+s+\ldots=m} a_r^* b_s^* c_t^* \ldots\right\}$ is symmetrical and non-increasing for $m \geq 0$ ".

Let now φ be a permutation of the indices such that $B_m^* = B_{\varphi(m)}$. The fact that $\{X_m\}$ is symmetrical and non-increasing for $m \geq 0$ allows us to continue (2.9) as follows :

(2.10)
$$\sum a_{r_1} a_{-r_2} b_{s_1} b_{-s_2} c_{t_1} c_{-t_2} \ldots \leq \sum_m X_m B_m^*$$
$$= \sum_m X_{\varphi^{-1}(m)} B_m$$
$$= \sum b_{s_1} b_{-s_2} X_{\varphi^{-1}(-m)}$$
$$\leq \sum b_{s_1}^+ {}^+b_{s_2} X_m^*$$
$$= \sum b_{s_1}^+ {}^+b_{s_2} X_m$$
$$= \sum a_{r_1}^+ {}^+a_{r_2} b_{s_1}^+ {}^+b_{s_2} C_n^* \ldots$$

The second equality above follows from the symmetry of $\{X_m\}$ and the fact that $\varphi(m) = -\varphi(-m)$ (a consequence of the symmetry of $\{B_n\}$) and the third from the fact that $X_m = X_m^*$. The second inequality is a consequence of (2.7).

Repeating the same argument we can replace in the second member of (2.10) C_n^* by $C_{t_1}^+ \, {}^+C_{t_2}$. Continuing this way we obtain after a finite number of steps the desired inequality (2.6).

Step 2. We now assume that

(2.11) $$\Sigma \, a_r \, b_s \, c_t \le \Sigma \, a_r^+ \, {}^+b_s \, c_t^*$$

and we shall prove (2.7) by induction on the number k of symmetrical sequences in it.

(2.11) is the case $k = 1$. Assuming the result for $k-1$ and writing

$$P_m = \sum_{r+s=m} a_r b_s , \qquad Q_n = \sum_{t+\ell+\ldots=n} c_t^* \, d_\ell^* \ldots , \qquad {}^+P_m = P_{q(m)}$$

we have

$$\begin{aligned} \Sigma \, a_r \, b_s \, c_t \, d_\ell \ldots &= \Sigma \, P_m \, c_t \, d_\ell \ldots \\ &\le \Sigma \, {}^+c_t \, P_m^+ \, d_\ell^* \ldots \\ &= \sum_m {}^+P_m \, Q_m \\ &= \sum_m P_m \, Q_{\varphi^{-1}(m)} \\ &= \Sigma \, a_r \, b_s \, Q_{\varphi^{-1}(-m)} \\ &\le \Sigma \, a_r^+ \, {}^+b_s \, Q_m^* \\ &= \Sigma \, a_r^+ \, {}^+b_s \, c_t^* \, d_\ell^* \ldots \end{aligned}$$

which is the desired inequality.

The first inequality above follows from our induction hypothesis, the second inequality from the symmetry of $\{c_t\}$, the second inequality again from our induction hypothesis, and the last but one equality from the fact that $Q_m = Q_m^*$, which is a consequence of lemma 2.1.

<u>Step</u> 3. A simple continuity argument shows that the a_r , b_s , c_t in (2.11) can be taken non-negative rationals. Multiplying now by a suitable integer we may further assume that they are non-negative integers.

We write now the sequence $\{a_r\}$ as a sum of sequences $\{a'_r\}$, $\{a''_r\}$, ...
consisting of zeros and ones only

$$a_r = a'_r + a''_r + \dots$$

by setting

$$a'_r = \begin{cases} 0 & \text{if } a_r = 0 \\ 1 & \text{if } a_r > 0 \end{cases} , \quad a''_r = (a_r - a'_r)' \dots$$

We define similarly the sequences : b'_s , b''_s , ... ; c'_t , c''_t , ... ; ...

It is easy to show that

$$a^+_r = (a'_r)^+ + (a''_r)^+ + \dots, \quad {}^+b_s = {}^+(b'_s) + {}^+(b''_s) + \dots, \quad c^*_t = (c'_t)^* + (c''_t)^* + \dots$$

Assuming that (2.11) is true when the sequences $\{a_r\}$, $\{b_s\}$, $\{c_t\}$ consist of zeros and ones only and adding the inequalities which result from it when we replace $\{a_r\}$, $\{b_s\}$, $\{c_t\}$ by $\{a^{(k)}_r\}$, $\{b^{(\ell)}_s\}$, $\{c^{(n)}_t\}$ for all possible values of k , ℓ , n , we obtain the general case of (2.11).

<u>Step</u> 4. It remains to prove (2.11) when the $\{a_r\}$, $\{b_s\}$, $\{c_t\}$ consist of zeros and ones only.

We write

$$f(x) = \sum_r a_r \exp(irx) \quad , \quad g(x) = \sum_s b_s \exp(isx) \quad , \quad h(x) = \sum_t \exp(itx)$$

$$f^+(x) = \sum_r a^+_r \exp(irx) = \sum_{r=-R}^{R'} \exp(irx) \quad , \quad R \le R' \le R+1$$

$$^+g(x) = \sum_s {}^+b_s \exp(isx) = \sum_{s=-S'}^{S} \exp(isx) \quad , \quad S \le S' \le S+1$$

$$h^*(x) = \sum_t c^*_t \exp(itx) = \sum_{t=-T}^{T} \exp(itx)$$

and we have to prove

$$\int f\,g\,h \le \int f^{++}g\,h^{*}. \tag{2.12}$$

Since h is an exponential sum the left-hand side of (2.12) is not greater than the sum of all the Fourier coefficients of $f(x)\,g(x)$, i.e.

$$\int f\,g\,h \le f(0)\,g(0) = (R+1+R')(S+1+S').$$

If $T \ge \max\{R+S', R'+S\}$ then $f^{++}g$ is a trigonometric polynomial of degree $\le T$, which implies that the right-hand member of (2.12) is equal to $(R+1+R')(S+1+S')$ and hence (2.12) holds.

If $R' = 0$ or $S' = 0$ then (2.12) reduces to a special case of the inequality

$$\int fg \le \int f^{++}g, \quad f, g \text{ exponential sums}. \tag{2.13}$$

(2.13) is an immediate consequence of the obvious fact that $\int f^{++}g$ equals the minimum of the number of non-zero coefficients of f and g respectively.

It follows that we may assume

$$R' > 0, \quad S' > 0, \quad \max(R+S', R'+S) = M > T. \tag{2.14}$$

We shall use induction on M. We assume that (2.12) is valid when $\max\{R'+S, R+S'\} < M$.

Let u be the greatest index such that $a_u \ne 0$ and v the smallest such that $b_v \ne 0$. We set

$$F(x) = f(x) - \exp(iux) \quad , \quad G(x) = g(x) - \exp(ivx)$$

and have

$$F^{+}(x) = \sum_{r=-(R'-1)}^{R} \exp(irx) = f^{+}(-x) - \exp(-iR'x)$$

$$^{+}G(x) = \sum_{s=-S}^{S'-1} \exp(isx) = {}^{+}g(-x) - \exp(iS'x).$$

Since $\max\{R+(S'-1), (R'-1)+S\} = M-1$, our induction hypothesis implies

(2.15)
$$\int F\,G\,h \le \int F^{+}(x)\,{}^{+}G(x)\,h^{*}(x)\,dx$$
$$= \int F^{+}(-x)\,{}^{+}G(-x)\,h^{*}(-x)\,dx$$
$$= \int f^{+}\,{}^{+}g\,h^{*} - \int h^{*}(x)\Big\{f^{+}(x)\exp(-iS'x) + {}^{+}g(x)\exp(iR'x) + \exp\big[i(R'-S')x\big]\Big\}dx$$
$$= \int f^{+}\,{}^{+}g\,h^{*} - (2T+1).$$

The second equality follows from the symmetry of h and the third from the fact that the expression in curly brackets is a connected exponential sum containing h^{*} (this is an immediate consequence of (2.14)).

We observe now that

$$f\,g\,h = F\,G\,h + h\Big\{\exp(ivx)F(x) + \exp(iux)\,G(x) + \exp\big[i(u+v)\,x\big]\Big\}$$

and that the expression in curly brackets is an exponential sum. We conclude that the contribution of $h\{\dots\}$ in the integral $\int f\,g\,h$ is at most $2T+1$ (= the number of non zero coefficients of h). Hence

(2.16)
$$\int f\,g\,h \le \int F\,G\,h + (2T+1).$$

Combining (2.15) and (2.16) we obtain (2.12).

3. COMMENTS ON THEOREM 2.1. (i) The remarkable theorem proved in 2 is due to Gabriel ([12]). Here we have reproduced his proof with some modifications taken from ([18]).

To a large extent the structure of Gabriel's proof is modeled on that of Hardy and Littlewood in ([15]), where they give the important special case of (2.7) corresponding to symmetrical sequences $\{a_r\}, \{b_s\}, \{c_t\}, \dots$

It appears however that the argument used in ([15]) was not conclusive. The passage from the case of 3 sequences to the general case is based on the assertion : "$P_m = \sum_{r+s=m} a_r b_s$ is symmetrical when the sequences $\{a_r\}$ and $\{b_s\}$ are symmetrical". The example : $a_o = a_1 = a_{-1} = 1$, $a_r = 0$ if $r \neq 0, \pm 1$; $b_o = b_2 = b_{-2} = b_3 = b_{-3} = 1$, $b_s = 0$ if $s \neq 0, \pm 2, \pm 3$; gives $P_o = P_4 = P_{-4} = 1$, $P_1 = P_{-1} = P_2 = P_{-2} = P_3 = P_{-3} = 2$, $P_m = 0$ if $m \neq 0, \pm 1, \pm 2, \pm 3, \pm 4$. Since the largest value 2 in $\{P_m\}$ appears an even number of times $\{P_m\}$ is not symmetrical. This problem does not arise in Gabriel's version of the proof (see also [18], p. 274 footnote a).

(ii) For the case of two equal sequences in (2.6) see also ([3]) and ([6]), where this special case is attributed to Pisot and Schoenberg.

(iii) The above mentioned paper of Gabriel also contains a partial converse of theorem (2.1) which we state without proof.

THEOREM 2.2. <u>If</u> $\{a_r\}$ <u>is a sequence of non-negative numbers with finitely many non-zero terms and</u>

$$a_r = a_{-r} \quad , \quad a_o \geq a_r \quad \underline{\text{for all}} \quad r,$$

$$\Sigma\, a_{n_1} \ldots a_{n_{2k}} = \Sigma\, a^*_{n_1} \ldots a^*_{n_{2k}}$$

<u>then</u> $a_r = \lambda a^*_r$ <u>for some integer</u> λ, <u>i. e. the</u> r <u>for which</u> $a_r \neq 0$ <u>form an arithmetic progression containing</u> 0.

(iv) Easier proofs of (2.4) can be obtained if we allow a constant factor in the second member ([17], see also II 4). Part of the difficulty of the present proof is to show that this constant can be taken equal to 1.

(v) An estimate for $\|D_N\|_{2k}$, $k = 1, 2, \ldots$ would be very useful in connection with formula (2.4). In the next section we shall obtain such estimates for more general exponents and in the next chapter we shall prove an exact formula for $\|D_N\|_1$.

An exact, but rather complicated, formula for $\|D_N\|_{2k}$ can also be given. Indeed $\|D_N\|_{2k}^{2k}$ is obtained from (2.17) if we put $n = 2k$ and $m = 0$. Denoting, as usual, by $\hat{f}(m)$ the Fourier coefficients of f we have :

$$\widehat{D_N^n}(m) = \binom{A_m}{n-1} - \binom{n}{1}\binom{A_m-(2N+1)}{n-1} + \dots + (-1)^k \binom{n}{k}\binom{A_m-k(2N+1)}{n-1} + \dots, \quad m \geq 0 \tag{2.17}$$

where : $A_m = nN + n - 1 - m$ and the summation stops at the first k such that : $A_m - (k+1)(2N+1) < n - 1$, i. e. $nN - m < (k+1)(2N+1)$.

