FOURIER SERIES AND WAVELETS

Jean-Pierre Kahane
Pierre Gilles Lemarié-Rieusset

PART II. - WAVELETS

FOURIER SERIES AND WAVELETS

Jean-Pierre Kahane
Pierre Gilles Lemarié-Rieusset

PART II. - WAVELETS

CONTENTS

PART II. - WAVELETS

Preface

Like the others in this collection, the present book has different aspects : history, classical mathematics, contemporary mathematics. The period of time extends from 1807, when Fourier wrote his first memoir on the Analytic Theory of Heat, to 1994 and the last developments on wavelets. The work is divided into two parts. The first, written by Jean-Pierre Kahane, deals with Fourier series in the classical sense, decomposition of a function into harmonic components. The second, by Pierre-Gilles Lemarié-Rieusset, expounds the modern theory of wavelets, the most recent tool in pure and applied harmonic analysis. There is an interplay between these two topics. Some common features appear in their history, their linkage with physics and numerical computation, their role and impact in mathematics. Since the first part is more classical, emphasis was put on the historical aspect ; how problems appear and move in the course of time. The history being shorter in the second part, the purely mathematical exposition - including original contributions - plays a central role.

From the Fourier point of view mathematical analysis originates from the study of Nature and expresses natural laws in the most general and powerful way. At first, Fourier series are a general method, including a good numerical algorithm, to describe and to compute the functions which occur in the heat diffusion and equilibrium. Then they become an interesting object by themselves and the germ of new theories, developed by the followers of Fourier. In succession we see Dirichlet and the convergence problem, Riemann and real analysis, Cantor and set theory, Lebesgue and functional analysis, probabilistic methods, algebraic structures. Classical Fourier series are still a seminal branch of modern mathematics, as well as a tool of constant use by physicists and engineers. The fast Fourier transform extended this use enormously in the past thirty years.

Interaction with physics and construction of efficient algorithms for numerical computation, which appear in Fourier series from the very beginning, are also at the heart of wavelet theory. Here the initiators were engineers and physicists, and mathematicians came later. But in no time wavelets became a unifying language and method outside and inside mathematics. Now they play a decisive role in the new network which expands between mathematical analysis, theoretical physics, signal analysis, image analysis, telecommunications, fast methods of computation, thanks to which new applications were found for purely mathematical theories.

The book is meant to give an idea of these movements as well as solid information on Fourier series and the state of the art about wavelets. On these matters the authors have personal experience and personal views. This is clear in the choice of the original papers by Fourier, Dirichlet, Riemann, Cantor, reproduced in the first part of the book, as in the choice and treatment of purely mathematical questions, both in the first and second parts.

The authors are grateful to a number of colleagues and collaborators for their help in scientific, linguistic or bibliographic matters, among others Fan Ai-hua, Olivier Gebuhrer, Monique Hakim, Geoffrey Howson, Lee Lorch, Yves Meyer, Hélène Nocton, Hervé Queffélec, Jean-Bernard Robert, Jan Stegeman, Guido Weiss.

Josette Dumas had to convert our handwritten manuscripts into a real book. If the reader appreciates the presentation of our work the merit belongs to her.

The figures of Part II haven been drawn with help of MICRONDE, a software developed at Orsay by Y. and M. Misiti, G. Oppenheim and J. M. Poggi as a preliminary version of a MATLAB wavelet toolbox.

PART II

WAVELETS

Pierre-Gilles Lemarié-Rieusset

Mathematics
University of Paris-Sud
Orsay (France)

Chapter 0

WAVELETS : A BRIEF HISTORICAL ACCOUNT

Before proceeding in the following chapters to a detailed mathematical presentation of wavelet theory, we begin with a historical sketch of the how and why of wavelets.

Wavelets have a very short history ; they appeared some ten years ago and from the very beginning they became a popular and promising tool for various scientific applications. Ten years later, a striking feature of wavelet history is the following point : as we shall see, most of wavelet theory (according both to theoretical ideas and to actual applications) has been known for long before its appearance as a new mathematical tool. The reason why wavelets became so popular is therefore not that much the novelty of the theory itself, but the easy way it gave to the unification of former ideas which were used in a wide variety of scientific fields (as e.g. Littlewood-Paley decomposition in real harmonic analysis, coherent states expansion in constructive quantum field theory, pyramidal algorithms and multiscale filtering in computer vision, subband coding schemes in signal processing, refinement schemes in computer-aided design, and so on). This unification has its own dynamics, allowing scientific cooperation between searchers from very various fields and modifying deeply the scientific perception of many issues in each of these fields. An excellent introduction to this aspect of wavelet theory is the little book by Y. Meyer *"Ondelettes : algorithmes et applications"* [MEY7], and his recent survey paper in the Bulletin of AMS [MEY8].

1. Jean Morlet and the beginning of wavelet theory (1982).

In the beginning of the 1980's, J. Morlet developped a new time-frequency analysis, using what he called "wavelets of constant shape".

Time-frequency analysis has always been a challenge in signal processing. In the study of transient signals, which are evolving in time in an unpredictible way, the notion of frequency analysis can only be local in time. Functions cannot be represented any more as a superposition of waves (i.e. of sinusoids with infinite duration), but as a superposition of wavelets (waves of short duration).

A strong limitation for the development of a time-frequency analysis is the fact that a wavelet cannot have a frequential dispersion and a time duration both arbitrarily small. This is related to Heisenberg's inequality and thus often presented as an *uncertainty principle* in books on signal processing. This limitation is expressed by the inequality

$$\sigma_t \sigma_\xi \geq \frac{1}{2}$$

where σ_t is the time dispersion of ψ :

$$\sigma_t = \inf_{t_0} \left(\frac{\int (t-t_0)^2 \mid \psi(t) \mid^2 dt}{\int \mid \psi(t) \mid^2 dt} \right)^{1/2}$$

and σ_ξ its frequency dispersion (defined by the same formulas as σ_t, where $\psi(t)$ is replaced by its Fourier transform $\hat{\psi}(\xi) = \int e^{-iu\xi}\psi(u)du$, and t by ξ). The minimum value is attained only for Gaussian functions

$$\psi = Ce^{i\omega t}e^{-\lambda(t-t_0)^2}.$$

[Note that $\sigma_t\sigma_\xi$ is invariant under the three following operations :

- shift in time : $\psi \to \psi(t-t_0)$
- modulation (i.e. shift in frequency) : $\psi \to e^{i\omega t}\psi(t)$
- dilation : $\psi \to \psi(\lambda t) \quad (\lambda > 0)$].

In 1946, D. Gabor introduced a time-frequency analysis, which is often called the *short-time Fourier transform.* The idea is to use a window function g in order to localize the Fourier analysis, then to shift the window to another position, and so on. The striking features of this analysis are the *local aspect* of the Fourier analysis (with resolution in time equal to the size of the window) and the fact that it deals only with a *discrete* set of coefficients (which allows numerical processing of those coefficients). In Gabor's formalism, one analyses the signal $f(t)$ by help of the window $g(t)$ and compute the coefficients

$$C(m,n) = \int f(t)g(t-nt_0)e^{-im\omega_0 t}dt, \quad m \in \mathbb{Z},\ n \in \mathbb{Z}. \tag{1}$$

It means that we have localized f around nt_0 with help of the window $g(t-nt_0)$ (where t_0 is roughly the "size" of the window), and then computed the Fourier coefficients of our localized $f(t)g(t-nt_0)$ corresponding to the frequencies $m\omega_0$ (where the wavelength $\frac{2\pi}{\omega_0}$ corresponds to the size of the window). If we want the representation to be complete, it is necessary to choose $t_0\omega_0 \leq 2\pi$. Gabor's choice for the window function g was the Gaussian function $g(t) = 2^{\frac{1}{4}}e^{-\pi t^2}$ with $t_0 = 1$ and $\omega_0 = 2\pi$; the choice of the Gaussian function was motivated by the desire of minimizing the joint resolution $\sigma_t\sigma_\xi$. Other windows have been proposed in signal processing, mainly compactly supported and regular windows (the regularity is required in order to avoid the creation of important high frequencies in the signal $f(t)g(t-nt_0)$).

Another very popular tool for time-frequency analysis is the so-called Wigner-Ville transform :

$$W_f(t,\omega) = \int f\left(t+\frac{s}{2}\right)\bar{f}\left(t-\frac{s}{2}\right)e^{-i\omega s}ds. \tag{2}$$

Though not positive-valued, the Wigner-Ville transform is often looked at as a density in the time-frequency plane, since it satisfies

$$\int W_f(t,\xi)dt = \mid \hat{f}(\xi) \mid^2 \quad \text{and} \quad \int W_f(t,\xi)\frac{d\xi}{2\pi} = \mid f(t) \mid^2 .$$

But the non-linearity of the transform creates interactions between distant times or distant frequencies which makes this "density" very delicate to use. However, the Wigner-Ville transform has remained since its creation by J. Ville in 1947 [VIL] a very popular tool in signal analysis (and seemingly a very efficient one for short duration signals).

In 1982, J. Morlet modified the Gabor wavelets in order to study the layering of sediments in the geophysics of oil exploration. He was a French engineer, a former student of the Ecole Polytechnique, and was working for the French oil company *Elf Aquitaine.* The problem he was working on was the following one : one generates acoustic waves at the surface of the earth and records then the reflected waves ; in the data which are thus received, one tries to find the influence of each sediment layer, which can be found by the instantaneous frequency of the reflected waves (since some waves are trapped inside the layer and some not). Since there are different layers, one seeks to determine different instantaneous frequencies. Morlet tried to use the Gabor wavelets, but he was dissatisfied for many reasons : the Gabor wavelets oscillate too much at high frequencies (leading to significant numerical instability in the computation of the coefficients) and too little at low frequencies, while they don't allow a practical reconstruction formula.

Then Morlet had the idea of using dilation instead of modulation. He decided to use analytic signals $F(t)$ instead of real valued $f(t)$ (F is related to f by $\hat{F}(\xi) = \hat{f}(\xi)$ if $\xi > 0$ and 0 if $\xi < 0$) ; if f belongs to L^2 then F belongs to the *Hardy space* H^2 of complex valued square integrable functions F such that $\operatorname{Supp} \hat{F} \subset [0, +\infty]$; this is an usual tool for defining the instantaneous frequency (if $F(t) = A(t)e^{i\varphi(t)}$, A being the modulus and φ the phase of $F(t)$, then the instantaneous frequency is defined as $\frac{d\varphi}{dt}$). Morlet worked with the wavelet ψ defined as $\hat{\psi}(\xi) = \xi^2 e^{-\xi^2/2}$ if $\xi > 0$ (which is the analytic signal related to the second derivative of the Gaussian) or as $\hat{\psi}(\xi) = \frac{1}{\xi_0^2}\left(-e^{-(\xi-\xi_0)^2/2} + e^{-\xi_0^2/2}e^{-\xi^2/2}\right)$ for $\xi > 0$ (related to the difference of Gaussian functions $\frac{1}{\xi_0^2}\Re e(-e^{-i\xi_0 x} + e^{-\xi_0^2/2})e^{-x^2/2}$, a good approximation to the second derivative $(x^2-1)e^{-x^2/2}$ of $e^{-x^2/2}$ if ξ_0 is near to 5) ; for practical computations, $e^{-\xi_0^2/2}$ is neglictible and thus the Morlet wavelet can be seen as a modulated Gaussian function. Then Morlet decided to filter the signal $F(t)$ by help of the filters $\hat{\psi}(a_0^m \xi)$ $(m \in \mathbb{Z})$:

$$F(t) \to F_m(t) = \int F(t-s) a_0^{-m} \bar{\psi}(-a_0^{-m} s)\, ds.$$

The operator $F \to (F_m)_{m\in\mathbb{Z}}$ could be seen as a filtering with a filter bank of band-pass filters $\hat{\psi}(a_0^m \xi)$ with constant quality factor : the filters kills nearly all the frequencies outside of a band $[\omega_m - \frac{\lambda_m}{2}, \omega_m + \frac{\lambda_m}{2}]$ where $\omega_m = \frac{\omega_0}{a_0^m}$, $\lambda_m = \frac{\lambda_0}{a_0^m}$ so that the ratio $\frac{\lambda_m}{\omega_m}$ is constant. Moreover, the key observation of J. Morlet was that for small enough a_0 the quantity $\sum_{m\in\mathbb{Z}} \mid \hat{\psi}(a_0^m \xi) \mid^2$ was near to be constant : indeed if we define ρ as $\rho = \frac{\sup_{\xi>0} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m \xi)|^2}{\inf_{\xi>0} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m \xi)|^2}$, then for $a_0 = 2$ (a convenient value for computer processing) $\rho \leq 1.1$ in case of the second derivative of Gaussian and $\rho \leq 1.3$ for the difference of Gaussian (with $\xi_0 \approx 5.336$). Those values of ρ allowed stable and fast reconstruction algorithms of F from F_m's, even for $a_0 = 2$.

The second idea of J. Morlet was the way how to get a discrete sample (for numerical processing) providing a good representation of the $F_m(t)$. Since $\hat{\psi}(2^m\xi)$ was to be seen as a band-pass filter with band-width λ_m proportional to $\frac{1}{2^m}$, Morlet decided (according to Shannon's rule for the sampling of band-limited signals) to sample $F_m(t)$ on a regular grid with a mesh proportional to 2^m. He found experimentally that the mesh $b_m = 2^m$ was small enough to allow still a good reconstruction for the choices of his wavelets, so that he dealt finally with the coefficients

$$C(m,n) = F_m(n2^m) =< F(t) \mid 2^{-m}\psi(2^{-m}t - n) > . \tag{3}$$

The wavelet transform was born.

2. Alex Grossmann and the Marseille team (1984).

In 1982, J. Morlet had already constructed his wavelet transform, which he found to have remarkable numerical efficiency. But he wondered to which extent this transform was mathematically correct, and in order to answer this question he began to work with Alex Grossmann, a theoretical physicist working at the *Centre de Physique Theorique* (C.P.T.), a C.N.R.S. laboratory located in Marseille.

A. Grossmann and his co-workers (as e.g. Thierry Paul and Ingrid Daubechies) related the Morlet wavelet transform to the theory of coherent states in quantum physics. Instead of looking only at a discrete set of dilations, they introduced the so-called *continuous wavelet transform* [GROS] where all dilations are to be considered :

$$f \to C(a,b) =< f \mid \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a}) >, \ a > 0, \ b \in \mathbb{R}.$$

Now, the mapping $U_{a,b} : \psi \to \frac{1}{\sqrt{a}} \cdot \psi(\frac{x-b}{a})$ is unitary on H^2 and $(a,b) \to U_{a,b}$ is an irreducible representation of the affine group $ax+b$ into $\mathcal{L}(H^2,H^2)$. Then it was easy to see that there was an unbounded operator A on H^2 such that for all $\psi \in \mathcal{D}(A)$ and all $f_1, f_2 \in H^2$

$$\int_G < f_1 \mid U_{a,b}(\psi) >< U_{a,b}(\psi) \mid f_2 > d\mu(a,b) =< A\psi \mid \psi >< f_1 \mid f_2 > \tag{4}$$

where G is the affine group $ax+b$ and $d\mu$ is the Haar measure on $G : d\mu = \frac{da}{a^2}db$. [Of course, formula (4) holds for a more general class of representations of groups ; it allows the decomposition of $f \in H$ as a superposition $f = \frac{1}{<A\psi|\psi>} \int_G < f \mid U_g(\psi) > U_g(\psi) d\mu(g)$ of elementary "states" $U_g(\psi)$, the so-called *coherent states*]. In the $ax+b$ case, the operator A is defined by $\widehat{Af} = \mid \xi \mid^{-1} \hat{f}$, and ψ satisfies (4) with a finite $< A\psi \mid \psi >$ if and only if ψ satisfies the *admissibility condition :*

$$\int_0^\infty \mid \hat{\psi}(\xi) \mid^2 \frac{d\xi}{\xi} < +\infty.$$

Thus, the Morlet transform appeared to be a sampling of the coherent states expansion $f \to < f \mid U_{a,b}(\psi) >$, and a way to compute back f from this sampling. If coherent states for the affine group were not new (they had been introduced in 1968 by Aslaksen and Klauder [ASL]), the efficient algorithmic reconstruction formulas of J. Morlet were however both new and far-reaching.

The second important idea introduced by A. Grossmann was that the problem of stable reconstruction of f from its discrete wavelet coefficients (as processed in the Morlet algorithm) was to be related to the theory of *frames*. The notion of a frame was introduced in 1952 by Duffin and Schaeffer for the study of non-harmonic Fourier series [DUF]. They dealt with the problem of non-uniform sampling for band-limited functions : if f satisfies $\mathrm{Supp}\,\hat{f} \subset [-\gamma, \gamma]$ and $f \in L^2$, when can we have stable reconstruction from a sample $(f(\lambda_n))_{n\in\mathbb{Z}}$? They showed that this problem was equivalent to the existence of two positive constants A such that for all $f \in L^2([-\gamma,\gamma])$

$$A\int_{-\gamma}^{\gamma} \mid f(t) \mid^2 dt \leq \sum_{n\in\mathbb{Z}} \left| \int_{-\gamma}^{\gamma} f(t)e^{+i\lambda_n t}dt \right|^2 \leq B\int_{-\gamma}^{\gamma} \mid f(t) \mid^2 dt \ ;$$

then they defined a frame in a Hilbert space H as a familly $(e_n)_{n\in\mathbb{Z}}$ such that for some $A, B > 0$ and for all $f \in H \ : A \parallel f \parallel_H^2 \leq \sum_{\mathbb{Z}} |< f \mid e_n >|^2 \leq B \parallel f \parallel_H^2$. At that point, the problem of stable reconstruction of f from its wavelet coefficients $\left\langle f \mid \frac{1}{\sqrt{a_0^m}}\psi\left(\frac{x-nb_0a_0^m}{a_0^m}\right)\right\rangle$ was reformulated as the problem of estimating the quantity $\sum_m \sum_n \left|\left\langle f \mid \frac{1}{\sqrt{a_0^m}}\psi\left(\frac{x-nb_0a_0^m}{a_0^m}\right)\right\rangle\right|^2$. This could be done successfully by I. Daubechies in 1986, which could even compute numerical estimates for the frame bounds A, B for a large variety of wavelets ψ [DAU2].

This cooperation between J. Morlet and the Marseille University went much further than reshaping the Morlet wavelet transform in the language of coherent states and establishing its mathematical safety. Physicists understood very quickly that the Morlet algorithm, decomposing a function on the whole family of scales, could be an efficient tool for multiscale analysis. The need for such an analysis emerged in the 1970's and the 80's in the field of non-linear physics, where chaotic dynamical systems lead to the study of the multifractal structure of strange attractors (see [ARN] for an introduction to the case of wavelets in this setting). Around Morlet and Grossmann, a network of scientists from various fields began to explorate the use of wavelets as a multiscale tool ; among them, we may quote Richard Kronland-Martinet, working on acoustics at the *Laboratoire de mécanique et d'acoustique* (Marseille), who wrote a software for wavelet analysis which has been extensively used, Marie Farge, working on turbulence at the *Laboratoire de météorologie dynamique* (Paris), and Alain Arneodo, working on critical phenomena at phase transition at the *Centre de Recherche Paul Pascal* (Bordeaux).

Marseille has remained an important center for wavelet theory, with an important financial support of the *C.N.R.S.*, organizing two international conferences on wavelets in 1987 and 1989 [COM], [MEY6] and regular workshops at the *C.I.R.M. (Centre International de Rencontres Mathématiques)*. One of the most impressive achievements of the

Marseille team is the development of the ridge-skeleton algorithm, a method for extracting the modulation laws of asymptotic signals [TCH]. In the meantime, A. Arneodo and his co-workers have developped an impressive program for proving the hypothesis of Frisch and Parisi (1984) on the multifractal structure of the singularities developped in turbulent fluids. In 1989, they published in *Nature* a paper entitled "Wavelet analysis of turbulence reveals the multifractal nature of the Richardson cascade", [ARG], and now are developping the numerical setting for a wavelet-based calculus of the spectrum of singularities of the turbulent signals [BAC], [JAF4].

3. Yves Meyer and the triumph of harmonic analysis (1985).

In 1985, wavelet analysis entered the mathematical field. Y. Meyer was teaching mathematics at the Ecole Polytechnique, and his researches were centered on the theory of Calderón-Zygmund operators, a generalization of a class of operators studied by A. Calderón and A. Zygmund in the late 50's. The class studied by Calderón and Zygmund can be roughly described as (in dimension d) convolution operators with distributions which are homogeneous of exponent $-d$; equivalently, we may describe them as operators commuting with dilations and translations. Jean Lascoux, who was chairman of the *Centre de physique théorique* at the *Ecole polytechnique,* then drew the attention of Y. Meyer to the wavelet transform of A. Grossmann and J. Morlet, an analysis which is itself invariant under dilations and translations.

Y. Meyer understood immediately the deep connection between his sophisticated mathematics and the algorithm developped by J. Morlet. The formula

$$(5) \qquad f = \frac{1}{\int_0^{+\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi}} \int\int < f \mid \psi_{a,b} > \psi_{a,b} \frac{da}{a^2} db$$

was a well-known formula in real harmonic analysis, the so-called Calderón's resolution of identity introduced in 1964 in the context of Banach space interpolation [CAL] and extensively used for providing "atomic decompositions" in Banach spaces of distributions. The striking novelty, however, was the vindication that this formula led to efficient algorithms in order to explore the multifractal structure of nature. Meyer was deeply impressed and, according to his own words, "took the first train to Marseille."

Y. Meyer was the right man at the right time. His mathematical knowledge, and his ascertained acquaintance with Calderón's formula, allowed him to give a mathematical foundation to wavelet theory, mainly based on the Littlewood-Paley-Stein theory of spaces of regular functions. This theory is presented in the books of E. Stein [STE1], [STE2] who developped it in the 1950's and the 60's, using as its main tools the Littlewood-Paley decomposition of functions (generalized from the work of Littlewood and Paley in the 30's), the Lusin area integral (another variant of the wavelet decomposition formula (5) using the wavelet $\frac{1}{(x+i)^2}$), the Calderón-Zygmund convolution operators and covering lemmas for open sets using dyadic cubes (those lemmas are connected to dyadic martingales and to the Haar basis (1909)).

The first contribution of Y. Meyer (with Daubechies and Grossmann) was the construction of a "painless" wavelet decomposition : he constructed a (real-valued) wavelet ψ

such that :

$$\forall f \in L^2(\mathbb{R}), \quad f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} < f \mid \psi_{j,k} > \psi_{j,k} \tag{6}$$

where $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$, $\hat{\psi} \in C_c^\infty$ and $0 \notin \mathrm{Supp}\,\psi$ [DAUG]. This is easily done by adapting the Littlewood-Paley-Stein decomposition $\hat{f} = \sum_{j\in\mathbb{Z}} \widehat{\Delta_j f} = \sum_{j\in\mathbb{Z}} \hat{\omega}(\frac{\xi}{2^j})\hat{f}(\xi)$ where $\hat{\omega} = \mid \hat{\psi} \mid^2$ and $\sum \hat{\omega}(\frac{\xi}{2^j}) = 1$. Such a construction had been independently explored by Frazier and Jawerth under the name of the φ-transform [FRJ] in the context of harmonic analysis.

In this construction, the frame $(\psi_{j,k})$ was a redundant system of analyzing functions. Y. Meyer believed this redundancy to be unavoidable (by analogy with the Balian-Low uncertainty theorem for Fourier windows [BAL]) and he tried to prove that there was no Hilbertian basis $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with ψ localized in space and frequency. Then he was very much astonished to discover such a basis, with $\hat{\psi} \in C_c^\infty$ and $0 \notin \mathrm{Supp}\,\psi$: ψ was as regular, oscillating and localized as possible ! (He was very lucky in the finding of his basis : if he had tried to find a basis of $H^2(\mathbb{R})$ instead of $L^2(\mathbb{R})$ or a basis with another integer dilation factor than 2, he would have failed : there is no such basis with $\psi \in \mathcal{S}$!). Using the machinery of Calderón-Zygmund operators and of Littlewood-Paley theory, he was able to prove that his basis was an unconditional basis for several classes of Banach spaces used in harmonic analysis. He exposed his construction at the *Colloque Peccot* organized in September 1985 at the *Collège de France* and at the *Séminaire Bourbaki* in February 1986. Thus, a large echo was given to wavelet theory among mathematicians.

This connection between wavelets and functional analysis has been very important, not only for mathematical foundation of the theory, but also for applications. The 2-microlocalization of J. M. Bony, a tool introduced for the study of the propagation of singularities in non-linear PDE's, has been turned in an efficient tool for the analysis of the chirps in signal processing or of the multi-fractal singularities in turbulence (S. Jaffard [JAF4], Y. Meyer [JAM]). The analysis of Calderón-Zygmund operators has been turned in an efficient algorithm for the calculus of the matrix of parametrices of differential operators (G. Beylkin, R. Coifman and V. Rokhlin [BEY1]). The scale of Besov spaces, which was mainly used in approximation theory or in PDE's, has been introduced in statistics as a convenient tool for estimation or compression problems (D. Donoho [DON1]).

From 1985 up to now, Yves Meyer (who moved to the *Université Paris IX)* and Ronald Coifman (Yale University) accompanied all developments of wavelet theory and applications in every direction, and their strong commitment to meet scientists from various fields and countries and discuss with them about the specificical issues of their field surely played a great rôle in the rapidity of the diffusion of wavelet theory among scientists.

4. Stéphane Mallat and the fast wavelet transform (1986).

In September 1985, Y. Meyer tried to extend his construction of a wavelet basis to a multi-dimensional setting, in order to have a nice and complete theory established before his talk at the *Colloque Peccot* at the end of the month. Guided by the analogy with the Haar basis (in dimension 1, the basis is given by $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ with $\psi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$ and in dimension d by $(\psi_{j,k,\epsilon} = 2^{jk/2}\psi_\epsilon(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ where $\epsilon \in \{0,1\}^d \backslash \{(0,\cdots,0)\}$, $\psi_\epsilon = \psi_{(\epsilon_1)} \otimes \cdots \otimes \psi_{(\epsilon_d)}$, $\psi_{(1)} = \psi$ and $\psi_{(0)} = \varphi = \chi_{[0,1]}$), he postulated the existence of a function φ completing his wavelet ψ. This function φ was constructed by P. G. Lemarié and the analysis described by the new system was reformulated by R. Coifman and Y. Meyer as a double family of orthogonal projection operators : in dimension 1, the operators P_j are given by :

$$P_j f = \sum_{k\in\mathbb{Z}} < f \mid \varphi_{j,k} > \varphi_{j,k} \qquad (\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)) \tag{7}$$

and the operators Q_j by

$$Q_j f = \sum_{k\in\mathbb{Z}} < f \mid \psi_{j,k} > \psi_{j,k} \qquad (\psi_{j,k} = 2^{j/2}\psi(2^j x - k)) \ ; \tag{8}$$

the main relationships are then :

$$P_j f \to 0 \quad \text{as} \quad j \to -\infty, \qquad P_j f \to f \quad \text{as} \quad j \to +\infty$$

$$Q_j = P_{j+1} - P_j, \quad \text{so that} \quad f = P_0 f + \sum_{j=0}^{+\infty} Q_j f = \sum_{j\in\mathbb{Z}} Q_j f.$$

Some months after, G. Battle (working on constructive quantum field theory) and P. G. Lemarié (a former student of Y. Meyer) announced independently the construction of spline orthonormal wavelets with exponential decay. One more time, one constructed a "mother wavelet" ψ generating an Hilbertian basis $\psi_{j,k}$ of $L^2(\mathbb{R})$ and thereafter a "father wavelet" φ allowing the extension of the construction to $\mathbb{R}^d$. The connection of wavelet theory to spline functions seems afterwise very natural : in a way, this can be described as a primitivation of the Haar basis followed by a Gram-Schmidt orthonormalization. As a matter of fact, this had already been done in 1981 by J. O. Strõmberg, who exhibited spline wavelet bases as unconditional bases for real Hardy spaces H^p $(0 < p \leq 1)$; but Strõmberg's construction had little echo outside of the circle of specialists of Hardy spaces. Five years later, the quick expansion of wavelet theory in applied mathematics gave an important echo to the Battle-Lemarié construction.

The first consequence of this sudden blooming of mother and father wavelets was produced in a seemingly remote scientific field. S. Mallat was preparing a Ph. D. in electrical engineering science in Philadelphia. He understood very quickly the structure of the underlying algorithms in these "ψ and φ" wavelet transforms. If we assume $P_0 f$ to be given and we want to compute its decomposition as $P_0 f = P_{-N} f + Q_{-N} f + Q_{-N+1} f + \cdots + Q_{-1} f$, this can be quickly performed by the cascade of decompositions

$$P_0 f = P_{-1} f + Q_{-1} f, P_{-1} f = P_{-2} f + Q_{-2} f, \cdots, P_{-N+1} f = P_{-N} f + Q_{-N} f,$$

and the decomposition

$$P_{-j}f = P_{-j-1}f + Q_{-j-1}f$$

has a very simple structure : since $P_{-j-1}f = P_{-j-1}(P_{-j}f)$, we have if we write $P_{-j}f = \sum_{k\in\mathbb{Z}} s_{j,k}\varphi_{-j,k}$

$$\begin{aligned} s_{j+1,k} &= < \sum_{\ell\in\mathbb{Z}} s_{j,\ell}\varphi_{-j,\ell} \mid \varphi_{-j-1,k} > = \sum_{\ell\in\mathbb{Z}} s_{j,\ell} < \varphi(x-\ell+2k) \mid \frac{1}{\sqrt{2}}\varphi(\frac{x}{2}) > \\ &= \sum_{\ell\in\mathbb{Z}} s_{j,\ell}h_{2k-\ell}. \end{aligned}$$

Thus the projection $P_{-j}f \to P_{-j-1}f$ was, in terms of the coefficients $(s_{j,k})$ and $(s_{j+1,k})$, a convolution followed by an undersampling. Such algorithms were already used in computer vision (as e.g. the Laplacian pyramidal algorithm of Burt and Adelson (1983) [BUR]) or in signal processing (the subband coding scheme of Esteban and Galand (1977 [EST]). Even the orthogonality of the projection operators could be formalized in former settings ; the filters used in the decomposition $P_{-j}f \to P_{-j-1}f$ and $P_{-j}f \to Q_{-j-1}f$ were quadrature mirror filters with perfect reconstruction (Smith and Barnwell (1984) [SMI]).

Thus, S. Mallat inserted the wavelet transform in a field which was at that time in a full expansion. Following the Laplacian pyramidal algorithm, he was able to describe a fast wavelet transform, which has deeply modified the algorithmic structure of the wavelet transform. Natural applications of this new algorithm were data compression in signal or image processing, edge detection or texture analysis in computer vision. Wavelets became a common tool in signal processing, and the many journals edited by the IEEE began to publish papers on wavelets (up to a special issue of the *Transactions of the IEEE on Information Theory* in 1992).

S. Mallat developped with Y. Meyer the formalism describing the sequence of projection operators P_j. This is the formalism of the so-called *Multi-resolution analysis :* P_j is the orthogonal projection operator onto a closed subspace V_j of $L^2(\mathbb{R})$, where the spaces V_j satisfy :

(i) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in L^2
(ii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
(iii) V_0 has an orthonormal basis $(\varphi(x-k))_{k\in\mathbb{Z}}$ with φ rapidly decaying at infinity ($\forall k \in \mathbb{N}$, $x^k\varphi \in L^2$).

φ is called "the" (orthonormal) *scaling function* associated to (V_j). It satisfies

$$\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} < \varphi\left(\frac{x}{2}\right) \mid \varphi(x-k) > \varphi(x-k).$$

The *scaling filter* m_0 associated to φ is

$$m_0(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} < \varphi\left(\frac{x}{2}\right) \mid \varphi(x-k) > e^{-ik\xi},$$

and "the" orthonormal wavelet ψ is given by

$$\hat{\psi}(\xi) = e^{-i\xi/2}\bar{m}_0\left(\frac{\xi}{2}+\pi\right)\hat{\varphi}\left(\frac{\xi}{2}\right).$$

5. Ingrid Daubechies and the FIR filters (1987).

Mallat's work focused the attention on the quadrature mirror filters which were used in the fast wavelet transform. In order to produce compactly supported mother and father wavelets (or, in the new language, wavelet ψ and scaling function φ), it was equivalent to seek for a quadrature mirror filter (with perfect reconstruction) with a finite impulse response ; it means to find a trigonometric polynomial m_0 such that $m_0(0) = 1$, $\mid m_0(\xi) \mid^2 + \mid m_0(\xi+\pi) \mid^2 = 1$ and such that $\prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right)\chi_{[-\pi,\pi]}\left(\frac{\xi}{2^N}\right)$ converges in L^2 to $\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$, such a condition being requested in order to obtain that φ, defined by $\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right) = \hat{\varphi}(\xi)$, inherits the property $\sum_{k\in\mathbb{Z}} \mid \hat{\varphi}(\xi+2k) \mid^2 = 1$ [i.e. $<\varphi(x) \mid \varphi(x-k)> = \delta_{k,0}$] from the partial products $\prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right)\chi_{[-\pi,\pi]}\left(\frac{\xi}{2^N}\right)$.

I. Daubechies showed that it was enough to deal with $\mid m_0(\xi) \mid^2$ instead of m_0 (and then use the Riesz lemma on the "polynomial square root" of non-negative-valued trigonometric polynomials), and that the so-called maximally flat filters of O. Herrmann (introduced in 1970 [HEM]) could be taken as such filters $\mid m_0(\xi) \mid^2$. In that case, we have

$$m_1(\xi) = \mid m_0(\xi) \mid^2 = \sum_{k=0}^{N-1} \binom{2N-1}{k} \left(\frac{1+\cos\xi}{2}\right)^{2N-1-k} \left(\frac{1-\cos\xi}{2}\right)^{k} ;$$

it is a polynomial of degree $2N-1$ such that

$$m_1(\pi) = \frac{\partial}{\partial\xi}m_1(\pi) = \cdots = \left(\frac{\partial}{\partial\xi}\right)^{N-1} m_1(\pi) = 0,$$

while $m_1(0) = 1$, $\frac{\partial}{\partial\xi}m_1(0) = \cdots = \left(\frac{\partial}{\partial\xi}\right)^{N-1} m_1(0) = 0$ (this flatness of m_1 at π and at 0 up to order N characterizes m_1 among all trigonometric polynomials of degree $2N-1$). Hermann's filters were intended to mimick the ideal cut-off filter $m_1(\xi) = 0$ if $\frac{\pi}{2} < \mid \xi \mid \leq \pi$, 1 if $\mid \xi \mid < \frac{\pi}{2}$, $\frac{1}{2}$ if $\mid \xi \mid = \frac{\pi}{2}$. They were designed in the purpose of signal processing, and the fact that $m_1(\xi) + m_1(\xi+\pi) = 1$ was in a way accidental (Hermann introduced indeed a more general class, in order to mimick other ideal cut-off filters such as $m_1(\xi) = 1$ if $\mid \xi \mid < c_0$, 0 if $c_0 < \mid \xi \mid \leq \pi$).

I. Daubechies showed that this choice of $\mid m_0(\xi) \mid^2$ of degree $2N-1$ leads to a compactly supported orthonormal scaling function φ_N of support[0,2N−1] and whose regularity was increasing linearly with N ($\varphi_N \in C^{\alpha_N}$ with $\alpha_N \sim \left(1 - \frac{\mathrm{Log}\,3}{2\,\mathrm{Log}\,2}\right)N$ when $N \to +\infty$). Moroever the flatness of $\mid m_0(\xi) \mid^2$ at π could be rewritten in terms of the celebrated Strang-Fix conditions in numerical analysis : every polynomial $P(x)$ of degree $\leq N-1$ could be written as

$$P(x) = \sum_{k \in \mathbb{Z}} < P \mid \varphi_N(x-k) > \varphi_N(x-k).$$

Her paper presenting her construction was published in 1988 in the journal *Communications on Pure and Applied Mathematics* [DAU1] and has remained the basic reference about wavelets for most numerical applications (since it provided tables of coefficients of the filters for the first values of N and presented an algorithm to plot the scaling function φ_N). I. Daubechies, who began as a theoretical physicist sharing her time between Belgium and Marseille, joined the AT&T Bell Laboratories at Murray Hill (New Jersey) and developped the application of compactly supported scaling functions to signal or image processing. The Daubechies functions are now intensively studied through the world and enter the family of special functions (with quadrature formulas, algorithms for tabulating the functions and their derivatives or their primitives, and so on ...), and a special section of the celebrated book of mathematical routines *Numerical recipes* has been devoted to the wavelet transform.

Thus, between 1982 and 1988, wavelet theory was deeply and constantly modified by its interaction with more and more scientific fields, and each time the focus was shifted from one field to another one connections were found with most recent and promising developments in the new field : the multifractal hypothesis of Frisch and Parisi was published in 1984, the Calderón-Zygmund operator theory has been deeply renewed and illuminated by the David-Journé theorem in 1983, the Laplacian pyramid of Burt and Adelson appeared in 1983, the quadrature mirror filters with perfect reconstruction were introduced in 1984. Wavelet theory doesn't unify these theories or algorithms, which have their intrinsic motivations and applications, but it gives them a common language, which allows quick interactions between pure and applied mathematicians, between mathematicians and physicists, or between theoretical physicists and engineers, and so on.

The history of wavelet theory doesn't end in 1988. Much work has been done since 1988, which is exposed in the following chapters. Bi-orthogonal bases instead of orthogonal bases have been introduced in 1990 by J. C. Feauveau, in his thesis in computer science prepared at Orsay (France), and became very useful in image processing (because the filters involved in the calculus could be chosen with linear phase). Other bases were introduced, which were no more wavelet bases (in the sense of the Grossmann-Morlet-Meyer wavelets of constant shape); e.g. the wavelet packets (a full library of orthonormal bases constructed from a quadrature mirror filter and equipped with an algorithm for best basis selection) or the local sine basis of Malvar and Meyer.

Wavelet theory has become a scientific field. The third edition of the *Wavelet Literature Survey* (by Pittner, Schreid and Ueberhuber) in August 1993 contains around 1 000 bibliographical references ; the authors underline the exponential increase of publications on wavelets, the number of which doubles each year. A journal devoted to wavelets has been founded in 1993 (*Applied Computational Harmonic Analysis)*, while the journals of the SIAM and of the IEEE publish regularly papers on wavelets applied to numerical analysis or to electronical engineering. The *Revista de Matemática Iberoamericana* publishes most of the pure mathematical papers on wavelets. Many books have been published recently on wavelets (the Wavelet Literature Survey quotes 19 of them). A "Wavelet Digest" is diffused by e-mail (subscribe by sending an e-mail with the subject "subscribe" to wavelet at bigcheese.math.scarolina.edu).

Wavelet theory has thus encountered a very large echo. But it is still a very young theory and it is very delicate to foresee which form it will take in the future (even in the very next future) and in which field it will be really successful. We try to give in the following chapters a mathematical "state of the art" as it is now in the first half of 1994.

Chapter 1

THE NOTION OF WAVELET REPRESENTATION

The wavelet representation was introduced by Jean Morlet in the early 80's as an efficient time-frequency representation algorithm. The main difference with the Gabor representation is the fact that the wavelet representation has a more and more acute spatial resolution (and a less and less acute frequency resolution) as the frequency grows higher ; the reason of this specific stress on acute spatial resolution for high frequencies is the connection between high frequency components and existence of singularities in the analyzed signals (which is mainly a local phenomenon). Thus, the wavelet representation was introduced as an algorithm for detecting singularities (as edges in image processing or attacks in signal analysis) and has turned to be a very efficient tool in functional analysis.

In this chapter, we will introduce the integral wavelet transforms, which deal with a continuous set of analyzing functions. Discrete wavelet transforms will be discussed in the next chapter. We consider only univariate square-integrable functions (i.e. functions in $L^2(\mathbb{R})$) and postpone the study of multivariate wavelets to chapter 7.

1. Time-frequency localization and Heisenberg's inequality.

An important feature for the Fourier transform $f(x) \to \hat{f}(\xi)$, where

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx, \tag{1}$$

is the fact that the integration of the analyzed function f is performed over the whole line, so that every point contributes to the calculus of $\hat{f}(\xi)$ and that it is difficult to recover local information on f from $\hat{f}$. A very simple but enlightening example is given by the reconstruction formula :

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{ix\xi} d\xi \; ; \tag{2}$$

in this formula the integrand $\hat{f}(\xi)e^{ix\xi}$ has the same modulus for all x and the local information is therefore contained in the phase. Assume by instance that f is vanishing on a neighborhood of a point x_0 but has some irregularities for away from x_0 ; $| \hat{f}(\xi) |$ remains important for big values of ξ and the flatness of f around x_0 is due only to cancellation between big integrands of opposite signs, a property which is known to lead to highly unstable numerical computations.

This is a reason why local Fourier analysis has been developped, in order to limit this numerical unstability to the neighborhood of singularities instead of spreading it over the whole line. Moreover, the notion of instantaneous frequency, as in frequency modulated signals, requires also local Fourier transforms.

In order to catch some local Fourier information, we may use an analyzing function ψ which we require to be localized both in frequency (around some mean frequency ξ_ψ) and in time (around some mean time x_ψ). Such a function is generally called a *wavelet*, which means a wave of finite duration. As a matter of fact, due to the Plancherel formula, we have :

$$< f \mid \psi > = \frac{1}{2\pi} < \hat{f} \mid \hat{\psi} > \tag{3}$$

(where $< \cdot \mid \cdot >$ is the scalar product on $L^2(\mathbb{R})$) ; therefore, the number $< f \mid \psi >$, which is an average information about f on a neighborhood of x_ψ, is also an average information about $\hat{f}$ on a neighborhood of ξ_ψ.

If we define x_ψ and ξ_ψ as the mean values of x and ξ for the probability measures

$$\frac{1}{\| \psi \|_2^2} \mid \psi(x) \mid^2 dx \quad \text{and} \quad \frac{1}{\| \hat{\psi} \|_2^2} \mid \hat{\psi}(\xi) \mid^2 d\xi \ :$$

$$x_\psi = \int_{-\infty}^{+\infty} x \mid \psi(x) \mid^2 \frac{dx}{\| \psi \|_2^2} \quad , \quad \xi_\psi = \int_{-\infty}^{+\infty} \xi \mid \hat{\psi}(\xi) \mid^2 \frac{d\xi}{\| \hat{\psi} \|_2^2}, \tag{4}$$

a measurement of the resolution of the analyzing function ψ in time or frequency is commonly given by the quadratic deviations :

$$\begin{aligned} \Delta x_\psi &= \left(\int_{-\infty}^{+\infty} \mid x - x_\psi \mid^2 \mid \psi(x) \mid^2 \frac{dx}{\| \psi \|_2^2} \right)^{1/2}, \\ \Delta \xi_\psi &= \left(\int_{-\infty}^{+\infty} \mid \xi - \xi_\psi \mid^2 \mid \hat{\psi}(\xi) \mid^2 \frac{d\xi}{\| \hat{\psi} \|_2^2} \right)^{1/2} \end{aligned} \tag{5}$$

and the *joint resolution* is given by the product $\Delta x_\psi \cdot \Delta \xi_\psi$. This means that in the time-frequency space (or phase space) the information conveyed by ψ is located on the rectangle

$$R_\psi = \left[x_\psi - \frac{1}{2}\Delta x_\psi, x_\psi + \frac{1}{2}\Delta x_\psi \right] \times \left[\xi_\psi - \frac{1}{2}\Delta \xi_\psi, \xi_\psi + \frac{1}{2}\Delta \xi_\psi \right]$$

and the joint resolution is the area of R_ψ. [We have to use a definition of localization in terms of momenta, since ψ and $\hat{\psi}$ cannot be both compactly supported : if ψ has compact support, $\hat{\psi}$ is analytical and cannot vanish identically on an interval].

Heisenberg's inequality states that the joint resolution of ψ cannot be arbitrarily small:

THEOREM 1 *(Heisenberg's inequality). - Let $\psi \in L^2(\mathbb{R})$, $\psi \neq 0$. Then :*

$$\Delta x_\psi \Delta \xi_\psi \geq \frac{1}{2} \tag{6}$$

and the minimal value $\frac{1}{2}$ is attained only for Gaussian functions.

Proof. Define H to be the Hilbert space $\{f \in L^2 / xf \in L^2 \text{ and } \xi\hat{f} \in L^2\}$, equipped with the norm $(\| f \|_2^2 + \| xf \|_2^2 + \| \xi\hat{f} \|_2^2)^{1/2}$. Then (6) is meaningful only for $\psi \in H$ (otherwise Δx_ψ or $\Delta \xi_\psi$ is infinite). If $\psi \in H$, we may change ψ in $\psi(x+x_\psi)e^{-ix\xi_\psi}$, which doesn't change the values of Δx_ψ and $\Delta\xi_\psi$, and thus suppose $x_\psi = \xi_\psi = 0$. We have thus to show that $\| \psi \|_2 \| \hat{\psi} \|_2 \leq 2 \| x\psi \|_2 \| \xi\hat{\psi} \|_2$, or equivalently that :

$$\| \psi \|_2^2 \leq 2 \| x\psi \|_2 \| \psi' \|_2 . \tag{7}$$

If $\psi \in C_c^\infty$, this is obvious since :

$$\| \psi \|_2^2 = \int \psi\bar{\psi} dx = -\int x\frac{d}{dx}(\psi\bar{\psi})dx = -2\mathcal{R}\mathrm{e}\int x\psi\frac{d\bar{\psi}}{dx}dx$$

so that we may conclude by the Cauchy-Schwarz inequality. Moreover, we may extend inequality (7) to any $\psi \in H$ because C_c^∞ is dense in H : just pick some $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int \varphi\, dx = 1$ and $\varphi \equiv 1$ in the neighborhood of 0, then for any $f \in H$ we have :

$$\lim_{N\to+\infty} \| \{f\varphi(\frac{x}{N})\} * N\varphi(Nx) - f \|_H = 0.$$

Now (7) is an equality if and only if the Cauchy-Schwarz inequality turns to be an equality, and so if and only if the functions $x\psi$ and $\frac{d\psi}{dx}$ are homothetic ($\frac{d\psi}{dx} = \lambda x\psi$ with $\lambda \in \mathbb{R}$). It occurs if and only if $\psi = C\exp(-\lambda\frac{x^2}{2})$ for some positive λ. ∎

In some cases, it may be useful to modify the definition of the resolution of ψ in time or frequency. For instance, we may define for $\epsilon > 0$, $\Delta x_{\psi,\epsilon}$ as :

$$\Delta x_{\psi,\epsilon} = \inf_{y\in\mathbb{R}} \left(\int_{-\infty}^{+\infty} | x-y |^{2\epsilon} | \psi(x) |^2 \frac{dx}{\| \psi \|_2^2} \right)^{\frac{1}{2\epsilon}} \tag{8}$$

and similarly $\Delta\xi_{\psi,\epsilon}$. We still have an Heisenberg inequality for those modified joint resolutions :

PROPOSITION 1. - *For any positive ϵ and ϵ' there exists a positive constant $C_{\epsilon,\epsilon'}$ such that for any $\psi \in L^2(\mathbb{R})$, $\psi \neq 0$,*

$$\Delta x_{\psi,\epsilon} \cdot \Delta\xi_{\psi,\epsilon'} \geq C_{\epsilon,\epsilon'}.$$

Proof. Proposition 1 is equivalent to show the following inequality :

$$\| \psi \|_2^{\frac{1}{\epsilon}+\frac{1}{\epsilon'}} \leq \gamma_{\epsilon,\epsilon'} \| | x |^\epsilon \psi \|_2^{\frac{1}{\epsilon}} \| | \xi |^{\epsilon'} \hat{\psi} \|_2^{\frac{1}{\epsilon'}} \tag{9}$$

where $\gamma_{\epsilon,\epsilon'} = (2\pi)^{-\frac{1}{2\epsilon'}} \frac{1}{C_{\epsilon,\epsilon'}}$. We define $H_{\epsilon,\epsilon'}$ as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that f and $\hat{f}$ are locally integrable, $| x |^{\epsilon} f \in L^2$ and $| \xi |^{\epsilon'} \hat{f} \in L^2$, equipped with the norm :

$$\| f \|_{\epsilon,\epsilon'} = \| | x |^{\epsilon} f \|_2 + \| | \xi |^{\epsilon'} \hat{f} \|_2 .$$

LEMMA. - *$H_{\epsilon,\epsilon'}$ is complete.*

To prove the lemma, we consider a Cauchy sequence f_n in $H_{\epsilon,\epsilon'}$ and we want to prove that it is convergent. We first show that the sequence converges in $\mathcal{S}'(\mathbb{R})$. We pick some N such that N is bigger than ϵ and ϵ' and some $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\int \varphi\, dx = 1$, $\int x^p \varphi\, dx = 0$ for any $p > 0$. For $\omega \epsilon \mathcal{S}$, we define

$$\omega_1 = \sum_{p=0}^{N-1} x^N \frac{\varphi^{(N+p)}(x)}{(N+p)!}(-1)^{N+p} \int y^p \omega(y) dy \quad \text{and} \quad \omega_2 = \omega - \omega_1 \ ;$$

by construction we have $\frac{\omega_1}{x^N} \in \mathcal{S}(\mathbb{R})$ and (since $\int x^p \omega_2(x) dx = 0$ for $0 \leq p \leq N-1$) $\frac{\hat{\omega}_2}{\xi^N} \in \mathcal{S}(\mathbb{R})$. Now we write

$$< \omega \mid f_n > = < \omega_1 \mid f_n > + \frac{1}{2\pi} < \hat{\omega}_2 \mid \hat{f}_n > ;$$

the convergence of $| x |^{\epsilon} f_n$ in L^2 implies the convergence of $< \omega_1 \mid f_n >$ and the convergence of $| \xi |^{\epsilon'} \hat{f}_n$ in L^2 implies the convergence of $< \hat{\omega}_2 \mid \hat{f}_n >$. Thus f_n converges in $\mathcal{S}'$ to a distribution f. Clearly f is locally integrable outside of $\{0\}$ and $\| | x |^{\epsilon} | f_n - f \|_{L^2(\mathbb{R}^*)} \to 0$ as $n \to +\infty$, and similarly for $\hat{f}$; we just have to prove that f and $\hat{f}$ are indeed locally integrable on the whole line to conclude that $f \in H_{\epsilon,\epsilon'}$ and $f_n \to f$ in $H_{\epsilon,\epsilon'}$ as $n \to +\infty$. In that purpose, we write $\hat{f} = \varphi \hat{f} + (1 - \varphi)\hat{f}$ where $\varphi \in \mathcal{C}_c^\infty$ is identically 1 in a neighborhood of 0 ; the inverse Fourier transform of $\varphi \hat{f}$ is a $\mathcal{C}^\infty$ function and the inverse Fourier transform of $(1-\varphi)\hat{f}$ belongs to L^2, so that f is locally integrable. We prove in the same way the local integrability of $\hat{f}$. The lemma is then proved.

Moreover the proof of the lemma shows that any f in $H_{\epsilon,\epsilon'}$ is indeed square-integrable: we already know that f is square integrable far away from 0 and we have seen that f is locally square integrable (as a sum of a $\mathcal{C}^\infty$ function and a square integrable function). Since $H_{\epsilon,\epsilon'}$ is a Banach space, we may apply the closed graph theorem to the inclusion $H_{\epsilon,\epsilon'} \subset L^2$ to get :

$$\| f \|_2 \leq C \left(\| | x |^{\epsilon} f \|_2 + \| | \xi |^{\epsilon'} \hat{f} \|_2 \right) \tag{10}$$

for any $f \in H_{\epsilon,\epsilon'}$, where C is a positive constant depending only on ϵ and ϵ'. Applying (10) to $f_R = \frac{1}{\sqrt{R}} f(\frac{x}{R})$, we obtain :

$$\| f \|_2 \leq C \left(R^{\epsilon} \| | x |^{\epsilon} f \|_2 + R^{-\epsilon'} \| | \xi |^{\epsilon'} \hat{f} \|_2 \right)$$

for any positive R. Now we get (9) by choosing $R = \left(\frac{\| |\xi|^{\epsilon'} \hat{f} \|_2}{\| |x|^{\epsilon} f \|_2}\right)^{\frac{1}{\epsilon+\epsilon'}}$. ■

We may now define the notion of a wavelet representation :

DEFINITION 1. - *A wavelet representation is a family $(\psi_\omega)_{\omega \in \Omega}$ of functions in L^2, indexed by a measured space (Ω, μ), such that :*

(i) *each ψ_ω has norm 1 : $\int |\psi_\omega|^2 \, dx = 1$;*
(ii) *$\omega \to \psi_\omega$ is weakly measurable (i.e. for any $f \in L^2$, $\omega \to < f \mid \psi_\omega >$ is measurable) ;*
(iii) *there exists two positive constants A, B such that for any $f \in L^2$,*

$$A \| f \|_2^2 \leq \int_\Omega |< f \mid \psi_\omega >|^2 \, d\mu(\omega) \leq B \| f \|_2^2 .$$

(iv) *For some fixed ϵ, ϵ' we have : $\sup_\omega \Delta x_{\psi_\omega, \epsilon} \Delta \xi_{\psi_\omega, \epsilon'} < +\infty$.*

Condition (i) is only a normalisation. Conditions (ii) and (iii) express that we may recover f from the coefficients $< f \mid \psi_\omega >$ in a stable manner : this point will be developped in the next section. Condition (iv) expresses that the coefficients $< f \mid \psi_\omega >$ are time-frequency informations and that the joint resolution of these informations remains bounded as ω runs in Ω.

2. Almost orthogonal families, frames and bases in a Hilbert space.

In this section, we will precise the meaning of conditions (ii) and (iii) in the definition of a wavelet representation.

DEFINITION 2. - *An almost orthogonal family in a Hilbert space H (with scalar product $< \mid >_H$ and norm $\| \; \|_H$) is a family $(\psi_\omega)_{\omega \in \Omega}$ of vectors in H indexed by a measured space (Ω, μ) such that :*
(i) *$\omega \to \psi_\omega$ is weakly measurable (i.e. for any $f \in H$, $\omega \to < f \mid \psi_\omega >_H$ is measurable) ;*
(ii) *there exists a positive constant C_0 such that for any $f \in H$:*

$$\int_\Omega |< f \mid \psi_\omega >_H|^2 \, d\mu(\omega) \leq C_0 \| f \|_H^2 . \tag{11}$$

The reason why we call such a family an almost orthogonal one is the following : if we consider a linear combination $\int_\Omega m(\omega) \psi_\omega d\mu(\omega)$ of the ψ_ω with coefficients $m(\omega) \in L^2(\Omega)$, then the combination takes sense as a weak integral :

$$< \int_\Omega m(\omega) \psi_\omega d\mu(\omega) \mid f >_H \equiv \int_\Omega m(\omega) < f \mid \psi_\omega > d\mu(\omega)$$

and defines an element of H ; moreover we have :

$$\| \int_\Omega m(\omega)\psi_\omega d\mu(\omega) \|_H \leq \sqrt{C_0} \| m \|_{L^2(\Omega)} .$$

It is easy to see that if (ψ_ω) is an almost orthogonal family in H_1 and T a bounded operator from H_1 to another Hilbert space H_2, then $(T(\psi_\omega))_{\omega\in\Omega}$ is also an almost orthogonal family. Moreover, if $(\psi_\omega)_{\omega\in\Omega}$ is an almost orthogonal family in H_1 and $(\varphi_\omega)_{\omega\in\Omega}$ an almost orthogonal family in H_2, then the operator $S \ : \ f \to Sf = \int_\Omega < f \mid \psi_\omega > \varphi_\omega d\mu(\omega)$ is a bounded operator from H_1 to H_2. (If H_1 is separable, every bounded operator S from H_1 to H_2 can be written in that form : take $\Omega = \mathbb{N}$, μ the counting measure, $(\psi_\omega)_{\omega\in\Omega}$ a Hilbertian basis of H_1 and $\varphi_\omega = S(\psi_\omega)$). We may also define a bounded operator U by inserting in the integral a bounded multiplier $m(\omega) \in L^\infty(\Omega)$:

$$Uf = \int_\Omega m(\omega) < f \mid \psi_\omega > \varphi_\omega d\mu(\omega).$$

An important case is the case of Ω being a countable discrete set and μ the counting measure $\left(\int f\, d\mu = \sum_{\omega\in\Omega} f(\omega)\right)$. If $(\psi_\omega)_{\omega\in\Omega}$ is an almost orthogonal family in H and if $(\lambda_\omega)_{\omega\in\Omega} \in \ell^2(\Omega)$ then $\sum_{\omega\in\Omega} \lambda_\omega\psi_\omega$ is not only weakly convergent but is also summable in H : indeed for any set $J \subset \Omega$ we have

$$\| \sum_{\omega\in J} \lambda_\omega\psi_\omega \|_H \leq \sqrt{C_0} \left(\sum_{\omega\in J} \mid \lambda_\omega \mid^2 \right)^{1/2} .$$

We thus are dealing with strongly convergent series.

DEFINITION 3. -

(i) *A frame in a Hilbert space H is an almost orthogonal family $(\psi_\omega)_{\omega\in\Omega}$ of vectors in H such that there exist two positive constants A, B such that for any $f \in H$,*

$$A \| f \|_H^2 \leq \int_\Omega |< f \mid \psi_\omega >_H|^2 \, d\mu(\omega) \leq B \| f \|_H^2 . \tag{12}$$

(ii) *A dual frame of a frame $(\psi_\omega)_{\omega\in\Omega}$ in a Hilbert space H is a frame $(h_\omega)_{\omega\in\Omega}$ such that for any $f \in H$,*

$$f = \int_\Omega < f \mid \psi_\omega > h_\omega d\mu(\omega). \tag{13}$$

Formula (13) is to be meant as a stable reconstruction formula. The following proposition expresses that every frame has a dual frame :

PROPOSITION 2. - *Let $(\psi_\omega)_{\omega\in\Omega}$ be an almost orthogonal family in H and G be the Gram operator defined by :*

$$Gf = \int_\Omega < f \mid \psi_\omega >_H \psi_\omega d\mu(\omega). \tag{14}$$

Then :

(i) *G is a non-negative self-adjoint bounded operator on H.*
(ii) *G is invertible if and only if $(\psi_\omega)_{\omega\in\Omega}$ is a frame.*
(iii) *The family $(h_\omega)_{\omega\in\Omega}$, with $h_\omega = G^{-1}(\psi_\omega)$, is a dual frame for (ψ_ω).*
(iv) *Moreover G^{-1} can be computed as a Von Neumann series. More precisely, if A, B are the frame constants in (11), we have*

$$\| \frac{2}{A+B}G - \mathrm{Id} \|_{op} \leq \rho < 1 \text{ with } \rho = \frac{B-A}{B+A}$$

so that

$$G^{-1} = \frac{2}{A+B}\sum_{k=0}^{\infty}(I - \frac{2}{A+B}G)^k.$$

Remark. - Whenever $A = B$, we have $f = \frac{1}{A}Gf$. Such a frame is called a *tight frame* and has $(\frac{1}{A}\psi_\omega)_{\omega\in\Omega}$ as a dual frame.

Proof. The proposition is quite obvious. By definition, G is self-adjoint and non-negative. If G is invertible, we have

$$< Gf \mid f >_H \geq \frac{1}{\| G^{-1} \|_{op}} \| f \|_H^2$$

and the ψ_ω, $\omega \in \Omega$, are a frame. Moreover we have :

$$\begin{aligned}\int_\Omega < f \mid \psi_\omega >_H < G^{-1}\psi_\omega \mid g >_H d\mu(\omega) &= \int_\Omega < f \mid \psi_\omega >_H < \psi_\omega \mid G^{-1}g >_H d\mu(\omega) \\ &= < Gf \mid G^{-1}g >_H = < f \mid g >_H,\end{aligned}$$

so that $(G^{-1}(\psi_\omega))$ is a dual frame of (ψ_ω). Last, we have :

$$\frac{A-B}{A+B} \| f \|_H^2 \leq < \frac{2}{A+B}Gf - f \mid f >_H \leq \frac{B-A}{B+A} \| f \|_H^2 \ ;$$

since $\frac{2}{A+B}G - \mathrm{Id}$ is self-adjoint, we get $\| \frac{2}{A+B}G - \mathrm{Id} \|_{op} \leq \frac{B-A}{B+A}$. ■

Of course, frames have not in general a unique dual frame. If $(\psi_\omega)_{\omega\in\Omega}$ is a frame with dual frame $(h_\omega)_{\omega\in\Omega}$, we have the *redundancy formula* (or reproducing formula)

$$\psi_\alpha = \int_\Omega < \psi_\alpha \mid h_\omega >_H \psi_\omega d\mu(\omega) \tag{15}$$

(since the operator $\int_\Omega < \cdot \mid h_\omega >_H \psi_\omega d\mu(\omega)$ is the adjoint of $\int_\Omega < \cdot \mid \psi_\omega >_H h_\omega d\mu(\omega)$ and is therefore the identity operator).

In case there is some α and β, $\alpha \neq \beta$, such that $< \psi_\alpha \mid h_\beta > \neq 0$ and Ω is a countable discrete set with counting measure μ, we may express ψ_β as a linear combination of the $\psi_\omega, \omega \neq \beta$:

$$\psi_\beta = \frac{1}{< \psi_\alpha \mid h_\beta >} \left\{ \psi_\alpha - \sum_{\omega \neq \beta} < \psi_\alpha \mid h_\omega > \psi_\omega \right\}$$

and we have another dual frame $(k_\omega)_{\omega \in \Omega}$ for (ψ_ω) with $k_\beta = 0$,

$$k_\alpha = h_\alpha + \frac{1 - < \psi_\alpha \mid h_\alpha >}{< \psi_\alpha \mid h_\beta >} h_\beta \quad \text{and} \quad k_\omega = h_\omega - \frac{< \psi_\alpha \mid h_\omega >}{< \psi_\alpha \mid h_\beta >} h_\beta$$

elsewhere. We have thus proved the following proposition :

PROPOSITION 3. - *Let H be a separable Hilbert space, Ω a countable discrete set with counting measure μ and $(\psi_\omega)_{\omega \in \Omega}$ a family of vectors in H. Then the following assertions are equivalent :*

(i) *$(\psi_\omega)_{\omega \in \Omega}$ is a frame and has a unique dual frame in H.*
(ii) *$(\psi_\omega)_{\omega \in \Omega}$ is a Riesz basis, i.e. the operator $(\lambda_\omega) \to \sum_{\omega \in \Omega} \lambda_\omega \psi_\omega$ is an isomorphism of $\ell^2(\Omega)$ onto H.*

Moreover, in that case, the dual frame $(h_\omega)_{\omega \in \Omega}$ is given by the coefficient functions :

(16) $$< \sum_{\omega \in \Omega} \lambda_\omega \psi_\omega \mid h_\alpha > = \lambda_\alpha .$$

3. Fourier windows, Gabor wavelets and the Balian-Low theorem.

A very simple and useful way to perform local Fourier analysis is the following one : in order to compute local Fourier coefficients for a function f around a point x_0, choose a window function g which is rapidly decreasing far away from 0 and compute the usual Fourier coefficients for the windowed function $\bar{g}(x - x_0) f(x)$. Then moving the center x_0 of the window $g(x - x_0)$ along the real line allows one to catch informations at any point.

DEFINITION 4. - *A Fourier window representation, with window function $g \in L^2(\mathbb{R})$ $(g \neq 0)$, is the family of analyzing functions (g_{x_0,ξ_0}), $(x_0, \xi_0) \in \mathbb{R}^2$, defined by :*

(17) $$g_{x_0,\xi_0}(x) = g(x - x_0) e^{i \xi_0 x} .$$

The point is that this representation is a tight frame :

(18) $$\int\int_{\mathbb{R}^2} |< f \mid g_{x_0,\xi_0} >|^2 \, dx_0 d\xi_0 = 2\pi \parallel g \parallel_2^2 \parallel f \parallel_2^2 .$$

This is very easy to prove. Indeed, $< f \mid g_{x_0,\xi_0} >$ is the Fourier coefficient at $\xi = \xi_0$ of the windowed function $f(x)\bar{g}(x - x_0)$, so that we get by Plancherel equality :

$$\int\int_{\mathbb{R}^2} |< f \mid g_{x_0,\xi_0} >|^2 \, dx_0 d\xi_0 = 2\pi \int\int_{\mathbb{R}^2} | f(x)\bar{g}(x - x_0) |^2 \, dx_0 dx$$
$$= 2\pi \| g \|_2^2 \| f \|_2^2 .$$

(Notice that for almost all x_0, $f(x)g(x - x_0)$ is square integrable in x).

Of course the requirement on g that $g \in L^2$ is a minimal one. If we really want to get a local Fourier analysis, g has to be localized both in time (so that $f(x)\bar{g}(x - x_0)$ is localized around x_0) and in frequency (so that the windowing by g doesn't introduce unwanted high frequencies). The Fourier window representation is then a wavelet representation (according to definition 1, up to normalization $\| g \|_2 = 1$).

Time and frequency play exactly the same rôle in the Fourier windows formalism. Just notice that $< f \mid (g)_{x_0,\xi_0} > = \frac{e^{ix_0\xi_0}}{2\pi} < \hat{f} \mid (\hat{g})_{\xi_0,-x_0} >$. This symmetry between time and frequency and the minimum joint resolution property of the Gaussian functions has turned those Gaussian functions

$$g(x) = \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{\sigma}} e^{-\frac{1}{4}\frac{x^2}{\sigma^2}} \tag{19}$$

(where $\sigma = \Delta x_g$) into a very popular choice for Fourier windows.

In 1946, D. Gabor proposed to use a family of Gaussian wavelets $g_{m,n} = g(x - an)e^{ibmx}$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$, with a Gaussian g (as given by (19)) and with time sampling mesh a and frequency sampling mesh b satisfying the Nyquist rule $ab = 2\pi$ [GAB]. Instead of dealing with the whole family of wavelet coefficients $< f \mid g_{x_0,\xi_0} >$ ($x_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}$), one deals with a discrete sample of them, for which the sampling points are located on the regular grid $a\mathbb{Z} \times b\mathbb{Z}$. As a matter of fact, every square-integrable function may be represented as a superposition of Gabor wavelets but this representation is unstable. M. J. Baastians has computed explicitly the coefficient functions of the Gabor basis and exhibited the singularities of these functions [BAA]. The point is that the $g_{m,n}$ don't constitute a frame. If we take more functions ($ab < 2\pi$) we obtain a redundant frame and if we take fewer functions ($ab > 2\pi$) the system is no more total in L^2, as shown by I. Daubechies in her thorough discussion of the subject [DAU3].

In order to get rid of unstability in the Gabor wavelet representation, R. Balian tried in the early 80's to construct an orthonormal wavelet basis $(g_{m,n})_{m\in\mathbb{Z},n\in\mathbb{Z}}$ with $g_{m,n} = g(x - am)e^{inbx}$ for some fixed function g. It is very easy to see that we must have

$ab = 2\pi$. [By instance, let $\chi_N = \chi_{[N\frac{2\pi}{b},(N+1)\frac{2\pi}{b}]}$ and write

$$\begin{aligned}\chi_N &= \chi_N \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} < \chi_N \mid g_{m,n} > g_{m,n} \\ &= \frac{b}{2\pi}\chi_N(x)\sum_{m\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\chi_N(x+\frac{2k\pi}{b})\bar{g}(x-am+\frac{2k\pi}{b})\right)g(x-am) \\ &= \frac{b}{2\pi}\chi_N(x)\sum_{m\in\mathbb{Z}} \mid g(x-am)\mid^2 .\end{aligned}$$

We get $\sum_{m\in\mathbb{Z}} \mid g(x-am) \mid^2 = \frac{2\pi}{b}$ a. e., and $ab = 2\pi$ by integration on $x \in [0,a]$]. A very simple example of such a basis is given by $g = \chi_{[0,1]}$, $a = 1$, $b = 2\pi$: we just cut the real line in disjoint intervals and use for each interval the ordinary Fourier basis ; this window g however is not localized in frequency : $\Delta\xi_g = +\infty$. Another example is given by inverting the role of time and frequency : we cut the frequency axis in disjoint intervals of length2π ($b = 2\pi$) and then use the cardinal sine basis $\left(g = \frac{\sin \pi x}{\pi x},\ a = 1\right)$; of course, for this window we have $\Delta x_g = +\infty$. This unlocalization property ($\Delta x_g = +\infty$ or $\Delta\xi_g = +\infty$) is in fact a general feature of Gabor wavelet bases, as it was noticed independently by R. Balian and Low [BAL], [LOW] :

THEOREM 2 *(Balian-Low theorem). - If g is a function such that*

$$(g(x-m)e^{2i\pi nx})_{m\in\mathbb{Z},n\in\mathbb{Z}}$$

is a Hilbertian basis for $L^2(\mathbb{R})$ then $\Delta x_g = +\infty$ or $\Delta\xi_g = +\infty$.

Proof. There is a very short proof by G. Battle [BAT3]. Let us assume that such a basis exists with finite Δx_g and $\Delta\xi_g$. We then have :

$$< xg \mid g' >=$$

$$\begin{aligned}&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} < xg \mid g_{m,n} >< g_{m,n} \mid g' > \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \{< g \mid (xg)_{m,n} > +m < g \mid g_{m,n} >\}\{- < (g')_{m,n} \mid g > -2i\pi n < g_{m,n} \mid g >\} \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} < g \mid (xg)_{m,n} >< -(g')_{m,n} \mid g > \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} < g_{-m,-n} \mid xg >< -g' \mid g_{-m,-n} > \\ &= - < g' \mid xg > .\end{aligned}$$

But if $f \in C_c^\infty$, we have $< xf \mid f' >= - < f' \mid xf > - \parallel f \parallel_2^2$ and we already know that C_c^∞ is dense in $H = \{f \in L^2 / xf \in L^2,\ f' \in L^2\}$. We thus get $\parallel g \parallel_2^2 = 0$, which is absurd. ∎

I. Daubechies has shown how to adapt Battle's proof to show that if $\Delta x_g \Delta \xi_g < +\infty$ the $g_{m,n}$ cannot even constitute a frame [DAU3]. One may circonvene the Balian-Low theorem in two ways : on one hand, R. Balian has constructed for every $\epsilon < 1$ a basis $(g_{m,n} = g(x-m)e^{2i\pi nx})_{m\in\mathbb{Z},n\in\mathbb{Z}}$ with $\Delta x_{g,\epsilon}\Delta \xi_{g,\epsilon} < +\infty$ [BAL] ; on the other hand, J. Bourgain has constructed an orthonormal basis $(\psi_n)_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with uniformly bounded time and frequency resolution ($\sup_{n\in\mathbb{Z}} \Delta x_{\psi_n} < +\infty$ and $\sup_{n\in\mathbb{Z}} \Delta \xi_{\psi_n} < +\infty$) [BOU]. This result cannot be improved with better uniform localization : Stegers has proved that for any orthonormal basis $(\psi_n)_{n\in\mathbb{Z}}$of $L^2(\mathbb{R})$ and any $\epsilon > 1$, we have $\sup_{n\in\mathbb{Z}} \Delta x_{\psi_n,\epsilon} = +\infty$ or $\sup_{n\in\mathbb{Z}} \Delta \xi_{\psi_n,\epsilon} = +\infty$ (a result quoted in [BOU]).