Proof. We shall use induction on n. Let first $n = 1$. If $m \geqslant N$ then $nN - m = N - m < 0$ and hence (2.17) gives $\widehat{D_N}(m) = 0$ as should be expected. If $m \leq n$ then the summation stops at the first term (since $2N+1 < N-m$) and hence (2.17) gives $D_N(m) = \binom{A_m}{0} = 1$, again as should be expected.

Assume now that (2.17) has been proved for n. Using the well known formula

$$\binom{k}{\ell-1} + \binom{k}{\ell} = \binom{k+1}{\ell}$$

we obtain :

$$\widehat{D_N^{n+1}}(m) = \sum_{k=-N}^{N} \widehat{D_N^n}(m+k)$$

$$= \binom{A_m+N}{n-1} + \binom{A_m+N-1}{n-1} + \dots + \binom{A_m-N}{n-1} -$$

$$- \binom{n}{1}\left\{\binom{A_m-N-1}{n-1} + \dots + \binom{A_m-3N-1}{n-1}\right\} + \binom{n}{2}\{\dots\} - \dots$$

$$= \left\{\binom{A_m+N+1}{n} - \binom{A_m-N}{n}\right\} - \binom{n}{1}\left\{\binom{A_m-N}{n} - \binom{A_m-3N-1}{n}\right\} + \dots$$

$$= \binom{A_m+N+1}{n} - \left\{\binom{n}{1}+1\right\}\binom{A_m-N}{n} + \left\{\binom{n}{1}+\binom{n}{2}\right\}\binom{A_m-3N-1}{n} - \dots$$

$$= \binom{A_m+N+1}{n} - \binom{n+1}{1}\binom{A_m+N+1-(2N+1)}{n} + \binom{n+1}{2}\binom{A_m+N+1-2(2N+1)}{n} - \dots$$

Since $A_m + N + 1 = nN + n - 1 - m + N + 1 = (n+1)N + (n+1) - 1 - m$ the proof of (2.17) is complete.

4. ARBITRARY EXPONENTS.

We examine now L^p norms of exponential sums when the exponent p is not necessarily an even integer.

We assume first that $p > 2$. It is very easy to extend (2.4) in this case if we allow a constant factor in the second member.

Let f be an exponential sum with N terms and let $f(x) = \sum_{m=1}^{N} \exp(imx)$. We have

$$\|f\|_p^p = \int |f|^p$$
$$= \int |f|^{p-2} |f|^2 \leq N^{p-2} \int |f|^2 = N^{p-1}.$$

If $|x| < \frac{\pi}{3N}$ then $|F(x)| \geq \left| \sum_{m=1}^{N} \cos mx \right| \geq \frac{N}{2}$ and hence :

$$\|F\|_p^p = \int |F|^p \geq \frac{1}{2\pi} \int_{|x| \leq \frac{\pi}{3N}} |F|^p \geq \frac{1}{3N} \left(\frac{N}{2}\right)^p = \frac{N^{p-1}}{3.2^p} \geq \frac{1}{3.2^p} \|f\|_p^p$$

or

(2.18) $$\|f\|_p \leq C \|F\|_p, \qquad p > 2.$$

Here and in the sequel C will denote an absolute positive constant (not always the same). In (2.18) C can be taken, say, less than 4 by the argument we used in order to obtain (2.18). We already know that if $p = 2q$, $q = 1, 2, \ldots$, then C can be taken equal to 1. When p tends to infinity then both $\|f\|_p$ and $\|F\|_p$ tend to N $(= \|f\|_\infty = \|F\|_\infty)$ and hence the least value of C in (2.18) tends to 1 as $p \to \infty$.

The problem of finding the best value for the constant C in (2.18) was cited in ([17]) as an open one. It appears to be still open.

We observe that in the other direction there are exponential sums f such that $\|f\|_p$ is considerably smaller than $\|F\|_p$. If for instance

$$f(x) = 1 + 2\cos 10x + 2\cos 10^2 x + \ldots + 2\cos 10^N x$$

then $\|f\|_4 \le C N^{1/2}$ while $\|D_N\|_4 \gtrsim N^{3/4}$. This property of f is shared by the so-called lacunary exponential sums which we shall examine in III.3.

We pass now to the case $p < 2$. Let again f be an exponential sum with N terms and $F(x) = \sum_{m=1}^{N} \exp(imx)$. We have

$$N = \int |f|^2 = \int |f|^{2-p} |f|^p \le N^{2-p} \int |f|^p$$

and hence

(2.19) $$\|f\|_p \ge N^{(p-1)/p}, \qquad p < 2.$$

We observe now that

$$|F(x)| = \left| \frac{\exp(iNx)-1}{e^{ix}-1} \right| \le \min\left\{N, \frac{2}{x}\right\}, \qquad |x| \le \pi .$$

Using this inequality we have

(2.20) $$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} |F|^p &= \frac{1}{2\pi}\int_{|x|<\frac{1}{N}} |F|^p + \frac{1}{2\pi}\int_{|x|\geq\frac{1}{N}} |F|^p \\ &\le \frac{1}{2\pi} N^p \cdot \frac{1}{N} + 2 . 2^p \int_{1/N}^{\pi} \frac{dx}{x^p} \\ &\le \frac{C}{p-1} N^{p-1} \end{aligned}$$

(the use of the absolute constant C is justified since $p < 2$).

Combining (2.19) and (2.20) we obtain

(2.21) $$\|f\|_p \ge C(p-1)\|F\|_p, \qquad 1 < p < 2.$$

As in the case of (2.18) the best constant in (2.21) is not known. Certainly the constant $C(p-1)$ we obtained is very far from the best (if nothing else because it gives a bound less than 1 when $p < 1 + \frac{1}{CN^{1/2}}$, while, as we shall see in the next chapter, even the L^1 norm of an exponential sum with N terms tends to infinity with N).

The same example as before, $f(x) = 1+2\cos 10x + .. + 2\cos 10^N x$, shows that $\|f\|_p$ (even $\|f\|_1$) can be of order of $\|f\|_2$ i. e. $N^{1/2}$ (see III.3 for the proof).

The very crude arguments we used in order to obtain (2.18) and (2.21) cannot give best constants (compare for instance with the fine Gabriel-Hardy-Littlewood argument used in II.2). However the inequalities we reached raise a number of interesting questions (for a rather long list of such questions see [17]). We have already mentioned two of them (the best constants in (2.18) and (2.21)). A rather strong conjecture (suggested by the case of an even exponent) is that in both cases the best constant is 1.

We mention here that in (2.18) and (2.21) if we replace f by a symmetrical trigonometric polynomial and F by the polynomial f* corresponding to the non-increasing rearrangement of the absolute values of the coefficients of f then the constants cannot be taken equal to 1. The function $f(x) = 1 - \frac{3}{4}\cos x + \frac{3}{4}\cos 2x - \frac{1}{2}\cos 3x + \frac{3}{4}\cos 4x$, for instance, shows that (i) $f \geq 0$ (hence $\|f\|_1 = 1$) and that (ii) $\frac{\pi}{3}$ is a simple root of f^* (hence $\|f^*\| > \int f^* = 1$). This example is due to Lehmer ([25]). For these problems see ([23]) and ([26]).

Returning to exponential sums we mention finally that, despite a considerable effort made by several mathematicians during the last 30 years in order to settle at least the limiting case $p = 1$, it appears that much remains to be done. This limiting case will be examined in detail in the next chapter.

III. L^1 NORM.

In this chapter we examine the problem of finding a lower bound for the L^1 norm of exponential sums which depends on the number of terms N only. The main result is theorem 3.5 which gives such a lower bound of the order of $(\log N/\log\log N)^{1/2}$.

1. INTRODUCTION.

Let $n_1, \ldots, n_N$ be N (≥ 3) distinct positive integers and write :

$$(3.1) \quad \begin{aligned} f(x) &= \sum_{k=1}^{N} \exp(in_k x) \quad , \quad & F_N(x) &= \sum_{k=1}^{N} \exp(ikx) \\ g(x) &= 1 + 2\sum_{k=1}^{N} \cos n_k x \quad , \quad & D_N(x) &= 1 + 2\sum_{k=1}^{N} \cos kx . \end{aligned}$$

The restrictions that $N \geq 3$ and that $n_1, \ldots, n_N$ are positive are made for technical purposes only. (After some trivial modifications almost all the results we shall present here remain valid for general exponential sums). In most of the cases (but not always) it will be of no importance whether we deal with f or with the "Cosine" sums g. The latter are real even functions and it is sometimes more convenient to work with them.

From what was said in II 4 one is led to the question of the validity of the inequalities :

$$(3.2) \qquad \|f\|_1 \geq C\|F_N\|_1 \quad , \quad \|g\|_1 \geq C\|D_N\|_1 .$$

It is obvious that the two inequalities in (3.2) are equivalent. However we cannot exclude a priori the possibility that the constant C can be taken 1 in the second inequality but not in the first.

(3.2) appeared for the first time in ([17]) and later in a collection of research problems by J. Littlewood. Usually it is referred to as "Littlewood's conjecture".

If in (3.2) we replace f by $a_o + 2 \sum_{n=1}^{N} \alpha_n \cos n_x$ and F_N by $a_o + 2 \sum_{n=1}^{N} a_n^* \cos nx$ ($a_o \geq a_1^* \geq a_2^* \geq \ldots$ is the non-increasing rearrangement of a_o, $|a_1|, \ldots |a_N|$) then (3.2) may be false ([23]).

2. THE DIRICHLET KERNEL D_N.

An equivalent form of (3.2) is

(3.3) $$\|g\|_1 \geq C \log N.$$

To see this we examine in detail the so called Lebesgue constant $\|D_N\|_1$. We have

$$\begin{aligned}
\|D_N\|_1 &= \int |1 + 2\cos x + \ldots + 2\cos Nx|\, dx \\
&\geq \frac{1}{\pi} \int_o^{\pi} (1 + 2\cos x + \ldots + 2\cos Nx) \sin (N + \tfrac{1}{2})\, x\, dx \\
&= \frac{1}{\pi} \left\{ \frac{2}{2N+1} + \sum_{k=1}^{N} \frac{2}{2N+1+2k} + \sum_{k=1}^{N} \frac{2}{2N+1-2k} \right\} \\
&\geq \frac{1}{\pi} \left(1 + \frac{1}{3} + \ldots + \frac{1}{2N-1}\right) \\
&\geq C \log N
\end{aligned}$$

from which the equivalence of (3.2) and (3.3) follows.

Much more accurate estimates of $\|D_N\|_1$ are known. D_N, being a sum of consecutive terms of a geometric progression, can be given by the explicit formula

(3.4) $$D_N(x) = \frac{\sin(N + \frac{1}{2})\, x}{\sin \frac{x}{2}}.$$

Using (3.4) we can prove easily ([40], II, 12.1)

(3.5) $$\|D_N\|_1 = \frac{4}{\pi^2} \log N + 0(1) \quad \text{as} \quad N \to \infty .$$

An exact formula for $\|D_N\|_1$ has been obtained by L. Féjer ([11]) :

(3.6) $$\|D_N\|_1 = \frac{1}{2N+1} + \frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k} \tan \frac{\pi k}{2N+1} .$$

Another exact formula, due to Szegö ([36]), is

(3.7) $$\|D_N\|_1 = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \left\{ 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k(2N+1)-1} \right\}$$

Here is Szegö's proof of (3.7) :

Expanding $|\sin x|$ in a series of cosines we have :

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2-1}$$

$$= |\sin x| - |\sin 0|$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1 - \cos 2kx}{4k^2 - 1} .$$

Hence :

$$|\sin(2N+1)x| = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1 - \cos 2k(2N+1)x}{4k^2-1}$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(1-\cos 2x)+(\cos 2x - \cos 4x)+\ldots+(\cos [2k(2N+1)-2]x - \cos 2k(2N+1)x)}{4k^2-1}$$

$$= \left(\frac{4}{\pi} \sin x\right) \sum_{k=1}^{\infty} \frac{\sin x + \sin 3x + \ldots + \sin [2k(2N+1)-1]x}{4k^2-1} .$$

It follows :

$$|D_N(2x)| = (\operatorname{sgn} \sin x) \left\{ \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin x + \sin 3x + \ldots + \sin [2k(2N+1)-1]x}{4k^2-1} \right.$$

from which we obtain (3.7) by termwise integration.

An immediate corollary of (3.7) is

(3.8) $$\|D_{N+1}\|_1 > \|D_N\|_1 ,$$

that is "the Lebesgue constants form an increasing sequence". This property of the $\|D_N\|_1$'s is due to T. H. Gronwall ([13]), but his proof was more complicated than that of Szegö.

A simple change of variables shows that the estimate (3.3) remains valid for any exponential sum whose frequencies form an arithmetic progression (here and in the sequel "arithmetic progression" means "unbroken part of an infinite arithmetic progression"). The L^1 Norm of exponential sums corresponding to arithmetic progressions shows some sort of "stability", which sometimes yields a satisfactory estimate for the L^1 Norm of other exponential sums. This is a consequence of the fact that such exponential sums are Fourier-Stieltjes series of measures and hence conversion factors (multipliers) for L^1 ([40], IV, 11). More precisely we have :

THEOREM 3.1. <u>Let</u> $f(x) = \sum_{k=1}^{N} \exp(in_k x)$, <u>where</u> $\{n_k\}$ <u>is an arithmetic progression, and let</u> $h(x)$ <u>be another exponential sum with no frequencies in the infinite arithmetic progression containing</u> $\{n_k\}$. <u>Then</u>

(3.9) $$\|f + h\|_1 \geq \|f\|_1 .$$

Proof. Without loss of generality we may assume that $n_1 = a$, $n_2 = 2a$, ..., $n_N = Na$, for some integer a. We have :

$$\sum_{b=1}^{a} f(x + b\frac{2\pi}{a}) = a\, f(x) \qquad , \qquad \sum_{b=1}^{m} h(x + b\frac{2\pi}{a}) = 0 .$$

These equalities follow, say, from the fact that for each frequency m of f or h, $\exp im(x + b\frac{2\pi}{a})$, $b = 1, \ldots a$, correspond to the vertices of a regular polygon. Using the above relation and the periodicity of $f + h$ we obtain

$$a\|f\|_1 = \|\Sigma(f+h)(. + b\tfrac{2\pi}{a})\|_1$$
$$\leq \sum_{b=1}^{a} \|(f+h)(. + b\tfrac{2\pi}{a})\|_1$$
$$= a\|f + h\|_1$$

which proves (3.9).

3. LACUNARY SEQUENCES.

The ease with which we were able to obtain such exact estimates for $\|D_N\|_1$ is due to the special form of the Dirichlet kernel. If we try for instance to repeat the argument which gave us $\|D_N\|_1 \geq C \log N$ for the exponential sum f in (3.1), we obtain

(3.10) $$\|f\|_1 \geq C \left\{\frac{1}{n_2-n_1} + \ldots + \frac{1}{n_N-n_1}\right\}.$$

(3.10), which is a special case of a more general inequality of Hardy and Littlewood ([40], VII, 8.7), is in many cases not satisfactory. Assume for example that $f(x) = 1 + 2\left\{\cos 3x + \ldots + \cos 3^N x\right\}$. We shall see later that $\|f\|_1 \geq C\, N^{1/2}$ while the second member of (3.9), as a function of N, remains bounded. The same is true for a large class of exponential sums. We examine them in some detail.

$\|f\|_4^4$ is the constant term of $f^2(x)\,\overline{f^2(x)}$ and hence it is equal to the number of distinct quadruples (n_k, n_ℓ, n_m, n_t) which satisfy the equation

(3.11) $$n_k + n_\ell = n_m + n_t \qquad , \qquad k, \ell, m, n = 1, 2, \ldots, N.$$

A glance at (3.11) shows that this number is less than N^3 (in fact it is less than $\frac{2}{3}(N^3 + 1)$, as an application of II(2.4) shows). Using now Hölder's inequality we obtain :

$$N = \int |f|^2 = \int |f|^{2/3} |f|^{4\frac{1}{3}} \le \|f\|_1^{\frac{2}{3}} (\|f\|_4^4)^{1/3}$$

and hence

(3.12) $$\|f\|_1 \ge \left\{N^3/\|f\|_4^4\right\}^{1/2}$$

Of course there is nothing special about the exponent 4. Any even exponent can serve the same purpose as well. We insist however on even exponents since otherwise we do not know of any simple arithmetic characterization of the Norms.

It follows from (3.12) that the smaller the number of relations of the form (3.11) the better estimate for $\|f\|_1$ we get. If for instance $\{n_k\}$ is lacunary, i. e. $\frac{n_{k+1}}{n_k} \ge q > 1$, $q = 1, 2, \dots N-1$, and if b is the smaller integer such that $q^b > 3$ then

(3.13) $$\|f\|_1 \ge C b^{-1} N^{1/2} \quad (\sim C(q-1) N^{1/2}, \quad \text{as} \quad q \to 1^+).$$

To see this we observe that $\{n_k\}$ is the union of the b disjoint sequences $\{n_m, n_{m+b}, \dots\}$, $m = 1, \dots, b$, each of which has no more than $1 + \frac{N}{b}$ elements which satisfy the relations

$$\frac{n_{m+(k+1)b}}{n_{m+kb}} > 3, \quad k = 0, 1, \dots$$

It follows that for each of these sequences (3.11) has only trivial solutions $(k = m, \ell = t$ or $k = t, \ \ell = m)$ and hence the corresponding exponential sums f_m satisfy the inequality

$$\|f_m\|_4^4 \le C(\tfrac{N}{b})^2.$$

Observing now that $|f|^4 \le b^3 \sum_{m=1}^{b} |f_m|^4$ and using (3.12) we obtain (3.13).

We note that there are sequences other than lacunary for which the number of solutions of (3.11) is as small as $C N^2$. If $\{n_k\}$ is lacunary and $\{m_k\}$ is such that $m_k \equiv n_k \mod A$, for some integer $A > n_N$, then $\{m_k\}$ is an example of such a sequence.

The above technique of dealing with lacunary sequences is very common not only in harmonic analysis ([40], V, 6) but also in other branches of mathematics, e. g. in probability ([22]). Unfortunately it leaves much to be desired in some important cases. Thus, for example, in the case of the Dirichlet kernel it gives the poor estimate $\|D_N\|_1 \geq C$. However, if a sequence contains a long enough lacunary subsequence (although it need not be lacunary itself) then a relatively satisfactory estimate for the L^1 Norm can be derived (e. g. $(\log N)^{1/2}$ for the Dirichlet kernel). This is a corollary of an elegant theorem due to Paley (40, XII, 7.8).

THEOREM 3.2. If $\frac{m_{j+1}}{m_j} \geq q > 1$ then

$$\left\{\sum_j |a_{m_j}|^2\right\}^{1/2} \leq C\|a_o + a_1 e^{ix} + a_2 e^{i2x} + \ldots + a_n e^{inx}\|_1. \tag{3.14}$$

In particular if the sequence $\{n_k\}$ contains a lacunary subsequence of length M, then the L^1 Norm of f (defined in (3.1)) exceeds a positive constant multiple of $M^{1/2}$. (We remark that here it is very important that the frequencies appearing in f be non-negative).

Proof. The proof is based on the fact that $F(z) = a_o + a_1 z + \ldots + a_n z^n$, $|z| < 1$, can be written as a product $F_1(z) F_2(z)$ where $F_1(z) = \sum_{n=0}^{\infty} b_n z^n$, $F_2(z) = \sum_{n=0}^{\infty} c_n z^n$, $|z| < 1$, are analytic and $\|F_1\|_2^2 = \|F_2\|_2^2 = \|F\|_1$ (for a function $G(z)$, analytic in $|z| < 1$, $\|G\|_p$ is defined as the p th Norm of its boundary limit $G(e^{ix})$). In all cases we shall consider here, e. g. when G is a polynomial, the existence of boundary limits will offer no difficulty and so it will be taken for granted).

This well-known factorization ([40], VII, 7.22) is due to F. Riesz and can be proved as follows : If $F(z)$ has roots on $|z| = 1$, we consider the function $F(rz)$, $r < 1$, which certainly has no roots on $|z| = 1$ except for a finite number of r's. Thus, using a simple limiting argument, we may assume that $F(z) \neq 0$, $|z| = 1$. If $F(z) \neq 0$ on $|z| < 1$ we very simply take $F_2 = F_1 = F^{1/2}$ (any analytic branch of $F^{1/2}$). Otherwise let $z_1, z_2, \ldots, z_m$ $(m \leq n)$ be the roots of F in $|z| < 1$ (multiple roots will be repeated according to their multiplicity). For each z_k we consider the Möbius transformation which sends z_k to 0 and maps the unit circle onto itself. We call $B(z)$ the product of these transformations. Obviously $|B(z)| = 1$ on $|z| = 1$ and the function $F(z)\{B(z)\}^{-1} = G(z)$ is analytic and zero-free on $|z| \leq 1$. It is clear now that the functions $F_1 = G^{1/2}$, $F_2 = B\,G^{1/2}$ (any analytic branch of the square root) have the desired properties.

From the representation $F = F_1 F_2$ we get

$$|a_{n_j}| = \Big|\sum_{k=0}^{n_{j-1}} b_k c_{n_j-k} + \sum_{k=n_{j-1}+1}^{n_j} b_k c_{n_j-k}\Big|$$

$$\leq \Big\{\sum_{0}^{\infty} |b_k|^2\Big\}^{1/2} \Big\{\sum_{n_j-n_{j-1}}^{n_j} |c_k|^2\Big\}^{1/2} + \Big\{\sum_{0}^{\infty} |c_k|^2\Big\}^{1/2} \Big\{\sum_{n_{j-1}+1}^{n_j} |b_k|^2\Big\}^{1/2}$$

and hence

$$(3.15) \qquad |a_{n_j}|^2 \leq 2\Big(\sum_{0}^{\infty} |b_k|^2\Big)\Big(\sum_{n_j-n_{j-1}}^{n_j} |c_k|^2\Big) + 2\Big(\sum_{0}^{\infty} |c_k|^2\Big)\Big(\sum_{n_{j-1}+1}^{n_j} |b_k|^2\Big)$$

$$= 2\|F\|_1 \Big\{\sum_{n_j-n_{j-1}}^{n_j} |c_k|^2 + \sum_{n_{j-1}+1}^{n_j} |b_k|^2\Big\}.$$

We observe that the second terms inside the curly brackets have sum (over j) less than $\sum_{0}^{\infty} |b_k|^2 = ||F||_1$. The first terms also have sum less than $C \sum_{0}^{\infty} |c_k|^2 = C||F||_1$. This follows from the fact that the union of the intervals $[n_j - n_{j-1}, n_j]$ covers any integer at most K times, where K depends only on q. To see this it suffices to observe that

$$n_j < n_k - n_{k-1} \quad \text{if} \quad k-j > -\frac{\log(1-\frac{1}{2})}{\log q} = K$$

and hence the sequence $[n_j - n_{j-1}, n_j]$ is a union of no more than K disjoint subsequences of intervals. Adding now the inequalities (3.15) we obtain

$$\sum_j |\alpha_{n_j}|^2 \leq 2||F||_1 (K+1)\, ||F||_1$$

which implies (3.14) with $C = O(\frac{1}{q-1} \log \frac{1}{q-1})$, as $q \to 1^+$.

REMARKS. (i) The analyticity of F was crucial in the above proof. This is the reason why the hypothesis $n_j \geq 0$ was needed.