4. Morlet wavelets.

We have seen there was no basis for $L^2(\mathbb{R})$ with good uniform localization both in time and frequency. Moreover, there are another disturbing points in the Fourier window formalism : the analyzing functions $g_{x,\xi}$ are more and more oscillating as ξ tends to ∞ and this leads to numerical unstability ; the functions $g_{x,\xi}$ are still mixing the contributions of every point lying inside the window and this is an obstacle to the isolation of the singularities of the analyzed signal ; the coefficients $< f \mid g_{x,\xi} >$ give information on the mean values of f at one fixed scale (the size of the window g), and those coefficients will be strongly correlated for the low frequencies (for wave lengths much bigger than the window size) and will suffer from important computational noise for the high frequencies (for wave lengths much smaller than the window size).

Therefore, J. Morlet has defined in the early 80's another wavelet representation, namely the wavelets of constant shape [GROS].

DEFINITION 5. - *A Morlet wavelet is a real-valued function $\psi \in L^2(\mathbb{R})$ ($\psi \neq 0$) satisfying the admissibility condition $C_\psi < +\infty$, where C_ψ is defined as :*

(20)
$$C_\psi = \int_0^{+\infty} \mid \hat{\psi}(\xi) \mid^2 \frac{d\xi}{\xi}.$$

The Morlet wavelet representation associated to ψ is the family of analyzing functions $(\psi_{a,b})_{a>0,b\in\mathbb{R}}$ defined by :

(21)
$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right).$$

Just like the Fourier window representation, the Morlet wavelet representation provides a tight frame :

(22)
$$\int_0^{+\infty}\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >|^2 \frac{da}{a^2} db = C_\psi \parallel f \parallel_2^2 .$$

This is very easy to prove. Just notice that if $g_a(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{-x}{a}\right)$, then $f * g_a(b) = < f \mid \psi_{a,b} >$ so that

$$\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >|^2 \, db = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mid \hat{f}(\xi) \mid^2 \mid \sqrt{a}\hat{\psi}(a\xi) \mid^2 \, d\xi \; ;$$

since

$$\int_0^{+\infty} \mid \sqrt{a}\hat{\psi}(a\xi) \mid^2 \frac{da}{a^2} = C_\psi,$$

(22) is proved.

In the Morlet representation, the wavelets have no uniformly bounded time or frequency resolution ($\Delta x_{\psi_{a,b}} = a\Delta x_\psi$ and $\Delta \xi_{\psi_{a,b}} = \frac{1}{a}\Delta\xi_\psi$) but the joint resolution remains constant. The requirement for a Morlet wavelet to be real-valued has two reasons : the first one is that in signal analysis most signals are real-valued and therefore it is natural to deal with real-valued analyzing functions ; the second one is that dilation acts similarly on negative and positive frequencies so that we cannot separate a frequency ω from its opposite $-\omega$ by mean of a Morlet transform (unless we deal also with dilations with a negative factor a) ; if we only deal with real-valued functions, $\hat{f}(\omega)$ and $\hat{f}(-\omega)$ are linked by the relationship $\hat{f}(-\omega) = \bar{\hat{f}}(\omega)$ and the coupling of ω and $-\omega$ is no trouble any more for time-frequency representations.

The admissibility condition $C_\psi < +\infty$ means roughly speaking that $\int \psi \, dx = 0$ (it does really mean it if $\mid x \mid^\epsilon \psi \in L^1$ for some $\epsilon > 0$, for instance if $\Delta x_{\psi,\frac{1}{2}+\epsilon} < +\infty$ for some $\epsilon > 0$). Parameter b is a time-position parameter and parameter a a scale parameter. This is illustrated by the following proposition :

PROPOSITION 4. - *Let ψ be a Morlet wavelet. Then :*

(i) *there exists a real-valued function h such that $h \in \mathcal{C}_c^\infty$, $\int h \, dx = 0$ and the family $(h_{a,b})_{a>0, b\in\mathbb{R}}$ is a dual frame of $(\psi_{a,b})$:*

$$(23) \qquad \int_0^{+\infty} \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db = f \quad for\ any\ f \in L^2(\mathbb{R}) \; ;$$

(ii) *similarly, there exists a real-valued function k such that $\hat{k} \in \mathcal{C}_c^\infty(\mathbb{R})$, $0 \notin \operatorname{Supp} \hat{k}$ and the family $(k_{a,b})_{a>0, b\in\mathbb{R}}$ is a dual frame of $(\psi_{a,b})$.*

Proof. We first characterize dual wavelet frames.

LEMMA. - *If ψ and φ are two Morlet wavelets, then $(\psi_{a,b})$ and $(\varphi_{a,b})$ are dual frames if and only if we have : $\int_0^{+\infty} \hat{\psi}(\xi)\bar{\hat{\varphi}}(\xi)\frac{d\xi}{\xi} = 1$.*

The lemma is proved in the very same way as equality (22). We find for any wavelets

ψ and φ and any square-integrable functions f and g :

$$\int_0^{+\infty}\int <f \mid \psi_{a,b}><\varphi_{a,b}\mid g>\frac{da}{a^2}db = \frac{1}{2\pi}\int_{-\infty}^{0}\left[\int_{-\infty}^{0}\bar{\hat{\psi}}(\eta)\hat{\varphi}(\eta)\frac{d\eta}{|\eta|}\right]\hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi$$
$$+\frac{1}{2\pi}\int_0^{+\infty}\left[\int_0^{+\infty}\bar{\hat{\psi}}(\eta)\hat{\varphi}(\eta)\frac{d\eta}{\eta}\right]\hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi$$

by first integrating on b and using a Plancherel formula. The lemma is then proved.

Part (ii) of proposition 4 is obvious. $C_c^\infty((0,+\infty))$ is dense in $L^2\left((0,+\infty),\frac{d\xi}{\xi}\right)$, so we may find $\omega \in C_c^\infty((0,+\infty))$ such that

$$\int_0^{+\infty}\hat{\psi}(\xi)\bar{\omega}(\xi)\frac{d\xi}{\xi}=1.$$

N. w just define k by $\hat{k}=\omega$ if $\xi>0$ and $\hat{k}(\xi)=\bar{\omega}(-\xi)$ if $\xi<0$.

Part (i) is quite so easy. Let k be the function described by point (ii), K the primitive of k in the Schwartz class $\mathcal{S}(\mathbb{R})$ ($\hat{K}=\frac{\hat{k}}{i\xi}$), H_n be real-valued compactly supported smooth functions converging to K in $\mathcal{S}(\mathbb{R})$ and $h_n=\frac{d}{dx}H_n$. Then it is easy to see that $\hat{h}_n \to \hat{k}$ in $L^2(\frac{d\xi}{\xi})$ as $n\to+\infty$, so that

$$\lim_{n\to+\infty}\int_0^{+\infty}\hat{\psi}\bar{\hat{h}}_n\frac{d\xi}{\xi}=1.$$

Let γ_n be $\gamma_n=\int_0^{+\infty}\hat{\psi}\bar{\hat{h}}_n\frac{d\xi}{\xi}$. If some γ_n is a positive real number, define h as $h=\frac{1}{\gamma_n}h_n$; if no γ_n is a positive real number, there are at least two complex numbers γ_{n_1} and γ_{n_2} which are independent over the real field and we may define $h=\lambda_1 h_{n_1}+\lambda_2 h_{n_2}$ where λ_1 and λ_2 are two real numbers such that $\lambda_1\gamma_{n_1}+\lambda_2\gamma_{n_2}=1$. Proposition 4 is then proved.

Proposition 4 is of constant use in wavelet theory, because it shows that the coefficients $<f\mid\psi_{a,b}>$ which are useful to reconstruct f at a point x_0 are located on a cone $\{(a,b)/\mid b-x_0\mid\leq Ma\}$ over x_0 ; if b is too far from x_0 (relatively to the scale a), $<f\mid\psi_{a,b}>$ will not convey information about x_0. Moreover high frequency information is located at small scales : the coefficients $<f\mid\psi_{a,b}>$ which are useful for reconstructing $\hat{f}$ at ξ_0 are located on a band $\alpha\mid\xi_0\mid<\frac{1}{a}<\beta\mid\xi_0\mid$. Thus singularities will be characterized by big coefficients $<f\mid\psi_{a,b}>$ for small scales a and for points b close to the singular points ; they don't affect the coefficients of regular points, as we shall see later.

A related class of wavelet transforms is the so-called *analytic wavelet transform* where the wavelet ψ, instead of being real-valued, has no negative frequencies (supp $\hat{\psi}\subset[0,+\infty)$). We still define C_ψ as $C_\psi=\int_0^{+\infty}\mid\hat{\psi}(\xi)\mid^2\frac{d\xi}{\xi}$ and $\psi_{a,b}$ as $\psi_{a,b}=\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$. The $(\psi_{a,b})_{a>0,b\in\mathbb{R}}$ are then a tight frame for the subspace of L^2 of the so-called *analytic signals*: for any $f\in L^2$ such that Supp $\hat{f}\subset[0,+\infty]$ we have :

$$f=\frac{1}{C_\psi}\int_0^{+\infty}\int_{-\infty}^{+\infty}<f\mid\psi_{a,b}>\psi_{a,b}\frac{da}{a^2}db.$$

There is a one-to-one correspondence between real-valued functions $f \in L^2(\mathbb{R})$ and analytic signals F given by : $f = 2\mathcal{R}e\, F$ and $\hat{F} = \chi_{[0,+\infty)}\hat{f}$; if f is real-valued and ψ is an analytic wavelet, then

$$f = \frac{2}{C_\psi}\mathcal{R}\mathrm{e}\left(\int_0^{+\infty}\int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > \psi_{a,b}\frac{da}{a^2}db\right)$$

and

$$\| f \|_2^2 = \frac{2}{C_\psi}\int_0^{+\infty}\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >|^2 \frac{da}{a^2}db.$$

The modulus $|< f \mid \psi_{a,b} >|$ is therefore used for defining a time-frequency density of the energy of f and that's the reason why analytic wavelets are sometimes preferred to real-valued one : there is a dephasage between real and imaginary parts of $\psi_{a,b}$ which avoids the pulsatory cancellation which is observed for real-valued wavelets ; the vanishing of an analytic wavelet coefficient is therefore more physically significant than the one of a real-valued wavelet coefficient.

5. Wavelet analysis of global regularity.

We will see in this section how to apply the Morlet wavelet transform to the study of the global regularity of a function ; the study of the pointwise regularity will be discussed in the next section.

The main requirement on ψ for regularity analysis will be that ψ has enough vanishing momenta. We begin with the easy example of the Sobolev spaces H^s. (Recall that $f \in H^s$ if and only if $(1+\xi^2)^{s/2}\hat{f}(\xi) \in L^2$).

PROPOSITION 5. - *Let $s > 0$ and ψ be a Morlet wavelet such that :*

$$\int_0^{+\infty} \mid \hat{\psi}(\xi) \mid^2 \frac{d\xi}{\xi^{1+2s}} = C_{\psi,s} < +\infty. \tag{24}$$

Then for any $f \in L^2(\mathbb{R})$ the following assertions are equivalent :
(i) $f \in H^s$
(ii) $\int_0^{+\infty}\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >|^2 \frac{da}{a^{2+2s}}db < +\infty.$
Moreover $\| f \|_{H^s}$ *is equivalent to* $\| f \|_2 + \left(\int_0^{+\infty}\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >|^2 \frac{da}{a^{2+2s}}db\right)^{1/2}$.

Remark. - If $| x |^\sigma \psi \in L^1$ for some $\sigma > s$, condition (24) is fulfilled if and only if for all $p \in \{0, 1, \cdots, [s]\}$, $\int x^p \psi\, dx = 0$. Thus (24) is mainly a vanishing momenta condition.

Proof. Since the integrand $|< f \mid \psi_{a,b} >|^2 \frac{da}{a^{2+2s}}db$ is non-negative, we may use Fubini's theorem and first integrate with respect to b. We obtain :

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} |< f \mid \psi_{a,b} >| \frac{da}{a^{2+2s}}db = \frac{1}{2\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \mid \hat{f}(\xi \mid^2 \mid \sqrt{a}\hat{\psi}(a\xi) \mid^2 \frac{da}{a^{2+2s}}d\xi$$

$$= C_{\psi,s}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \hat{f}(\xi) \mid^2 \mid \xi \mid^{2s} d\xi$$

and the proposition is obvious. ■

We thus see clearly that the regularity of a function is related to the decaying of small-scale coefficients. As a matter of fact, condition (24) ensures that $\hat{\psi} = |\xi|^s \hat{\omega}$ where ω is still a Morlet wavelet, and "integration by parts" gives for any $f \in H^s$:

$$< f \mid \psi_{a,b} >= a^s < \Lambda^s f \mid \omega_{a,b} >$$

where Λ^s is the fractional derivative operator : $\widehat{\Lambda^s f} = |\xi|^s \hat{f}$.

A second example of regularity analysis is given by the Hölder spaces C^α. For $\alpha \in (0,1)$, a function f belongs to C^α if f is bounded and if $|||f|||_\alpha$ is finite, where $|||f|||_\alpha = \sup_{x,y;x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$; C^α is a Banach space for the norm $\| f \|_{C^\alpha} = \| f \|_\infty + |||f|||_\alpha$. When $\alpha = m + \rho$ with $m \in \mathbb{N}$ and $\rho \in (0,1)$, then $f \in C^\alpha$ if f is of class C^m, if its derivatives up to order m are bounded and if $f^{(m)} \in C^\rho$; C^α is then normed by $\| f \|_{C^\alpha} = \sum_{k=0}^m \| f^{(k)} \|_\infty + |||f^{(m)}|||_\rho$.

For integral values of α, we shall use the Zygmund space C^α_*. The space C^1_* is the space of bounded functions f such that $|||f|||_{1,*}$ is finite, where

$$|||f|||_{1,*} = \sup_{x,h;h\neq 0} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|}.$$

The space C^{m+1}_* is the space of functions f such that f is of class C^m, its derivatives are bounded up to order m and $f^{(m)} \in C^1_*$; it is normed with $\| f \|_{C^{m+1}_*} = \sum_{k=0}^m \| f^{(k)} \|_\infty + |||f^{(m)}|||_{1,*}$.

For technical reasons, we need also to introduce the Hardy space H^1 of integrable functions $f \in L^1(\mathbb{R})$ such that the Hilbert transform Hf of f is also integrable (Hf is defined as $Hf = VP\frac{1}{\pi x} * f$, or equivalently $\widehat{Hf} = -i \,\mathrm{sgn}\, \xi \hat{f}$). If f is such that $f \in L^p(\mathbb{R})$ for some $p > 1$ and $|x|^\epsilon f \in L^1(\mathbb{R})$ for some $\epsilon > 0$, then $f \in H^1$ if and only if $\int f\, dx = 0$.

PROPOSITION 6. - *Let $\alpha = m + \rho$ be a positive real number with $m \in \mathbb{N}$ and $0 < \rho \leq 1$. Let ψ be a Morlet wavelet. Then :*

(i) *if $\alpha \notin \mathbb{N}$, if $|x|^\alpha \psi \in L^1$ and if $\int_{-\infty}^{+\infty} x^p \psi(x) dx = 0$ for $p \in \{0, \cdots, m\}$ then for any $f \in L^2$ we have :*
- *$f \in C^\alpha$ if and only if $a^{-1/2-\alpha} < f \mid \psi_{a,b} > \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$*
- *$|||f^{(m)}|||_\rho$ is equivalent to $\| a^{-1/2-\alpha} < f \mid \psi_{a,b} > \|_\infty$*

(ii) *if $\alpha \in \mathbb{N}$, if $x^\alpha \psi \in L^1$ and the even part of $x^\alpha \psi$ belongs to the Hardy space H^1 and if $\int x^p \psi \, dx = 0$ for $p \in \{0, \cdots, \alpha\}$ then for any $f \in L^2$ we have :*
- *$f \in C^\alpha_*$ if and only if $a^{-1/2-\alpha} < f \mid \psi_{a,b} > \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$*
- *$|||f^{(m)}|||_{1,*}$ is equivalent to $\| a^{-1/2-\alpha} < f \mid \psi_{a,b} > \|_\infty$.*

Moreover results (i) and (ii) are still valid if hypothesis "$f \in L^2$" is changed into "$f \in L^\infty$".

Remarks.

• hypothesis "$f \in L^\infty$" is more natural than "$f \in L^2$" since the spaces C^α and C^α_* are included in L^∞ and not in L^2 ;

• in case of α being integer, the extra requirement $\frac{x^\alpha\psi(x)+(-x)^\alpha\psi(-x)}{2} \in H^1$ is satisfied whenever $\int x^\alpha \psi\, dx = 0$ and $| x |^{\alpha+\epsilon} \psi \in L^1$ for some $\epsilon > 0$ (since $\psi \in L^2$).

Proof. We will begin by the sufficiency part. We consider the Besov space $B_\infty^{N+\epsilon,1}$ defined for $N \in \mathbb{N}$ and $\epsilon \in (0,1)$ as : $h \in B_\infty^{N+\epsilon,1}$ if and only if $h^{(p)} \in L^1$ for $0 \leq p \leq N$ and $||| h^{(N)} |||_{[\epsilon]}$ if finite where

$$||| f |||_{[\epsilon]} = \sup_{y\neq 0} \frac{1}{| y |^\epsilon} \int | f(x) - f(x+y) | \, dx.$$

LEMMA 1. - *Let $\alpha = m + \rho$ and let h be a Morlet wavelet such that $h \in B_\infty^{N+\epsilon,1}$ where $N = m$, $\rho < \epsilon < 1$ if $\rho \neq 1$ and $N = m+1$, $\epsilon > 0$ if $\rho = 1$. If $\mu(a,b)$ is a bounded measurable function on $\mathbb{R}_+ \times \mathbb{R}$ such that $| \mu(a,b) | \leq C_\mu a^{1/2+\alpha}$ then the distribution I_μ defined as*

$$< f \mid I_\mu > = \int_0^{+\infty} \int_{-\infty}^{+\infty} \bar{\mu}(a,b) < f \mid h_{a,b} > \frac{da}{a^2} db$$

belongs to C^α if $\alpha \notin \mathbb{N}$ and to C^α_* if $\alpha \in \mathbb{N}$.

Moreover $||| I_\mu^{(m)} |||_\rho$ if $\rho < 1$ and $||| I_\mu^{(m)} |||_{1,*}$ if $\rho = 1$ are bounded by $C_h \| a^{-1/2-\alpha}\mu \|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}$ where C_h depends only on h.

The lemma is quite obvious. We first see easily that I_μ is well-defined and has bounded derivatives up to order m :

$$I_\mu^{(p)} = \int_0^{+\infty} \int_{-\infty}^{+\infty} \mu(a,b)(h^{(p)})_{a,b} \frac{1}{a^p} \frac{da}{a^2} db.$$

Indeed, for $0 \leq p \leq m$, we have :

$$\begin{aligned} | I_\mu^{(p)}(x) | &\leq C \int_0^{+\infty} \int_{-\infty}^{+\infty} \inf(1, a^{1/2+\alpha}) \frac{1}{\sqrt{a}} \, | h^{(p)}(\frac{x-b}{a}) | \, \frac{1}{a^p} \frac{da}{a^2} db \\ &= C \, \| h^{(p)} \|_1 \int_0^{+\infty} \inf(1, a^{1/2+\alpha}) \frac{da}{a^{\frac{3}{2}+p}} < +\infty. \end{aligned}$$

We may therefore suppose $m = 0$ (by replacing μ with $a^{-m}\mu(a,b)$ and h with $h^{(m)}$) and try to control $||| I_\mu |||_\rho$ or $||| I_\mu |||_{*,1}$. If $\rho < 1$, we have :

$$\begin{aligned} &| I_\mu(x) - I_\mu(x+y) | \\ &\leq C_\mu \int_0^{+\infty} \int_{-\infty}^{+\infty} a^{1/2+\rho} \frac{1}{\sqrt{a}} \left| h(\frac{x-b}{a}) - h(\frac{x+y-b}{a}) \right| \frac{da}{a^2} db \\ &\leq C_\mu \int_0^{|y|} a^{1/2+\rho} a^{1/2} 2 \, \| h \|_1 \, \frac{da}{a^2} + C_\mu \int_{|y|}^{+\infty} a^{1/2+\rho} a^{1/2} \, ||| h |||_{[\epsilon]} \, \frac{| y |^\epsilon}{a^\epsilon} \frac{da}{a^2} \\ &= C_\mu \, | y |^\rho \left\{ \frac{2}{\rho} \, \| h \|_1 + \frac{1}{\epsilon - \rho} \, ||| h |||_{[\epsilon]} \right\}. \end{aligned}$$

If $\rho = 1$, we have :

$$| I_\mu(x+y) + I_\mu(x-y) - 2I_\mu(x) |$$

$$\leq C_\mu \int_0^{+\infty} \int_{-\infty}^{+\infty} a^{\frac{3}{2}} \frac{1}{\sqrt{a}} \left| h(\frac{x+y-b}{a}) + h(\frac{x-y-b}{a}) - 2h(\frac{x-b}{a}) \right| \frac{da}{a^2} db \ ;$$

but we have :

$$\int | h(z+b) + h(-z+b) - 2h(b) | \, db$$

$$= \int \left| \int_b^{b+z} (h'(t) - h'(t-z)) dt \right| db$$

$$\leq \int \int_{t-z \leq b \leq t} | h'(t) - h'(t-z) | \, dt \, db \leq z^{1+\epsilon} \ ||| \ h' \ |||_{[\epsilon]}$$

so that :

$$| I_\mu(x+y) + I_\mu(x-y) - 2I_\mu(x) |$$

$$\leq C_\mu \int_0^{|y|} a^{\frac{3}{2}} \sqrt{a} 4 \ \| \ h \ \|_1 \ \frac{da}{a^2} + C_\mu \int_{|y|}^{+\infty} a^{\frac{3}{2}} \ ||| \ h' \ |||_{[\epsilon]} \ a^{1/2} \frac{| \ y \ |^{1+\epsilon}}{a^{1+\epsilon}} \frac{da}{a^2}$$

$$= C_\mu \ | \ y \ | \left\{ 4 \ \| \ h \ \|_1 + \frac{1}{\epsilon} \ ||| \ h' \ |||_{[\epsilon]} \right\}$$

and the lemma is proved.

The sufficient part is now straightforward if $f \in L^2$, because we know that there is a Morlet wavelet h which is $\mathcal{C}^\infty$ and compactly supported such that $f = \int_0^{+\infty} \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$ and we know also that $< f \mid \psi_{a,b} >$ is bounded (by $\| \ f \ \|_2 \| \ \psi \ \|_2$). We may then apply lemma 1 to get that if $a^{-1/2-\alpha} < f \mid \psi_{a,b} >$ is bounded, then f belongs to C^α ($\alpha \notin \mathbb{N}$) or C^α_* ($\alpha \in \mathbb{N}$).

If f is assumed to be bounded instead of being square-integrable, we know that $\int_0^1 \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$ will be well-defined and belong to C^α or C^α_* if $a^{-1/2-\alpha} < f \mid \psi_{a,b} >$ is bounded. We may then conclude that f belongs to C^α or C^α_* by the following lemma :

LEMMA 2 *(The infra-red cut-off lemma). - Let $f \in L^\infty$, ψ be a Morlet wavelet such that $| \ x \ |^\alpha \ \psi \in L^1$ and h be a $\mathcal{C}^\infty$ compactly supported Morlet wavelet such that $(h_{a,b})$ is a dual frame for $(\psi_{a,b})$. Then*

$$\tilde{f} = \int_0^1 \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$$

is well-defined and $f - \tilde{f}$ belongs to C^α ($\alpha \notin \mathbb{N}$) or C^α_* ($\alpha \in \mathbb{N}$).

To prove the lemma, it is sufficient to check that for every interval I of length 2 ($I = [x_I - 1, x_I + 1]$), $f - \tilde{f}$ belongs to $C^\alpha(I)$ or $C^\alpha_*(I)$ and that the derivatives of $f - \tilde{f}$ are controlled independently of I. Let M be such that $\mathrm{Supp}\, h \subset [-M, M]$; we choose $\varphi \in C_c^\infty$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $[-2(M+1), 2(M+1)]$ and define $f_I = f\varphi(x - x_I)$. On I, we have $f = f_I$ and therefore :

$$f-\tilde{f} = f_I - \tilde{f} = \int_1^{+\infty}\int_{-\infty}^{+\infty} < f_I \mid \psi_{a,b} > h_{a,b}\frac{da}{a^2}db + \int_0^1\int_{-\infty}^{+\infty} < f_I - f \mid \psi_{a,b} > h_{a,b}\frac{da}{a^2}db.$$

Since $|< f_I \mid \psi_{a,b} >| \le 2\sqrt{M+1} \parallel f \parallel_\infty \parallel \psi \parallel_2$, the first integral is a C^∞ function. In the second integral, we may restrict integration on b to the domain $\mid b - x_I \mid \le 1 + aM$ (for other values of b, $h_{a,b} \equiv 0$ on I) ; in that case we find :

$$|< f_I - f \mid \psi_{a,b} >| \le \parallel f \parallel_\infty \int_{(x-x_I)\ge 2(M+1)} \mid \psi_{a,b} \mid dx \le a^{1/2+\alpha}\frac{\parallel f \parallel_\infty}{(M+1)^\alpha} \parallel \mid x \mid^\alpha \psi \parallel_1$$

and thus

$$\int_0^1\int_{x_I-1-aM}^{x_I+1+aM} < f_I - f \mid \psi_{a,b} > h_{a,b}\frac{da}{a^2}db$$

belongs to C^α (or C_*^α). Lemma 2 is proved, as well as the sufficiency part of proposition 6.

For the necessary part, we begin with the case $m = 0$. If $\rho < 1$, the result is straightforward : we just write

$$< f \mid \psi_{a,b} > = \int (f(x) - f(b))\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)dx$$

to get

$$\begin{aligned}|< f \mid \psi_{a,b} >| &\le ||| f |||_\rho \int \mid x - b \mid^\rho \frac{1}{\sqrt{a}} \mid \psi\left(\frac{x-b}{a}\right) \mid dx \\ &= a^{1/2+\rho} ||| f |||_\rho \parallel \mid x \mid^\alpha \psi \parallel_1 .\end{aligned}$$

If $\rho = 1$ and ψ is an even function, this is also obvious :

$$\begin{aligned}< f \mid \psi_{a,b} > &= \int (f(x) - f(b))\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)dx \\ &= \sqrt{a}\int (f(b+ay) - f(b))\psi(y)dy \\ &= \sqrt{a}\frac{1}{2}\int (f(b+ay) + f(b-ay) - 2f(b))\psi(y)dy\end{aligned}$$

and thus

$$|< f \mid \psi_{a,b} >| \le a^{\frac{3}{2}} ||| f |||_{1,*} \parallel x\psi \parallel_1 .$$

We now have to study the case of ψ an odd function such that $x\psi \in H^1$. We may write $\psi = H\omega$; ω is then computed as $\omega = -H\psi$ and is obviously a Morlet wavelet and an even function ; moreover $x\omega \in L^1$ because

$$x\omega = -xH\psi = -\frac{1}{\pi}\left(\int \psi\, dx\right) - H(x\psi) = -H(x\psi).$$

Now we have

$$< f \mid \psi_{a,b} >=< f \mid (H\omega)_{a,b} >=< f \mid H(\omega_{a,b}) >=< Hf \mid \omega_{a,b} >$$

since H commutes with translations and dilations and is a unitary operator. The only thing we have to check is that if $f \in L^2 \cap C^1_*$ then Hf belongs to $L^2 \cap C^1_*$ and that $||| Hf |||_{1,*} \leq C ||| f |||_{1,*}$. This is a well-known result, but we may easily prove it : we know that f can be written as

$$f = \frac{1}{C_h} \int_0^{+\infty} \int_{-\infty}^{+\infty} < f \mid k_{a,b} > k_{a,b} \frac{da}{a^2} db$$

where k is an even Morlet wavelet such that $\hat{k}$ is C^∞ and compactly supported ; we have (in the distribution sense)

$$Hf = \frac{1}{C_h} \int_0^{+\infty} \int_{-\infty}^{+\infty} < f \mid k_{a,b} > (Hk)_{a,b} \frac{da}{a^2} db$$

with $< f \mid k_{a,b} >$ bounded by $\| f \|_2 \| k \|_2$ and by $a^{\frac{3}{2}} ||| f |||_{1,*} \| xk \|_1$ and Hk belongs to $B^{1+\epsilon,1}_\infty$ for $\epsilon > 0$, so that Hf belongs to C^1_* and $||| Hf |||_{1,*} \leq C ||| f |||_{1,*}$ by lemma 1. If f is not square-integrable, this proof doesn't work any more, since H is not bounded onto C^1_* ; but we may approximate f by $f\varphi(\frac{x}{R})$ where φ is C^∞, compactly supported and $\varphi \equiv 1$ on a neighborhood of 0 ; as $R \to +\infty$ we have

$$< f\varphi(\frac{x}{R}) \mid \psi_{a,b} > \to < f \mid \psi_{a,b} >$$

and for $R \geq 1$,

$$\| f\varphi(\frac{x}{R}) \|_{C^1_*} \leq C(\| f \|_\infty + ||| f |||_{1,*}),$$

so that we obtain

$$|< f \mid \psi_{a,b} >| \leq Ca^{\frac{3}{2}}(\| f \|_\infty + ||| f |||_{1,*}) ;$$

we get rid of the $\| f \|_\infty$ term by homogeneity : if $f(\frac{x}{R}) = f_R$, we have obtained

$$\begin{aligned} |< f_R \mid \psi_{a,b} >| = \sqrt{R}\, |< f \mid \psi_{\frac{a}{R},\frac{b}{R}} >| &\leq Ca^{\frac{3}{2}}(\| f_R \|_\infty + ||| f_R |||_{1,*}) \\ &= Ca^{\frac{3}{2}}(\| f \|_\infty + \frac{1}{R} ||| f |||_{1,*}) \end{aligned}$$

so that

$$\| a^{-\frac{3}{2}} < f \mid \psi_{a,b} > \|_{L^\infty(\mathbb{R}_+,\mathbb{R})} \leq C\,(R \| f \|_\infty + ||| f_{1,*} |||) ;$$

if R goes to 0, we obtain the required estimate.

The case of a general m is then proved by integration by parts. All we have to do is to show that under the hypothesis of proposition 6 ψ can be written as $\psi = (-1)^m \varphi^{(m)}$ where φ satisfies the same assumptions with m replaced by 0 : if $\rho < 1$, then φ is a Morlet wavelet such that $| x |^\rho \varphi \in L^1$, if $\rho = 1$ then φ is a Morlet wavelet such that $x\varphi \in L^1$ and the even part of $x\varphi$ belongs to H^1. We will then have

$$< f \mid \psi_{a,b} > = < f^{(m)} \mid \varphi_{a,b} > a^{-m} \text{ and } |< f \mid \psi_{a,b} >| \leq C a^{-m} a^{-\rho-1/2} ||| f^{(m)} |||_\rho \text{ if } \rho < 1$$

and

$$|< f \mid \psi_{a,b} >| \leq C a^{-m} a^{-\frac{3}{2}} ||| f^{(m)} |||_{1,*} \quad \text{if} \quad \rho = 1.$$

We first integrate ψ one time. We define ω as $\omega = \int_x^{+\infty} \psi(t)dt$. We have as well $\omega = -\int_{-\infty}^x \psi(t)dt$, since $\int \psi \, dx = 0$. Thus :

$$\int | x |^{m+\rho-1} | \omega(x) |$$

$$\leq \int_{-\infty}^0 \int_{-\infty}^x | x |^{m+\rho-1} | \psi(u) | \, du + \int_0^{+\infty} \int_x^{+\infty} | x |^{m+\rho-1} | \psi(u) | \, du$$
$$= \int_{-\infty}^{+\infty} \frac{| u |^{m+\rho}}{m+\rho} | \psi(u) | \, du.$$

We get $| x |^{m+\rho-1} \omega \in L^1$ and, since ω is bounded by $\| \psi \|_1$, we have also $\omega \in L^2 \cap L^1$. We may compute in a similar way $\int x^p \omega \, dx$ and find for $0 \leq p \leq m-1$,

$$\int x^p \omega \, dx = \int \frac{x^{p+1}}{p+1} \psi \, dx = 0.$$

In particular, ω is still a Morlet wavelet. If $\rho < 1$, we have finished : it is enough reiterating m-times the integration to get φ. If $\rho = 1$, we have still to prove that the even part of $x^m \omega$ belongs to H^1 ; we may as well suppose that $x^{m+1}\psi$ belongs to $H^1$and try to prove that $x^m \omega$ belongs to H^1. This is straightforward : since $\int x^p \psi \, dx = 0$ for $0 \leq p \leq m$, we have $H(x^{m+1}\psi) = x^{m+1} H\psi$; if Ω is defined as

$$\Omega = \int_x^{+\infty} H\psi(u)du = -\int_{-\infty}^x H\psi(u)du,$$

we know that

$$\| x^m \Omega \|_1 \leq \frac{1}{m+1} \| x^{m+1} H\psi \|_1 \; ;$$

now it is obvious that $\Omega = H\omega$ by use of the Fourier transform and that $x^m \Omega = H(x^m \omega)$ since $\int x^p \omega \, dx = 0$ for $0 \leq p \leq m-1$. We may then reiterate m-times the integration to get φ. Proposition 6 is entirely proved.

6. Wavelet analysis of pointwise regularity.

The spaces C^α or H^σ have given us a criterion for global regularity of functions. The wavelet analysis allows us to characterize as well the pointwise regularity. S. Jaffard has proved the following theorem [JAF2] (see also M. Holschneider and Ph. Tchamitchian [HOT]).