(ii) It is to be noted also that the lacunarity of $\{n_j\}$ was used in a way different from the usual one (see e. g. the proof of 3.13).

In the case of real trigonometric polynomials there is a substitute for theorem (3.2) which neither implies nor is implied by it.

THEOREM 3.3. <u>If</u> $F(x) = \sum_{k=1}^{N} \alpha_k \cos kx + b_k \sin kx$ <u>and</u> $\frac{n_{j+1}}{n_j} \geq q > 1$, $j = 1, \ldots, N-1$, <u>then</u>

$$\text{(3.16)} \qquad \left\{ \sum_k |a_{n_k}|^2 + |b_{n_k}|^2 \right\}^{1/2} \leq C \left\| \int |F| (\log |F|)^{1/2} \right\|_1 + C.$$

Theorem (3.3) is due to Zygmund and its proof can be found in ([40], XII, 7.6). The assumption of lacunarity can be relaxed a little (it suffices to assume the usual condition on the number of solutions of (3.11).

The results we have proved up to now, although interesting in themselves, leave essentially intact the general case of the problem raised in III.1. For instance, if a sequence of non negative n_k's is such that most of its terms lie in an interval $(A,(1+\varepsilon)A)$, where ε is a small positive number, we cannot expect very much from theorems (3.2) and (3.3).

However, borrowing again an argument from the theory of lacunary series, we can show that it is only a small proportion of terms in $\{n_k\}$ which prevents us from obtaining easily a lower bound of the order of $(\log N)^{1/2}$ for $\|\sum_{k=1}^{N} \exp(in_k x)\|_1$. The argument we have in mind is the use of the so-called "Riesz products" and the precise formulation of this result is the following :

THEOREM 3.4. Given f as in (3.1) and $\varepsilon > 0$ we can subtract less than $0(N^{\varepsilon})$ terms from $\{n_k\}$ so that the L^1 Norm of the exponential sum corresponding to the remaining sequence exceeds $C(\varepsilon)(\log N)^{1/2}$.

Here $C(\varepsilon)$ means a positive constant depending only on ε.

Proof. We shall construct a subsequence $\{m_1, \ldots, m_r\}$ of $\{n_1, \ldots n_N\}$ such that :

(i) $\frac{\varepsilon}{\log 3} \log N - 1 < r \le \frac{\varepsilon}{\log 3} \log N$

(ii) For all k, $1 \le k \le N$, all sequences of distinct indices $j_1, \ldots, j_t$, $1 < t \le r$ and all choices of signs $\pm$ we have $m_k \ne \pm m_{j_1} \pm m_{j_2} \pm \ldots \pm m_{j_t}$.

To avoid trivialities we assume, as we may, that N is large enough to have $(\varepsilon/\log 3) \log N > 1$.

We take for the moment the existence of such a sequence for granted and argue as follows :

The number of integers of the form $\pm m_{j_1} \pm m_{j_2} \pm \dots \pm m_{j_t}$ is at most

$$2^r + \binom{r}{1} 2^{r-1} + \dots + \binom{r}{r-2} 2^2 < 3^r < 3^{(\varepsilon/\log 3)\log N} = N^{\varepsilon}.$$

We delete the integers of this form from the sequence $\{n_k\}$ and call h the exponential sum corresponding to the remaining sequence. It suffices to prove that $\|h\|_1 \geq C\, r^{1/2}$.

To this end we use the "Riesz product"

$$P(x) = \prod_{t=1}^{r} \left(1 + \frac{i}{r^{1/2}} \cos m_t x\right)$$

and observe that $P(x)$, written in full as a trigonometric polynomial, will have in common with h only the r frequencies $m_1, m_2, \dots, m_r$ and the corresponding coefficients will be equal to $\frac{i}{r^{1/2}}$.

It follows that

$$\int h(-x)\, P(x)\, dx = \frac{i}{\sqrt{r}} \cdot r\, \frac{1}{2} = C r^{1/2}. \tag{3.17}$$

Since $|P(x)| \leq \prod_{t=1}^{r} (1 + \frac{1}{r})^{1/2} = (1 + \frac{1}{r})^{r/2} \leq C$, (3.17) implies the desired inequality $\|h\|_1 \geq C r^{1/2}$.

It remains to prove the existence of the subsequence $\{m_1, \dots, m_r\}$.

We take $m_1 = n_1$ and suppose that $m_1, \dots, m_s$ have been chosen so that (ii) holds with $r = s$. Let S be the set of integers of the form $\pm m_{j_1} \pm m_{j_2} \pm \dots \pm m_{j_t}$, $t \leq s$ and $\bar{S}$ the union of the sets : $\{m_1, \dots, m_s\}$, $\{x : 2x \in S\}$, S, $S+m_1$, $S-m_1$, $S+m_2$, $S-m_2$, $\dots$, $S+m_s$, $S-m_s$. As usual $A+x$ means the set $\{a+x : a \in A\}$.

The number of elements in S is at most

$$2^s + \binom{s}{1} 2^{s-1} + \dots + \binom{s}{s-1} 2 < 3^s$$

and hence $\bar{S}$ contains less than $s + 2.3^s + 2s.3^s \leq 10^{s+1}$ elements.

If $10^{s+1} < N$ then we can choose $m_{s+1} \in \{n_1, \ldots, n_N\} - \bar{S}$ and it is trivial to verify that $\{m_1, \ldots, m_{s+1}\}$ satisfies (ii). So the construction can be continued as long as the condition $10^{s+1} < N$ is satisfied. It follows that condition (i) can certainly be satisfied if ε is small enough. Since such an assumption on ε is obviously harmless the proof is complete.

REMARK. The construction used in the above proof is rather crude. As we shall see in the next section a more careful choice of the subsequence $\{m_k\}$ and a much more careful choice of the factor $P(x)$ in (3.17) will allow us to obtain a satisfactory estimate for $\|f\|_1$.

4. THE GENERAL CASE.

IN 1960 P. Cohen introduced a method of estimating the L^1 Norm of exponential sums which gives positive results without any condition whatsoever on the sequence of frequencies in them.

Using a refinement of this method we shall prove in this section the following

THEOREM 3.5. <u>If</u> $n_1, \ldots, n_N$ <u>are</u> N (≥ 2) <u>distinct integers then</u>

(3.18) $$\|\exp(in_1x) + \ldots + \exp(in_Xx)\|_1 \geq C(\log N/\log \log N)^{1/2}.$$

<u>Proof</u>. To avoid some trivial verifications we shall assume that N is large ($> e^{10^5}$, say, will suffice). This assumption obviously does not affect the validity of (3.18). One characteristic point of the proof is the selection of a suitable subsequence $\{m_k\}$ of $\{n_k\}$. The selection begins with the largest frequency in $\{n_k\}$ and continues downwards.

Thus it seems appropriate (again without loss of generality) to assume, and we shall do so, that

$$n_1 > n_2 > \ldots > n_N > 0.$$

We write : $f(x) = \exp(in_1x) + \ldots + \exp(in_Nx)$.

We shall construct a sequence $g_1(x) = \exp(in_1x)$, $g_2(x)$, $\ldots$, $g_r(x)$ of trigonometric polynomials such that

$$(3.19) \qquad |g_k(x)| \leq 1, \quad I_k = (1 - \tfrac{1}{r})\, I_{k-1} + C\, r^{-1/2}, \qquad k = 2, \ldots, r,$$

where

$$I_k = \int f(-x)\, g_k(x)\, dx.$$

Assuming for the moment that this construction has been carried out up to an $r \geq C(\log N/\log\log N)$ and observing that $I_1 = 1$ we have

$$\begin{aligned} \|f\|_1 &\geq I_r \\ &= (1 - \tfrac{1}{r})\, I_{r-1} + Cr^{-1/2} \\ &= C\, r^{-1/2}\left\{1 + (1-\tfrac{1}{r}) + \ldots + (1-\tfrac{1}{r})^{r-2}\right\} + (1 - \tfrac{1}{r})^{r-1} \\ &\geq C\, r^{-1/2}(r-1)(1 - \tfrac{1}{r})^{r-2} \\ &\geq C\, r^{1/2} \\ &\geq C(\log N/\log\log N)^{1/2}. \end{aligned}$$

The last but one inequality follows from the fact that $(1 - \frac{1}{r})^r$ converges to e^{-1}, as $r \to \infty$, and hence it remains greater than a fixed positive constant.

Thus our task has been reduced to the construction of a sequence $g_1, \ldots, g_r$ of trigonometric polynomials satisfying (3.19) up to an r of the order of $\log N/\log\log N$.

We shall need two lemmas.

LEMMA 3.1. *If* $r \geq 100$, $-\frac{r}{2} \leq P \leq \frac{r^2}{2}$, $-\frac{r^2}{2} \leq Q \leq \frac{r^2}{2}$, *then*

$$\left|1 - \frac{1}{r} - \frac{P+iQ}{r^2} + \frac{(P+iQ)^2}{r^4}\right| + \frac{1}{4r^{3/2}}(r+2P)^{1/2} \leq 1. \tag{3.20}$$

The assumption $r \geq 100$ is made for technical purposes only. It could be weakened but this would be of no importance to the proof of theorem (3.5).

LEMMA 3.2. *Let* S *be a set of positive integers containing* n_1 *and let* $N(p)$, $p \in S$, *be the number of values* k *such that* $n_k \geq p$. *If* r *is a positive integer such that*

$$r^4 \sum_{p \in S} N(p) \leq N \tag{3.21}$$

then there are r *integers* $m_1 \geq m_2 \geq \ldots \geq m_r$ *among the* $n_1, \ldots, n_N$ *such that*

$$P + (m_i - m_j) + (m_k - m_\ell) \neq n_s, \quad \text{for all } p \in S, \quad i \leq j, \quad k < \ell, \quad 0<i,j \quad \ell \leq r \tag{3.22}$$

and

$$m_t = n_{q(t)} \quad \text{with} \quad q(t) \leq t^4 \sum_{p \in S} N(p). \tag{3.23}$$

We take first these two lemmas for granted and complete the proof of theorem (3.5).

We write r for the integral part of $\frac{1}{10}(\log N/\log\log N)$ and observe that if N is large enough, $r \geq 100$. Under this restriction we shall show that there are sets S_k and functions g_k which, for $k \leq r$, satisfy the following recursive relations :

$$S_{k+1} = S_k \cup T_k \cup R_k \quad , \quad S_1 = \{n_1\},$$

where

$T_k = \{m_1^{(k)}, \ldots, m_r^{(k)}\}$ is constructed as in lemma 3.2 with $S = S_k$.

$$R_k = \{p + (m_i^{(k)} - m_j^{(k)}) + (m_s^{(k)} - m_\ell^{(k)}) : p \in S_k,\ i \leq j,\ s < \ell,\ 0 < i,j,s,\ell \leq r\},$$

and

$$g_{k+1}(x) = \frac{1}{4r^{3/2}} h_k(x) + g_k(x) \left\{ 1 - \frac{1}{r} - \frac{P_k + iQ_k}{r^2} + \frac{(P_k+iQ_k)^2}{r^4} \right\}$$

$$g_1(x) = \exp(in_1 x),$$

where

$$h_k(x) = \exp(im_1^{(k)}x) + \exp(im_2^{(k)}x) + \ldots + \exp(im_r^{(k)}x)$$

$$P_k + iQ_k = \sum_{\ell<j} \exp\left\{ i(m_\ell^{(k)} - m_j^{(k)})x \right\}.$$

The only requirement, according to lemma 3.2, which is needed for the construction of S_{k+1}, g_{k+1} from S_k, g_k is that (3.21) holds with $S = S_k$. This is trivially true for S_1, if N is large enough. (3.23), the definition of S_{k+1}, and the trivial relations

$$N(p + m_i^{(k)} - m_j^{(k)} + m_s^{(k)} - m_\ell^{(k)}) \le N(p), \quad i \le j, \quad s < \ell, \quad 0 < i,j,s,\ell \le r$$

$$N(m_j^{(k)}) = q^{(k)}(j), \quad \text{where} \quad q^{(k)} \quad \text{is as in (3.23)} \quad \text{with} \quad S = S_k,$$

$$\sum_{p \in S_k} N(p) \ge 1$$

imply

$$\sum_{p \in S_{k+1}} N(p) \le \left\{ \sum_{p \in S_k} N(p) \right\} \left\{ 1 + (1^4 + \ldots + r^4) + \frac{1}{2} r^4 \right\} \le r^5 \sum_{p \in S_k} N(p)$$

and hence

$$\sum_{p \in S_k} N(p) \le r^{5(k-1)}$$

Thus, if $k \le r$, the assumption on r guarantees the validity of (3.21) and consequently the existence of g_k.

It is clear that r, P_k, Q_k satisfy the hypothesis of lemma 3.1 and that

$|h_k(x)|^2 = r + 2P_k$. We also have $|g_1(x)| \leq 1$ and by induction $|g_k(x)| \leq 1$. Observing now that the frequencies appearing in $P_k + iQ_k$ and $(P_k + iQ_k)^2$ do not appear in $f(x)\, g_k(-x)$ we obtain (3.19).

Thus, modulo lemmas (3.1) and (3.2), the proof of theorem (3.5) is complete.

Proof of lemma (3.1). Computing the square root of the absolute value in (3.20) we find

$$\left(1 - \frac{1}{r} - \frac{Pr^2 - P^2}{r^4}\right)^2 + \frac{Q^2}{r^8}\left[Q^2 - 2r^4\left(1 - \frac{1}{r} - \frac{Pr^2-P^2}{r^4}\right) + r^4\left(1 - \frac{2P}{r^2}\right)^2\right] =$$

$$= A^2 + \frac{Q^2}{r^8} B.$$

Since

$$\frac{d}{dP}(Pr^2 - P^2) = r^2 - 2P \geq 0$$

we have

$$|Pr^2 - P^2| \leq \max\left\{ \left|-\frac{r}{2} r^2 - \left(-\frac{r}{2}\right)^2\right|, \left|\frac{r^2}{2} r^2 - \left(\frac{r^2}{2}\right)^2\right| \right\} = \frac{r^4}{4}.$$

The last inequality and the hypotheses on r, P, Q imply

$B \leq r^4 \left[0.25 - 2(1 - 0.01 - 0.25) + 1.01^2\right] \leq 0$

$A \geq 1 - 0.01 - 0.25 \geq 0.$

Collecting the above results we see that it suffices to show

$$r^{5/2}(r + 2P)^{1/2} \leq 4r^3 + 4(Pr^2 - P^2). \tag{3.24}$$

If $P \leq 0$ the right hand side of (3.24) exceeds $4r^3 - 2r^3 - r^2 \geq r^3$ which obviously implies (3.24). If $P \geqslant 0$ then $Pr^2 - P^2 = P(r^2-P) \geq P^2 \frac{r^2}{2}$ and (3.24) reduces to the trivial inequality

$(4r + 2P)^2 \geq r(r + 2P).$

Proof of lemma (3.2). We take $m_1 = n_1$ and assume that $m_1, \ldots m_t$ have been constructed in such a way that (3.22) and (3.23) hold for all $p \in S$, $i \leq j$, $k < \ell$, $0 < i,j,k,\ell \leq t$. It will be enough to show that if $t < r$ then we can find a m_{t+1} in such a way that (3.22) and (3.23) hold with $0 < i,j,k,\ell \leq t+1$.

To prove this assertion it suffices to show that the ineligible choices for m_{t+1}, i.e. those for which (3.22), with $0 < i,j,k,\ell \leq t+1$, fails, are at most

$$(t+1)^3 \sum_{p \in S} N(p).$$

Indeed, if this is the case, then (note that $\sum_{p \in S} N(p) \geq 1$)

$$\begin{aligned} q(t) + (t+1)^3 \sum_{p \in S} N(p) + 1 &\leq \left\{t^4 + (t+1)^3\right\} \sum_{p \in S} N(p) + 1 \\ &< (t+1)^4 \sum_{p \in S} N(p) \\ &\leq r^4 \sum_{p \in S} N(p) \\ &\leq N, \end{aligned}$$

and hence there exists a $y = q(t+1)$, with

$$q(t) < y = q(t+1) \leq (t+1)^4 \sum_{p \in S} N(p) \leq N ,$$

such that (3.23) and (3.22) hold with $m_{t+1} = n_y$, $0 < i,j,k,\ell \leq t+1$.

It remains to show that the ineligible choices for m_{t+1} are at most

$$(t+1)^3 \sum_{p \in S} N(p).$$

For fixed p and $n_s > p$ it can easily be seen that there are at most $t^2(t-1) + 2t^2 + t < (t+1)^3$ expressions of the form $p - n_s + m_i - m_j + m_k - m_\ell$, $i \leq j$, $k < \ell$, $0 < i,j,k,\ell \leq t+1$ containing m_{t+1} (once, twice or at most three times) and that the vanishing of such an expression uniquely determines m_{t+1} in terms of the remaining variables appearing in this expression. (Note that if m_{t+1} does not exist or, if it cancels in such an expression, then, by hypothesis, the letter cannot vanish).

Since there are at most $N(p)$ possible choices for n_s, for a fixed p, we shall have at most $(t+1)^3 N(p)$ ineligible choices for m_{t+1} for each p, and hence at most $(t+1)^3 \sum_{p \in S} N(p)$ for all p.

5. COMMENTS ON THEOREM 3.5.

(i) The proof given in III.4 is taken from ([29]). (3.18) with exponent 1/8 was proved by P. Cohen ([7]). This was the first positive result concerning Littlewood's conjecture (up to that time it was not known even if $\|f\|_1$ tends to infinity with N). A few months later H. Davenport ([8]), following Cohen's method, reduced the exponent to 1/4. A new element in Davenport's paper was the use of an inequality similar to (3.20), on which the important estimate $|g_k| \leq 1$ is based. Davenport's inequality was (we use the same notation as in lemma 3.1)

$$(3.25) \qquad \left|1 - \frac{2}{r^2} - \frac{P+iQ}{r^3}\right| + \frac{1}{r^{5/2}}(r+2P)^{1/2} \leq 1.$$

Any attempt to improve on the exponent 1/4 by changing the factors $\frac{1}{r^2}, \frac{1}{r^s}, \frac{1}{r^{s/2}}$ leads nowhere. In fact if $\varepsilon > 0$ we find that for some $a \geqslant 0$

$$(3.26) \qquad \left|1 - \frac{a}{r^\varepsilon} - \frac{P+iQ}{r^{2+\frac{\varepsilon}{2}}}\right| + \frac{1}{r^{1+\frac{3\varepsilon}{4}}}(r+2P)^{1/2} \leq 1$$

Repeating the proof (using now (3.26) instead of (3.25)) we again obtain the same estimate $C(\log N/\log\log N)^{1/4}$ (see next remark). We observe that the exponent d of the coefficient r^{-d} of $P+iQ$ is always greater than 2 $(d = 2 + \frac{\varepsilon}{2})$ and this for a very good reason : if $d = 2$ then the point $z = \frac{P+iQ}{r^2}$, for $P = 0$, may be at a distance greater than 1 from the point $1 - \frac{a}{r^\varepsilon}$, and this makes (3.26) impossible. However, since z lies (almost) in the right half of the unit disc, the point $z - z^2$, even with $P = 0$, will be close enough to 1 so that an inequality of the form (3.25), with $z - z^2$ instead of $\frac{P+iQ}{r^3}$, may be possible. That this is really the case is the content of lemma (3.1). The changes needed for the rest of the proof are more or less automatic and lead finally to the improved estimate $C(\log N/\log \log N)^{1/2}$.

(ii) (3.20) and (3.25) are of the form

(3.27) $$\left|1 - \frac{C}{r^a} - \left\{\text{terms in } (P+iQ)^t, \ t = 1, 2, \ldots, k\right\}\right| + \frac{1}{Cr^b}(r+2P)^{\frac{1}{2}} \leq 1.$$

The argument used in this paper yields

(3.28) $$\|f\|_1 \geq C(\log N/k \log \log N)^{(a+1-b)/a}.$$

It is very easy to see that always $(a+1-b)/a \geq 1$, and hence the best bound we can expect from this argument will be greater than $C(\log N/\log \log N)$.

(iii) Assume that the terms $(P+iQ)^t$ in (3.27) form a polynomial in $z = \frac{P+iQ}{r^2}$. We further assume that the coefficients of this polynomial are independent of r and the coefficient of z is positive On examining (3.27) for the real values of z we find that always $(a+1-b)/a \leq \frac{1}{2}$. Hence no further improvement can be obtained from Cohen's method.*

However it is to be noted that although these assumptions are satisfied in the case of (3.20) they are not in that of (3.25).

(iv) With slight modifications in the proof we can show that (3.18) remains valid if we replace $\sum_{k=1}^{N} \exp(in_k x)$ by $\sum_{k=1}^{N} a_k \exp(in_k x)$, provided that $|a_k| \geq 1$. Indeed if

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* This remark was communicated to the author by P. Cohen.

we put $g_1(x) = (\operatorname{sgn} a_1)\exp(in_1 x)$, and $h_k(x) = \sum_{k=1}^{N} (\operatorname{sgn} b_j^{(k)}) \exp(im_j^{(k)} x)$, where $\operatorname{sgn} y = |y|/y$ and $b_j^{(k)} = n_{q^{(k)}(j)}$, in the recursive definition of $\{g_k\}$ we obtain

$$I_s = (1 - \frac{1}{r}) I_{s-1} + \frac{1}{4r^{3/2}} \sum_j \left| a_{q^{(k)}(j)} \right|$$

$$\geq (1 - \frac{1}{r}) I_{s-1} + C r^{-1/2},$$

which is sufficient for the proof.

(v) Our main effort in the proof of theorem (3.5) was to construct a function g_r, with $|g_r| \leq 1$, such that the integral $\int f(x)\, g_r(-x)\, dx$ is large. The function g_r was constructed, in a rather complicated way, from some trigonometric polynomials closely related to the even powers of f. It is not hard to obtain an exact formula for $\|f\|_1$ by using even powers of f.

Assume for simplicity that f is the cosine sum $1 + 2 \sum_{k=1}^{N} \cos n_k x$ and write $F(x) = f(x)/(2N+1)$. We shall have

$$(3.29) \qquad |f| = (2N+1)|F|$$
$$= (2N+1)\left\{1 - (1-F^2)\right\}^{1/2}$$
$$= (2N+1)\left\{1 - \tfrac{1}{2}(1 - F^2) - \left|\binom{1/2}{2}\right|(1-F^2)^2 - \ldots - \left|\binom{1/2}{k}\right|(1-F^2)^k - \ldots\right\}.$$

Of course there are other formulas for giving $|f|$ in terms of the powers of f. If e. g. K_m is what is usually called an approximate identity (i. e. $K_m \geq 0$, $\int_{-\infty}^{\infty} K_m = 1$, $K_m(t) = K_m(-t)$, $\sup_{|t|>a} K_m(t) \to 0$, as $m \to \infty$ for every $a > 0$) then

$$(3.30) \qquad |f(x)| = \lim_{m \to \infty} 2 f(x) \int_0^{f(x)} K_m(t)\, dt.$$

It is not hard to see that (3.29) corresponds to the choice : $K_m(t) = a_m(1-t^2)^m$

for a suitable a_m if $|t| \leq 1$ and 0 otherwise. In general if we chose for K_n an even polynomial we obtain $\|f\|_1$ from (3.30) as a limit of linear combinations of the even Norms of f. Coming back to (3.29) and writing $G(t) = \frac{1}{2N+1} D_N(t)$ $(D_N(t) = 1 + 2 \sum_{k=1}^{N} \cos kx)$ we have

$$\int (1-F^2) = \int (1 - G^2)$$

$$\int (1-F^2)^2 = \int (1-2F^2) + \int F^4$$

$$\leq \int (1-2G^2) + \int G^4 = \int (1-G^2)^2$$

(The above inequality follows from II(2.4)).