THEOREM 3 *(Pointwise regularity analysis).* -

(i) Let $\alpha \in (0,1)$, f *be a function such that* $| f(t) | \leq C(1+ | t |)^\alpha$ a. e. *and* ψ *a Morlet wavelet such that* $| x |^\alpha \psi \in L^1$. *Then :*

* *if for some* $x_0 \in \mathbb{R}$ *we have* $| f(x) - f(x_0) | \leq C | x - x_0 |^\alpha$ *where* C *doesn't depend on* x *then for all* $a > 0$ *and* $b \in \mathbb{R}$:

$$|< f | \psi_{a,b} >| \leq C\sqrt{a}(| a |^\alpha + | x_0 - b |^\alpha) ;$$

** *conversely, if* $|< f | \psi_{a,b} >| \leq C\sqrt{a}\left(| a |^\alpha + \frac{|x_0-b|^\alpha}{1+\log^+ \frac{1}{|x_0-b|}}\right)$ *and if moreover* $f \in C^\epsilon$ *for some* $\epsilon > 0$ *then* $| f(x) - f(x_0) | \leq C | x - x_0 |^\alpha$.

ii) Let f *be a function such that* $| f(t) | \leq C(1+ | t |)$ a. e. *and* ψ *be a Morlet wavelet such that* $x\psi \in L^1$ *and* $\int x \psi \, dx = 0$. *Then :*

* *if* f *is differentiable at some point* x_0, *then for all* $a > 0$ *and* $b \in \mathbb{R}$

$$|< f | \psi_{a,b} >| \leq \sqrt{a}(a+ | b - x_0 |)\theta(a+ | b - x_0 |) \quad with \quad \theta(x) \to 0 \quad as \quad x \to 0 ;$$

** *conversely, if* $|< f | \psi_{a,b} >| \leq \sqrt{a}(a\,\omega(a)+ | b - x_0 | \omega(| b - x_0 |))$ *with* ω *a continuous non-decreasing non-negative function on* $[0,+\infty)$ *such that* $\int_0^1 \omega(t)\frac{dt}{t} < +\infty$, *if moreover* f *is* C^ϵ *and* $| x |^{1+\epsilon} \psi \in L^1$ *for some* $\epsilon > 0$ *then* f *is differentiable at* x_0.

Remark. - The extra requirement $f \in C^\epsilon$ cannot be dropped in the converse part of theorem 3, as shown by a counter-example by Y. Meyer given in [JAM].

Proof. i) * Since $\psi \in L^1$ and ψ is a Morlet wavelet, we have $\int \psi \, dx = 0$. Thus

$$< f | \psi_{a,b} >= \int (f(x) - f(x_0))\psi_{a,b} dx$$

and

$$\begin{aligned} < f | \psi_{a,b} > &\leq C \int \{| x - b |^\alpha + | b - x_0 |^\alpha\} | \psi_{a,b}(x) | dx \\ &= C\{\| | x |^\alpha \psi \|_1 \, a^{1/2+\alpha} + \| \psi \|_1 \sqrt{a} \, | b - x_0 |^\alpha\}. \end{aligned}$$

** Since f is bounded ($f \in C^\epsilon \subset L^\infty$) and $| x |^\alpha \psi \in L^1$, we may apply the infra-red cut-off lemma (lemma 2) to get that if h is a C^∞ compactly supported Morlet wavelet such that the $(h_{a,b})$ family is a dual frame for $(\psi_{a,b})$ and if $\tilde{f} = \int_0^1 \int_{-\infty}^{+\infty} < f | \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$ then $f - \tilde{f} \in C^\alpha$. So we have only to prove $| \tilde{f}(x_0) - \tilde{f}(x_0 + y) | \leq C | y |^\alpha$. Moreover $\tilde{f}$ is bounded (since $f \in C^\epsilon$ and $f - \tilde{f} \in C^\alpha$) and we may therefore assume $| y | \leq 1$.

We have :

$$| \tilde{f}(x_0) - \tilde{f}(x_0 + y) | \leq \int_0^1 \int_{-\infty}^{+\infty} |< f | \psi_{a,b} >|| h_{a,b}(x_0) - h_{a,b}(x_0 + y) | \frac{da}{a^2} db$$

and we have already seen that :

$$\int_0^1 \int_{-\infty}^{+\infty} a^{\alpha+1/2} \mid h_{a,b}(x_0) - h_{a,b}(x_0+y) \mid \frac{da}{a^2} db \leq C \mid y \mid^\alpha .$$

We know that

$$|< f \mid \psi_{a,b} >| \leq C a^{\alpha+1/2} + C \inf \left(a^{1/2+\epsilon}, a^{1/2} \frac{\mid b - x_0 \mid^\alpha}{1 + \log^+ \frac{1}{|b-x_0|}} \right).$$

We fix M such that $\operatorname{Supp} h \subset [-M, M]$; we have :

$$\mid \tilde{f}(x_0) - \tilde{f}(x_0+y) \mid \leq$$

$$C \mid y \mid^\alpha + C \int_0^1 \int_{|b-x_0| \geq 100 Ma} \inf \left(a^{1/2+\epsilon}, a^{1/2} \frac{\mid b - x_0 \mid^\alpha}{1 + \log^+ \frac{1}{|b-x_0|}} \right) \mid h_{a,b}(x_0+y) \mid \frac{da}{a^2} db.$$

In the integral, we have $\mid h_{a,b}(x_0+y) \mid \neq 0$ only if $\mid b - x_0 - y \mid \leq Ma$, so that

$$\mid b - x_0 \mid \frac{99}{100} \leq \mid y \mid \leq \frac{101}{100} \mid b - x_0 \mid$$

and

$$\mid \tilde{f}(x_0) - \tilde{f}(x_0+y) \mid \leq C \mid y \mid^\alpha + C \int_0^1 \inf \left(a^\epsilon, \frac{\mid y \mid^\alpha}{1 + \log^+ \frac{1}{|y|}} \right) \frac{da}{a} \leq C' \mid y \mid^\alpha .$$

Thus, we have shown

$$\mid f(x_0) - f(x_0+y) \mid \leq C \mid y \mid^\alpha .$$

ii) * : If f is differentiable at x_0, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \epsilon(x)(x - x_0)$$

where ϵ is bounded on $\mathbb{R}$ and tends to 0 as x tends to x_0. Therefore :

$$< f \mid \psi_{a,b} > = \int \epsilon(x)(x - x_0)\psi_{a,b}(x) dx = \sqrt{a} \int \epsilon(b + ay)(ay + b - x_0)\psi(y) dy,$$

and we get :

$$|< f \mid \psi_{a,b} >| \leq \| \epsilon \|_\infty \sqrt{a} \left(a \int_{|y| \geq \frac{1}{\sqrt{a}}} \mid y \mid \mid \psi(y) \mid dy + \mid b - x_0 \mid \int_{|y| \geq \frac{1}{\sqrt{a}}} \mid \psi(y) dy \right)$$
$$+ \sqrt{a}(a \| x\psi \|_1 + \mid b - x_0 \mid \| \psi \|_1) \sup_{|t-x_0| \leq \sqrt{a} + |b-x_0|} \mid \epsilon(t) \mid$$
$$\leq a^{1/2} o(a + \mid b - x_0 \mid).$$

** We may again replace f by $\tilde{f} = \int_0^1 \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$ where h is a smooth compactly supported Morlet wavelet such that the $(h_{a,b})$ family is a dual frame for $(\psi_{a,b})$: since $f \in L^\infty$ and $\mid x \mid^{1+\epsilon} \psi \in L^1$, the infra-red cut-off lemma tells us that $f - \tilde{f}$ is $C^{1+\epsilon}$, hence differentiable at any point. Moreover $\tilde{f} \in L^\infty$ and we have only to estimate the behaviour of $\tilde{f}(x_0) - \tilde{f}(x_0 + y)$ for small y.

First, we notice that

$$\lambda = \int_0^1 \int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > \frac{d}{dx}(h_{a,b})(x_0) \frac{da}{a^2} db,$$

which is a natural candidate for $\frac{d}{dx}\tilde{f}(x_0)$, is well-defined, the integral being absolutely convergent : if h has its support contained in $[-M, M]$, the integration domain can be restricted to $\mid b - x_0 \mid \leq Ma$, so that

$$\mid < f \mid \psi_{a,b} > \mid \leq C\, a^{\frac{3}{2}} \omega((M+1)a)$$

and

$$\int_0^1 \int_{-\infty}^{+\infty} \mid < f \mid \psi_{a,b} > \mid \mid \frac{d}{dx}(h_{a,b})(x_0) \mid \frac{da}{a^2} db \leq C \parallel h' \parallel_1 \int_0^{M+1} \omega(a) \frac{da}{a}.$$

This proof gives us that

$$\int_0^1 \int_{\mid b - x_0 \mid \leq 100Ma} < f \mid \psi_{a,b} > h_{a,b} \frac{da}{a^2} db$$

is differentiable (everywhere) with derivative

$$\int_0^1 \int_{\mid b - x_0 \mid \leq 100Ma} < f \mid \psi_{a,b} > (h')_{a,b} \frac{da}{a^3} db.$$

Now if $\mid b - x_0 \mid \geq 100Ma$, $h_{a,b}(x_0) = 0$ while $h_{a,b}(x_0 + y) \neq 0$ implies that $\mid x_0 + y - b \mid \leq Ma$ so that $\frac{99}{100} \mid x_0 - b \mid \leq \mid y \mid \leq \frac{101}{100} \mid x_0 - b \mid$. We obtain

$$\begin{aligned} &\int_0^1 \int_{\mid b - x_0 \mid \geq 100Ma} \mid < f \mid \psi_{a,b} > \mid \mid h_{a,b}(x_0 + y) \mid \frac{da}{a^2} db \\ \leq &C \int_0^1 \inf\{a^\epsilon, \mid y \mid \omega(2 \mid y \mid)\} \frac{da}{a} \\ \leq &C \mid y \mid \omega(2 \mid y \mid) \left(1 + \log^+ \frac{1}{\mid y \mid \omega(2 \mid y \mid)}\right). \end{aligned}$$

It is straightforward that $\lim_{\gamma \to 0^+} \omega(\gamma) = \lim_{\gamma \to 0} \omega(\gamma) \log^+ \frac{1}{\gamma} = 0$; we just have to show that $\lim_{\gamma \to 0^+} \omega(\gamma) \log^+ \frac{1}{\gamma} = 0$: this is obvious since $\frac{1}{2}\omega(\gamma)\log\frac{1}{\gamma} \leq \int_\gamma^{\sqrt{\gamma}} \omega(t) \frac{dt}{t} \to 0$ as $\gamma \to 0^+$. Theorem 3 is now entirely proved. ■

EXAMPLE *(Lacunary Fourier series). - Let* $f = \sum_{k=0}^{+\infty} q_k \cos(n_k x + \varphi_k)$ *be a function such that* $\sum | q_k | < +\infty$ *and* $\frac{n_{k+1}}{n_k} \geq r > 1$. *Then* f *has the same regularity at every point :*

* *if* f *is* C^α *at one point* x_0 *for some* $\alpha \in (0,1)$ *(i.e.* $\sup_{y \neq x_0} \frac{|f(y)-f(x_0)|}{|y-x_0|^\alpha} < +\infty$*) then* f *is* C^α *globally (i.e.* $\sup_{x,y;x\neq y} \frac{|f(y)-f(x)|}{|y-x|^\alpha} < +\infty$*) ;*

** *if* f *is differentiable at one point* x_0 *then* $q_k = o\left(\frac{1}{n_k}\right)$, *and if* $| q_k | \leq \omega(n_k)\frac{1}{n_k}$ *where* ω *is decreasing, non-negative and satisfies the Dini condition* $\int_0^1 \omega(\frac{1}{t})\frac{dt}{t} < +\infty$, *then* f *is differentiable everywhere.*

Proof. Choose a wavelet $\omega \in \mathcal{S}(\mathbb{R})$ such that ω is even and $\operatorname{Supp} \hat{\omega} \subset [-r,-1] \cup [1,r]$ and define $\tilde{\omega} = H\omega$. We then have :

$$< f \mid \omega_{a,b} >= \sqrt{a} \sum_{k=0}^{\infty} q_k \omega(a n_k) \cos(\varphi_k + n_k b)$$

$$< f \mid \tilde{\omega}_{a,b} >= \sqrt{a} \sum_{k=0}^{\infty} q_k \omega(a n_k) \sin(\varphi_k + n_k b).$$

In these sums, at most one term is non-zero : if k_0 is such that $1 < a n_{k_0} < r$ then

$$< f \mid \omega_{a,b} >= \sqrt{a} q_{k_0} \omega(a n_{k_0}) \cos(\varphi_{k_0} + n_{k_0} b)$$

and

$$< f \mid \tilde{\omega}_{a,b} >= \sqrt{a} q_{k_0} \omega(a n_{k_0}) \sin(\varphi_{k_0} + n_{k_0} b)$$

(while $< f \mid \omega_{a,b} >=< f \mid \tilde{\omega}_{a,b} >= 0$ in the case where there is no such k_0) ; thus

$$|< f \mid \omega_{a,b} >|^2 + |< f \mid \tilde{\omega}_{a,b} >|^2 = a \mid q_{k_0} \mid^2 \mid \omega(a n_{k_0}) \mid^2$$

doesn't depend on b and the regularity will be the same at all points.

Now we choose $\theta \in [1,r]$ such that $\omega(\theta) \neq 0$. For $k \in \mathbb{N}$, we fix $a_k = \frac{\theta}{n_k}$. If f is C^α at some point x_0, we find $| q_k | \leq \frac{C}{\omega(\theta)} \left|\frac{\theta}{n_k}\right|^\alpha$, so that $(n_k^\alpha q_k) \in \ell^\infty$; this in turn implies for every a and b, $|< f \mid \omega_{a,b} >| \leq C a^{1/2+\alpha}$ and f is C^α at every point.

Similarly, if f is differentiable at some point x_0, we have $|< f \mid \omega_{a,x_0} >| = o(a)$ and $|< f \mid \tilde{\omega}_{a,x_0} >| = o(a)$ so that $q_k = o\left(\frac{1}{n_k}\right)$. The converse part is given by point ii) ** of theorem 3 (since that if $q_k = o\left(\frac{1}{n_k}\right)$ then $q_k \in C_*^1$). ■

In particular, we see that the Zygmund class C_*^1 is very different from the class of the Lipschitz functions : the function $\sum_0^\infty \frac{1}{2^k} \sin 2^k x$ belongs to C_*^1 and is differentiable nowhere (while a Lipschitz function is differentiable almost everywhere).

Chapter 2

DISCRETE WAVELET TRANSFORMS

In this chapter, we will see the various ways of defining a discrete wavelet transform. From a functional point of view, it means to turn a continuous representation (by mean of a function of two continuous parameters) into a discrete one (as a series, or a two-parameters sequence). This is easily handled by sampling the Morlet wavelet coefficients on a hyperbolic grid, such a grid being suited to non over-redundant representations of functions by mean of affine wavelets $\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$. The discretization of the Morlet wavelet representation leads to numerical wavelet processing, but introduces as a new difficulty the proeminent part played by the dual frame (which is no more arbitrary, as it was in the integral transform).

1. Sampling theorems for the Morlet wavelet representation.

In order to turn his wavelet representation into a numerically efficient algorithm, J. Morlet proposed to approximate this representation by sampling the wavelet coefficients on a "regular hyperbolic" grid, i.e. by keeping the wavelet coefficients $< f \mid \psi_{a,b} >$ for $(a,b) = (a_0^m, na_0^m b_0)$, where m and n are integers and the numbers $a_0 > 1$ and $b_0 > 0$ are fixed. Morlet's idea was that if the grid was fine enough (a_0 close enough to 1 and b_0 to 0) then the family $(\psi_{(m,n)} = \psi_{a_0^m, na_0^m b_0})_{m\in\mathbb{Z}, n\in\mathbb{Z}}$ would be a frame in $L^2(\mathbb{R})$ for the counting measure on $\mathbb{Z}\times\mathbb{Z}$. This idea was proved to be good in a very illuminating paper by I. Daubechies [DAU2] under some slight restrictions on ψ. Such restrictions are unavoidable because not every Morlet wavelet can be sampled into a discrete frame :

THEOREM 1 (Regular sampling theorem). - *Let ψ be a (non identically zero) real-valued square integrable function. For fixed $a_0 > 1$ and $b_0 > 0$, we note*

$$\psi_{(m,n)} = a_0^{-m/2}\psi(a_0^{-m}x - nb_0) \quad (m \in \mathbb{Z},\ n \in \mathbb{Z}).$$

Then :

i) if the $(\psi_{(m,n)})$ family is a frame in $L^2(\mathbb{R})$ for the counting measure on $\mathbb{Z}^2$, i.e. if for two positive constants A and B we have

$$\text{(1)} \qquad \textit{for all } f \in L^2(\mathbb{R}), \quad A \| f \|_2^2 \le \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{(m,n)} >|^2 \le B \| f \|_2^2,$$

then we must have almost everywhere on $\mathbb{R}$:

$$\text{(2)} \qquad A \le \frac{1}{b_0}\sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m\xi)|^2 \le B \ ;$$

in particular, ψ is a Morlet wavelet :

(3) $$\int_0^{+\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < +\infty ;$$

ii) if for some $\epsilon > 0$, ψ satisfies $|x|^{1/2+\epsilon}\psi \in L^2$ and $|\xi|^\epsilon \hat{\psi}(\xi) \in L^2$ and if $\int \psi\, dx = 0$ (so that ψ is a Morlet wavelet) then ψ satisfies

(4) $$\text{ess.inf.} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m\xi)|^2 > 0 \ \ and \ \ \text{ess.sup.} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m\xi)|^2 < +\infty$$

for any a_0 close enough to 1 [i.e. there exists $\alpha(\psi) > 1$ such that (4) is satisfied for any $a_0 \in (1, \alpha(\psi))$] ; moreover for any a_0 such that ψ satisfies (4) a sufficient condition for the family $(\psi_{(m,n)})_{m\in\mathbb{Z},n\in\mathbb{Z}}$ associated to a_0 and to some positive b_0 to be a frame is that b_0 is close enough to 0 (i.e. $b_0 \in (0, \beta(a_0, \psi))$ for some positive $\beta(a_0,\psi)$ depending on a_0 and ψ).

Part i) of theorem 1 was proved by C. K. Chui and X. Shi [CHS]. Part ii) was proved by I. Daubechies under different hypotheses on ψ [roughly speaking, $\hat{\psi}$ was required to be continuous, to be $0(|\xi|^\epsilon)$ for some $\epsilon > 0$ in the neighborhood of 0 and to be $0(|\xi|^{-1-\epsilon})$ in the neighborhood of infinity]. We choose a different requirement on ψ not only for the sake of originality, but mainly for three reasons. The first one is that we shall deal in the following chapters with wavelets which don't satisfy Daubechies' requirement while they satisfy ours (for instance, the celebrated Haar basis). The second one is that the main tool in our proof will be a theorem (the "vaguelettes lemma") which is of constant use in wavelet-related operator theory. The third one is that this proof works as well for an irregular sampling as for a regular one :

THEOREM 2 (Irregular sampling theorem). - *Let ψ be a Morlet wavelet such that for some $\epsilon > 0$, $|x|^{1/2+\epsilon}\psi \in L^2$ (so that $\int \psi\, dx = 0$) and $|\xi|^\epsilon \hat{\psi} \in L^2$. Let $(a_\alpha, b_\alpha)_{\alpha\in A}$ be a countable family of points in $(0,+\infty) \times \mathbb{R}$ and let $\psi_{a_\alpha,b_\alpha} = a_\alpha^{-1/2}\psi\left(\frac{x-b_\alpha}{a_\alpha}\right)$.*

i) A necessary condition for $(\psi_{a_\alpha,b_\alpha})_{\alpha\in A}$ to be a frame in $L^2(\mathbb{R})$ for the counting measure on A is that there exist numbers $\lambda > 1$, $\theta > 0$ and $N \in \mathbb{N}^*$ such that

(5) $$\forall a > 0,\ \forall b \in \mathbb{R},\ \ 1 \le \text{Card}\left\{\alpha \in A / \frac{1}{\lambda}a \le a_\alpha \le \lambda a \ \ and \ \ |b - b_\alpha| \le \theta a\right\} \le N$$

(where Card E is the cardinal of the set E).

ii) Conversely, a sufficient condition for $(\psi_{a_\alpha,b_\alpha})_{\alpha\in A}$ to be a frame is that the family $(a_\alpha, b_\alpha)_{\alpha\in A}$ satisfies (5) for some $N \in \mathbb{N}^*$ and some λ close enough to 1 and some θ close enough to 0 [i.e. there exist $\lambda_0 > 1$ and $\theta_0 > 0$ such that $(\psi_{a_\alpha,b_\alpha})_{\alpha\in A}$ is a frame whenever it satisfies (5) for some $N \in \mathbb{N}^*$, some $\lambda \in (1, \lambda_0)$ and some $\theta \in (0, \theta_0)$].

This sampling theorem shows that Morlet's choice of a regular hyperbolic grid to discretize his wavelet representation is somehow optimal. Irregular sampling has been recently discussed by several authors, see e.g. K. H. Gröchenig [GROE2], as a robustness criterion for the regular sampling which thus is proved to be poorly affected by a small perturbation of the sampling points.

Theorem 2 shows also that the information conveyed by $< f \mid \psi_{a,b} >$ can be viewed as an average information about f on $[b-a\theta, b+a\theta]$ and about $\hat{f}$ on $\left[-\frac{\lambda}{a}, -\frac{1}{\lambda a}\right] \cup \left[\frac{1}{\lambda a}, \frac{\lambda}{a}\right]$, as we pointed it in the preceding chapter : in order to have a frame, we need a complete but not over-redundant information, and this is the meaning of condition (5).

We postpone the proof of theorems 1 and 2 to sections III and IV and devote section II to the "vaguelettes lemma".

2. The vaguelettes lemma and related results for the $\mathbf{H}_{\epsilon,\epsilon'}$ spaces.

The vaguelettes lemma is an almost orthonormality criterion for functions generated through dilations and translations :

THEOREM 3 (The vaguelettes lemma). *Let ϵ and ϵ' be positive real numbers such that $\epsilon > \frac{1}{2}$. Let $(f_{(j,k)})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be a bounded family in $H_{\epsilon,\epsilon'}$:*

(6.1) $$\sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} \left(\| \mid x \mid^{\epsilon} f_{(j,k)} \|_2 + \| \mid \xi \mid^{\epsilon'} \hat{f}_{(j,k)} \|_2 \right) < +\infty$$

such that

(6.2) $$\text{for all } j \in \mathbb{Z} \text{ and all } k \in \mathbb{Z}, \quad \int f_{(j,k)} dx = 0.$$

Let A be a positive real number greater than 1 $(A > 1)$. Then the family $(\psi_{(j,k)})$ defined by

(6.3) $$\psi_{(j,k)} = A^{j/2} f_{(j,k)}(A^j x - k)$$

is an almost orthogonal family in $L^2(\mathbb{R})$. More precisely, there exists a constant $C(A, \epsilon, \epsilon')$ such that for all sequences $(\lambda_{j,k}) \in \ell^2(\mathbb{Z}^2)$ and all such families $(\psi_{(j,k)})$:

(6.4) $$\| \sum_j \sum_k \lambda_{j,k} \psi_{(j,k)} \|_2 \leq C(A, \epsilon, \epsilon') \left(\sum_j \sum_k \mid \lambda_{j,k} \mid^2 \right)^{1/2} \sup_j \sup_k \| f_{(j,k)} \|_{H_{\epsilon,\epsilon'}} .$$

Remark. - The vaguelettes lemma has been introduced for the analysis of singular integral operators (see e. g. Yves Meyer [MEY3]) with slightly stronger requirements on the $f_{(j,k)}$: they were required to satisfy for some positive ϵ_0 and ϵ'_0 :

$$\sup_{j,k} \left\{ \| (1+ \mid x \mid)^{1+\epsilon_0} f_{(j,k)} \|_\infty + \sup_{x,y,x\neq y} \frac{\mid f_{(j,k)}(x) - f_{(j,k)}(y) \mid}{\mid x - y \mid^{\epsilon'_0}} \right\} < +\infty$$

(and of course $\int f_{(j,k)} dx = 0$). Such functions $f_{(j,k)}$ satisfy (6.1) for any $\epsilon \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon_0\right)$ and $\epsilon' \in \left(0, \frac{1+2\epsilon_0}{2+2\epsilon_0}\epsilon'_0\right)$. ■

Proof. The proof is very simple. We will first check the one-scale almost orthonormality (and need for that only the space localization $\sup_{j,k} \| (1+ | x |)^\epsilon f_{(j,k)} \|_2 < +\infty$ with $\epsilon > \frac{1}{2}$). We will thereafter address the almost orthogonality between scales with a very simple tool: integration by parts (and need for "differentiation" the regularity requirement

$$\sup_{j,k} \| | \xi |^{\epsilon'} f_{(j,k)} \|_2 < +\infty$$

and for "integration" the vanishing mean condition $\int f_{(j,k)} dx = 0$).

LEMMA 1. - *For any $\epsilon > \frac{1}{2}$ there is a positive constant $C(\epsilon)$ such that for every $(\lambda_k)_{k\in\mathbb{Z}} \in \ell^2$ and every sequence (f_k) in $L^2\left((1+ | x |)^{2\epsilon} dx\right)$*

$$(7) \qquad \| \sum_{k\in\mathbb{Z}} \lambda_k f_k(x-k) \|^2_{L^2(dx)} \leq C(\epsilon) \sum_{k\in\mathbb{Z}} | \lambda_k |^2 \sup_{k\in\mathbb{Z}} \| (1+ | x |)^\epsilon f_k \|_2^2 .$$

This is obvious

$$\int | \sum_{k\in\mathbb{Z}} \lambda_k f_k(x-k) |^2 \, dx$$

$$\leq \int \sum_{k\in\mathbb{Z}} | \lambda_k |^2 (1+ | x-k |)^{2\epsilon} | f_k(x-k) |^2 \sum_{k\in\mathbb{Z}} \frac{1}{(1+ | x-k) |^{2\epsilon}} dx$$

$$\leq \| \sum_{k\in\mathbb{Z}} \frac{1}{(1+ | x-k |)^{2\epsilon}} \|_\infty \sum_{k\in\mathbb{Z}} | \lambda_k |^2 \| (1+ | x |)^\epsilon f_k \|_2^2 . \quad ■$$

LEMMA 2. - *For all positive ϵ and ϵ', and for all $\alpha \in (0, \epsilon')$, the fractional derivation operator D^α (defined by $\widehat{D^\alpha f}(\xi) = | \xi |^\alpha \hat{f}(\xi)$) is bounded from $H_{\epsilon,\epsilon'}$ to $L^2((1+ | x |)^{2\eta} dx)$ if $\eta < \frac{1}{2} + \alpha$ and $\eta \leq (1 - \frac{\alpha}{\epsilon'})\epsilon$. If $\epsilon > \frac{1}{2}$ and $\alpha < \frac{\epsilon'\epsilon}{\epsilon+1/2}$, we thus may choose $\eta > \frac{1}{2}$.*

Let σ be a smooth function on $\mathbb{R}$ such that $\sigma(x) > 0$ for all x and $\sigma(x) = | x |$ for $| x | \geq 1$. If $f \in H_{\epsilon,\epsilon'}$, we know that $\hat{f}$ belongs to the Sobolev space H^ϵ and that $| \xi |^{\epsilon'} \hat{f}$ belongs to $L^2 = H^0$; hence we may use complex interpolation between H^ϵ and H^0 to get that $\sigma^\alpha \hat{f}$ belongs to $H^{(1-\frac{\alpha}{\epsilon'})\epsilon}$ if $\alpha \in (0, \epsilon')$. But we have $\widehat{D^\alpha f} = \frac{|\xi|^\alpha}{\sigma^\alpha} \sigma^\alpha \hat{f}$ and it is easy to see that $\frac{|\xi|^\alpha}{\sigma(\xi)^\alpha}$ is a multiplier for H^η if and only if $| \eta | < \frac{1}{2} + \alpha$. ■

LEMMA 3. - *Let ϵ be a positive number such that $\epsilon > \frac{1}{2}$. Let $\alpha \in (0,1)$ be such that $\alpha < \epsilon$. Then the fractional integration operator I^α (defined by $\widehat{I^\alpha f}(\xi) = | \xi |^{-\alpha} \hat{f}(\xi)$) is bounded from $L^2((1+ | x |)^{2\epsilon} dx) \cap \{f \in L^1 / \int f \, dx = 0\}$ to $L^2((1+ | x |)^{2\eta} dx)$ whenever $\eta < \min(\epsilon - \alpha, \frac{3}{2} - \alpha)$.*

In particular, if $\alpha < \epsilon - \frac{1}{2}$, one may choose $\eta > \frac{1}{2}$.

In order to show that $|\xi|^{-\alpha}\hat{f}(\xi)$ belongs to H^η, we have no problem for ξ far away from 0, since far from $\{0\}$ $|\xi|^{-\alpha}$ is a smooth multiplier with bounded derivatives. We thus fix a smooth function $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 2$ and we will show that $|\xi|^{-\alpha}\varphi\hat{f}$ belongs to H^η. Let's call γ the inverse Fourier transform of $|\xi|^{-\alpha}\varphi$, so that the problem is now to show that $f * \gamma$ belongs to $L^2((1+|x|)^{2\eta}dx)$.

It is easy to check that :

$$|\gamma(x)| \leq C(1+|x|)^{-1+\alpha} \tag{8.1}$$

$$\left|\frac{d}{dx}\gamma(x)\right| \leq C(1+|x|)^{-2+\alpha}. \tag{8.2}$$

Moreover, since $\int f\,dx = 0$, we may write

$$\gamma * f(x) = \int f(y)(\gamma(x-y) - \gamma(x))dy,$$

and hence

$$|\gamma * f(x)| \leq C\left(I_1(x) + I_2(x) + I_3(x)\right),$$

where :

$$I_1(x) = \int_{|y|<\frac{|x|}{2}} |f(y)| \frac{|y|}{(1+|x|)^{2-\alpha}}dy$$

$$I_2(x) = \int_{|y|>\frac{|x|}{2}} |f(y)| \frac{dy}{(1+|x|)^{1-\alpha}}$$

$$I_3(x) = \int_{|x-y|<\frac{|x|}{2}} |f(y)| \frac{1}{(1+|x-y|)^{1-\alpha}}dy.$$

Now, it is easy to check that :

$$\int |I_1(x)|^2 (1+|x|)^{2\eta}dx$$

$$\leq \|(1+|x|)^\epsilon f\|_2^2 \int\int_{|y|<\frac{1}{2}|x|} \frac{(1+|x|)^{2\eta}}{(1+|x|)^{4-2\alpha}} \frac{|y|^2}{(1+|y|)^{2\epsilon}}dx\,dy$$

and that the last integral converges if and only if $\alpha + \eta < \frac{3}{2}$ and $\alpha + \eta < \epsilon$;

$$\int |I_2(x)|^2 (1+|x|)^{2\eta}dx$$

$$\leq \|(1+|x|)^\epsilon f\|_2^2 \int\int_{|y|>\frac{|x|}{2}} \frac{(1+|x|)^{2\eta}}{(1+|x|)^{2-2\alpha}} \frac{1}{(1+|y|)^{2\epsilon}}dx\,dy$$

and the last integral converges if and only if $\alpha+\eta<\epsilon$; finally, we have for any $\delta>0$,

$$I_3(x)(1+|x|)^\eta \leq \int_{|x-y|<\frac{|x|}{2}} 2^\eta(1+|y|)^{\eta+\delta}|f(y)|\frac{1}{(1+|x-y|)^{1-\alpha+\delta}}dy\ ;$$

if $\delta>\alpha$, $(1+|x|)^{-1+\alpha-\delta}\in L^1$ and if $\eta+\delta<\epsilon$, $(1+|x|)^{\eta+\delta}f\in L^2$ so that if $\alpha+\eta<\epsilon$ we have $I_3(x)(1+|x|)^\eta\in L^2$. ■

Proof of the vaguelettes lemma.

We may now end up the proof of theorem 3. We define

$$F_j=\sum_{k\in\mathbb{Z}}\lambda_{j,k}f_{(j,k)}(x-k)$$

and

$$G_j=\sum_{k\in\mathbb{Z}}\lambda_{j,k}\psi_{(j,k)}=A^{j/2}F_j(A^jx).$$

We choose a positive α such that I^α and D^α are bounded from $H_{\epsilon,\epsilon'}\cap\{f\in L^1/\int f\,dx=0\}$ to $L^2((1+|x|)^{2\eta}dx)$ for some $\eta>\frac{1}{2}$ (as we may do by lemmas 2 and 3). Now, we have the following invariance properties for I^α and D^α :

$$I^\alpha\{f(x-k)\}=\{I^\alpha f\}(x-k)\ \ ,\ \ D^\alpha\{f(x-k)\}=\{D^\alpha f\}(x-k),$$
$$I^\alpha\{f(Ax)\}=A^{-\alpha}\{I^\alpha f\}(Ax)\ \ ,\ \ D^\alpha\{f(Ax)\}=A^\alpha\{D^\alpha f\}(Ax).$$

Hence, we have :

$$\| D^\alpha G_j\|_2=A^{\alpha j}\| D^\alpha F_j\|_2\leq C\,A^{\alpha j}\left(\sum_k|\lambda_{j,k}|^2\right)^{1/2}\quad\text{(by lemma 1)}$$

$$\| I^\alpha G_j\|_2=A^{-\alpha j}\| I^\alpha F_j\|_2\leq C\,A^{-\alpha j}\left(\sum_k|\lambda_{j,k}|^2\right)^{1/2}$$

and therefore, for $j\geq\ell$:

$$|<G_j\mid G_\ell>|=|<I^\alpha G_j\mid D^\alpha G_\ell>|\leq C\,A^{-\alpha|j-\ell|}\left(\sum_k|\lambda_{j,k}|^2\right)^{1/2}\left(\sum_k|\lambda_{\ell,k}|^2\right)^{1/2}$$

and this last estimate gives us :

$$\begin{aligned}\|\sum_j\sum_k\lambda_{j,k}\psi_{(j,k)}\|_2^2&=\sum_j\sum_\ell<G_j\mid G_\ell>\\&\leq C\sum_j\sum_\ell A^{-\alpha|j-\ell|}\left(\sum_k|\lambda_{j,k}|^2\right)^{1/2}\left(\sum_k|\lambda_{\ell,k}|^2\right)^{1/2}\\&\leq C\left(1+2\frac{A^{-\alpha}}{1-A^{-\alpha}}\right)\sum_j\sum_k|\lambda_{j,k}|^2\,.\end{aligned}$$

We thus have shown the almost orthonormality of the family $(\psi_{(j,k)})$. ∎

Remark. - In case of $\psi_{(j,k)} = A^{j/2}\psi(A^j x - k)$, the vaguelettes lemma can be greatly simplified :

PROPOSITION 1. - *For the family* $(A^{j/2}\psi(A^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ *(where* $\psi \in L^2$ *and* $A > 1$ *are fixed) to be almost orthonormal in* $L^2(\mathbb{R})$, *it is enough to have for some positive* α :

(9.1) $$\text{ess. sup.} \sum_{k\in\mathbb{Z}} | \xi + 2k\pi |^{2\alpha} | \hat{\psi}(\xi + 2k\pi) |^2 < +\infty$$

(9.2) $$\text{ess. sup.} | \xi |^{-\alpha} | \hat{\psi}(\xi) | < +\infty.$$

Proof. This is based on the following remark : for any $f \in L^2(\mathbb{R})$, the following assertions are equivalent :

(10.1) The family $(f(x-k))_{k\in\mathbb{Z}}$ is almost orthonormal in $L^2(\mathbb{R})$;

(10.2) the function $\sum_{k\in\mathbb{Z}} | \hat{f}(\xi + 2k\pi) |^2$ is essentially bounded.

Now (9.1) and (9.2) ensure us that the families $(D^\alpha\psi(x-k))_{k\in\mathbb{Z}}$ and $(I^\alpha\psi(x-k))_{k\in\mathbb{Z}}$ are almost orthonormal, and the proof of the proposition is similar to the end of the proof of the vaguelettes lemma. ∎

Of course, if ψ belongs to $H_{\epsilon,\epsilon'}$ (with $\epsilon > 1/2$ and $\epsilon' > 0$) and satisfies $\int \psi\, dx = 0$, it satisfies (9.1) and (9.2) for some positive α, as we have seen it in lemmas 1 to 3.

3. Proof of the regular sampling theorem.

We now prove theorem 1. We begin with the necessary condition. We suppose that

$$(\psi_{(m,n)} = a_0^{-m/2}\psi(a_0^{-m}x - nb_0))_{m\in\mathbb{Z},n\in\mathbb{Z}}$$

is a frame :

(11) $$\forall f \in L^2(\mathbb{R}),\ \ A \| f \|_2^2 \le \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |< f | \psi_{(m,n)} >|^2 \le B \| f \|_2^2$$

and we want to prove

(12) $$A \le \frac{1}{b_0} \sum_{m\in\mathbb{Z}} | \hat{\psi}(a_0^m \xi) |^2 \le B \text{ a.e.}$$

This will be easily done due to the following lemma :

LEMMA 4. - *We note* $f_y(x) = f(x-y)$. *If* $(\psi_{(m,n)})$ *is a frame then for all* $f \in L^2(\mathbb{R})$ *we have :*
(13)

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f_y \mid \psi_{(m,n)} >|^2\, dy = \frac{1}{b_0}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m \xi)|^2\, d\xi.$$

Proof. The family of operators $(L_T)_{T>0}$ defined by

$$L_T(f) = \left\{\frac{1}{2T}\int_{-T}^{T} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f_y \mid \psi_{(m,n)} >|^2\, dy\right\}^{1/2}$$

is clearly an equicontinuous family of sublinear operators from $L^2(\mathbb{R})$ to $\mathbb{R}^+$ (since $L_T(f) \le \sqrt{B}\, \| f \|_2$). Therefore, it is enough to prove (13) for a dense subset of $L^2(\mathbb{R})$.

We define $Q_m(f)$ as $Q_m(f) = \sum_{n\in\mathbb{Z}} |< f \mid \psi_{(m,n)} >|^2$. Then we have :

$$Q_m(f_y) = \frac{1}{2\pi b_0}\int_0^{2\pi\, 1/b_0 a_0^m} \left|\sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2n\pi\frac{1}{b_0 a_0^m}) e^{-iy(\xi+2n\pi\frac{1}{b_0 a_0^m})}\, \hat{\psi}(a_0^m\xi + \frac{2n\pi}{b_0})\right|^2 d\xi$$

and we get, by Fubini's theorem :

$$\begin{aligned}
&\frac{b_0 a_0^m}{2\pi}\int_X^{X+2\pi/b_0 a_0^m} Q_m(f_y)dy \\
&\quad = \frac{1}{2\pi b_0}\int_0^{2\pi\, 1/b_0 a_0^m} \sum_{n\in\mathbb{Z}} |\hat{f}(\xi + 2n\pi\frac{1}{b_0 a_0^m})|^2 |\hat{\psi}(a_0^m\xi + \frac{2n\pi}{b_0})|^2\, d\xi \\
&\quad = \frac{1}{2\pi b_0}\int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 |\hat{\psi}(a_0^m\xi)|^2\, d\xi.
\end{aligned}$$

We have thus proved a one-scale version of equality (13) :

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} Q_m(f_y)dy = \frac{1}{2\pi b_0}\int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 |\hat{\psi}(a_0^m\xi)|^2\, d\xi.$$

Now, Fatou's lemma gives us :

$$\begin{aligned}
&\frac{1}{2\pi b_0}\int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 \sum_m |\hat{\psi}(a_0^m\xi)|^2\, d\xi \\
&\quad = \sum_m \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} Q_m(f_y)dy \\
&\quad \le \liminf_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \sum_m Q_m(f_y)dy \le B\, \| f \|_2^2
\end{aligned}$$

and we get therefore :

$$\sum_m \mid \hat{\psi}(a_0^m \xi) \mid^2 \leq b_0 B \quad \text{a. e.}$$

We will now prove (13) for f such that $\hat{f}$ is C^∞, $\hat{f}$ has compact support and $0 \notin \operatorname{supp} \hat{f}$. In that case, we have :

$Q_m(f_y) \leq$

$$\frac{1}{2\pi b_0} \int_0^{2\pi\, 1/b_0 a_0^m} \sum_{n\in\mathbb{Z}} \mid \hat{f}(\xi + 2n\pi \frac{1}{b_0 a_0^m}) \mid^2 \sum_{n\in\mathbb{Z}} \mid \varphi(\xi + 2n\pi \frac{1}{b_0 a_0^m}) \mid^2 \mid \hat{\psi}(a_0^m \xi + \frac{2n\pi}{b_0}) \mid^2 d\xi$$

$$\leq \frac{1}{2\pi b_0} \| \sum_{n\in\mathbb{Z}} \mid \hat{f}(\xi + 2n\pi \frac{1}{a_0^m b_0}) \mid^2 \|_\infty \int_{-\infty}^{+\infty} \mid \varphi(\xi) \mid^2 \mid \hat{\psi}(a_0^m \xi) \mid^2 d\xi$$

where φ is a C^∞ function with compact support such that $0 \notin \operatorname{Supp} \varphi$ and $\varphi \equiv 1$ on $\operatorname{Supp} f$. Thus we have :

$$Q_m(f_y) \leq C_0 (1 + a_0^m) \int_{-\infty}^{+\infty} \mid \hat{\varphi}(\xi) \mid^2 \mid \hat{\psi}(a_0^m \xi) \mid^2 d\xi$$

where the constant C_0 doesn't depend on m (but depends on f). But we have :

$$\sum_m \int_{-\infty}^{+\infty} \mid \hat{\varphi}(\xi) \mid^2 \mid \hat{\psi}(a_0^m \xi) \mid^2 (1 + a_0^m) d\xi$$

$$= \int_{-\infty}^{+\infty} \mid \hat{\varphi}(\xi) \mid^2 \sum_m \mid \hat{\psi}(a_0^m \xi) \mid^2 d\xi + \int_{-\infty}^{+\infty} \mid \hat{\psi}(\xi) \mid^2 \sum_m \mid \hat{\varphi}(a_0^m \xi) \mid^2 d\xi.$$

The last two integrals are convergent since we already know that $\sum_m \mid \hat{\psi}(a_0^m \xi) \mid^2$ is bounded and since the boundedness of $\sum_m \mid \hat{\varphi}(a_0^m \xi) \mid^2$ is obvious. Thus, we may apply Lebesgue's dominated convergence theorem to conclude that (13) is satisfied for any f such that $\hat{f}$ is a C^∞ compactly supported function with $0 \notin \operatorname{Supp} \hat{f}$, hence for all $f \in L^2$. ∎

The necessary part of theorem 1 has thus been proved. For the sufficient part, we may just notice that, if $\mid x \mid^{1/2+\epsilon} \psi$ is square-integrable, the Sobolev injection theorem tells us that $\hat{\psi}$ is C^ϵ (we will suppose $\epsilon \in (0,1)$) ; since $\hat{\psi}(0) = 0$ ($\int \psi\, dx = 0$), $\hat{\psi}$ is $0(\mid \xi \mid^\epsilon)$ near 0. We know also (lemma 2 in section II) that for some positive α and η, $\mid \xi \mid^\alpha \hat{\psi}$ belongs to the Sobolev space $H^{1/2+\eta}$, hence $\hat{\psi}$ is $0(\mid \xi \mid^{-\alpha})$ in the neighborhood of infinity. Thus, for any $a > 1$, $\sum_{m\in\mathbb{Z}} \mid \hat{\psi}(a^m \xi) \mid^2$ converges uniformly on every compact of $\mathbb{R}\backslash\{0\}$. Moreover $\hat{\psi}$ is not identically 0 and therefore $\hat{\psi}$ has no zero on some interval $[A_0, B_0]$ $(0 < A_0 < B_0)$ (and on $[-B_0, -A_0]$ as well by Hermitian symmetry) ; hence $\sum_{m\in\mathbb{Z}} \mid \hat{\psi}(a^m \xi) \mid^2$ doesn't vanish on $\mathbb{R}^*$ for any $a \in \left(1, \frac{B_0}{A_0}\right)$. Therefore, ψ satisfies (4) for small enough a_0.

Now we fix a_0 such that (4) is satisfied. We will write $\psi^{b_0}_{(m,n)}$ for $\psi_{a_0^m, nb_0 a_0^m}$; the problem is to show that we control $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi^{b_0}_{(m,n)} >|^2$ for small enough b_0.

LEMMA 5. - *For any $\epsilon > 1/2$, $\epsilon' > 0$, $a_0 > 1$, there exists a constant C_0 such that for every $\psi \in H_{\epsilon,\epsilon'}$ such that $\int \psi\, dx = 0$ and every $f \in L^2$ we have :*

$$(14.1) \qquad \forall b_0 \in (0,1), \quad \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, b_0 a_0^m} >|^2 \leq \frac{C_0}{b_0} \| \psi \|^2_{H_{\epsilon,\epsilon'}} \| f \|_2^2$$

$$(14.2) \qquad \lim_{b_0\to 0} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} b_0\, |< f \mid \psi_{a_0^m, nb_0 a_0^m} >|^2 = \int \sum_m a_0^{-m} |< f \mid \psi_{a_0^m, b} >|^2\, db.$$

Proof. We first suppose $b_0 \in [1/2, 1)$. Then we may write $\psi_{a_0^m, nb_0 a_0^m}$ as $\psi_{a_0^m, nb_0 a_0^m} = a_0^{-m/2}\psi\left(b_0(a_0^{-m}\frac{x}{b_0} - n)\right)$. The vaguelettes lemma (theorem 3) gives us :

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, nb_0 a_0^m} >|^2 \leq C \| f \|_2^2 \| \psi(b_0 x) \|^2_{H_{\epsilon,\epsilon'}}$$

and (12.1) is proved for $b_0 \in [1/2, 1)$.

Now if $b_0 \in \left[\frac{1}{2^{N+1}}, \frac{1}{2^N}\right]$, we may write $b_0 = \frac{b_1}{2^N}$, so that :

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, nb_0 a_0^m} >|^2 = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, n\frac{b_1}{2^N}{}_0 a_0^m} >|^2 .$$

If $0 \leq r < 2^N$, we write $\psi_{[r,N]} = \psi(x - r\frac{b_1}{2^N})$; then

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m (n2^N + r)\frac{b_1}{2^N} a_0^m} >|^2 &\leq C \| f \|_2^2 \| \psi_{(r,N)} \|^2_{H_{\epsilon,\epsilon'}} \\ &\leq C_0 \| f \|_2^2 \| \psi \|^2_{H_{\epsilon,\epsilon'}}\end{aligned}$$

and thus :

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, nb_0 a_0^m} >| \leq C_0 2^N \| f \|_2^2 \| \psi \|^2_{H_{\epsilon,\epsilon'}}$$

and (14.1) is proved.

Because of (14.1), it is enough to prove (14.2) for a dense subset of $L^2(\mathbb{R})$. We use the same proof as for lemma 4 and thus we suppose $\hat{f} C^\infty$, compactly supported and $0 \notin \operatorname{Supp} \hat{f}$. Then :

$$\begin{aligned}&2\pi b_0 \sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, nb_0 a_0^m} >|^2 \\ &\qquad \leq \| \sum_{n\in\mathbb{Z}} | \hat{f}(\xi + 2n\pi\frac{1}{b_0 a_0^m}) |^2 \|_\infty \int_{-\infty}^{+\infty} | \varphi(\xi) |^2 | \hat{\psi}(a_0^m \xi) |^2\, d\xi \\ &\qquad \leq C(1 + a_0^m) \int_{-\infty}^{+\infty} | \varphi(\xi) |^2 | \hat{\psi}(a_0^m \xi) |^2\, d\xi\end{aligned}$$

where C doesn't depend neither on m nor on $b_0 \in (0,1)$ (and where $0 \notin \operatorname{Supp}\varphi$ and $\varphi\hat{f} = \hat{f}$). By dominated convergence, we get :

$$\lim_{b_0 \to 0} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |< f \mid \psi_{a_0^m, nb_0a_0^m} >|^2\, b_0 = \sum_{m\in\mathbb{Z}} \left(\lim_{b_0\to 0} \sum_{n\in\mathbb{Z}} b_0 \,|< f \mid \psi_{a_0^m, nb_0a_0^m} >|^2 \right).$$

But

$$\sum_{n\in\mathbb{Z}} 2\pi b_0 \,|< f \mid \psi_{a_0^m, nb_0a_0^m} >|^2$$

$$= \int_{-\pi\, 1/b_0a_0^m}^{\pi\, 1/b_0a_0^m} \mid \sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2n\pi \frac{1}{b_0a_0^m}) \hat{\psi}(a_0^m\xi + \frac{2n\pi}{b_0}) \mid^2 d\xi$$
$$= \int \mid \hat{f}(\xi) \mid^2 \mid \hat{\psi}(a_0^m\xi) \mid^2 d\xi$$

as soon as $\operatorname{Supp}\hat{f} \subset \left[-\pi\frac{1}{b_0a_0^m}, \pi\frac{1}{b_0a_0^m}\right]$. Now, (14.2) is proved since

$$\int \mid \hat{f}(\xi) \mid^2 \mid \hat{\psi}(a_0^m\xi) \mid^2 d\xi = 2\pi a_0^{-m} \int |< f \mid \psi_{a_0^m, b} >|^2\, db. \quad \blacksquare$$

We may now easily end the proof of theorem 1. If $\psi \in H_{\epsilon,\epsilon'}$ $(\epsilon > 1/2, \epsilon' > 0)$ is such that $\int \psi\, dx = 0$ and if a_0 is such that for some positive A, B we have (everywhere outside from $\{0\}$)

$$A \leq \sum_{m\in\mathbb{Z}} \mid \hat{\psi}(a_0^m\xi) \mid^2 \leq B$$

then we have :

$$A \parallel f \parallel_2^2 \leq \int \sum_{m\in\mathbb{Z}} a_0^{-m} \,|< f \mid \psi_{a_0^m, b} >|^2\, db \leq B \parallel f \parallel_2^2$$

and we have just to check at which rate the left-hand side of (14.2) goes to the right-hand side as b_0 goes to 0. For

$$\mid S(b_0, f) - S(\frac{b_0}{2}, f) \mid$$

where

$$S(b_0, f) = \left\{ \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} b_0 \,|< f \mid \psi_{a_0^m, nb_0a_0^m} >|^2 \right\}^{1/2},$$

it is a direct consequence of Minkowski's inequality to state that :

$$\mid S(b_0, f) - S(\frac{b_0}{2}, f) \mid \leq \left\{ \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \frac{b_0}{2} \,|< f \mid \psi_{a_0^m, nb_0a_0^m} - \psi_{a_0^m, (n+1/2)b_0a_0^m} >|^2 \right\}^{1/2}.$$

By lemma 5, we have, since

$$\psi_{a_0^m, nb_0 a_0^m} - \psi_{a_0^m,(n+1/2)b_0 a_0^m} = a_0^{-m/2}\left\{\psi([a_0^{-m}x - nb_0]) - \psi([a_0^{-m}x - nb_0] - \frac{b_0}{2})\right\},$$

the following estimate :

$$| S(b_0, f) - S(\frac{b_0}{2}, f) | \leq C_0(\eta, \eta') \| \psi(x) - \psi(x - \frac{b_0}{2}) \|_{H_{\eta,\eta'}} \| f \|_2$$

where $\eta \in (1/2, \epsilon)$ and $\eta' \in (0, \epsilon')$. Now, it is easy to check that, if $b_0 \in (0,1)$, there exists $\gamma > 0$ (which doesn't depend on b_0) such that
$\| \psi(x) - \psi(x - \frac{b_0}{2}) \|_{H_{\eta,\eta'}} \leq C\, b_0^\gamma \| \psi \|_{H_{\epsilon,\epsilon'}}$: it is enough to see that

$$\| \psi(x) - \psi(x - \frac{b_0}{2}) \|_2 \leq C \| | \xi |^{\epsilon'} \psi \|_2\, b_0^{\epsilon'} \quad \text{if} \quad \epsilon' \in (0,1)$$

while

$$\| (1 + | x |)^\epsilon (\psi(x) - \psi(x - \frac{b_0}{2})) \|_2 \leq C \| (1 + | x |)^\epsilon \psi \|_2$$

and

$$\| (1 + | \xi |)^{\epsilon'} \hat{\psi}(\xi)(1 - e^{-ib_0\epsilon/2}) \|_2 \leq 2 \| (1 + | \xi |)^{\epsilon'} \hat{\psi} \|_2,$$

and then to interpolate.

Now, if we define $S_0(f)$ as $\left(\int \sum_{m\in\mathbb{Z}} a_0^{-m} |< f \mid \psi_{a_0^m, b} >|^2 \, db\right)^{1/2}$, we obtain :

$$| S(b_0, f) - S_0(f) | \leq C \frac{b_0^\gamma}{1 - (1/2)^\gamma} \| \psi \|_{H_{\epsilon,\epsilon'}} \| f \|_2$$

and thus if b_0 is small enough the family $(\psi_{a_0^m, nb_0 a_0^m})$ is a frame.

4. Proof of the irregular sampling theorem.

We begin the proof of theorem 2 with the following sparsity lemma :

LEMMA 6. - *Let $\psi \in L^2(\mathbb{R})$ ($\psi \neq 0$) and let $(a_\alpha, b_\alpha)_{\alpha\in A}$ be such that the family $\psi_{a_\alpha, b_\alpha} = a_\alpha^{-1/2}\psi\left(\frac{x - b_\alpha}{a_\alpha}\right)$ is almost orthonormal :*

(15) $$\| \sum_{\alpha\in A} \lambda_\alpha \psi_{a_\alpha, b_\alpha} \|_2^2 \leq C_0 \sum_{\alpha\in A} | \lambda_\alpha |^2$$

then there exists N_0 such that :

(16) $$\forall a > 0, \ \forall b \in \mathbb{R}, \ \text{Card}\left\{\alpha \in A / \frac{1}{2}a \leq a_\alpha \leq 2a \ \text{and} \ | b - b_\alpha | \leq a\right\} \leq N_0.$$

Proof. The map $(a,b) \to \psi_{a,b} = \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a})$ is a continuous one from $(0,+\infty) \times \mathbb{R}$ to L^2. Hence if a is close enough to 1 ($| a-1 |< \epsilon_0$) and b close enough to 0 ($| b |< \epsilon_0$), we have $\| \psi - \psi_{a,b} \|_2 \leq \frac{1}{2} \| \psi \|_2$ and $\mathcal{R}\mathrm{e} < \psi, \psi_{a,b} > \geq \frac{1}{2} \| \psi \|_2^2$. Now fix a and b in $\mathbb{R}$ and call $K_{a,b} = \{(a_\alpha, b_\alpha)/(1-\epsilon_0)a \leq a_\alpha \leq (1+\epsilon_0)a \text{ and } | b_\alpha - b |< a\epsilon_0\}$; for $(a_\alpha, b_\alpha) \in K_{a,b}$, we have

$$\mathcal{R}\mathrm{e} < \psi_{a,b} \mid \psi_{a_\alpha,b_\alpha} >= \mathcal{R}\mathrm{e} < \psi \mid \psi_{\frac{a_\alpha}{a},\frac{b_\alpha - b}{a}} > \geq \frac{1}{2} \| \psi \|_2^2,$$

hence

$$\frac{1}{4} \| \psi \|_2^4 \,\mathrm{Card}\, K_{a,b} \leq \sum_\alpha |< \psi_{a,b} \mid \psi_{a_\alpha,b_\alpha} >|^2 \leq C_0 \| \psi \|_2^2$$

and we see that

$$\mathrm{Card}\, K_{a,b} \leq \frac{2C_0}{\| \psi \|_2^2}.$$

Thus we get (16). ∎

LEMMA 7. - *Let $\psi \in H_{\epsilon,\epsilon'}$ ($\epsilon > \frac{1}{2}, \epsilon' > 0$) be such that $\int \psi \, dx = 0$. Let $\gamma_{m,n} = \sup_{2^m \leq a \leq 2^{m+1}, 2^m n \leq b \leq 2^m(n+1)} |< \psi \mid \psi_{a,b} >|$ ($m, n \in \mathbb{Z}$). Then $\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \gamma_{m,n}^2 < +\infty$.*

Proof. Let $(a_{m,n}, b_{m,n})$ be a point in $[2^m, 2^{m+1}] \times [2^m n, 2^m(n+1)]$ such that

$$\gamma_{m,n} = |< \psi \mid \psi_{a_{m,n},b_{m,n}} >| \,.$$

We have :

$$\psi_{a_{m,n},b_{m,n}} = \frac{1}{2^{m/2}} \sqrt{\frac{2^m}{a_{m,n}}} \psi \left(\frac{2^m}{a_{m,n}} (\frac{x}{2^m} - n) + \frac{2^m n - b}{a_{m,n}} \right).$$

Now $(\sqrt{\theta}\psi(\theta x + \lambda))_{1/2 \leq \theta \leq 1, 0 \leq \lambda \leq 1}$ is a bounded set in $H_{\epsilon,\epsilon'}$ and we may apply the vaguelettes lemma (theorem 3) to conclude :

$$\forall f \in L^2, \;\; \sum_{m,n} |< f \mid \psi_{a_{m,n},b_{m,n}} >|^2 \leq C \| f \|_2^2 \,.$$

Let $f = \psi$ to conclude. ∎

End of necessary part of theorem 2 :

Let $\psi \in H_{\epsilon,\epsilon'}$, ($\epsilon > 1/2$, $\epsilon' > 0$) with $\int \psi \, dx = 0$ and let $(\psi_{a_\alpha,b_\alpha})$ be a frame. We call N_0 the number in inequality (16) and Γ_0 the number $\sum\sum \gamma_{m,n}^2$ in lemma 7 ; and we call A_0 a positive number such that for all $f \in L^2$, $A_0 \| f \|_2^2 \leq \sum_\alpha |< f \mid \psi_{a_\alpha,b_\alpha} >|^2$.

For $a > 0$, $b \in \mathbb{R}$ and $N \in \mathbb{N}^*$ we call $K_{a,b,N} = \{(\alpha,\beta) \in (0,+\infty) \times \mathbb{R}/2^{-N} \leq \frac{a_\alpha}{a} \leq 2^{N+1} \text{ and } | \frac{b - b_\alpha}{a} | \leq (N+1)2^{N+1}\}$. Then we have :

i) $A_0 \| \psi \|_2^2 \leq \sum_\alpha |< \psi_{a,b} \mid \psi_{a_\alpha,b_\alpha} >|^2$

$$\text{ii)}\quad \sum_{(a_\alpha,b_\alpha)\notin K_{a,b,N}} |< \psi_{a,b} \mid \psi_{a_\alpha,b_\alpha} >|^2 = \sum_{(\frac{a_\alpha}{a},\frac{b_\alpha-b}{a})\notin K_{1,0,N}} |< \psi, \psi_{\frac{a_\alpha}{a},\frac{b_\alpha-b}{a}} >|^2$$

$$\leq N_0 \sum_{|m|\geq N} \sum_{|n|\geq N} \gamma_{m,n}^2.$$

This last estimate doesn't depend on a or b and goes to 0 as N goes to $+\infty$. Thus if N is big enough, we obtain $\sum_{K_{a,b,N}} |< \psi_{a,b} \mid \psi_{a_\alpha,b_\alpha} >|^2 > 0$ and $\operatorname{Card} K_{a,b,N} \geq 1$. Thus we have proved inequality (5). ■

Proof of the sufficient part :

We fix $a_0 > 1$ and $b_0 > 0$ such that the family $(\psi_{a_0^m,nb_0a_0^m})$ $(m \in \mathbb{Z},\, n \in \mathbb{Z})$ is a frame. Now, if for each $a > 0$ and $b \in \mathbb{R}$, $\operatorname{Card}\{\alpha / \frac{1}{\lambda}a \leq a_\alpha \leq \lambda a$ and $\mid b - b_\alpha \mid \leq \theta a\} \geq 1$, we may pick $\alpha_{m,n}$ such that $\frac{1}{\lambda}a_0^m \leq a_{\alpha_{m,n}} \leq \lambda a_0^m$ and $\mid b_0 n a_0^m - b_{\alpha_{m,n}} \mid \leq a_0^m \theta$. If $\lambda^2 < a_0$ and $\theta < \frac{1}{2}$, the map $(m,n) \to \alpha_{m,n}$ is one-to-one and thus we have :

$$\sum_m \sum_n |< f \mid \psi_{a_{\alpha_{m,n}},b_{\alpha_{m,n}}} >|^2 \leq \sum_\alpha |< f \mid \psi_\alpha >|^2 .$$

But we have :

$$\sum_m \sum_n |< f \mid \psi_{a_0^m,nb_0a_0^m} >|^2 \geq A_0 \parallel f \parallel_2^2 \quad \text{with} \quad A_0 > 0$$

while :

$$\left| \left(\sum_m \sum_n |< f \mid \psi_{a_0^m,nb_0a_0^m} >|^2 \right)^{1/2} - \left(\sum_m \sum_n |< f \mid \psi_{a_{\alpha_{m,n}},b_{\alpha_{m,n}}} >|^2 \right)^{1/2} \right|$$

$$\leq \left\{ \sum_m \sum_n |< f \mid \psi_{a_0^m,nb_0a_0^m} - \psi_{a_{\alpha_{m,n}},b_{\alpha_{m,n}}} >|^2 \right\}^{1/2}$$

$$\leq C \parallel f \parallel_2 \sup \left\{ \parallel \psi(x) - \sqrt{\frac{1}{a}}\psi(\frac{x}{a} - b) \parallel_{H_{\epsilon,\epsilon'}} / \frac{1}{\lambda} \leq a \leq \lambda, \mid b \mid \leq \lambda\theta \right\}$$

by the vaguelettes lemma. This last estimate goes to 0 as $\lambda \to 1$ and $\theta \to 0$ and theorem 2 is proved. ■

5. Some remarks on dual frames.

We have seen that sampling the Morlet wavelet representation on a fine enough hyperbolic grid gives a discrete frame : we have turned a functional representation (as a function in $L^2\left(\frac{da}{a}db\right)$) into a sequential representation (as a sequence in $\ell^2(\mathbb{Z} \times \mathbb{Z})$). Such a discrete representation is better suited to computing than the former continuous one. However, we have lost a very convenient tool of the continuous wavelet transform, namely the possibility of choosing a nice dual wavelet (such as a smooth compactly supported one): if our new discrete frame has no redundancy but is a Riesz basis of $L^2(\mathbb{R})$, the dual frame is then unique and there is no reason that this dual frame should well behave, whatever the behaviour of our wavelet ψ is. The problem of the dual frame is then two-fold :

• while the frame $(\psi_{a_0^m, nb_0 a_0^m})_{m\in\mathbb{Z}, n\in\mathbb{Z}}$ is easily described by a finite set of data (the function ψ, the scale mesh a_0 and the position mesh b_0), the dual frame $(h_{m,n})$ might be of a more complex structure : one sees easily that $h_{m,n}(x) = \frac{1}{a_0^{m/2}} h_{0,n}\left(\frac{x}{a_0^m}\right)$ but the functions $h_{0,n}$, $n \in \mathbb{Z}$, are not in general generated from $h_{0,0}$ by shifts. By instance, let $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ be an orthonormal wavelet basis of $L^2(\mathbb{R})$ ($\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}$) with a smooth rapidly decaying wavelet ψ (ψ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$; we will see in chapter 4 that such wavelet bases exist) and define $\tilde{\psi} = \psi(x) - r\psi(2x)$ where $|r| < \sqrt{2}$; then the $\tilde{\psi}_{j,k} = 2^{j/2}\tilde{\psi}(2^j x - k)$ are still a Riesz basis of $L^2(\mathbb{R})$ and for such a basis the dual basis ($h_{j,k} = 2^{j/2} h_k(2^j x)$) is easily computed :

$$h_k = \sum_{j\leq 0,\ 2^j k \in \mathbb{Z}} \left(\frac{\sqrt{2}}{r}\right)^j \psi_{j,2^j k}.$$

We thus see that

$$h_{2^N(2k_0+1)} = h_{2^N}(x - 2^N 2k_0),$$

but that the functions h_0 and $h_{2^N}(x+2^N)$ are all different. Moreover, h_0 is badly behaved: if $|r| \geq 1$, it isn't integrable on $\mathbb{R}$.

• the second point is the following one : even if the dual frame $(h_{m,n})$ is a wavelet frame $h_{m,n} = a_0^{-m} h(a_0^{-m}x - nb_0)$, there is no reason for h to have good decay at infinity or to have good regularity properties. So we cannot hope to characterize functional spaces or regular points (as in the preceding chapter) without knowing the behaviour of h. That means that, in order to have a characterization of functional spaces in terms of sequences of wavelet coefficients, one needs to deal not only with the analyzing wavelet ψ but with the reconstruction wavelet h as well.

By instance, let us describe the characterization of Sobolev spaces by mean of an oscillating analyzing wavelet ψ and a regular reconstruction wavelet h :

PROPOSITION 2. - *Let a_0 be greater than 1 and b_0 be a positive real number. Let ψ and h generate dual frames in $L^2(\mathbb{R})$, $(\psi_{m,n} = a_0^{m/2}\psi(a_0^m x - nb_0))_{m\in\mathbb{Z},n\in\mathbb{Z}}$ and $(h_{m,n} = a_0^{m/2} h(a_0^m x - nb_0))_{m\in\mathbb{Z},n\in\mathbb{Z}}$. Let s be a positive real number and D^s be the fractional derivative operator defined by $\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi)$.*

i) In order to get that for some constant C

(17.1) $$\forall f \in H^s, \quad \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_0^{2ms} |< f \mid \psi_{m,n} >|^2 \leq C \, \| D^s f \|_2^2$$

a necessary condition is

$$(17.2) \qquad |\,\xi\,|^{-s}\,\hat{\psi} \in L^\infty$$

and a sufficient condition is

(17.3) *for some positive* δ,

$$|\,\xi\,|^{-s-\delta}\,\hat{\psi} \in L^\infty.$$

ii) In order to get that for some constant C

(18.1) *for all finite sequence* $(\lambda_{m,n})$,

$$\| D^s \left(\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \lambda_{m,n} h_{m,n} \right) \|_2^2 \leq C \sum_m \sum_n a_0^{2ms} \,|\, \lambda_{m,n} \,|^2$$

a necessary condition is

$$(18.2) \qquad \sum_{k\in\mathbb{Z}} |\, \hat{h}(\xi + \frac{2k\pi}{b_0})\,|^2 |\, \xi + \frac{2k\pi}{b_0}\,|^{2s} \in L^\infty$$

and a sufficient condition is

(18.3) *for some positive* δ,

$$\sum_{k\in\mathbb{Z}} |\, \hat{h}(\xi + \frac{2k\pi}{b_0})\,|^2 |\, \xi + \frac{2k\pi}{b_0}\,|^{2s+2\delta} \in L^\infty.$$

Remark. - Condition (17.3) is satisfied as soon as we have for some $\sigma > s$, $|\,x\,|^\sigma\,\psi \in L^1$ and $\int x^p \psi\, dx = 0$ for $0 \leq p \leq [s]$. Condition (18.3) sounds classical (for wavelet bases, it can be found in [GRI1]) ; it is satisfied as soon as we have for some $\sigma > s$, $h \in H^\sigma$ and h has enough decay at infinity in order to ensure $D^s h \in L^2(|\,x\,|^{1+\eta}\, dx)$ for some positive η, i.e. as soon as $h \in H^\sigma$ and $|\,x\,|^\epsilon\, h \in L^2$ for some $\sigma > s$ and some $\epsilon > \frac{\sigma}{2(\sigma - s)}$. ■

Proof. (17.1) is equivalent to the statement that the family

$$\left((I^s\psi)_{m,n} = a_0^{m/2} (I^s\psi)(a_0^m x - nb_0) \right)_{m\in\mathbb{Z}, n\in\mathbb{Z}}$$

(with $\widehat{I^s\psi} = |\,\xi\,|^{-s}\,\hat{\psi}$) is almost orthonormal in L^2, while (18.1) is equivalent to the almost orthonormality of the family

$$\left((D^s h)_{m,n} = a_0^{m/2} (D^s h)(a_0^m x - nb_0) \right).$$

The necessary parts are then straightforward by looking just at the scale $a_0^0 = 1$: for $f \in L^2$ the almost orthonormality of $(f(x - kb_0))_{k \in \mathbb{Z}}$ is equivalent to the boundedness of $\sum_{k \in \mathbb{Z}} | \hat{f}(\xi + \frac{2k\pi}{b_0}) |^2$. The sufficient parts are corollaries of proposition 1. ■

6. Wavelet theory and modern Littlewood-Paley theory.

The Morlet wavelet theory has many common points with the modern Littlewood-Paley theory, as developped in the years 60's and 70's. A. P. Calderón introduced in the 60's a resolution of identity, for the purpose of interpolating Banach spaces [CAL], which is similar to the integral wavelet transform. Let $\psi \in L^2(\mathbb{R})$ be such that :

i) ψ is real-valued ;
ii) $\hat{\psi}$ is C^∞, has compact support and $0 \notin \operatorname{Supp} \hat{\psi}$;

and let Ψ_t be the convolution operator $\Psi_t f = f * \frac{1}{t}\psi(\frac{x}{t})$. We write Ψ_t^* for the adjoint operator of Ψ_t ; then :

(19) $\int_0^{+\infty} \Psi_t \circ \Psi_t^* \frac{dt}{t} = C_\psi \mathrm{Id}_{L^2}$ (Calderón formula).

The constant C_ψ is the Morlet constant :

$$C_\psi = \int_0^\infty | \hat{\psi}(\xi) |^2 \frac{d\xi}{\xi}, \tag{20}$$

the connexion between Calderón formula and the Morlet wavelet transform being given by $\int_{-\infty}^{+\infty} < f \mid \psi_{a,b} > \psi_{a,b} db = a\Psi_a \circ \Psi_a^*(f)$.

The Calderón formula provides a Parseval equality

$$\| f \|_2^2 = \frac{1}{C_\psi} \int_0^{+\infty} \| \Psi_t f \|_2^2 \frac{dt}{t} \tag{21}$$

but the point is that is also suited to many other spaces, as by instance Lebesgue spaces L^p $(1 < p < +\infty)$, Hardy spaces H^p $(0 < p < +\infty)$, Sobolev spaces H^s, Besov spaces $B_q^{s,p}$, and so on. We have seen in the preceding chapter the case of the Sobolev spaces H^s, the Hõlder spaces C^s or the Zygmund spaces C_*^s :

- for $s > 0$ and $f \in L^2$, $f \in H^s \Leftrightarrow \int_0^{+\infty} \| \Psi_t f \|_2^2 \frac{dt}{t^{1+2s}} < +\infty$
- for $s > 0$ $(s \notin \mathbb{N})$ and $f \in L^\infty$, $f \in C^s \Leftrightarrow \sup_{t>0} t^{-s} \| \Psi_t f \|_\infty < +\infty$
- for $s \in \mathbb{N}^*$ and $f \in L^\infty$, $f \in C_*^s \Leftrightarrow \sup_{t>0} t^{-s} \| \Psi_t f \|_\infty < +\infty$.

Thus, the Calderón formula provides a representation of functions as a continuous superposition of functions which are frequentially well localized.