Were it true in general that $\int (1-F^2)^k \leq \int (1-G^2)^k$ then the strong conjecture $\|f\|_1 \geq \|D_N\|_1$ would be a consequence of (3.29). However this is false. To see this we argue as follows* :

The roots of D_N are all simple as can easily be seen from the explicit formula (3.4). If f has a double root b, then the integral of $(1-F^2)^k$ over a neighborhood of b exceeds $\int (1-G^2)^k$ provided that k is large enough. The example

$$f(x) = 1 + 2\left\{\cos x + \cos 3x + \cos 7x + \cos 9x + \cos 13x + \cos 19x + \cos 40 x\right\}$$

shows that such an f exists (it has the double root $x = \frac{\pi}{3}$).

(vi) We use the same notation as in the previous remark. The best choice for a function g so that $|g| \leq 1$ and $\int f(x)\, g(-x)\, dx$ is large if of course $g(x) = \text{sgn}\, f(x)$. $\text{sgn}\, f(x)$ can be expressed in terms of the odd powers of $f(x)$ (using 3.30 for instance). But, as in the case of $|f(x)|$, it seems that no representation is known which can give positive results for our purposes.

We note in passing that Littlewood ([27], problem 22) asks for a lower bound of the

(*) This argument was orally communicated to the author by Y. Domar.

number of real roots of $f(x)\ (= 1 + 2 \sum_{k=1}^{N} \cos n_k x)$, and adds : "Possibly $N-1$, or not much less". It appears that the problem is still open. Progress in this direction would possibly provide useful information on the problems treated in these Notes.

(vii) Let f be an integrable function such that $|\hat{f}(n_k)| \geq 1$, $k = 1, 2, \ldots, N$, and $|\hat{f}(m)| \leq \varepsilon_N$ otherwise. It is easy to see that we can choose $\varepsilon_N > 0$, depending only on N, so that the proof of theorem 3.5 holds good and yields : $\|f\|_1 \geq b_N$, where $b_N \to \infty$ as $N \to \infty$.

J.-P. Kahane has used this fact to show the existence of a sequence $\{c_n\}$ such that $c_n \to 0$, as $n \to \infty$, and for all permutations φ the sequence $c_{\varphi(n)}$ is not the sequence of Fourier coefficients of an integrable function ([21]).

IV. ONE SIDED L^∞ NORMS.

In this chapter we examine the lower bound of the absolute value of the minimum of real exponential sums and its relation to the lower bound of the L^1 Norm of exponential sums.

1. INTRODUCTION.

The L^∞ Norm of an exponential sum f having N terms presents no problem at all. In fact $||f||_\infty = f(0) = N$.

The problem becomes more interesting if we examine more general trigonometric polynomials. If, for instance, we consider polynomials whose non-zero coefficients are complex numbers of absolute value 1 then the Hardy Littlewood example $\sum_{k=1}^{N} \exp(ik \log k) \exp(i k x)$ shows that the L^∞ Norm can be of the same order as the L^2 Norm $N^{1/2}$ ([27], see also [33] for some related results). Although we shall not examine this aspect of the problem we mention a nice example, found independently by H. Shapiro and W. Rudin ([35], [32]), of trigonometric polynomials P_n, Q_n of degree $N = 2^n - 1$ whose non-zero coefficients are 1 or -1 and whose L^∞ Norm is less than $C N^{1/2}$ $(= C||P_n||_2 = C||Q_n||_2)$. P_n and Q_n, usually referred to as the Rudin-Shapiro polynomials, are defined recursively as follows :

$$P_o = Q_o = 1$$

$$P_{n+1}(x) = P_n(x) + \exp(i2^n x)\, Q_n(x)$$

$$Q_{n+1}(x) = P_n(x) - \exp(i2^n x)\, Q_n(x).$$

On observing that $|P_n|^2 + |Q_n|^2 = 2(|P_{n-1}|^2 + |Q_{n-1}|^2)$ we immediately obtain the desired properties of P_n and Q_n.

Throughout this chapter f will denote the cosine sum

(4.1) $$f(x) = \cos n_1 x + \ldots + \cos n_N x, \quad 0 < n_1 < \ldots < n_N \text{ integers}$$

and for any real 2π-periodic function g, we define

(4.2) $$M(g) = |\inf g(x)|.$$

We always have $\|f\|_\infty = N$. On observing that $\|f\|_\infty = \max f(x)$ it is natural to ask : What about $M(f)$?

The problem of showing that $M(f)$ tends to infinity with N was raised by Ankeny and Chowla ([5]). More generally, as in the case of the L^1 Norm, we may ask for a lower bound of $M(f)$ depending on N only. In the following sections we examine this problem in detail.

2. THE MINIMUM OF REAL EXPONENTIAL SUMS.

Let g be any real polynomial without constant term. We have ([36])

(4.3) $$\|g\|_1 = \int g \leq \int |g + M(g)| + M(g) = 2M(g).$$

Thus any lower bound for $\|g\|_1$ automatically yields a lower bound for $M(g)$. Using (4.3) and III.5 (iv) we obtain :

THEOREM 4.1. Let g be a real trigonometric polynomial with N non-zero coefficients of absolute value ≥ 1. Then $M(g) \geq C(\log N/\log\log N)^{1/2}$.

In some special cases it is easy to obtain satisfactory lower bounds for $M(g)$ without any appeal to theorem 3.5, e. g. when we can find by other means a lower bound for $\|g\|_1$ by other means. We give a few examples of this sort.

If g is the sum of N sines then $M(g) \geq (\frac{1}{2}N)^{1/2}$. For, g is odd and hence $M(g) = \|g\|_\infty \geq \|g\|_2 = (\frac{1}{2}N)^{1/2}$.

If f (recall our convention (4.1)) has only odd frequencies then $M(f) = -f(\pi) = N$.

If the frequencies $\{n_k\}$ of f satisfy the condition : $n_N \leq 2n_1$, then $f(\frac{2\pi}{3n_1}) \leq (-\frac{1}{2}) + \ldots + (-\frac{1}{2}) = -\frac{N}{2}$. This example shows that a suitable translation (say by $2n_N$) of the sequence $\{n_k\}$ makes the problem of $M(f)$ trivial while any translation is of no importance at all for the problem of $\|f\|_1$.

It is not hard to see that $M(f)$ can be as small as $CN^{1/2}$. Since for any N there is N_1 such that $N = \frac{1}{2}(N_1^2 - N_1) + 0(N^{1/2})$ we may assume that N is of the form $\frac{1}{2}(N_1^2 - N_1)$. Let $g(x) = \exp(in_1x) + \ldots + \exp(in_{N_1}x)$, $0 < n_1 < \ldots < n_{N_1}$, and write

$$2f(x) = |g(x)|^2 - N = \sum_{k\neq j} \exp\{i(n_k - n_j)x\}$$

$$= 2\sum_{k>j} \cos(n_k - n_j)x.$$

Since f is a cosine sum with $N = \frac{1}{2}(N_1^2 - N_1)$ and $M(f) \geq \frac{1}{2}N_1 \geq C\,N^{1/2}$ our assertion is proved.

It is not known if there exists an f with $M(f)$ of order smaller than $N^{1/2}$. The last example and theorem (4.1) show that the order of the smallest $M(f)$, when f ranges over all cosine sums with N terms, is greater than $(\log N/\log\log N)^{1/2}$ and less than $N^{1/2}$. This result, despite the very large gap between $(\log N/\log\log N)^{1/2}$ and $N^{1/2}$, appears to be the best known result up to now.

Although we do not know if $M(f) \geq C N^{1/2}$ for every f it is easy to show that there is always a subset of $\{n_k\}$ such that $M(g) > C N^{1/2}$, where g is the cosine sum corresponding to that subset . This is an immediate consequence of the following assertion : There is a choice of signs $\pm$ such that

$$(4.4) \qquad \| \pm \cos n_1 x \pm \ldots \pm \cos n_N x \|_1 \geq C N^{1/2}.$$

Indeed $\pm \cos n_1 x \pm \ldots \pm \cos n_N x$ is the difference of two cosine sums corresponding to subsets of $\{n_k\}$ and (4.4) implies that at least one of them has L^1 Norm which exceeds $CN^{1/2}$ (hence the same is true for the absolute value of its minimum).

The method of proof of (3.13) (convexity of the L^p Norms) shows that (4.4) is a consequence of the assertion : "There is a choice of signs $\pm$ such that

$$(4.5) \qquad \| \pm \cos n_1 x \pm \ldots \pm \cos n_N \|_4 \leq C N^{1/2} \text{''}.$$

(4.5), as well as (4.4), is a well known result in the theory of Rademacher series (see, e. g. |22|) and we prove it by the so called "randomization" method. Taking into account some obvious cancellations we have

$$(4.6) \qquad \Sigma \, | \pm \cos n_1 x + \ldots + \cos n_N x |^4 = 2^N \Big\{ \sum_k (\cos n_k x)^4 + 6 \sum_{k<j} (\cos n_k x)^2 (\cos n_j x)^2 \Big\}$$

$$\leq C . 2^N \Big\{ \sum_k (\cos n_k x)^2 \Big\}^2$$

$$\leq C . 2^N N^2 ,$$

where the summation on the left is taken over the 2^N possible choices of signs $\pm$. Integrating now both members of (4.6) we see that the average of $\| \pm \cos n_1 x + \ldots + \cos n_N x \|_4^4$ is less than $C N^2$ which obviously implies (4.5).

For a result more precise than (4.4) see ([38]).

3. AN ALTERNATIVE APPROACH TO THEOREM (4.1).

K. F. Roth gave a proof of theorem (4.1), for the case of cosine sums, which does not depend on theorem (3.5) ([31]). His method, interesting in itself, is different from that of Cohen, used in the proof of theorem (3.5) (in any case it is not clear what, if any, is the relation between the two proofs). We shall give in this section an outline of Roth's proof.

We introduce first some notation. We recall the definitions (4.1) and (4.2) and write $A = \{n_1, \ldots, n_N\}$ and use the capital letters B, D, E, ... for sets of positive integers. We further write $A^* = A \cup (-A)$ and similarly for B^*, D^*, ... (as usual $-A = \{x : -n \in A\}$). For any set of integers Y we put $f_Y(x) = \sum_{m \in Y} \exp(imx)$. Thus in particular $f_{A^*} = 2f$. We write also $M = 2M(f)$ for simplicity. Finally for any sets Y and Z, $|Y|$ will denote the cardinality of Y and $Y+Z$ the set $\{y+z : y \in Y$ and $z \in Z\}$.

Our objective is to show that $M \geqslant C(\log N/\log\log N)^{1/2}$. To this end it obviously suffices to show that

$$(8M)^{16M^2} > N. \tag{4.7}$$

Roth's argument is indirect. Assume that (4.7) is false and call t the largest integer such that $2^t < 4M^2$ (hence $2^t \geq 2M^2$). Then

$$(8M)^{2^t} \leq N^{1/4}. \tag{4.8}$$

The proof is now divided into two steps. In the first we use (4.8) to show the existence of a set E of positive integers such that

(i) E is symmetric with centre of symetry a positive integer e not belonging to E, i. e. for every $x \in E$ we have $2e - x \in E$.

(ii) $|E| \geq 10 M^4$.

(iii) For any subset D of E such that $|D| \geq 2M^2 + 1$ there is a set B, depending in general on D, such that $|B| = 2M^2$ and $D + B^* \subset A$.

In the second step it is shown that if a set E satisfies (i), (ii) and (iii) then the same is true for the set $e + E$, and hence for the sets $2e + E, \ldots, 2^k e + E, \ldots$. Since (iii) implies that A contains elements greater than e we conclude that A must contain elements greater than $2^k e$ for any k. This contradicts the finiteness of A and completes the proof.

Outline of step 2. The verification of (iii) is the only non trivial point, and this can be achieved by a counting argument. More precisely : given $D \subset E$ with $|D| \leq 2M^2+1$ we must find a set B such that $|B^*| = 2M^2$ and $e + D + B^* \subset A$. To do this we count the ineligible elements of $E - e$, i. e. those $y \in E - e$ for which $e + D + y \not\subset A$. It turns out that they are less than $|E| - 2M^2$ and this guarantees the existence of B. This counting is a more or less routine matter if we observe the following property of the set E :

"For any $y \in E$ there are at most $2M^2 - 1$ elements of $y+E$ not belonging to A".

Were this property false, then, using (iii), one could find two sets B , D satisfying the hypotheses but not the conclusion of the following simple consequence of the definition of M (and hence finish step 2).

LEMMA 4.1. <u>If</u> $B \subset A$, $D \cap A = \emptyset$ <u>and</u> $B - D \subset A$, <u>then</u> $|B| < 2M^2$.

Proof. Recalling our definitions we have

$$\frac{|B|^2}{M^2} = \left\{\int (f_B - f_D)(1 + \frac{f_{A*}}{M})\right\}^2 \tag{4.9}$$

$$< \int |f_B - f_D|^2 (1 + \frac{f_{A*}}{M})$$

$$= 2|B| + \frac{1}{M}\int |f_B|^2 f_{A*} + \frac{1}{M}\int |f_D|^2 f_{A*} - \frac{1}{M}\int f_B \bar{f}_D f_{A*} - \frac{1}{M}\int \bar{f}_B f_D f_{A*}$$

$$\leq 2|B| + \frac{1}{M}|B|^2 + \frac{1}{M}|B|^2 - \frac{1}{M}|B|^2 - \frac{1}{M}|B|^2 = 2|B|$$

from which $|B| < 2M^2$ follows immediately.

The first inequality in (4.9) follows from the Cauchy-Schwarz inequality and the fact that if $\frac{f_{A*}}{M} \geqslant 0$. The last inequality is a consequence of the relations $\int f_B \bar{f}_D f_{A*} = \int \bar{f}_B f_D f_{A*} = |B|^2$ which follow immediately from our hypothesis $B\text{-}D \subset A$.

<u>Outline of step 1</u>. For the construction of a set having the properties (i), (ii) and (iii) we shall need another simple consequence of the definition of M.

LEMMA 4.2. <u>If</u> $B \subset A$ <u>and</u> $|B| \geq 2M^2$ <u>then</u> $\int |f_B|^2 f_A \geq \frac{|B|^2}{4M}$.

Proof. We have

$$2\int |f_B|^2 f_A = \int |f_B|^2 f_{A*} = \int |f_B|^2 (f_{A*} + M) - M|B|$$

$$\geq \frac{1}{M}\left\{\int f_B (f_{A*} + M)\right\}^2 - M|B|$$

$$= \frac{|B|^2}{M} - M|B| \geq \frac{|B|^2}{2M}.$$

The first inequality follows from the Cauchy-Schwarz inequality and the fact that $f_{A*} + M \geq 0$. The last inequality is a consequence of our hypothesis $|B| \geq 2M^2$.

We observe that $M(f_A) = M(f_{2A})$, where $2A = \{2a : a \in A\}$, so that we may assume that A contains only even integers. This assumption is made because in the course of the proof we need the fact that all the frequencies of $|f_A|^2$, i. e. the numbers $n_k - n_\ell$, are even.

We set $p_o = 0$, $E_o = A$, $G_o = \{0\}$. Lemma 4.2 yields

$$\int |f_{E_o}|^2 f_A \geq \frac{N^2}{4M}$$

Since the integral on the left is a sum of no more than N of the Fourier coefficients of $|f_{E_o}|^2$, there is a positive integer h_1 such that (recall that $|f_{E_o}|^2$ has only even frequencies)

$$\widehat{|f_{E_o}|^2}(2h_1) \geq \frac{N}{4M}.$$

On observing that $|f_{E_o}|^2 = N + \sum_{k \neq \lambda} \exp\{i(n_k - n_x)x\}$ we conclude that the set $E_1 = \{n_k : n_\ell - n_k = 2h_1 \text{ for some } \ell\}$ has at least $\frac{N}{4M}$ elements. We set $p_1 = h_1$, $G_1 = \{-h, h\}$ and observe that if $a \in E_1$, $b \in G_1$ then $p_1 + a + b$ equals a or $a + 2h_1$. In either case $p_1 + a + b$ belongs to A, by the definition of E_1, and hence $p_1 + E_1 + G_1 \subset A$.

Applying again lemma 4.2 we obtain

$$\begin{aligned} \int |f_{E_1}|^2 f_{A - \{G_1 - G_1\}} &\geq \int |f_{E_1}|^2 f_A - |G_1 - G_1| N \\ &\geq \frac{|E_1|^2}{4M} - 4N \\ &\geq \frac{N}{2(8M)^3}. \end{aligned}$$

The first inequality follows from the fact that $\widehat{|f_{E_1}|^2}(k) \leq N$ for all k, the second from lemma 4.2 and the fact $|G_1 - G_1| \leq |G_1|^2 = 4$ and the last from (4.8) (assuming $t \geq 2$).

As before the above inequality allows us to define a subset E_2 of A and a number h_2 such that $|E_2| \geq \frac{N}{2(8M)^3}$, $2h_2 \notin G_1 - G_1$ and $p_2 + E_2 + G_2 \subset A$, where $p_2 = p_1 + h_2$, $G_2 = \{-h_2+G_1\} \cup \{h_2+G_1\}$.

(4.8) enables us to continue this construction for t steps. This leads to a triple (p, E', G) where p is a positive integer, G is a set such that $G = -G$ and $|G| = 2^t$, $E' \subset A$ and $|E'| \geq \frac{N}{2(8M)^{2t-1}}$. Moreover we have $p + V + G \subset A$.

The set E' constructed this way satisfies the properties (ii) and (iii) (with $B^* = C$ for every $D \subset V$) but not necessarily (i). However, again because of (4.8), $|E_1| > \frac{N}{2(8M)^{2t-1}} > 8MN^{3/4}$, which means that (ii) is satisfied with a large margin. Using this fact and an argument similar to the one used before (this time we choose a large coefficient of $f^2_{E'}$) in order to define $E_1, E_2, \ldots$ we can find a subset E of E which satisfies the desired condition (i), (ii) and (iii).

Remark. It is interesting to observe that the definition of M was used only in order to prove the lemmas 4.1 and 4.2, on which the rest of the proof is based.

4. A RELATION BETWEEN $\|F\|_1$ AND $M(\mathrm{Re}\, F)$.

The result of this section applies not only to exponential sums but also to trigonometric polynomials having coefficients not less than one in absolute value (compare III 5 (iv)). Our exposition follows ([30]).

Let $F(x) = c_1\exp(in_1x) + \ldots + c_N\exp(in_Nx) = f(x) + ig(x)$, $|c_i| \geq 1$, $i = 1, 2, \ldots N$.

If we impose no condition on the sequence $n_1, \ldots, n_N$ of distinct positive integers then the best lower bounds of $||F||_1$ and $M(f)$ that we know are of the order of $(\log N/\log\log N)^{1/2}$ (see theorems 3.5 and 4.1).

In our next theorem we show that

$$M \log M + ||F||_1 \geq C \log N. \tag{4.10}$$

It follows that either $||F||_1 \geq C \log N$ or $M > C \log N/\log\log N$. Were it true that $||F||_1 < CM$ then we could deduce that $M > C \log N/\log\log N$. We shall return to this point later (V.2, remark (ii)).

(4.10) is an immediate corollary of the following more general theorem whose proof, although very simple, is based on rather deep results of Fourier analysis.

THEOREM 4.2. If c_k^* denotes the sequence $|c_k|$, $k = 1, \ldots, N$, rearranged in non increasing order then

$$M \log M + ||F||_1 \geq C \sum_{k=1}^{N} \frac{c_k^*}{k} - C. \tag{4.11}$$

<u>Proof</u>. The function

$$G(z) = 2M + c_1 z^{n_1} + \ldots + c_N z^{n_N}$$

is holomorphic and its real part is not less than M on the circle $|z| = 1$. It follows that its real part is greater than 0 in $|z| < 1$ and hence we have

$$G(z) = |G| \exp(iq), \quad |q| = |\operatorname{Arg} G| \leq \frac{\pi}{2}, \quad |z| \leq 1.$$

The function $f \log |G| - qg = \operatorname{Re}(F \log G)$ is harmonic and has the value 0 at the origin. An application of the mean value property for harmonic functions yields

$$\int f \log |G| = \int q\, g < C ||F||_1 .$$

Writing $f^- = \max(0, -f)$, $\log^+ |f| = \max(0, \log |f|)$, observing that $f = |f| - 2f^-$, $\log^+ |f| \leq \log |G|$ and remembering that $\log |G|$ is harmonic with value

$\log(2M)$ at the origin we obtain

$$\begin{aligned}\int |f| \log^+ |f| &\leq \int |f| \log |G| \\ &\leq C\|F\|_1 + 2\int f^- \log |G| \\ &\leq C\|F\|_1 + 2M \int \log |G| \\ &= C\|F\|_1 + 2M \log 2M \\ &\leq C(\|F\|_1 + M \log M + C).\end{aligned}$$

Using now the inequality (see remark iii)

$$(4.12) \qquad \sum_{k=1}^{N} \frac{c_k^*}{k} < C \int |f| \log^+ |f| + C$$

we obtain (4.11).

(4.10) follows immediately from (4.11) if we observe that $|c_k| \geq 1$ implies that the left hand side of (4.12) exceeds $C \log N$.

<u>Remarks</u> (i). The argument used in the above proof is essentially the same as the one leading to a classical inequality of M. Riesz for the conjugate function ([40] VII, 2.10).

(ii) A general remark concerning the above proof is that it is not based on the L^2 Parceval formula (which appear to be the case in the proof of theorem 3.5 and in the argument of Roth given in II.3) but on inequalities property belonging to the space $L \log^+ L$, that is a space much closer to L^1 than the space L^2. This is probably the reason that we have now reached (although in a conditional form) a lower bound of the order of $\log N$ and not $(\log N)^{1/2}$.

(iii) (4.12) is due to Hardy and Littlewood ([16]) . A proof of this inequality can be found in ([40] , XII, Ex. 8i) (the result we are looking for is not stated explicitly but is contained in the last line of the hint given there). It can also be considered as the limiting case of a better known inequality concerning rearrangements of Fourier coefficients of functions in L^p ([40], XII, 5.10) and proved by the so-called extrapolation method ([40], XII, 4.41).

V. SEQUENCES GROWING SLOWER THAN POWERS.

In this chapter we examine briefly the problems of $\|f\|_1$ and $M(f)$ when the sequence $\{n_k\}$ satisfies the condition $n_k < k^A$.

I INTRODUCTION.

The difficulty of the problems treated in the previous chapters is due to a large extent to the arbitrariness of the sequence $\{n_k\}$. We have seen for instance that in the case of arithmetic progressions or lacunary sequences it was relatively easy to obtain good estimates for the various L^p Norms of the corresponding exponential sums.

Arithmetic progressions, after a trivial change of variables, correspond to the most "dense" sequences $\{n_k\}$, while lacunary sequences correspond to "sparse" sequences. There are results on some intermediate cases scattered in the literature. In the following sections we isolate and examine briefly one of these cases. For some other cases see ([24]) and ([29]).