The *Littlewood-Paley decomposition* provides a similar decomposition, but the continuous parameter t is replaced by a discrete one and the integral by a series. Let ψ be such that :

(i) ψ is real valued
(ii) $\hat{\psi}$ is $\mathcal{C}^\infty$, has compact support and $0 \notin \operatorname{Supp} \hat{\psi}$
(iii) $\sum_{j\in\mathbb{Z}} \mid \hat{\psi}(2^j\xi) \mid^2 = 1.$

Then if we define $\Delta_j = \Psi_{1/2^j} \circ \Psi^*_{1/2^j}$, the Calderón formula (19) can be turned into the Littlewood-Paley decomposition :

$$\forall f \in L^2, \quad f = \sum_{j\in\mathbb{Z}} \Delta_j f \tag{22}$$

where the series converges in the L^2 sense. We then have

$$\| f \|_2^2 = \sum_{j\in\mathbb{Z}} \| \Psi_{1/2^j} f \|_2^2 \approx \sum_{j\in\mathbb{Z}} \| \Delta_j f \|_2^2$$

(where $\approx$ stands for equivalence). This decomposition allows the same characterizations of Banach spaces as the Calderón formula ; we have for instance :

- for $s > 0$ and $f \in L^2$, $f \in H^s \Leftrightarrow \sum_{j\in\mathbb{Z}} 4^{js} \| \Delta_j f \|_2^2 < +\infty$
- for $s > 0$ ($s \notin \mathbb{N}$) and $f \in L^\infty$, $f \in C^s \Leftrightarrow \sup_{j\in\mathbb{Z}} 2^{js} \| \Delta_j f \|_\infty < +\infty$
- for $s \in \mathbb{N}^*$ and $f \in L^\infty$, $f \in C^s_* \Leftrightarrow \sup_{j\in\mathbb{Z}} 2^{js} \| \Delta_j f \|_\infty < +\infty$.

Such characterizations have proved to be very useful for the analysis of non-linear partial differential equations (as for instance in the para-differential calculus of J. M. Bony [BON]).

A further discretization of the Calderón formula has been introduced in the mid 80's by Frazier, Jawerth and Weiss under the name of the *φ-transform* [FRJ]. They consider a function ψ such that :

(i) ψ is real-valued
(ii) $\hat{\psi}$ is $\mathcal{C}^\infty$, has compact support and $0 \notin \operatorname{Supp} \hat{\psi}$
(iii) for $f \in L^2$, $\| f \|_2^2 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f \mid \psi_{j,k} \rangle|^2$ (with $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$).

Such functions are easily constructed, as shown in the "painless non-orthogonal wavelet expansions" paper by Daubechies, Grossmann and Meyer [DAUG]. Now formulas (19) and (22) become

$$\forall f \in L^2, \quad f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} < f \mid \psi_{j,k} > \psi_{j,k} \tag{23}$$

and the characterization of H^s and C^s are turned into :

- for $s > 0$ and $f \in L^2$, $f \in H^s \Leftrightarrow \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 4^{js} |< f \mid \psi_{j,k} >|^2 < +\infty$

- for $s > 0$ ($s \notin \mathbb{N}$) and $f \in L^\infty$, $f \in C^s \Leftrightarrow \sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} 2^{j(s+1/2)} |< f \mid \psi_{j,k} >| < +\infty$

- for $s \in \mathbb{N}^*$ and $f \in L^\infty$, $f \in C^s_* \Leftrightarrow \sup_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} 2^{j(s+1/2)} |< f \mid \psi_{j,k} >| < +\infty.$

With the φ-transform, we thus are dealing only with numerical sequences, and not with functions any more.

In 1985, Y. Meyer constructed a function ψ satisfying the conditions i) to iii) required for the φ-transform and an extra condition : the family $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ was an Hilbertian basis of $L^2(\mathbb{R})$ [LEME]. In the following chapters, we will study and construct the wavelet bases. The main difference with the φ-transform is that we shall deal mainly with compactly supported wavelets (and show why they are well-suited to numerical computations).

Chapter 3

THE STRUCTURE OF A WAVELET BASIS

In this chapter, we will study the (bi-orthogonal) wavelet bases of $L^2(\mathbb{R})$, i.e. Riesz bases $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of the type :

(1) $$\psi_{j,k}(x) = a_0^{j/2}\psi(a_0^j x - kb_0)$$

such that their dual basis $(\tilde{\psi}_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$, defined by :

(2) $$\tilde{\psi}_{j,k} \in L^2 \text{ and } <\tilde{\psi}_{j,k} \mid \psi_{\ell,p}> = \delta_{j,\ell}\delta_{k,p} \text{ for all } \ell,p\in\mathbb{Z}$$

is of the same nature

(3) $$\tilde{\psi}_{j,k}(x) = a_0^{j/2}\tilde{\psi}(a_0^j x - kb_0)$$

(with the same parameters a_0 and b_0). We may always suppose $b_0 = 1$, since the isometry $U : f \to \sqrt{b_0}f(b_0x)$ transforms the basis $(\psi_{j,k})$ into $\left(U(\psi_{j,k}) = a_0^{j/2}(U\psi)(a_0^j x - k)\right)$ and the dual basis $(\tilde{\psi}_{j,k})$ into $\left(U(\tilde{\psi}_{j,k}) = a_0^{j/2}(U\tilde{\psi})(a_0^j x - k)\right)$. Whereas the parameter b_0 is thus arbitrary, a_0 cannot be chosen arbitrarily. If we want ψ and $\tilde{\psi}$ to be localized (i.e. if we require ψ and $\tilde{\psi}$ to be in $H_{1/2+\epsilon,\epsilon}$ for some positive ϵ) then, if moreover we ask a_0 to be integer, then necessarily $a_0 = 2$; as we will see, this is deeply connected to the fact that, for integer a_0, the spaces $V_0 = \text{Span}\,\{\psi_{\ell,k}/\ell < 0,\ k\in\mathbb{Z}\}$ and $\tilde{V}_0 = \text{Span}\left\{\tilde{\psi}_{\ell,k}/\ell < 0,\ k\in\mathbb{Z}\right\}$ (where Span stands for "closed linear span in L^2") are invariant under a translation by b_0 :

(4) $$f\in V_0 \Leftrightarrow f(x-b_0)\in V_0 \text{ and } f\in\tilde{V}_0 \Leftrightarrow f(x-b_0)\in\tilde{V}_0.$$

If a_0 is no more an integer, V_0 and $\tilde{V}_0$ need not be invariant under the translation by b_0 ; however, if we require ψ and $\tilde{\psi}$ to be localized and V_0 and $\tilde{V}_0$ to be invariant, a_0 has to belong to $\left\{1+\frac{1}{m}/m\in\mathbb{N}^*\right\}$.

As we will see, the space V_0 has a very specific structure and the description of its structure will help us in the following chapters to construct many wavelet bases.

1. General properties of shift-invariant spaces.

The main result of this section is the following one :

THEOREM 1 (The invariant projection theorem). - *Let P_0 a bounded projection operator on $L^2(\mathbb{R})$, $V = \text{Ran}\,P_0$ and $\tilde{V} = \text{Ker}\,P_0^\perp$ and let ϵ be some positive real number. Then the following assertions are equivalent (we assume $P_0 \neq 0$) :*

(i) P_0 satisfies the three following properties :

• *P_0 is invariant under integer shifts :*

$$\forall k \in \mathbb{Z},\ \forall f \in L^2, \quad P_0(f(x-k)) = (P_0 f)(x-k) \tag{5.1}$$

• *P_0 is an integral operator*

$$\exists p(x,y) \in L^1_{\ell oc}(\mathbb{R}\times\mathbb{R}),\ \forall f \in C_c^\infty, \quad P_0 f(x) = \int p(x,y) f(y)\,dy \ \ a.e. \tag{5.2}$$

• *$p(x,y)$ has enough decay off the diagonal to ensure :*

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} (1+\mid x-y\mid)^{1+\epsilon}(\mid p(x,y)\mid^2 + \mid p(y,x)\mid^2)\,dx\,dy < +\infty. \tag{5.3}$$

(ii) V has a Riesz basis $(\varphi_{\delta,k})_{1\le\delta\le D, k\in\mathbb{Z}}$ and $\tilde V$ has a Riesz basis $(\tilde\varphi_{\delta,k})_{1\le\delta\le D,k\in\mathbb{Z}}$ such that :

$$\forall \delta \in \{1,\cdots,D\},\ \forall k \in \mathbb{Z}, \quad \varphi_{\delta,k}(x) = \varphi_\delta(x-k) \ \ and \ \ \tilde\varphi_{\delta,k}(x) = \tilde\varphi_\delta(x-k) \tag{6.1}$$

$$\int (1+\mid x\mid)^{1+\epsilon} \mid \varphi_\delta(x)\mid^2 dx < +\infty \ \ and \ \ \int (1+\mid x\mid)^{1+\epsilon}\mid \tilde\varphi_\delta(x)\mid^2 dx < +\infty \tag{6.2}$$

$$< \varphi_{\delta,k} \mid \tilde\varphi_{\eta,\ell} >= \delta_{\delta,\eta}\delta_{k,\ell}. \tag{6.3}$$

Moreover, the number D doesn't depend on the choice of the Riesz basis $(\varphi_{\delta,k})$ and will be called the multiplicity of V.

Proof. We begin by the easy part ii) ⇒ i) : (6.3) means that $(\varphi_{\delta,k})$ and $(\tilde\varphi_{\delta,k})$ are dual Riesz bases, and therefore :

$$\forall f \in L^2, \quad P_0 f = \sum_{\delta=1}^{D}\sum_{k\in\mathbb{Z}} < f \mid \tilde\varphi_{\delta,k} > \varphi_{\delta,k}. \tag{7}$$

Now, (5.1) is obvious, while (5.2) and (5.3) are direct consequences of the following inequalities for $\varphi, \tilde\varphi \in L^2((1+\mid x\mid)^{1+\epsilon}dx)$:

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} (1+\mid x-y\mid)^{1+\epsilon}\left(\sum_{k\in\mathbb{Z}} \mid\varphi(x-k)\mid\mid\tilde\varphi(y-k)\mid\right)^2 dx\,dy$$

$$\le \int_{x\in[0,1]}\int_{y\in\mathbb{R}}\sum_{k\in\mathbb{Z}} \mid\varphi(x-k)\mid^2 (1+\mid x-k\mid)^{1+\epsilon}\mid\tilde\varphi(y-k)\mid^2(1+\mid y-k\mid)^{1+\epsilon}$$

$$\sum_{k\in\mathbb{Z}}\frac{(1+\mid x-y\mid)^{1+\epsilon}}{(1+\mid x-k\mid)^{1+\epsilon}(1+\mid y-k\mid)^{1+\epsilon}}dx\,dy$$

$$\le C_\epsilon\int_{x\in[0,1]}\int_{y\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\mid\varphi(x-k)\mid^2(1+\mid x-k\mid)^{1+\epsilon}\mid\tilde\varphi(y-k)\mid^2(1+\mid y-k\mid)^{1+\epsilon}dx\,dy$$

$$= C_\epsilon\int\mid\varphi(x)\mid^2(1+\mid x\mid)^{1+\epsilon}dx\int\mid\tilde\varphi(y)\mid^2(1+\mid y\mid)^{1+\epsilon}dy.$$

We thus have proved ii) $\Rightarrow$ i).

We now turn to the direct sense : i) $\Rightarrow$ ii). We first characterize shift-invariant Riesz bases.

DEFINITION 1. - *For $f, g \in L^2$, the correlation function $C(f,g)$ is defined by :*

$$C(f,g)(\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi)\bar{\hat{g}}(\xi + 2k\pi). \tag{8}$$

The series (8) converges almost everywhere and defines a 2π-periodic function which belongs to $L^1(0,2\pi)$.

DEFINITION 2. - *For $(f_\delta)_{1\leq\delta\leq D}$, $(g_\epsilon)_{1\leq\epsilon\leq E}$ two finite families of functions in L^2, the correlation matrix $M((f_\delta),(g_\epsilon))$ is defined as the $D\times E$ matrix $(C(f_\delta,g_\epsilon))_{1\leq\delta\leq D,1\leq\epsilon\leq E}$. Similarly, the auto-correlation matrix $M[(f_\delta)]$ is defined as $M[(f_\delta)] = M((f_\delta),(f_\delta))$.*

LEMMA 1. - *Let V be a shift-invariant closed linear subspace of $L^2(\mathbb{R})$ (i.e. $\forall f \in V$, $\forall k \in \mathbb{Z}$, $f(x-k) \in V$). Let $f_1, \cdots, f_D$ belong to V. Then the following assertions are equivalent :*

- (j) *the family $(f_\delta(x-k))_{1\leq\delta\leq D,k\in\mathbb{Z}}$ is a Riesz basis of V ;*
- (jj) *the functions $(f_\delta)_{1\leq\delta\leq D}$ satisfy the following three requirements :*
  - *$M[(f_\delta)]$ belongs to $M_D(L^\infty(0,2\pi))$*
  - *$M[(f_\delta)]$ is invertible in $M_D(L^\infty(0,2\pi))$*
  - $\forall f \in V$, $\det(M[(f_\delta)_{1\leq\delta\leq D} \cup (f)]) = 0$ a.e.

(j) $\Rightarrow$ (jj) is almost obvious. For $f \in L^2$, the almost orthonormality of the family $(f(x-k))_{k\in\mathbb{Z}}$ is equivalent to the essential boundedness of $C(f,f)$. Since $C(f,g) \leq \sqrt{C(f,f)}\sqrt{C(g,g)}$, we see that if the family $(f_\delta(x-k))$ is a Riesz basis of V, then the coefficients of $M[(f_\delta)]$ are essentially bounded. Moreover the dual basis $(f^*_{\delta,k})$ of $(f_\delta(x-k))$ in V is easily seen to be shift-invariant : $f^*_{\delta,k} = f^*_\delta(x-k)$, and we have $M[(f^*_\delta)]M[(f_\delta)] = \mathrm{Id}_D$, so that $M[(f_\delta)]$ is invertible in $M_D(L^\infty(0,2\pi))$. Finally, if f belongs to V, then we have $\hat{f} = \sum_{\delta=1}^D C(f,f^*_\delta)\hat{f}_\delta$ and thus for any $g \in L^2$, $C(f,g) = \sum_{\delta=1}^D C(f,f^*_\delta)C(f_\delta,g)$; this implies $\det(M[(f_\delta)\cup(f)]) = 0$.

(jj) $\Rightarrow$ (j) is easy as well. If $M[(f_\delta)]$ has bounded coefficients, the family $(f_\delta(x-k))$ is almost orthonormal. If moreover $M[(f_\delta)]$ is invertible, we may define a dual system by $M[(f_\delta)]^{-1} = (\lambda_{\delta,\epsilon}(\xi))_{1\leq\delta\leq D,1\leq\epsilon\leq D}$ and $\widehat{f^*_\delta} = \sum_{\epsilon=1}^D \lambda_{\delta,\epsilon}(\xi)\hat{f}_\epsilon$; we then have that the family $(f^*_\delta(x-k))$ is almost orthonormal and in duality with $(f_\delta(x-k))$; thus we may conclude that the family $(f_\delta(x-k))$ is a Riesz basis for a closed subspace of V. This subspace is the whole space V : for $f \in V$ define $f^0 \in V$ by $\widehat{f^0} = \sum_{\delta=1}^D C(f,f^*_\delta)\hat{f}_\delta$; then $C(f,f_\delta) = C(f^0,f_\delta)$, hence :

$$0 = \det M[(f_\delta)\cup(f-f^0)] = (\det M[f_\delta])C(f-f^0,f-f^0)$$

hence

$$C(f-f^0, f-f^0) = 0 \text{ a. e. };$$

thus

$$\| f - f^0 \|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} C(f-f^0, f-f^0)\, d\xi = 0$$

and $f = f^0$. The family $(f_\delta(x-k))$ spans V. ■

COROLLARY. - *If V admits finitely generated shift-invariant Riesz bases, the number of generators doesn't depend on the choice of the Riesz basis.*

Indeed, if $(f_\delta(x-k))_{1\leq\delta\leq D, k\in\mathbb{Z}}$ is a Riesz basis for V (with dual basis $(f_\delta^*(x-k))$ and if $(g_\epsilon)_{1\leq\epsilon\leq E}$ is a family of functions in V, then $\hat{g}_\epsilon = \sum_{\delta=1}^{D} C(g_\epsilon, f_\delta^*)\hat{f}_\delta$ so that :

$$M[(g_\epsilon)] = M((g_\epsilon),(f_\delta^*))M[(f_\delta)]M((f_\delta^*),(g_\epsilon)) \tag{9}$$

and

$$\operatorname{rank} M[(g_\epsilon)] \leq \operatorname{rank} M[(f_\delta)] = D \quad \text{a. e.} \ ■$$

LEMMA 2. - *Let ϵ be positive and $f_1, \cdots, f_D$ belong to $L^2((1+ \mid x \mid)^{1+\epsilon}dx)$.*

(j) *The series (8) defining $C(f_\delta, f_{\delta'})$ converges uniformly on $[0,2\pi]$ and $C(f_\delta, f_{\delta'})$ belongs to $H^{\frac{1+\epsilon}{2}}(\mathbb{R}/2\pi\mathbb{Z})$.*

(jj) *For all $\xi \in [0,2\pi]$, the function $f_\delta^\xi = \sum_{k\in\mathbb{Z}} e^{-i\xi(x-k)} f_\delta(x-k)$ belongs to $L^2(0,1)$.*

(jjj) *For all $\xi \in [0,2\pi]$, the matrix $M[(f_\delta)](\xi)$ is the Gram matrix of the functions $(f_\delta^\xi)_{1\leq\delta\leq D}$ in $L^2(0,1)$.*

This lemma is easy. If f belongs to $L^2((1+ \mid x \mid)^{1+\epsilon}dx)$, $\hat{f}$ belongs to $H^{1/2+\epsilon/2}(\mathbb{R})$; if $\omega \in C_c^\infty(\mathbb{R})$ is such that $\sum_{k\in\mathbb{Z}} \omega(x-2k\pi) = 1$, the norms

$$\| \hat{f} \|_{H^{1/2+\epsilon/2}} \quad \text{and} \quad \left(\sum_{k\in\mathbb{Z}} \| \omega(x-2k\pi)\hat{f} \|^2_{H^{1/2+\epsilon/2}} \right)^{1/2}$$

are equivalent, and thus we may conclude from the Sobolev injection $H^{1/2+\epsilon/2} \subset L^\infty$:

$$\sum_{k\in\mathbb{Z}} \sup_{\xi\in[2k\pi, 2\pi+2k\pi]} \mid \hat{f}(\xi) \mid^2 \leq C \| \hat{f} \|^2_{H^{1/2+\epsilon/2}} . \tag{10}$$

We have proved the uniform convergence of $C(f_\delta, f_{\delta'})$ on $[0,2\pi]$. Moreover, by the Poisson summation formula

$$C(f_\delta, f_{\delta'}) = \sum_{k\in\mathbb{Z}} < f_\delta(x) \mid f_{\delta'}(x-k) > e^{-ik\xi} \tag{11}$$

and we have :

$$\sum_{k\in\mathbb{Z}} |< f_\delta(x) \mid f_{\delta'}(x-k) >|^2 \, (1+ \mid k \mid)^{1+\epsilon}$$

$$\leq 2 \sum_{k\in\mathbb{Z}} \int_{|k|<10|x|} \mid f_\delta(x) \mid^2 \frac{1}{(1+ \mid x-k \mid)^{1+\epsilon}} dx \, \| f_{\delta'}(x)(1+ \mid x \mid)^{\frac{1+\epsilon}{2}} \|_2^2 \, (1+ \mid k \mid)^{1+\epsilon}$$

$$+ 2 \sum_{k\in\mathbb{Z}} \| f_\delta(x)(1+ \mid x \mid)^{\frac{1+\epsilon}{2}} \|_2^2 \int_{|k|<\frac{10}{9}|x-k|} \mid f_{\delta'}(x-k) \mid^2 \frac{1}{(1+ \mid x \mid)^{1+\epsilon}} dx (1+ \mid k \mid)^{1+\epsilon}$$

$$\leq 4 \| f_\delta(x)(1+10 \mid x \mid)^{\frac{1+\epsilon}{2}} \|_2^2 \| f_{\delta'}(x)(1+10 \mid x \mid)^{\frac{1+\epsilon}{2}} \|_2^2 \| \sum_k \frac{1}{(1+ \mid x-k \mid)^{1+\epsilon}} \|_\infty$$

so that $C(f_\delta, f_{\delta'}) \in H^{\frac{1+\epsilon}{2}}(\mathbb{R}/2\pi\mathbb{Z})$. Point (j) is proved.

Point (jj) is obvious :

$$\int_0^1 \left(\sum_{k\in\mathbb{Z}} \mid f_\delta(x-k) \right)$$

$$\leq \int_0^1 \sum_{k\in\mathbb{Z}} \mid f_\delta(x-k) \mid^2 (1+ \mid x-k \mid)^{1+\epsilon} dx \, \| \sum_k \frac{1}{(1+ \mid x-k \mid)^{1+\epsilon}} \|_\infty$$

$$= \| f_\delta(x)(1+ \mid x \mid)^{\frac{1+\epsilon}{2}} \|_2^2 \| \sum_k \frac{1}{(1+ \mid x-k \mid)^{1+\epsilon}} \|_\infty .$$

And point (jjj) is easily proved :

$$\int_0^1 f_\delta^\xi(x) \bar{f}_{\delta'}^\xi(x) dx = \int_{-\infty}^{+\infty} e^{-i\xi x} f_\delta(x) \bar{f}_{\delta'}^\xi(x) dx$$

$$= \int_{-\infty}^{+\infty} f_\delta(x) \sum_{k\in\mathbb{Z}} e^{-ik\xi} \bar{f}_{\delta'}(x-k) dx$$

$$= \sum_{k\in\mathbb{Z}} < f_\delta \mid f_{\delta'}(x-k) > e^{-ik\xi} = C(f_\delta, f_{\delta'})(\xi). \quad \blacksquare$$

LEMMA 3. - *If P_0 satisfies (5.1) to (5.3), then for all $\xi \in [0, 2\pi]$ the space $V^\xi = \{f^\xi / f \in V \cap L^2((1+ \mid x \mid)^{1+\epsilon} dx)\}$ is finite dimensional. Moreover $\dim V^\xi$ doesn't depend on ξ.*

In order to prove lemma 3, let us define P^ξ by

$$\forall f \in L^2(0,1), \quad P^\xi f = \int_0^1 p^\xi(x,y) f(y) dy \tag{12.1}$$

$$p^\xi(x,y) = \sum_{k\in\mathbb{Z}} p(x, y-k) e^{-i\xi(x-y+k)}. \tag{12.2}$$

We first notice that (5.3) ensures that $p^{\xi}(x,y) \in L^2([0,1]\times[0,1])$:

$$\int\int_{[0,1]^2} | p^{\xi}(x,y) |^2 \, dx\, dy \leq$$

$$\leq \| \sum_{k\in\mathbb{Z}} \frac{1}{(1+ | x-y+k |)^{1+\epsilon}} \|_{\infty} \int\int_{[0,1]^2} \sum_{k\in\mathbb{Z}} | p(x,y-k) |^2 \, (1+ | x-y+k |)^{1+\epsilon} dx < +\infty.$$

Thus P^{ξ} is a bounded operator on $L^2(0,1)$; since it has a square-integrable kernel, it is a Hilbert-Schmidt operator, hence a compact operator.

Now, for $f \in L^2(0,1)$ (extended by 0 outside from $[0,1]$), we have

$$P_0 f \in L^2((1+ | x |)^{1+\epsilon} dx) \ :$$

$$\int | P_0 f(x) |^2 \, (1+ | x |)^{1+\epsilon} dx \leq \| f \|_2^2 \int\int_{y\in[0,1]} | p(x,y) |^2 \, (1+ | x |)^{\epsilon} dx\, dy < +\infty$$

and

$$\{P_0(e^{i\xi x} f)\}^{\xi} = P^{\xi} f.$$

Thus, $\mathrm{Ran}\, P^{\xi} \subset V^{\xi}$. If $g^{\xi} \in V^{\xi}$, we have

$$\begin{aligned}
P^{\xi}(g^{\xi}) &= \int_{[0,1]} \sum_k e^{-i\xi(x-y+k)} p(x,y-k) \sum_p e^{-i\xi(y-p)} g(y-p) dy \\
&= \int_{[0,1]} \sum_k e^{-i\xi(x+k)} p(x+k,y) \sum_p e^{+i\xi p} g(y-p)\, dy \\
&= \sum_p \int_{[0,1]} \sum_k e^{-i\xi(x+k+p)} p(x+k+p, y-p) g(y-p) dy \\
&= \sum_p \int_{[0,1]} \sum_k e^{-i\xi(x+k)} p(x+k, y-p) g(y-p) dy \\
&= \sum_k e^{-i\xi(x+k)} (P_0 g)(x+k) = g^{\xi}.
\end{aligned}$$

Hence P^{ξ} is a projection operator onto V^{ξ}. Since it is compact, V^{ξ} is finite dimensional. Moreover, we have :

$$\dim V^{\xi} = \int\int_{[0,1]^2} p^{\xi}(x,y) p^{\xi}(y,x) dx\, dy \qquad (13)$$

since $p^{\xi}(x,y)$ may be rewritten as

$$p^{\xi}(x,y) = \sum_{i=1}^{\dim V^{\xi}} e_i(x) \overline{e}_i^{*}(y) \text{ a. e.}$$

where (e_i) is a basis of V^ξ and (e_i^*) the dual basis of (e_i) in $(\mathrm{Ker}\, P^\xi)^\perp$. Formula (13) proves that $\dim V^\xi$ is a continuous integer-valued function of ξ, hence is constant. ∎

LEMMA 4. - *Let V be a shift-invariant closed linear subspace of $L^2(\mathbb{R})$ such that $V \cap L^2((1+ \mid x \mid)^{1+\epsilon}dx)$ is dense in V, and let V^ξ be defined as in lemma 3. Then the following assertions are equivalent :*

(j) *V has an Hilbertian basis of the type $(f_\delta(x-k))_{1\le\delta\le D, k\in\mathbb{Z}}$ with $f_\delta \in L^2((1+ \mid x \mid)^{1+\epsilon}dx)$*
(jj) $\forall \xi \in [0, 2\pi]$, $\dim V^\xi = D$.

We already know (j) $\Rightarrow$ (jj) by lemmas 1 and 2. The implication (jj) $\Rightarrow$ (j) is proved by reccurence on D. We will show that we may find a function $g \in L^2((1+ \mid x \mid)^{1+\epsilon}dx) \cap V$ such that the family $(g(x-k))_{k\in\mathbb{Z}}$ is orthonormal. Then we call W the space spanned by the $g(x-k)$, $k \in \mathbb{Z}$, and $\tilde{V} = V \cap W^\perp$. Now $\tilde{V}$ is still shift-invariant (because both V and W are invariant), $\tilde{V} \cap L^2((1+ \mid x \mid)^{1+\epsilon}dx)$ is dense in $\tilde{V}$ (because if $f \in L^2((1+ \mid x \mid)^{1+\epsilon}dx) \cap V$, it is easy to see that

$$\sum_k < f(g(x-k) > g(x-k) \in L^2((1+ \mid x \mid)^{1+\epsilon}dx) \cap V)$$

and $V^\xi = \tilde{V}^\xi \oplus \mathbf{C}g^\xi$, hence $\dim \tilde{V}^\xi = D-1$.

Hence, we just have to show the existence of g. Since $\dim V^\xi = D$, we may find for all ξ a function $g_\xi \in L^2((1+ \mid x \mid)^{1+\epsilon}dx) \cap V$ such that $C(g_\xi, g_\xi)(\xi) \neq 0$, and we have also $C(g_\xi, g_\xi)(\eta) \neq 0$ for η in a neighborhood of ξ. By compacity of $[0, 2\pi]$, we may find a finite family of functions $g_1, \cdots, g_N$ and a subdivision $t_1 = 0 < t_2 < \cdots < t_N < t_{N+1} = 2\pi$ of $[0, 2\pi]$ such that : $\forall \xi \in [t_i, t_{i+1}]$, $C(g_i, g_i)(\xi) > 0$ ($i = 1, \cdots, N$). We will search for a function g of the type $\hat{g} = \sum_{i=1}^N \lambda_i(\xi)\hat{g}_i(\xi)$ (with λ_i a trigonometric polynomial) such that $\forall \xi$, $C(g,g)(\xi) \neq 0$.

We have :

$$C(g,g) = (\lambda_1, \cdots, \lambda_N) M[(g_i)] \begin{pmatrix} \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_N \end{pmatrix}.$$

Since the matrix $M[(g_i)]$ is hermitian and non-negative (as a Gram matrix), we have $C(g,g)(\xi) \neq 0$ if and only if $M[(g_i)] \begin{pmatrix} \bar{\lambda}_1 \\ \vdots \\ \bar{\lambda}_N \end{pmatrix} \neq 0$, or equivalently $\sum_{i=1}^n \bar{\lambda}_i(\xi)\vec{u}_i(\xi) \neq \vec{0}$ with $\vec{u}_i = \begin{pmatrix} C(g_1, g_i) \\ \vdots \\ C(g_N, g_i) \end{pmatrix}$. In order to prove the existence of such λ_i's, it is enough to look for continuous functions μ_i such that : $\forall \xi$, $\sum \mu_i(\xi)\vec{u}_i(\xi) \neq \vec{0}$; then we will conclude by the Stone-Weierstrass theorem. Let then φ_i, $1 \le i \le N$, be 2π-periodical C^∞ functions such that : $\varphi_i \equiv 1$ on $[t_i, t_{i+1}]$, $0 \le \varphi_i < 1$ outside from $\bigcup_{k\in\mathbb{Z}} [t_i + 2k\pi, t_{i+1} + 2k\pi]$,

$C(g_i,g_i)$ doesn't vanish on Supp φ_i and the support of φ_i is contained in $\left[\frac{t_{i-1}+t_i}{2},\frac{t_{i+1}+t_{i+2}}{2}\right]$ modulo 2π (with $t_0 = t_N - 2\pi$ and $t_{N+2} = t_2 + 2\pi$). Then we will choose μ_α to be $\mu_\alpha(\xi) = e^{i\omega_\alpha(\xi)}\frac{\hat{\varphi}_\alpha(\xi)}{\|\vec{u}_\alpha(\xi)\|}$ for some 2π-periodical C^∞ real-valued function ω_α. On $]t_\alpha,\frac{t_\alpha+t_{\alpha+1}}{2}]$ we have :

$$\sum \mu_k\vec{u}_k = \frac{\vec{u}_\alpha}{\|\vec{u}_\alpha\|}e^{i\omega_\alpha} + \varphi_{\alpha-1}\frac{\vec{u}_{\alpha-1}}{\|\vec{u}_{\alpha-1}\|}e^{i\omega_{\alpha-1}}$$

and, since $|\varphi_{\alpha-1}| < 1$, $\sum\mu_k\vec{u}_k \neq \vec{0}$. Similarly, $\sum\mu_k\vec{u}_k \neq \vec{0}$ on $[\frac{t_\alpha+t_{\alpha+1}}{2},t_{\alpha+1}[$. $\sum\mu_k\vec{u}_k$ can vanish only on the points $\xi = t_\alpha$; if we fix $\omega_\alpha(t_\alpha) = 0$, we thus have to avoid $\frac{\vec{u}_{\alpha+1}}{\|\vec{u}_{\alpha+1}\|} + \frac{\vec{u}_\alpha}{\|\vec{u}_\alpha\|}e^{i\omega_\alpha(t_{\alpha+1})} = \vec{0}$ and this can be easily done. ∎

We thus have a function $g \in V \cap L^2((1+|x|)^{1+\epsilon}dx)$ such that $C(g,g)$ doesn't vanish. If we define γ as $\hat{\gamma} = \frac{\hat{g}}{\sqrt{C(g,g)}}$, then $\gamma \in L^2((1+|x|)^{1+\epsilon}dx) \cap V$ (since $C(g,g) \in H^{1/2+\epsilon/2}(\mathbb{R}/2\pi\mathbb{Z})$, and $C(g,g)(\xi) \neq 0$ for all ξ, we have $C(g,g)^{-1/2} \in H^{1/2+\epsilon/2}(\mathbb{R}/2\pi\mathbb{Z})$ since we may use symbolic calculus on the algebra $H^{1/2+\epsilon/2}(\mathbb{R}/2\pi\mathbb{Z})$). Moreover

$$\sum_{k\in\mathbb{Z}} |\hat{\gamma}(\xi+2k\pi)|^2 = 1,$$

and this implies the orthonormality of $(\gamma(x-k))_{k\in\mathbb{Z}}$. Lemma 4 is proved. ∎

We may now prove (i) ⇒ (ii) in theorem 1. By lemmas 3 and 4, we know that V has an orthonormal basis $(\varphi_\delta(x-k))_{1\le\delta\le D,k\in\mathbb{Z}}$ with $\varphi_\delta \in L^2((1+|x|)^{1+\epsilon}dx)$. Then the dual basis $(\tilde{\varphi}_\delta(x-k))$ of $(\varphi_\delta(x-k))$ in $\tilde{V}$ is easily computed as

$$\tilde{\varphi}_\delta = P_0^*\varphi_\delta = \int \bar{p}(y,x)\varphi(y)dy$$

so that (5.3) and $\varphi_\delta \in L^2((1+|x|)^{1+\epsilon}dx)$ imply $\tilde{\varphi}_\delta \in L^2((1+|x|)^{1+\epsilon}dx)$. Theorem 1 is proved. ∎

We may give more results on the decay of the functions φ_δ and φ_δ^* in terms of the decay of $p(x,y)$:

PROPOSITION 1. - *Let P_0 satisfy the hypotheses and conclusions of theorem 1. Then :*

(i) one may choose φ_δ and $\tilde{\varphi}_\delta$ with rapid decay (i.e. $\forall k \in \mathbb{N}$, $x^k\varphi_\delta \in L^2$ and $x^k\tilde{\varphi}_\delta \in L^2$) if and only if :

$$\forall k \in \mathbb{N},\ \int_{x\in[0,1]}\int_{y\in\mathbb{R}} |x-y|^k\,(|p(x,y)|^2 + |p(y,x)|^2)dx\,dy < +\infty.$$

(ii) one may choose φ_δ and $\tilde{\varphi}_\delta$ with exponential decay (i.e. $\exists\epsilon > 0$, $e^{\epsilon|x|}\varphi_\delta(x) \in L^2$ and $e^{\epsilon|x|}\tilde{\varphi}_\delta(x) \in L^2$) if and only if :

$$\exists\alpha > 0,\ \int_{x\in[0,1]}\int_{y\in\mathbb{R}} e^{\alpha|x-y|}(|p(x,y)|^2 + |p(y,x)|^2)dx\,dy < +\infty.$$

(iii) one may choose φ_δ and $\tilde{\varphi}_\delta$ with compact support if and only if $p(x,y)$ is properly supported, i.e.

$$\exists M, \quad | x-y |> M \Rightarrow p(x,y)=0.$$

(iv) If P_0 is an orthogonal projection operator ($V = V^*$) then conclusions i), ii), and iii) are valid for orthonormal bases ($\varphi_\delta = \tilde{\varphi}_\delta^*$).

Proof. Except for the case of compactly supported functions, proposition 1 can be proved in exactly the same way as theorem 1 (because the integrability conditions are stable for the orthonormalization tool used in lemma 4 : $\hat{\gamma} = \frac{\hat{g}}{\sqrt{C(g,g}}$). The case of compactly supported functions is handled with the following lemma :

LEMMA 5. - *Let V be a shift-invariant closed subspace of $L^2(\mathbb{R})$ such that V contains non-trivial compactly supported functions. Then there exists two compactly supported functions g and h such that :*
- *$g \in V$ and $h \in L^2$;*
- *$< g(x-k) \mid h(x) >= \delta_{k,0}$ for all $k \in \mathbb{Z}$.*

For $\gamma \in V$ with compact support and $\eta \in L^2$ with compact support,

$$C(\gamma,\nu) = \sum_{k\in\mathbb{Z}} < \gamma(x) \mid \eta(x-k) > e^{-ik\xi}$$

is a trigonometric polynomial. We choose $g \in V$ with compact support such that :

$$\text{degree}\, C(g,g) = \min\{\text{degree}\, C(\gamma,\gamma)/\gamma \in V,\ \gamma \neq 0,\ \gamma \text{ has compact support}\}$$

and define I as $I = \{C(h,g) \mid h \in C_c^\infty\}$. I is an ideal of the ring of trigonometric polynomial:

$$C(h_1,g) + C(h_2,g) = C(h_1+h_2,g)$$

and

$$(\sum q_k e^{-ik\xi})C(h,g) = C(\sum q_k h(x-k), g).$$

This ring is principal, hence I has a generator Q_0. We say that $Q_0 = 1$. Indeed, let $Q_0 = R(e^{-i\xi})$ with $R(z) \in \mathbb{C}[X]$, $R(0) \neq 0$, and suppose that $R(z_0) = 0$ for some $z_0 \in \mathbb{C}^*$. We then have $\sum_{k\in\mathbb{Z}} z_0^k g(x+k) = 0$ in $\mathcal{D}'$; we define θ as

$$\theta = \sum_{k\geq 0} z_0^k g(x+k) = -\sum_{k<0} z_0^k g(x+k) \ ;$$

we have that θ has a compact support and therefore belongs to L^2 (as a locally finite sum of square-integrable functions) ; moreover $g(x) = \theta(x) - z_0\theta(x+1)$, hence

$$C(g,g) = | 1 - z_0 e^{i\xi} |^2 \ C(\theta,\theta) \ ;$$

we will have a contradiction if we prove $\theta \in V$. If $| z_0 | < 1$, the series $\sum_{k\geq 0} z_0^k g(x+k)$ converges strongly in V ; if $| z_0 | > 1$, the series $-\sum_{k<0} z_0^k g(x+k)$ converges strongly in V ; if $| z_0 | = 1$, the series $\sum_{k\geq 0} z_0^k g(x+k)$ has its partial sums $\sum_0^N z_0^k g(x+k)$ bounded in L^2

$$\left(\| \sum_0^N z_0^k g(x+k) \|_2 = \frac{1}{\sqrt{2\pi}} \| (z_0^{N+1} e^{i(N+1)\xi} - 1)\hat{\theta} \|_2 \leq 2 \| \theta \|_2 \right)$$

and converges to θ in $\mathcal{D}'$, hence weakly in L^2, but V as every closed subspace is weakly closed and $\theta \in V$. Thus $Q_0 = 1$ and lemma 5 is proved.

Point iii) of proposition 1 is now easily proved by recurrence on the multiplicity D of V. Just pick a g and a h as given by lemma 5 ; then the operator

$$Qf = \sum_{k\in\mathbb{Z}} < f \mid P_0^* h(x-k) > g(x-k)$$

is a projection operator such that $P_0 - Q$ is a projection operator ; both Q and $P_0 - Q$ have properly supported kernels, the multiplicity of $\operatorname{Ran} Q$ is 1 and the multiplicity of $\operatorname{Ran} P_0 - Q$ is $D - 1$.

Point iv) is then easily proved : it is enough to show that there exists $\gamma \in V$ such that γ is compactly supported and the family $(\gamma(x-k))_{k\in\mathbb{Z}}$ is orthonormal. Let $\omega \in V$ be a non-trivial compactly supported function and $\alpha = \operatorname{Inf} \operatorname{Supp} \omega$. We want to construct a function γ_0 defined by :

$$\| \gamma_0 \|_2 = \min\{\| g \|_2 / g \in V,\ \operatorname{Supp} g \subset [\alpha, +\infty],\ \int_\alpha^{\alpha+1} | g(x) |^2 \, dx = 1\}.$$

We define

$$K = \{g \in V / \operatorname{Supp} g \subset [\alpha, \alpha + 2M + 1],\ \int_\alpha^{\alpha+1} | g(x) |^2 \, dx = 1,$$

$$\| g \|_2 \leq \left(\int_\alpha^{\alpha+1} | \omega(x) |^2 \, dx \right)^{-1/2} \| \omega \|_2 \}.$$

We claim that K is a compact set : indeed, by point iii), we know that

$$U = \{g \in V / \operatorname{Supp} g \subset [\alpha, \alpha + 2M + 1]\}$$

is a finite-dimensional space, and K is a bounded and closed set in U. Moreover, if $g \in V$, $\operatorname{Supp} g \subset [\alpha, +\infty[$ and $\int_\alpha^{\alpha+1} | g(x) |^2 \, dx = 1$, we define $\tilde{g}$ as

$$\tilde{g} = g - P_0(\chi_{[\alpha+M+1,+\infty]} g) = P_0(\chi_{[\alpha,\alpha+M+1]} g) \ ;$$

it is easy to check that

$$\operatorname{Supp}\tilde{g} \subset [\alpha, \alpha + 2M + 1],\ \tilde{g} \in V,\ \tilde{g} = g \text{ on } [\alpha, \alpha + 1]$$

and

$$\| \tilde{g} \|_2 \leq \| \chi_{[\alpha,\alpha+M+1]} g \|_2 \leq \| g \|_2 \ ;$$

since K is compact, this proves the existence of γ_0 (with support in $[\alpha, \alpha + 2M + 1]$). Now if $k \geq 1$ we have

$$\operatorname{Supp}\gamma_0 + \lambda\gamma_0(x - k) \subset [\alpha, +\infty[$$

and

$$\gamma_0 + \lambda\gamma_0(x - k) = \gamma_0 \ \text{ on } \ [\alpha, \alpha + 1] \ ;$$

we obtain

$$\| \gamma_0 \|_2 \leq \| \gamma_0 + \lambda\gamma(x - k) \|_2 \ \text{ for all } \lambda \in \mathbf{C},$$

hence

$$< \gamma_0 \mid \gamma_0(x - k) >= 0.$$

The required function γ may then be defined as

$$\gamma = \frac{\gamma_0}{\| \gamma_0 \|_2}. \quad \blacksquare$$

2. The structure of a wavelet basis.

We will study in this section the structure of bi-orthogonal Riesz bases of $L^2(\mathbb{R})$

$$(a^{j/2}\psi_\alpha(a^j x - k))_{1 \leq \alpha \leq A, k \in \mathbb{Z}, j \in \mathbb{Z}}, (a^{j/2}\tilde{\psi}_\alpha(a^j x - k))_{1 \leq \alpha \leq A, k \in \mathbb{Z}, j \in \mathbb{Z}}.$$

THEOREM 2. - *Let $a > 1$.*

(i) Let $(\psi_\alpha)_{1 \leq \alpha \leq A}$ and $(\tilde{\psi}_\alpha)_{1 \leq \alpha \leq A}$ satisfy :

(14.1) $(a^{j/2}\psi_\alpha(a^j x - k))$ and $(a^{j/2}\tilde{\psi}_\alpha(a^j x - k))$ are bi-orthogonal Riesz bases of $L^2(\mathbb{R})$;

(14.2) *ψ_α and $\tilde{\psi}_\alpha$ belong to $H_{1/2+\epsilon,\epsilon'}$ for some positive ϵ, ϵ' ;*

(14.3) *the spaces*

$$V_0 = \operatorname{Span}(a^{j/2}\psi_\alpha(a^j x - k)/j < 0,\ 1 \leq \alpha \leq A,\ k \in \mathbb{Z})$$

and

$$\tilde{V}_0 = \operatorname{Span}(a^{j/2}\tilde{\psi}_\alpha(a^j x - k)/j < 0,\ 1 \leq \alpha \leq A,\ k \in \mathbb{Z})$$

are invariant under integer translations, *then the space* V_0 *has a Riesz basis* $(\varphi_\delta(x-k))_{1\le k\le D}$ *with dual basis* $(\tilde\varphi_\delta(x-k))_{1\le k\le D}$ *in* $\tilde V_0$ *such that* $\varphi_\delta, \tilde\varphi_\delta \in L^2((1+|x|)^{1+\epsilon}dx)$.

Moreover, we must have :

- $a=\frac{p}{q}$ *for two integers* $p>q\ge 1$ *with* $p\wedge q=1$;
- $(p-q)D=qA$.

When a is integer $(q=1)$, V_0 *and* $\tilde V_0$ *are always invariant and condition (14.3) is therefore always fulfilled.*

(ii) Conversely, if V_0 and $\tilde V_0$ *are closed linear subspaces of* $L^2(\mathbb{R})$ *such that :*

(15.1) V_0 *and* $\tilde V_0$ *have dual Riesz bases* $(\varphi_\delta(x-k))_{1\le\delta\le D,k\in\mathbb{Z}}$, $(\tilde\varphi_\delta(x-k))_{1\le\delta\le D,k\in\mathbb{Z}}$ *with* φ_δ *and* $\tilde\varphi_\delta$ *in* $H_{1/2+\epsilon,\epsilon'}$ *for some positive* ϵ *and* ϵ' ;

(15.2) $V_0\subset V_1=\{f(ax)/f\in V_0\}$ *and* $\tilde V_0\subset \tilde V_1=\{f(ax)/f\in\tilde V_0\}$;

(15.3) V_1 *and* $\tilde V_1$ *are shift-invariant :* $f\in V_1\Leftrightarrow f(x-k)\in V_1$, $f\in\tilde V_1\Leftrightarrow f(x-k)\in\tilde V_1$.

(15.4) *The spaces* $V_j=\{f(a^jx)/f\in V_0\}$ *and* $\tilde V_j=\{f(a^jx)/f\in\tilde V_0\}$ *satisfy :* $\bigcap_{j\in\mathbb{Z}}V_j=\bigcap_{j\in\mathbb{Z}}\tilde V_j=\{0\}$, $\bigcup_{j\in\mathbb{Z}}V_j$ *and* $\bigcup_{j\in\mathbb{Z}}\tilde V_j$ *are dense in* L^2

then $a\in\mathbb{Q}$, $a=\frac{p}{q}$ *with* $p\wedge q=1$, q *divides* D *and there exist* $A=(p-q)\frac{D}{q}$ *functions* ψ_α *and* A *functions* $\tilde\psi_\alpha$ *such that :*

(16.1) ψ_α *and* $\tilde\psi_\alpha$ *belong to* $H_{1/2+\epsilon,\epsilon'}$;

(16.2) $(\psi_\alpha(x-k))_{k\in\mathbb{Z}}$ *is a Riesz basis for* $V_1\cap\tilde V_0^\perp$ *and* $(\tilde\psi_\alpha(x-k))_{k\in\mathbb{Z}}$ *is a Riesz basis for* $\tilde V_1\cap V_0^\perp$;

(16.3) *the families* $(a^{j/2}\psi_\alpha(a^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z},1\le\alpha\le A}$ and $(a^{j/2}\tilde\psi_\alpha(a^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z},1\le\alpha\le A}$ *are dual Riesz bases of* $L^2(\mathbb{R})$.

This theorem has been proved in 1991-92 by P.-G. Lemarié-Rieusset [LEM6], [LEM9] and P. Auscher [AUS2]. The case of wavelet bases with non integer dilation parameter a has been studied in Auscher's thesis [AUS1].

Proof. We first explain why a has to be rational.

LEMMA 6. - *Let* V *be a closed linear subspace of* $L^2(\mathbb{R})$ *such that :*

(i) V *has a Riesz basis* $(f_\delta(x-k))_{1\le\delta\le D,k\in\mathbb{Z}}$ *with* $f_\delta\in L^2((1+|x|)^{1+\epsilon}dx)$ *for some positive* ϵ ;

(ii) V *is invariant under a shift by a factor* $a>1$: $f\in V\Leftrightarrow f(x-a)\in V$

then $a \in \mathbb{Q}$, $a = \frac{p}{q}$ *with* $p \wedge q = 1$, *and* D *is a multiple of* q.

The lemma 6 is easy. If V satisfies i) and ii), it is invariant for any translation $f \to f(x-h)$ where $h \in \mathbb{Z} + a\mathbb{Z}$; if $a \notin \mathbb{Q}$, $\mathbb{Z} + a\mathbb{Z}$ is dense in $\mathbb{R}$; since $h \to f(x-h)$ is continuous from $\mathbb{R}$ into $L^2(\mathbb{R})$, V is invariant under any translation. The orthogonal projection P of L^2 onto V commutes with translation, hence is a convolution with a distribution $p : Pf = p * f$, where $\hat{p}$ is bounded (since P is bounded on L^2) and satisfies $\hat{p}^2 = \hat{p}$ a.e. (since $P \circ P = P$) : we thus get $\hat{p} = \chi_E$ for a measurable set E and $V = \{f \in L^2 / \hat{f} = 0 \text{ a.e. outside from } E\}$. Now, V is supposed to have a basis $(f_\delta(x-k))$, which we may assume to be orthonormal as we saw in the preceding section. We thus have:

$$\forall f \in L^2 \quad \int |\hat{f}(\xi)|^2 \chi_E(\xi) d\xi = \sum_{\delta=1}^{D} \int_0^{2\pi} |\sum \hat{f}(\xi + 2k\pi) \bar{\hat{f}}_\delta(\xi + 2k\pi)|^2 \, d\xi$$

which gives, by considering $\hat{f}$supported in $[\xi_0 - \pi, \xi_0 + \pi]$ and letting ξ_0 run through $\mathbb{R}$:

$$\chi_E(\xi) = \sum_{\delta=1}^{D} |\hat{f}_\delta(\xi)|^2 \quad a.\ e.$$

which is absurd, since $\hat{f}_\delta$ is continuous. Thus a cannot be irrational. Now if $a = \frac{p}{q}$, the space $W = \{f(px)/f \in V\}$ and $Z = \{f(\frac{p}{q}x)/f \in V\}$ are shift-invariant ; moreover the orthogonal projectors on W and Z have kernels which satisfy the hypotheses (5.1) to (5.3) of theorem 1 (just use the basis of V to compute the kernels). Thus, W and Z have shift-invariant Riesz bases. If E is the multiplicity of W and F the multiplicity of Z, we have $E = pD = qF$, hence q divides D since $p \wedge q = 1$. ∎

The proof of theorem 2 is a direct consequence of theorem 1. Let's suppose that ψ_α and $\tilde{\psi}_\alpha$ satisfy (14.1) to (14.3). If Q_j is the projection operator from L^2 onto $\text{Span}(\psi_{j,k,\alpha}/1 \leq \alpha \leq A, k \in \mathbb{Z})$ in the direction of $(\text{Span}(\tilde{\psi}_{j,k,\alpha}/1 \leq \alpha \leq A, k \in \mathbb{Z}))^\perp$ (where $\psi_{j,k,\alpha} = a^{j/2}\psi_\alpha(a^j x - k)$ and $\tilde{\psi}_{j,k,\alpha} = a^{j/2}\tilde{\psi}_\alpha(a^j x - k)$), and if P_0 is the projection operator on V_0 in the direction of $\tilde{V}_0^\perp$, then :

$$P_0 = \sum_{j<0} Q_j = \text{Id}_{L^2} - \sum_{j \geq 0} Q_j. \tag{17}$$

(This proves that V_0 and $\tilde{V}_0$ are invariant if $a \in \mathbb{N}^*$ since for $j \geq 0$,

$$Q_j(f(x + \frac{1}{a^j})) = (Q_j f)(x + \frac{1}{a^j}),$$

hence

$$Q_j(f(x+1)) = (Q_j f)(x+1)).$$

If we look at the distribution kernels of $P_0 : p(x,y)$ and of $Q_j : q_j(x,y)$, we have :

$$p(x,y) = \sum_{j<0} q_j(x,y) = \delta(x-y) - \sum_{j\geq 0} q_j(x,y) \text{ in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}). \tag{18}$$

Now, since ψ_α and $\tilde{\psi}_\alpha$ belong to $L^2((1+ \mid x \mid)^{1+\epsilon}dx)$, we know that $q_0(x,y)$ is locally square-integrable and that

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} \left(\mid q_0(x,y)\mid^2 + \mid q_0(y,x)\mid^2\right) dx\,dy < +\infty \tag{19}$$

and for $j \geq 0$

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} \left(\mid q_j(x,y)\mid^2 + \mid q_j(y,x)\mid^2\right) dx\,dy$$

$$= \int_{x\in[0,a^j]}\int_{y\in\mathbb{R}} a^{-j(1+\epsilon)} \mid x-y \mid^{1+\epsilon} \left(a^{2j} \mid q_0(x,y)\mid^2 \mid q_0(y,x)\mid^2\right) \frac{1}{a^{2j}} dx\,dy$$

$$= a^{-j(1+\epsilon)} \int_{x\in[0,a^j]}\int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} \left(\mid q_0(x,y)\mid^2 + \mid q_0(y,x)\mid^2\right) dx\,dy.$$

Now, the function

$$x \longrightarrow \int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} (\mid q_0(x,y)\mid^2 + \mid q_0(y,x)\mid^2) dy$$

is 1-periodical, hence :

$$\int_{x\in[0,1]}\int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} (\mid q_j(x,y)\mid^2 + \mid q_j(y,x)\mid^2) dx\,dy$$

$$\leq a^{-j\epsilon}\left(a^{-j}[a^j+1]\right) \int_{x\in[0,1]}\int_{y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} \left(\mid q_0(x,y)\mid^2 + \mid q_0(y,x)\mid^2\right) dy$$

$$\leq C\ a^{-j\epsilon}.$$

Thus we obtain that outside the diagonal $x = y$, $p(x,y)$ is a locally square-integrable function such that :

$$\int\int_{x\in[0,1],y\in\mathbb{R}} \mid x-y \mid^{1+\epsilon} (\mid p(x,y)\mid^2 + \mid p(y,x)\mid^2) dx\,dy < +\infty. \tag{20}$$

Now, we prove that q_0 is locally in $L^{r_0}(\mathbb{R}\times\mathbb{R})$ for some $r_0 > 2$: we know that $\psi_\alpha \in L^2((1+ \mid x \mid)^{1+\epsilon}dx)$ and $\psi_\alpha \in H^{\epsilon'}$, hence $\psi_\alpha \in L^r(dx)$ for $\frac{1}{r} = \frac{1}{2} - \epsilon'$ (we suppose $\epsilon' < \frac{1}{2}$) ; we obtain by interpolation $\psi_\alpha \in L^\sigma((1+ \mid x \mid)^{\theta(1+\epsilon)}dx)$ for $0 < \theta < 1$ and $\frac{1}{\sigma} = \frac{1-\theta}{2} + \frac{\theta}{r}$.

We fix θ so that $\theta(1+\epsilon)>1$ and fix r_0 such that $2<r_0<\sigma$ and $\frac{2}{r_0}\theta(1+\epsilon)>1$. For a fixed compact set K, we get :

$$\int\int_{K\times K} \mid \psi_\alpha(x-k)\mid^{r_0}\mid \tilde{\psi}_\alpha(y-k)\mid^{r_0}\,dx\,dy$$

$$\leq C_K\frac{1}{(1+\mid k\mid)^{2\theta(1+\epsilon)}}\parallel \psi_\alpha\parallel^{r_0}_{L^{r_0}((1+|x|)^{\theta(1+\epsilon)}dx)}\parallel \tilde{\psi}_\alpha\parallel^{r_0}_{L^{r_0}((1+|x|)^{\theta(1+\epsilon)}dx)}$$

hence :

$$\sum_k \parallel \psi_\alpha(x-k)\tilde{\psi}_\alpha(y-k)\parallel_{L^{r_0}(K\times K)}\leq C\sum_k\frac{1}{(1+\mid k\mid)^{\frac{2}{r_0}\theta(1+\epsilon)}}<+\infty.$$

We have that $\overline{\bigcup_{j<0} a^j K}=\tilde{K}$ is bounded, hence compact, so that for $j<0$:

$$\int\int_{K\times K}\mid q_j(x,y)\mid^2\,dx\,dy=\int\int_{a^jK\times a^jK}\mid q_0(x,y)\mid^2\,dx\,dy$$

$$\leq\left(\int\int_{\tilde{K}\times\tilde{K}}\mid q_0(x,y)\mid^{r_0}\,dx\,dy\right)^{2/r_0}a^{j(1-\frac{2}{r_0})}\mid K\times K\mid^{1-\frac{2}{r_0}}$$

hence

$$\int\int_{K\times K}\mid p(x,y)\mid^2\,dx\,dy<+\infty.$$

We may apply theorem 1 to P_0 and thus prove the existence of shift-invariant dual Riesz bases for V_0 and $\tilde{V}_0$ with basic functions in $L^2((1+\mid x\mid)^{1+\epsilon}dx)$.

The converse implication (15) $\Rightarrow$ (16) is easy. Let P_1 be the projection operator onto V_1 in direction of $\tilde{V}_1^\perp$ and P_0 the projection operator onto V_0 in direction of $\tilde{V}_0^\perp$. Then $P_1-P_0=Q_0$ is itself a projection operator (we have $P_1\circ P_0=P_0$ since $V_0\subset V_1$ and $P_0\circ P_1=P_0$ since $\tilde{V}_0\subset\tilde{V}_1$) and we may apply (due to (15.1)) theorem 1 to Q_0. Hence $V_1\cap\tilde{V}_0^\perp=\operatorname{Ran}Q_0$ and $\tilde{V}_1\cap V_0^\perp=(\operatorname{Ker}Q_0)^\perp$ have dual Riesz bases $(\psi_\alpha(x-k))$ and $(\tilde{\psi}_\alpha(x-k))$ with ψ_α, $\tilde{\psi}_\alpha$ in $L^2((1+\mid x\mid^{1+\epsilon}dx)$. In particular, we have

$$\sum_{\delta=1}^{D}\sum_{k\in\mathbb{Z}}\mid<\psi_\alpha\mid\tilde{\varphi}_\delta^*(ax-k)>\mid<+\infty,$$

hence

$$\parallel\psi_\alpha\parallel_{H^{\epsilon'}}\leq\sum_{\delta=1}^{D}\sum_k a\mid<\psi_\alpha\mid\tilde{\varphi}_\delta(ax-k)>\mid\parallel\varphi_\delta(ax-k)\parallel_{H^{\epsilon'}}<+\infty$$

and ψ_α belongs to $H_{1/2+\epsilon,\epsilon'}$ (and $\tilde{\psi}_\alpha$ as well). By construction, we have

$$<\psi_{\alpha,j,k}\mid\tilde{\psi}_{\beta,\ell,p}>=\delta_{\alpha,\beta}\delta_{j,\ell}\delta_{k,p}$$

$$\text{(where } \psi_{\alpha,j,k} = a^{j/2}\psi_\alpha(a^j x - k) \text{ and } \tilde{\psi}_{\beta,\ell,p} = a^{\ell/2}\tilde{\psi}_\beta(a^\ell x - p))$$

and (due to (15.4))

$$\lim_{N\to+\infty} \| f - \sum_{|j|\le N} \sum_{\alpha=1}^{A} \sum_{k\in\mathbb{Z}} < f \mid \tilde{\psi}_{\alpha,j,k} > \psi_{\alpha,j,k} \|_2 = 0 \text{ for all } f \in L^2.$$

In order to prove that $(\psi_{\alpha,j,k})$ and $(\tilde{\psi}_{\alpha,j,k})$ are dual Riesz bases of $L^2(\mathbb{R})$, we just have to prove that they are almost orthonormal families. It is enough to show that $\int \psi_\alpha dx = 0$ and $\int \tilde{\psi}_\alpha dx = 0$, and then to apply the vaguelettes lemma.

LEMMA 7. - *Let ψ belong to $H_{1/2+\epsilon,\epsilon'}$ for some positive ϵ and ϵ'. Then for all $f \in L^2$,*

$$\lim_{j\to+\infty} \sum_{k\in\mathbb{Z}} a^j < f \mid \psi(a^j x - k) > \chi_{[0,1]}(a^j x - k) = (\int \psi dx) f \text{ in } L^2.$$

We just have to notice that the family $(\psi(x-k))$ is almost orthonormal, as well as $(\chi_{[0,1]}(x-k))$. The family of operators $(R_j)_{j\in\mathbb{Z}}$, defined by

$$R_j f = \sum_{k\in\mathbb{Z}} a^j < f \mid \psi(a^j x - k) > \chi_{[0,1]}(a^j x - k)$$

is then equicontinuous on $L^2(\mathbb{R})$. Thus, it is enough to prove the convergence only for a dense subset of L^2. We have seen that there exist $\sigma > 2$ and $\eta > 0$ such that ψ belongs to $L^\sigma((1+ \mid x \mid^{1+\eta} dx)$; thus we may write :

$| R_j f(x) | \le$
$\sum_{k\in\mathbb{Z}} \| \psi \|_{L^\sigma((1+|x|^{1+\eta})dx)} \left(\int \mid f \mid^{\frac{\sigma}{\sigma-1}} (1+ \mid a^j x - k \mid)^{-\frac{1}{\sigma-1}(1+\eta)} a^j dx\right)^{1-\frac{1}{\sigma}} \chi_{[0,1]}(a^j x - k)$;

if we choose σ close enough to 2 to ensure $\frac{1}{\sigma-1}(1+\eta) > 1$, we obtain easily :

$$| R_j f(x) | \le C M(| f |^{\frac{\sigma}{\sigma-1}})^{\frac{\sigma-1}{\sigma}},$$

where M is the Hardy-Littlewood maximal function ; but we have :

$$\| M\left(| f |^{\frac{\sigma}{\sigma-1}}\right)^{\frac{\sigma-1}{\sigma}} \|_2 = \| M\left(| f |^{\frac{\sigma}{\sigma-1}}\right) \|_{\frac{2(\sigma-1)}{\sigma}}^{\frac{\sigma-1}{\sigma}} \quad \left(\text{with } \frac{2(\sigma-1)}{\sigma} > 1, \text{ hence :}\right)$$

$$\le C_\sigma \| | f |^{\frac{\sigma}{\sigma-1}} \|_{\frac{2(\sigma-1)}{\sigma}}^{\frac{\sigma-1}{\sigma}} = C_\sigma \| f \|_2,$$

hence we may apply dominated convergence to conclude that when $f \in C_c^\infty$ the pointwise convergence of $R_j f$ to $(\int \psi\, dx) f$ (which is obvious since $| x |^\alpha \psi \in L^1$ for some positive α) implies the L^2 convergence of $R_j f$ to $(\int \psi\, dx) f$. ■

To end the proof of theorem 2, we just apply lemma 7 to $f = \tilde{\psi}_\alpha$ and $\psi = \psi_\alpha$ to conclude, since $< \tilde{\psi}_\alpha, \psi_{\alpha,j,k} >= 0$ for any $j > 0$ and $k \in \mathbb{Z}$, that $(\int \psi_\alpha dx)\tilde{\psi}_\alpha = 0$, hence $\int \psi_\alpha dx = 0$. By the same way, we have $\int \tilde{\psi}_\alpha dx = 0$ and theorem 2 is proved. ∎

Of course, we may have a more precise description of the shift-invariant Riesz bases by mean of proposition 1 :

PROPOSITION 2. - *In theorem 1, we may choose the* φ_δ *and* $\tilde{\varphi}_\delta$ *of the same decay as the* ψ_α *and* $\tilde{\psi}_\alpha$, *and conversely* ψ_α *and* $\tilde{\psi}_\alpha$ *of the same decay as* $\tilde{\varphi}_\delta$ *and* $\tilde{\varphi}_\delta$, *i.e. with rapid decay, exponential decay or compact support as described in Proposition 1. In case of orthonormal bases* ($\psi_\alpha = \tilde{\psi}_\alpha$ *or* $\varphi_\delta = \tilde{\varphi}_\delta$), *we may choose an associated orthonormal basis with the same decay properties.*

3. Definition and examples of multi-resolution analysis.

The notion of multi-resolution analysis has been introduced by S. Mallat in 1986 [MALL1] and plays a key part in the construction of wavelet bases.

DEFINITION 3. - *(i) A multi-resolution analysis of* $L^2(\mathbb{R})$ *is a sequence* $(V_j)_{j\in\mathbb{Z}}$ *of closed subspaces of* L^2 *such that :*

(21.1) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$

(22.2) $f \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

(22.3) V_0 *has a Riesz basis* $(\varphi(x-k))_{k\in\mathbb{Z}}$.

(ii) Two multi-resolution analyses of $L^2(\mathbb{R})$ (V_j), $(\tilde{V}_j)$ *are said to be bi-orthogonal if we have* $L^2(\mathbb{R}) = V_0 \oplus \tilde{V}_0^{\perp}$, *thus if there is a bounded projector on* L^2 *such that* $V_0 = \operatorname{ran} P_0$ *and* $\tilde{V}_0^{\perp} = \operatorname{Ker} P_0$.

(iii) A generalized multi-resolution of $L^2(\mathbb{R})$ *associated to a dilation factor* $a = \frac{p}{q}$ $(p \wedge q = 1)$ *and of multiplicity* D *is a sequence* $(V_j)_{j\in\mathbb{Z}}$ *of closed subspaces of* L^2 *such that :*

(22.1) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$

(22.2) $f \in V_j \Leftrightarrow f(ax) \in V_{j+1}$

(22.3) V_0 *has a shift-invariant Riesz basis* $(\varphi_\delta(x-k))_{1\leq\delta\leq D, k\in\mathbb{Z}}$

(22.4) V_1 *is shift-invariant :* $f \in V_1 \Leftrightarrow f(x-k) \in V_1$.

(Condition (22.4) is always satisfied if $a \in \mathbb{N}^*$*).*

We have seen that the bi-orthogonal wavelet bases whose wavelets have enough decay and regularity $(\psi_\alpha, \tilde{\psi}_\alpha \in H_{1/2+\epsilon,\epsilon'})$ can be derived from multi-resolution analyses. Now let us see some easy examples or counter-examples.

a) the Journé counter-example :

Let $E = \left[-\frac{8\pi}{7}, -\frac{4\pi}{7}\right] \cup \left[\frac{4\pi}{7}, \frac{6\pi}{7}\right] \cup \left[\frac{24\pi}{7}, \frac{32\pi}{7}\right]$ and $\hat{\psi} = \chi_E$. We check easily that $\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi+2k\pi) = 1$ a. e. and $\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j\xi) = 1$ a. e. ; so we get that $(\psi_{j,k} = 2^{j/2}\psi(2^j x-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an orthonormal basis of L^2. Let $V_0 = \mathrm{Span}(\psi_{\ell,k}/\ell < 0, k \in \mathbb{Z})$. If f belongs to V_0, $\mathrm{Supp}\,\hat{f}$ is contained in the closure of $\bigcup_{\ell<0} \mathrm{Supp}\,\hat{\psi}(\frac{\xi}{2^\ell})$, hence in $F = \left[-\frac{4\pi}{7}, \frac{4\pi}{7}\right] \cup \left[\frac{6\pi}{7}, \frac{8\pi}{7}\right] \cup \left[\frac{12\pi}{7}, \frac{16\pi}{7}\right]$. Now, if $\mathrm{Supp}\,\hat{f} \subset F$, the function $C(f,f) = \sum_{k\in\mathbb{Z}} \mid \hat{f}(\xi+2k\pi) \mid^2$ vanishes on $\left[\frac{8\pi}{7}, \frac{10\pi}{7}\right]$, and V_0 cannot have a shift-invariant Riesz basis (though V_0 is shift-invariant).

b) the "Littlewood-Paley" Meyer wavelet [LEME] :

Let $\hat{\varphi}$ be a non-negative even $\mathcal{C}^\infty$ function such that $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+2k\pi)^2 = 1$ and $\mathrm{Supp}\,\hat{\varphi} \subset \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$. We claim that the family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis for a space V_0 which generates by dyadic dilations a multi-resolution analysis (V_j) (with $V_j = \{f(2^j x)/f \in V_0\}$) : the orthonormality of the family $(\varphi(x-k))$ is equivalent to $\sum_{k\in\mathbb{Z}} \mid \hat{\varphi}(\xi+2k\pi) \mid^2 = 1$ a. e. ; now (21.2) and (21.3) are obvious for (V_j) ; since $f \in V_j$ implies $\mathrm{Supp}\,\hat{f} \subset \{\xi/ \mid \xi \mid \leq \frac{4\pi}{3}2^j\}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$; moreover if $\mathrm{Supp}\,\hat{f} \subset \{\xi/ \mid \xi \mid \leq \frac{2\pi}{3}\}$, we have $\hat{f} = C(f,\varphi)\hat{\varphi}$, hence $f \in V_0$: thus $\varphi(\frac{x}{2}) \in V_0$ (which implies $V_j \subset V_{j+1}$) and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in L^2. (V_j) is thus a multi-resolution analysis.

We may easily compute a wavelet basis associated to (V_j) by the following obvious lemma :

LEMMA 8. - *If V is a shift-invariant space with an Hilbertian basis $(\varphi_\delta(x-k))_{1\leq\delta\leq D,k\in\mathbb{Z}}$ and if $(\psi_\delta)_{1\leq\delta\leq D}$ are functions in V, then $(\psi_\delta(x-k))_{1\leq\delta\leq D,k\in\mathbb{Z}}$ is an Hilbertian basis of V if and only if the correlation matrix $M((\psi_\delta),(\varphi_\delta))(\xi)$ is unitary* a. e. *in ξ.*

We thus have to compute a function ψ with $\hat{\psi} = \frac{a(\xi)}{\sqrt{2}}\hat{\varphi}\frac{(\xi)}{2} + \frac{b(\xi)}{\sqrt{2}}e^{-i\frac{\xi}{2}}\hat{\varphi}(\frac{\xi}{2})$ (a, b 2π-periodical) such that $\begin{pmatrix} C(\varphi,\sqrt{2}\varphi(2x)) & a(\xi) \\ C(\varphi,\sqrt{2}\varphi(2x-1)) & b(\xi) \end{pmatrix}$ is unitary. We thus get a very easy solution with

$$a(\xi) = -\bar{C}(\varphi,\sqrt{2}\varphi(2x-1)) \text{ and } b(\xi) = \bar{C}(\varphi,\sqrt{2}\varphi(2x)),$$

which gives :

$$\hat{\psi}(2\xi) = \frac{1}{\sqrt{2}}a(2\xi)\hat{\varphi}(\xi) + \frac{1}{\sqrt{2}}b(2\xi)\hat{\varphi}(\xi)e^{-i\xi}$$

$$=\frac{1}{2}\left\{-\sum_k \bar{\hat{\varphi}}(2\xi+2k\pi)\hat{\varphi}(\xi+k\pi)e^{-i(\xi+k\pi)} + \sum_k e^{-i\xi}\bar{\hat{\varphi}}(2\xi+2k\pi)\hat{\varphi}(\xi+k\pi)\right\}\hat{\varphi}(\xi)$$

$$=\left\{\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+2k\pi+\pi)\bar{\hat{\varphi}}(2\xi+4k\pi+2\pi)\right\}e^{-i\xi}\hat{\varphi}(\xi)$$

$$=\frac{1}{2}C(\varphi,\varphi(\frac{x}{2}))(\xi+\pi)e^{-i\xi}\hat{\varphi}(\xi).$$

This wavelet ψ is smooth with $\hat{\psi}$ C^∞ and $\operatorname{Supp}\hat{\psi} \subset \left[-\frac{8\pi}{3}, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right]$ and generates a Hilbertian basis of $L^2(\mathbb{R})$ $(2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$.

c) The "Littlewood-Paley" David wavelet [MEY1] *:*

G. David slightly modified Meyer's construction to provide a Hilbertian basis

$$(a^{j/2}\psi(a^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

for $a = 1 + \frac{1}{m}$ with $\hat{\psi}$ C^∞ and compactly supported. Let $\hat{\varphi}$ be a non-negative $\mathbf{C}^\infty$ even function such that $\sum \mid \hat{\varphi}(\xi + 2k\pi) \mid^2 = 1$ and $\operatorname{Supp}\hat{\varphi} \subset \left[-\frac{2m+2}{2m+1}\pi, \frac{2m+2}{2m+1}\pi\right]$. Then the family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is orthonormal and any $f \in L^2$ such that $\operatorname{Supp}\hat{f} \subset \left[\frac{-2m}{2m+1}\pi, \frac{2m}{2m+1}\pi\right]$ satisfies $\hat{f} = C(f,\varphi)\hat{\varphi}$.