2. L^1 NORM.

Let $0 < n_1 < \ldots < n_N < \ldots$ be an infinite sequence of positive integers. In 1954 R. Salem ([34]) proved the following

THEOREM 5.1. If $n_k < k^a$, where a is a constant, then

$$(5.1) \qquad \limsup_{N\to\infty} \frac{1}{(\log N)^{1/2}} \int \left| \cos n_1 x + \ldots + \cos n_N x \right| dx > 0.$$

This result means that if $\{n_k\}$ grows slower than k^a then there is an infinite number of partial sums of the (infinite) exponential sum $\sum_1^\infty \cos n_k x$ for which the estimate $(\log N/\log\log N)^{1/2}$ can be slightly improved by dropping the factor log log N.

Outline of the proof. The main idea is to use the following result of Menchoff ([28]) : "Let q_k, $k = 1, 2, \ldots$ be an orthonormal system on $(0, 2\pi)$. The condition $\Sigma |c_k|^2(\log k)^2 < \infty$ implies that $\sum_{k=1}^{N} c_k q_k$ converges almost everywhere and this result cannot be improved".

The last clause means that given any function $q(k)$ such that $q(k) = 0(\log^2 k)$ then there is an orthonormal system q_k and a sequence c_k such that $\Sigma |c_k|^2 q(k) < \infty$ and $\sum_{k=1}^{N} c_k q_k$ diverges on a set of positive measure.

For any orthonormal system q_k Salem considers the two dimensional orthogonal system $\{q_k(x)(\cos n_k t)(\psi(t))^{1/2}\}$ where $\psi(t)(= \psi_n(t)) = 1 + \sup_{1\le k\le n} \{\cos n_1 x + \ldots + \cos n_k x\}$ and applies Bessel's inequality to a suitably chosen function $P(x,t)$. The result is an inequality which is equivalent (by duality) to :

$$(5.3) \qquad \int \sup_{k\le n} \left| \sum_{k=1}^{n} c_k q_k \right|^2 \le C\left(\sum_{k=1}^{n} |c_k|^2\right) \log m_n \max_{k\le n} \left(\int |\cos n_1 x + \ldots + \cos n_k x| \right)^2$$

for any sequence $\{c_k\}$.

Assuming now that (5.2) is false and using the hypothesis $m_k < k^a$ we see that the second member of (5.3) is dominated by : $C(\Sigma |c_n|^2)(\log n)^2 p(n)$, where $p(n) \to 0$ as $n \to \infty$. However from this form of (5.3) one can conclude that the condition

$$\Sigma |c_k|^2 (\log k)^2 p(k) < \infty$$

implies the almost everywhere convergence of $\sum_{k=1}^{N} c_k \varphi_k$ *. This contradicts Menchoff's theorem and completes the proof.

It is interesting to observe that if we consider exponential sums instead of cosine sums in theorem 5.1 then we can restrict ourselves to finite sequences and drop the hypothesis $n_k < k^a$. Moreover the proof will be an almost immediate corollary of Paley's theorem (III.th. 3.2). More precisely we have

THEOREM 5.2. <u>If</u> $0 < n_1 < \ldots < n_N$, <u>then</u>

$$\sup_{1 \le k \le n} \| \exp(in_1 x) + \ldots + \exp in_k x \|_1 \ge C(\log N)^{1/2}. \tag{5.4}$$

<u>Proof</u>. If $\{n_k\}$ contains a lacunary subsequence of length $\log N$ then (5.4) follows immediately from Paley's theorem. This is not in general true. However we can always find a number k, and a subsequence $\{m_1, m_2, \ldots\}$ of $\{n_1, \ldots n_N\}$ such that either $\{m_k - m_{k-1}, m_k - m_{k-2}, \ldots, m_k - m_1\}$ or $\{m_{k+1} - m_k, \ldots, m_N - m_k\}$ contains a lacunary subsequence of length greater than $C \log N$. Assuming this and using Paley's theorem we see that either $\| \exp(in_1 x) + \ldots + \exp(in_k x) \|_1$ or $\| \exp(in_1 x) + \ldots + \exp(in_N x) \|_1$

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* This follows from a standard argument in harmonic analysis. For many operations T_n defined on normed classes of functions the convergence almost everywhere of $T_n f$ will follow from the condition that $\| \sup_n T_n f \| \le C \|f\|$.

** As I learned from a conversation with N. Varopoulos this result was known to A. Beurling.

exceeds $C(\log N)^{1/2}$.

In order to prove the existence of n_k we write $A = \{n_1, \dots n_N\}$, $|B|$ for the cardinality of any set B, and argue as follows :

We partition the interval $I_o = [n_1, n_N]$ into 2, 4, 8,... equal closed intervals (These intervals are disjoint except when they are adjacent in this case they have exactly one point in common). We stop for the first time when k is the first integer such that the 2^k equal subintervals contain less than $\frac{N}{4}$ elements of A. We observe that we need at least two steps before reaching this stage. In the previous stage there is an interval containing more than $\frac{N}{4} - 1$ elements of A and hence we can choose an interval I_1 of the kth partition such that $\frac{N}{8} \leq |I_1 \cap A| < \frac{N}{4}$. The union of I_1 and its two neighbours contain at most $\frac{3N}{4}$ elements of A. Hence we can choose an element $m_1 \in A$ whose distance from I_1 is greater than $|I_1|$. We repeat now this construction starting from I_1 and continue in the same way. We obtain a sequence $I_o, I_1, I_2, \dots$ of intervals and a sequence of points $m_1, m_2, \dots$ of A such that : $I_o \geqslant I_1 > \dots,$ $|I_k| > 2|I_{k+1}|$, distance $\{m_k, I_k\} \geq |I_k|$, and $|I_k \cap A| \geqslant \frac{N}{8^k}$. It follows that $|I_k \cap A| > 1$ if $n = \left[\frac{1}{\log 8} \log N\right]$. It is clear now that either to the left or to the right of m_n there are $C \log N$ elements of A whose distances from m_n, as can easily be seen, form a lacunary sequence.

REMARKS. (i) Both theorems 5.1 and 5.2 remain valid if we replace the cosine or exponential sums by real or analytic polynomials with coefficients not less than 1 in absolute value.

(ii) It is not clear if theorem 5.1, without the hypothesis $n_k < k^a$, can be derived from theorem 5.2. This would obviously be the case if the second inequality in

(5.5) $C\|\cos n_1x+\ldots+\cos n_Nx\|_1 \leq \|\sin n_1x+\ldots+\sin n_Nx\|_1 \leq C\|\cos n_1x+\ldots+\cos n_Nx\|$

were true. It is a simple consequence of a théorem of Zygmund ([40], VII, 2.9) that (5.5) is true if we replace the constants by $C \log N$ to the right and $C(\log N)^{-1}$ to the left. It appears that, a part from some trivial cases (e. g. lacunary sequences $\{n_k\}$, $n_k \equiv 1 \pmod{4}$ etc), this is all that we know about the constants in (5.5). For some related problems concerning (5.5) with cosine and sine polynomials having positive coefficients (not necessarily equal to 1) see ([20]) and ([27], p. 11).

We note in passing that if (5.5) were true then we could conclude from (4.10) that $M > C\dfrac{\log N}{\log\log N}$.

3. ONE SIDED L^∞ NORMS OF COSINE SUMS.

Salem's theorem and (4.3) imply that if $0 < n_1 < n_2 < \ldots$ and $n_k < k^a$ then there are infinitely many partial sums $S_N(x) = \cos n_1x+\ldots+\cos n_Nx$ such that $M(S_N) > C(\log N)^{1/2}$. Much better results can be obtained if we assume in addition that a is close enough to 1. This will be a consequence of the following

THEOREM 5.3. Let $S_k(x) = b_0 + b_1\cos x+\ldots+b_k\cos kx$, $k = 1, 2, \ldots, N$, and assume that $b_k \geq 0$ and $S_k(x) \geq 0$ fo all k. There is an absolute constant a_0 (= 0,308...) such that for every $d \in (0, a_0)$ we have

(5.6) $$\sum_{k=1}^{N} \frac{b_k}{n_k^{1-d}} \leq \frac{c}{a_0 - d}\, b_0.$$

Theorem 5.3 is due to Selberg ([6]; see also [2] and [4]).

Fourier series with positive partial sums have been examined, from a different point of view, by other authors (see e. g. [40], XIII, § 4). Selberg's proof is based on an interesting argument which has also been used in order to characterize those series of the form $\sum_{n=1}^{\infty} \frac{1}{n^b} \cos nx$ whose partial sums are uniformly bounded below ([40], IV, 2).

We take theorem 5.3 temporarily for granted. Let $\{n_k\}$ be the sequence defined at the begining of this section and let us apply (5.6) with $b_o = M = \sup_{k=1,\dots N} M(S_k)$ (so that the hypothesis of the theorem is satisfied) and $b_n = 1$ if $n \in \{n_1, \dots, n_N\}$, $b_n = 0$ otherwise. We obtain

(5.7)
$$\begin{aligned} M &\geq C(a_o - d) \sum_{k=1}^{N} \frac{1}{n_k^{1-d}} \\ &\geq C(a_o - d) \sum_{k=1}^{N} \frac{1}{K^{a(1-d)}} \\ &\geq \frac{C(a_o - d)}{1-a(1-d)} N^{1-a(1-d)}. \end{aligned}$$

Thus the supremum of $M(S_k)$, $k = 1, \dots, N$, exceeds $N^{1-\alpha(1-d)}$. We observe that the exponent $1-a(1-d)$ can be equal to any number less than a_o, provided that a is close enough to 1.

<u>Proof of theorem 5.3</u>. We define first the number a_o. We write

$$q(t) = \int_o^{3\pi/2} \frac{\cos x}{x^t} \, dx, \quad 0 \leq t \leq 1$$

and observe that $q(0) < 0$, $q(1) = \infty$ We shall show that $q'(t) < 0$, $0 \leq t < 1$, and so q will have exactly one root a_o such that $0 < a_o < 1$. Moreover it will be clear that if $0 < t < a_o$ then $|q'(t)| < C$. Observing that $\frac{\cos x}{x^t}$ is a decreasing function of x in $[0, \frac{\pi}{2}]$ we obtain

$$q'(x) = \int_o^{3\pi/2} \log\frac{1}{x}\,\frac{\cos x}{x^t} \geqslant \int_o^{\pi/2} \log\frac{1}{x} = \left(\int_o^1 + \int_1^{\pi/2}\right)\log\frac{1}{x}\,\frac{\cos x}{x^t} \geq$$

$$\geq \cos 1 \left\{\int_o^1 \log\frac{1}{x} - \int_1^{\pi/2} \log x\right\} > \cos 1\left\{1 - (\tfrac{\pi}{2} - 1)\right\} \geqslant 0.$$

We observe that if $0 < d < a_o$ then $q(d) < 0$.

We define now the functions $h_o, h_1, \dots, h_N$ on $\left[0 , \frac{3\pi}{2}\right]$ by setting :

$h_o = 0$,

$h_N(x) = x^{-d}$ on $\left[0 , \frac{3\pi}{2N}\right]$ and zero elsewhere,

$h_k(x) = x^{-d}$ on $\left[\frac{3\pi}{2(k+1)} , \frac{3\pi}{2k}\right]$ and zero elsewhere.

Our hypotheses imply

$$(5.8) \qquad 0 \leq S_o h_o + S_1 h_1 + \dots + S_N h_N = b_o(h_1 + \dots + h_N) + \sum_{m=1}^{N} b_m(h_m + \dots + h_N).$$

Integrating (5.8) over $\left[0 , \frac{3\pi}{2}\right]$ we obtain

$$0 \leq b_o \int_o^{3\pi/2} \frac{dx}{x^d} + \sum_{m=1}^{N} b_m \int_o^{3\pi/2m} \frac{\cos mx}{x^d}\,dx$$

$$= C\, b_o + \left(\sum_{m=1}^{N} \frac{b_m}{m^{1-d}}\right) q(d)$$

from which (5.6) follows immediately.

REMARK. If $d \leq 0$ then (5.8) is still valid, provided that we remove the factor $\frac{1}{a_o - d}$ from the second member.

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