We now define a multi-resolution analysis (V_j) by : a Hilbertian basis of V_j is given by $(a^{j/2}\sqrt{m}\varphi(a^j mx - k))_{k\in\mathbb{Z}}$; in particular a basis of V_0 is given by $(\sqrt{m}\varphi(mx-k))_{k\in\mathbb{Z}}$ (and the multiplicity of V_0 is therefore m) and a basis of V_1 is given by $(\sqrt{m+1}\varphi((m+1)x-k))_{k\in\mathbb{Z}}$ (and the multiplicity of V_1 is $m+1$). We may easily compute a wavelet basis associated to (V_j) : the wavelet ψ is given by

$$\hat{\psi} = \sum_{k=0}^{m} C(\psi, \sqrt{m+1}\varphi((m+1)x-k)) \frac{1}{\sqrt{m+1}}\hat{\varphi}\left(\frac{\xi}{m+1}\right)e^{-ik\frac{\xi}{m+1}}$$

where the vector

$$\left\{C(\psi, \sqrt{m+1}\varphi((m+1)x-k))(\xi)\right\}_{0\le k\le m}$$

is the unique vector that completes the matrix

$$(C(\sqrt{m}\varphi(mx-r), \sqrt{m+1}\varphi((m+1)x-k))_{0\le r\le m-1, 0\le k\le m}$$

in $SU(m+1)$. Of course the wavelet ψ is C^∞, has its support contained in

$$\left[-\frac{2(m+1)^2}{m(2m+1)}\pi, -\frac{2m}{2m+1}\pi\right] \cup \left[\frac{2m}{2m+1}\pi, \frac{2(m+1)^2}{m(2m+1)}\pi\right]$$

and generates a Hilbertian basis $(a^{j/2}\psi(a^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$.

4. Non-existence of regular wavelets for the Hardy space $\mathbf{H}^{(2)}$.

The Meyer wavelet may be viewed as a smoothened version of the "Littlewood-Paley-Shannon" wavelet ψ defined by :

$$\hat{\psi} = \chi_{[-2\pi,-\pi]} \cup \chi_{[\pi,2\pi]}. \tag{23}$$

It is very easy to check that for such a wavelet ψ, we have

$$\sum_{k\in\mathbb{Z}} \mid \hat{\psi}(\xi+2k\pi) \mid^2 = \sum_{j\in\mathbb{Z}} \mid \hat{\psi}(2^j\xi) \mid^2 = 1 \text{ a. e.}$$

and therefore that $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a Hilbertian basis of $L^2(\mathbb{R})$.

We may define as well a Littlewood-Paley-Shannon wavelet for the space $H^{(2)} = \{f \in L^2/\text{Supp}\,\hat{f} \subset [0,+\infty]\}$ of analytical signals, by defining :

$$\hat{\psi} = \chi_{[2\pi,4\pi]}. \tag{24}$$

Then the family $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an Hilbertian basis of $H^{(2)}$. A very natural question is then to ask whether there is a smoothened version of the wavelet ψ. The answer was proved to be negative by P. Auscher in 1992 [AUS2]. The proof follows another proof by P. G. Lemarié-Rieusset [LEM5] of the existence of multi-resolution analyses associated to regular wavelets.

THEOREM 3 (Auscher's theorem). - *There is no wavelet $\psi \in H^{(2)} \cap H_{\epsilon,\epsilon'}$ (for some $\epsilon > 1/2$ and some $\epsilon' > 0$) so that $(2^{j/2}\psi(2^jx-k))$ is an Hilbertian basis of $H^{(2)}$.*

Proof. The proof begins by a discrete version of the result of C. K. Chui and X. Shi quoted in theorem 1 of the preceding chapter.

LEMMA 9. - *Let $\psi \in L^2(\mathbb{R})$ be such that :*

- (i) *the family $(\psi_{j,k} = 2^{j/2}\psi(2^jx-k))$ is orthonormal ;*
- (ii) *the closed linear span V_0 of the $\psi_{\ell,k}$, $\ell < 0$, $k \in \mathbb{Z}$ is shift-invariant : $f \in V_0 \Leftrightarrow f(x-1) \in V_0$.*

Then we have :

$$\forall f \in V_0,\ \ \| f \|_2^2 = \sum_{\ell<0} 2^\ell \sum_{k\in\mathbb{Z}} |< f \mid 2^{\ell/2}\psi(2^\ell(x-k)) >|^2 . \tag{25}$$

The meaning of formula (25) is the following one : the orthogonal projection operator P_0 from L^2 onto V_0 is shift-invariant. It can be written as

$$P_0 f = \sum_{j<0} \sum_{k\in\mathbb{Z}} < f \mid \psi_{j,k} > \psi_{j,k}$$

but the shift-invariance is hidden in this formula ; formula (25) gives

$$P_0 f = \sum_{\ell<0} 2^\ell \sum_{k\in\mathbb{Z}} < f \mid 2^{\ell/2}\psi(2^\ell(x-k)) > 2^{\ell/2}\psi(2^\ell(x-k)),$$

so that the shift-invariance is clearly expressed. The proof of lemma 9 is analog to the proof of the Chui-Shi result. We see easily that for any $f \in L^2$ we have :

$$\sum_{\ell<0} 2^{\ell} \sum_{k\in\mathbb{Z}} |< f \mid 2^{\ell/2}\psi(2^{\ell}(x-k)) >|^2 \leq P_0 f \|_2^2 :$$

if

$$Q_\ell = \sum_{k\in\mathbb{Z}} < f \mid \psi_{\ell,k} > \psi_{\ell,k}$$

and

$$\tau_h f(x) = \tau_h f(x-h),$$

then :

$$\sum_{-N}^{-1} 2^{\ell} \sum_{k\in\mathbb{Z}} |< f \mid 2^{\ell/2}\psi(2^{\ell}(x-k) >|^2 = \frac{1}{2^N} \sum_{r=0}^{2^N-1} \sum_{\ell=-N}^{-1} \| \tau_r Q_\ell \tau_{-r} f \|_2^2 \leq \| P_0 f \|_2^2$$

(since $\sum_{\ell<0} \| \tau_r Q_\ell \tau_r f \|_2^2 = \sum_{\ell<0} \| Q_\ell \tau_r f \|_2^2 = \| P_0 \tau_r f \|_2^2 = \| \tau_r P_0 f \|_2^2 = \| P_0 f \|_2^2$) ; thus lemma 9 is proved if we prove that for a dense subset of L^2 we have

$$\lim_{N\to+\infty} \sum_{-N}^{-1} 2^{\ell} \sum_{r=0}^{2^N-1} \| \tau_r Q_\ell \tau_{-r} f \|_2^2 = \| P_0 f \|_2^2 .$$

This is easily done for f such that $\operatorname{Supp} \hat{f}$ is compact and $0 \notin \operatorname{Supp} \hat{f}$: we saw that if $\hat{\varphi} \equiv 1$ on $\operatorname{Supp} \hat{f}$ and $0 \notin \operatorname{Supp} \hat{\varphi}$, then, for $\ell < 0$:

$$\| \tau_r Q_\ell \tau_{-r} f \|_2^2 \leq C\, 2^{-\ell} \int | \hat{\varphi}(\xi) |^2 | \hat{\psi}(2^{-\ell}\xi) |^2 \, d\xi,$$

hence :

$$\| P_0 f \|_2^2 - \sum_{-N}^{-1} 2^{\ell} \sum_{r=0}^{2^N-1} \| \tau_r Q_\ell \tau_{-r} f \|_2^2 \leq C \sum_{-\infty}^{-N-1} 2^{-\ell} \int | \hat{\varphi}(\xi) |^2 | \hat{\psi}(2^{-\ell}\xi) |^2 \, d\xi$$

and this majoration goes to 0 as N goes to $+\infty$. ■

LEMMA 10. - *Under the hypotheses of lemma 9, we have :*

(26) *For almost all* ξ, $\sum_{\ell\geq 1} \sum_{k\in\mathbb{Z}} | \hat{\psi}(2^{\ell}(\xi + 2k\pi) |^2 = 0$ *or is* ≥ 1.

Lemma 10 is an easy consequence of lemma 9. We write, due to formula (25) and the Poisson summation formula :

$$\forall f \in V_0, \quad \hat{f}(\xi) = \sum_{\ell\geq 1} \left(\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi) \bar{\hat{\psi}}(2^{\ell}(\xi + 2k\pi)) \right) \hat{\psi}(2^{\ell}\xi) \; a.e.$$

hence :

$$\sum_{k\in\mathbb{Z}} | \hat{f}(\xi+2k\pi) |^2 = \sum_{\ell\geq 1}\left|\sum_{k\in\mathbb{Z}} \hat{f}(\xi+2k\pi)\bar{\hat{\psi}}(2^\ell(\xi+2k\pi)\right|^2$$

$$\leq \sum_{k\in\mathbb{Z}} | \hat{f}(\xi+2k\pi) |^2 \sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi)) |^2$$

by the Cauchy-Schwarz inequality. Letting $\hat{f} = \hat{\psi}(2^\ell\xi)$ and summing on ℓ gives

$$\sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi)) |^2 \leq \left(\sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi) |^2\right)^{1/2}$$

and proves lemma 10. ■

LEMMA 11. - *Let ψ belong to $H_{\epsilon,\epsilon'}$ for some $\epsilon > 1/2$ and $\epsilon' > 0$. Then*

$$\sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi) |^2$$

defines a continuous function outside from $2\pi\mathbb{Z}$.

As a matter of fact, we know (by the proof of the vaguelettes lemma) that for some positive α and some $\eta > 1/2$ we have $| \xi |^{\alpha/2} \hat{\psi} \in H^\eta$. Hence we obtain

$$\sum_{k\in\mathbb{Z}} | \xi+2k\pi |^\alpha | \hat{\psi}(\xi+2k\pi) |^2 \in H^\eta \ ;$$

now if

$$A_\epsilon = \bigcup_{k\in\mathbb{Z}} [\epsilon+2k\pi, 2\pi-\epsilon+2k\pi],$$

we have for $\ell \geq 1$ and $\xi \notin A_\epsilon$:

$$\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi) |^2 \leq (2^\ell\epsilon)^{-\alpha} \sum_{k\in\mathbb{Z}} | 2^\ell(\xi+2k\pi) |^\alpha | \hat{\psi}(2^\ell(\xi+2k\pi) |^2$$

$$\leq (2^\ell\epsilon)^{-\alpha} \left\| \sum_{k\in\mathbb{Z}} | \eta+2k\pi |^\alpha | \hat{\psi}(\eta+2k\pi) |^2 \right\|_\infty$$

and lemma 11 is obvious. ■

Thus if ψ satisfies the hypotheses of lemma 9 and if moreover $\psi \in H_{\epsilon,\epsilon'}$ ($\epsilon > 1/2$, $\epsilon' > 0$), we must have

$$\sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} | \hat{\psi}(2^\ell(\xi+2k\pi) |^2 \geq 1 \quad \text{for} \quad \xi \notin 2\pi\mathbb{Z}$$

but

$$\int_0^{2\pi} \sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2^\ell(\xi+2k\pi)) \mid^2 d\xi = (\sum_{\ell\geq 1} 2^{-\ell}) \int_{-\infty}^{+\infty} \mid \hat{\psi}(\xi) \mid = 2\pi,$$

hence we get :

$$\sum_{\ell\geq 1}\sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2^\ell(\xi+2k\pi) \mid^2 = 1 \text{ for } \xi \notin 2\pi\mathbb{Z}.$$

Therefore, the Cauchy-Schwarz inequalities used for $\hat{f} = \hat{\psi}(2^\ell\xi)$ on the proof of lemma 10 were equalities, which means that for almost all ξ the vectors

$$e_\ell(\xi) = (\hat{\psi}(2^\ell(\xi+2k\pi)))_{k\in\mathbb{Z}}$$

are proportional one to each other when ℓ runs through $\mathbb{N}^*$; thus the function

$$U(\xi) = \frac{\mid \hat{\psi}(2^\ell\xi) \mid^2}{\sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2^\ell(\xi+2k\pi) \mid^2}$$

(where ℓ depends on ξ and is chosen such that $\sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2^\ell(\xi+2k\pi) \mid^2 > 0$) doesn't depend on the choice of ℓ. Moreover we have for all $\ell \geq 1$,

$$\mid \hat{\psi}(2^\ell\xi) \mid^2 = U(\xi) \sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2^\ell(\xi+2k\pi) \mid^2,$$

hence formula (26) gives

$$U(\xi) = \sum_{\ell\geq 1} \mid \hat{\psi}(2^\ell\xi) \mid^2 \quad \text{for } \xi \notin 2\pi\mathbb{Z}.$$

Now, we have

$$U(\xi) = \mid \hat{\psi}(2\xi) \mid^2 + U(2\xi),$$

hence

$$U(2\xi) = U(\xi)\left(1 - \sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2\xi+4k\pi) \mid^2\right) = U(\xi) \sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2\xi+2\pi+4k\pi) \mid^2 .$$

Let

$$\theta(\xi) = \sum_{k\in\mathbb{Z}} \mid \hat{\psi}(2\xi+2\pi+4k\pi) \mid^2 ;$$

we have the following properties for θ :

(27.1) $$\theta \in H^\epsilon \ (\epsilon > 1/2) \quad \text{and} \quad 0 \leq \theta \leq 1$$

(27.2) $$\theta(\xi)+\theta(\xi+\pi)=1 \quad \text{and} \quad \theta \text{ is } 2\pi\text{-periodical.}$$

(27.3) $$U(2\xi)=\theta(\xi)U(\xi) \quad \text{and} \quad U(\xi)=\sum_{\ell\geq 1} | \hat{\psi}(2^\ell\xi) |^2 .$$

We may define $V(\xi)=\prod_1^\infty \theta(\frac{\xi}{2^j})$. If $\theta(0)<1$, $V(\xi)=0$ and if $\theta(0)=1$, $V(\xi)$ is continuous and $V(0)=1$. Moreover we have :

$$\int_{-2^N\pi}^{2^N\pi} \prod_{j=1}^N \theta\left(\frac{\xi}{2^j}\right) d\xi = 2\pi \ ;$$

this is done by recurrence on N since that for any $2^N\pi$-periodical function f we have:

$$\int_{-2^N\pi}^{2^N\pi} f(\xi)\theta\left(\frac{\xi}{2^N}\right) d\xi = \int_0^{2^{N+1}\pi} f(\xi)\theta\left(\frac{\xi}{2^N}\right) d\xi =$$

$$\int_0^{2^N\pi} f(\xi)\left(\theta(\frac{\xi}{2^N})+\theta(\frac{\xi}{2^N}+\pi)\right) d\xi = \int_{-2^{N-1}\pi}^{2^{N-1}\pi} f(\xi)d\xi.$$

Since

$$\chi_{[-2^N\pi,2^N\pi]}(\xi)\prod_{j=1}^N \theta\left(\frac{\xi}{2^j}\right) \to V(\xi) \text{ for all } \xi \text{ as } N\to+\infty,$$

we have

$$\int_{-\infty}^{+\infty} V(\xi)d\xi \leq 2\pi$$

and then

$$\int_0^{+\infty} V(\xi)d\xi < 2\pi$$

(since $V=0$ or $V(0)=1$ and hence $\int_{-\infty}^0 V(\xi)d\xi>0$). But this is absurd since $0\leq U(\xi)\leq V(\xi)$ and

$$\int_0^{+\infty} U(\xi)d\xi = \sum_1^\infty \int_0^{+\infty} | \hat{\psi}(2^\ell\xi) |^2 \, d\xi = 2\pi. \ \blacksquare$$

Chapter 4

THE THEORY OF SCALING FILTERS

We have seen in the preceding chapter that bi-orthogonal wavelet bases $(\psi_{j,k} = 2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ and $(\psi^*_{j,k} = 2^{j/2}\psi^*(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ with regular and localized wavelets ψ, ψ^* are always provided by *multi-resolution analyses*. Therefore, before the construction of the wavelet bases which are now commonly used in wavelet theory (we will do this construction in the next chapter), we will describe some properties of multi-resolution analyses, and especially the properties of the so-called *scaling functions* and *scaling filters*. We will pay a particular interest to the compactly supported scaling functions, because such functions provide compactly supported wavelet bases (as we will see in the next chapter) and fast numerical algorithms (as we will see in chapter 8).

1. Multiresolution analysis, scaling functions and scaling filters.

We first recall the definition of a multi-resolution analysis :

DEFINITION 1. - *A multi-resolution analysis of $L^2(\mathbb{R})$ is a sequence of closed linear subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that :*

(1.1) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$

(1.2) $f \in V_j \Leftrightarrow f(2x) \in V_{j+1}$ *(dilation invariance)*

(1.3) V_0 *has a shift-invariant Riesz basis* $(\varphi(x-k))_{k\in\mathbb{Z}}$ *(shift-invariance).*

The function φ in (1.3) is called a scaling function for (V_j). A multi-resolution analysis is called ϵ-localized if one may choose its scaling function in $L^2(|\,x\,|^{2\epsilon}\,dx)$ (where $\epsilon > \frac{1}{2}$) ; it is called regular if one may choose its scaling function with rapid decay ($\varphi \in L^2(|\,x\,|^k\,dx)$ for all $k \in \mathbb{N}$).

We will say that a function φ in $L^2(\mathbb{R})$ is an ϵ-localized scaling function (or a regular scaling function) if there exists a multi-resolution analysis (V_j) such that φ is a scaling function for (V_j) and if moreover φ belongs to $L^2(|\,x\,|^{2\epsilon}\,dx)$ (or has rapid decay). This multi-resolution analysis is then uniquely defined by $V_0 = \mathrm{Span}(\varphi(x-k),\ k \in \mathbb{Z})$ and $V_j = \{f(2^jx)/f \in V_0\}$.

LEMMA 1. - *Let $\varphi \in L^2(\mathbb{R})$ and $\epsilon > \frac{1}{2}$. Then φ is an ϵ-localized scaling function if and only if φ satisfies the following three requirements :*

(i) $\varphi \in L^2(|\,x\,|^{2\epsilon}\,dx)$;

(ii) *The family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis for* $\mathrm{Span}(\varphi(x-k),\ k \in \mathbb{Z})$;

(iii) *There exists a sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that :*

(2) $$\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} a_k\varphi(x-k) \text{ (two-scale difference equation).}$$

Moreover, these three requirements are satisfied if and only if :

(j) $\hat{\varphi} \in H^\epsilon$;
(jj) $\inf_{\xi\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \mid \hat{\varphi}(\xi+2k\pi) \mid^2 > 0$;
(jjj) *There exists a 2π-periodical function $m_0(\xi) \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ such that :*

(3) $$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi).$$

Proof. The equivalence between (i), (ii), (iii) and (j), (jj), (jjj) is obvious : we have proved in the preceding chapter (lemma 2) that for f and g in $L^2((1+\mid x\mid)^{2\epsilon}dx)$ the series

$$C(f,g) = \sum_{k\in\mathbb{Z}} \hat{f}(\xi+2k\pi)\bar{\hat{g}}(\xi+2k\pi)$$

converges uniformly on $[0,2\pi]$ on a function belonging to $H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ (if $\epsilon > \frac{1}{2}$) ; since

$$m_0(\xi) = C(1/2\varphi(\frac{x}{2}),\varphi)/C(\varphi,\varphi),$$

the equivalence is obvious.

If φ is an ϵ-localized scaling function, it satisfies (i), (ii), by definition and (iii) because of the inclusion $V_{-1} \subset V_0$. Conversely, if φ satisfies (i), (ii) and (iii), we have to prove that

$$(V_j = \text{Span}(\varphi(2^j x-k)/k\in\mathbb{Z}))_{j\in\mathbb{Z}}$$

is a multi-resolution analysis. The dilation or shift invariance properties for (V_j) are obvious by definition of the V_j ; the inclusion $V_j \subset V_{j+1}$ follows from $\varphi\left(\frac{x}{2}\right) \in V_0$. Thus we just have to prove that $\cap V_j = \{0\}$ and $\cup V_j$ is dense.

By (ii), we know that there exists two positive constants A, B such that :

- for all $f \in L^2$ and all $N \in \mathbb{Z}$: $\sum_{k\in\mathbb{Z}} |< f \mid 2^{N/2}\varphi(2^N x-k) >|^2 \leq A \parallel f \parallel_2^2$;
- for all $f \in \bigcap_{j\in\mathbb{Z}} V_j$ and all $N \in \mathbb{Z}$: $\sum_{k\in\mathbb{Z}} |< f \mid 2^{N/2}\varphi(2^N x-k) >|^2 \geq B \parallel f \parallel_2^2$.

Now if there were some $f \in \bigcap_{j\in\mathbb{Z}} V_j$ such that $f \neq 0$, we could approximate f by a function $\omega \in C_c^\infty$ such that

$$\inf_{N\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |< \omega \mid 2^{N/2}\varphi(2^N x-k) >|^2 > 0.$$

But

$$\sum_{k\in\mathbb{Z}} |< \omega \mid 2^{N/2}\varphi(2^N x - k) >|^2 = \sum_{k\in\mathbb{Z}} |< 2^{-N/2}\omega(2^{-N}(x+k) \mid \varphi(x) >|^2$$
$$\leq \int_{-\infty}^{+\infty} |\varphi(x)|^2 \sum_{k\in\mathbb{Z}} \left|\omega(\frac{x+k}{2^N})\right| dx \, \| \omega \|_1$$

and this last term goes to 0 as N goes to $-\infty$. Thus $\cap V_j = \{0\}$.

The density of UV_j is almost obvious. We begin by proving that $\int \varphi \, dx \neq 0$: the function m_0 satisfies $| m_0(0) | \leq 1$ since

$$| m_0(0) |^2 = \frac{\sum_{k\in\mathbb{Z}} | \hat{\varphi}(4k\pi) |^2}{\sum_{k\in\mathbb{Z}} | \hat{\varphi}(2k\pi) |^2} ;$$

since $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ with $\epsilon > 1/2$ and is therefore Hölderian, the infinite product $\prod_{j=1}^{\infty} | m_0(\frac{\xi}{2^j}) |$ converges to a finite limit for all ξ ; now we have

$$| \hat{\varphi}(\xi) | = \prod_{j=1}^{N} | m_0(\frac{\xi}{2^j}) | \cdot | \hat{\varphi}(\frac{\xi}{2^N}) | = | \hat{\varphi}(0) | \prod_{j=1}^{\infty} | m_0(\frac{\xi}{2^j}) | ;$$

thus $\hat{\varphi}(0) \neq 0$. But $V = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is a closed subspace of L^2 which is invariant under any translation $\frac{k}{2^j}$ ($k \in \mathbb{Z}$, $j \in \mathbb{Z}$), hence under any real translation ; hence $V = \{f \in L^2/\text{Supp}\, \hat{f} \subset E\}$ for some measurable set E. This set E is invariant under dyadic dilations and contains a neighborhood of 0 (since $\hat{\varphi}(0) \neq 0$), hence $E = \mathbb{R}$ and $\overline{\cup V_j} = L^2$. ∎

DEFINITION 2. - *A 2π-periodical function $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ (where $\epsilon > \frac{1}{2}$) is called an (ϵ-localized) scaling filter if there is an ϵ-localized scaling function φ such that $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$.*

LEMMA 2. - *Let φ be an ϵ-localized scaling function and m_0 its scaling filter. Then $\hat{\varphi}(0) \neq 0$, $m_0(0) = 1$ and*

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right). \tag{4}$$

This lemma is obvious since we know already that $\hat{\varphi}(0) \neq 0$. ∎

Remark. - If $\theta(\xi)$ is a measurable function such that $\text{Supp}\,\theta \subset [-2,-1] \cup [1,2]$ and such that $\frac{1}{C} \leq \theta \leq C$ on $[-2,-1] \cup [1,2]$ for a positive constant C and if $\mu(\xi) = \sum_{j\in\mathbb{Z}} \theta(2^j\xi)$, and if φ is an ϵ-localized scaling function, we define ω by $\hat{\omega} = \mu(\xi)\hat{\varphi}$; then ω is a scaling function for some multi-resolution analysis of $L^2(\mathbb{R})$ and ω has the same scaling filter as φ ; but formula (4) can be applied only to φ. The knowledge of the scaling filter gives the knowledge of the scaling function only if the scaling function is ϵ-localized. ∎

2. Properties of the scaling filters.

In 1990, A. Cohen [COH] has given an useful criterion to characterize the scaling filters among the functions in $H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ ($\epsilon > \frac{1}{2}$).

THEOREM 1. - *Let $\epsilon > \frac{1}{2}$ and $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ such that $m_0(0) = 1$. Define the function $\hat{\varphi}$ and the operator T by :*

$$(5.1) \qquad \hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

$$(5.2) \qquad \forall f \in C^0(\mathbb{R}/2\pi\mathbb{Z}) \quad Tf(\xi) = | m_0(\frac{\xi}{2}) |^2 f(\frac{\xi}{2}) + | m_0(\frac{\xi}{2}+\pi) |^2 f(\frac{\xi}{2}+\pi)$$

(where $C^0(\mathbb{R}/2\pi\mathbb{Z})$ is the Banach space of 2π-periodical continuous functions, equipped with the norm $\| \cdot \|_\infty$). Then $\hat{\varphi}$ is the Fourier transform of an ϵ-localized scaling function φ if and only if m_0 satisfies the following requirements :

(i) $\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty < +\infty$;
(ii) *(A. Cohen's criterion) : there exists a compact set K which is a finite union of closed intervals and such that :*

$$(5.1) \qquad \sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1 \ a.\ e.$$

$$(5.2) \qquad \forall \xi \in K,\ \forall j \in \mathbb{N}^*, \quad m_0\left(\frac{\xi}{2^j}\right) \neq 0.$$

Proof. If φ is an ϵ-localized scaling function, we call γ the auto-correlation function

$$\gamma(\xi) = C(\varphi,\varphi)(\xi) = \sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) |^2 ;$$

we know that the series converges uniformly on any compact set of $\mathbb{R}$ (since $\varphi \in L^2(| x |^{2\epsilon}\, dx)$) and that γ is bounded by below ($\gamma(\xi) \geq C_0$ for a positive constant C_0) since $(\varphi(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis. Because of the relationship $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$, we have

$$\begin{aligned}\gamma(\xi) &= \sum_{k\in\mathbb{Z}} | m_0(\frac{\xi}{2}+k\pi) |^2 | \hat{\varphi}(\frac{\xi}{2}+k\pi) |^2 \\ &= | m_0(\frac{\xi}{2}) |^2 \gamma(\frac{\xi}{2}) + | m_0(\frac{\xi}{2}+\pi) |^2 \gamma(\frac{\xi}{2}+\pi),\end{aligned}$$

hence $T(\gamma)=\gamma$; T is a positive operator ($f\le g\Rightarrow Tf\le Tg$), and we may conclude from $T(\gamma)=\gamma$ and $1\le \frac{1}{\inf_{\eta\in\mathbb{R}}\gamma(\eta)}\gamma(\xi)$ that

$$0\le T^n(1)(\xi)\le \frac{1}{\inf\limits_{\eta\in\mathbb{R}}\gamma(\eta)}\gamma(\xi),$$

hence

$$\| T^n(1)\|_\infty \le \frac{\|\gamma\|_\infty}{\inf\limits_{\eta\in\mathbb{R}}\gamma(\eta)}.$$

(i) is thus proved. Moreover for all $\xi\in[0,2\pi]$, $\gamma(\xi)>0$, hence we may find a number $k(\xi)\in\mathbb{Z}$ such that $\hat\varphi(\xi+2k(\xi)\pi)\neq 0$; since $\hat\varphi$ is continuous, we may also find a positive number $\alpha(\xi)$ such that for all $\eta\in(\xi+2k(\xi)\pi-\alpha(\xi),\xi+2k(\xi)\pi+\alpha(\xi))$, $\hat\varphi(\eta)\neq 0$ (and hence for all $j\in\mathbb{N}^*$, $m_0(\frac{\eta}{2^j})\neq 0$). We have obtained an open covering $(\xi-\alpha(\xi),\xi+\alpha(\xi))_{\xi\in[0,2\pi]}$ of the compact set $[0,2\pi]$ and we may easily construct K with help of a finite subcovering. Thus, point (ii) is proved.

Conversely, let's assume that (i) and (ii) are satisfied. We choose a compact set K satisfying (5.1) and (5.2). The infinite product $\prod_1^\infty m_0(\frac{\xi}{2^j})$ converges uniformly on any compact set of $\mathbb{R}$ and thus $\hat\varphi$ is a continuous function ; moreover, since $\hat\varphi$ doesn't vanish on K, $\inf_{\xi\in K}|\hat\varphi(\xi)|>0$, and therefore

$$\sum_{k\in\mathbb{Z}}|\hat\varphi(\xi+2k\pi)|^2\ge \inf_{\xi\in K}|\hat\varphi(\eta)|^2>0.$$

We want to prove that $\operatorname{ess.sup}_{\xi\in\mathbb{R}}\sum_{k\in\mathbb{Z}}|\hat\varphi(\xi+2k\pi)|^2$ is finite. Let's define H as the space

$$H=\{f\in L^2/\sum_{k\in\mathbb{Z}}|f(\xi+2k\pi)|^2\in L^\infty\}$$

equipped with the norm

$$\| f\|_H=\|(\sum_{k\in\mathbb{Z}}|f(\xi+2k\pi)|^2)^{1/2}\|_\infty$$

and define on H the operator S by

$$Sf(\xi)=m_0(\frac{\xi}{2})f(\frac{\xi}{2})$$

(which is bounded from H to H) and V by

$$Vf(\xi)=(\sum_{k\in\mathbb{Z}}|f(\xi+2k\pi)|^2)^{1/2}$$

(a bounded sublinear operator from H to $L^\infty(\mathbb{R}/2\pi\mathbb{Z})$). Then we have the inequality :

$$(6) \qquad \forall n \in \mathbb{N}, \quad VS^n f(\xi) \leq \| Vf \|_\infty (T^n(1)(\xi))^{1/2} \ a.\ e.$$

which is easily proved by induction on n : just check that if $f \in H$ and $g \in C^0(\mathbb{R}/2\pi\mathbb{Z})$ satisfy $Vf(\xi)^2 \leq g(\xi)$ a. e. then $VSf(\xi)^2 \leq Tg(\xi)$ a. e. Now we choose $\omega \in C_c^\infty(\mathbb{R})$ with $\omega(0) = 1$, then $S^n\omega = \prod_{j=1}^n m_0(\frac{\xi}{2^j})\omega(\frac{\xi}{2^n})$ converges to $\hat{\varphi}$ pointwise and in the distribution sense. We have

$$\sum_{k\in\mathbb{Z}} | S^n\omega(\xi + 2k\pi) |^2 \leq \| \omega \|_H^2 \, T^n(1)(\xi) \ ;$$

by Fatou's lemma we get

$$\sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) |^2 \leq \| \omega \|_H^2 \sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty .$$

Now, let's suppose that $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$, $\epsilon > 1/2$: we want to show that $\hat{\varphi} \in H^\epsilon(\mathbb{R})$. We first notice that :

$$(7) \qquad \| m_0(\frac{\xi}{2}) \cdots m_0(\frac{\xi}{2^n}) \|_\infty \leq \| T^n(1) \|_\infty^{1/2} .$$

Now we will deal with the cases : $1/2 < \epsilon < 1$, $\epsilon \in \mathbb{N}^*$, $\epsilon = k + \alpha$ ($k \geq 1, 0 < \alpha < 1$) :

• for $1/2 < \epsilon < 1$, we want to show that $\int\int | \hat{\varphi}(\xi) - \hat{\varphi}(\xi + \eta) |^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\epsilon}}$ is finite ; by Fatou's lemma, it is enough to show that $\sup_n \int\int | S^n\omega(\xi) - S^n\omega(\xi + \eta) | \frac{d\xi\, d\eta}{|\eta|^{1+2\epsilon}} < +\infty$ but

$$\left\{\int\int | S^n\omega(\xi) - S^n\omega(\xi+\eta) |^2 \frac{d\xi\, d\eta}{| \eta |^{1+2\epsilon}}\right\}^{1/2}$$

$$\leq \sum_{j=1}^n \left\{\int\int \prod_{\ell=1}^{j-1} | m_0(\frac{\xi+\eta}{2^\ell}) |^2 | m_0(\frac{\xi}{2^j}) - m_0(\frac{\xi+\eta}{2^j}) |^2 \right.$$

$$\left. \prod_{\ell=j+1}^{n} | m_0(\frac{\xi}{2^\ell}) |^2 | \omega(\frac{\xi}{2^n}) |^2 \frac{d\xi\, d\eta}{| \eta |^{1+2\epsilon}} \right\}^{\frac{1}{2}}$$

$$+ \left\{\int\int \prod_1^n | m_0(\frac{\xi+\eta}{2^\ell}) |^2 | \omega(\frac{\xi}{2^n}) - \omega(\frac{\xi+\eta}{2^n}) |^2 \frac{d\xi\, d\eta}{| \eta |^{1+2\epsilon}}\right\}^{\frac{1}{2}}$$

$$\leq \sup_{p\in\mathbb{N}} \| T^p(1) \|_\infty^{1/2} \left\{ \sum_{j=1}^n 2^{j(1-2\epsilon)} \left[\int\int | m_0(\xi) - m_0(\xi+\eta) |^2 | S^{n-j}\omega(\xi) |^2 \frac{d\xi\, d\eta}{| \eta |^{1+2\epsilon}}\right]^{\frac{1}{2}}\right.$$

$$+2^{n(1-2\epsilon)}\left[\int\int |\omega(\xi)-\omega(\xi+\eta)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\epsilon}}\right]^{\frac{1}{2}}\Bigg\}$$

$$\leq \sup_{p\in\mathbb{N}} \| T^p(1)\|_\infty^{1/2} \left\{\sum_{j=1}^{n} 2^{j(1-2\epsilon)} \| S_0^{n-\ell}\omega\|_H\right.$$

$$\left[\int_{0\leq\xi\leq 2\pi}\int_{\eta\in\mathbb{R}} | m_0(\xi)-m_0(\xi+\eta)|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\epsilon}}\right]^{1/2}$$

$$\left. +2^{n(1-2\epsilon)}\left[\int\int |\omega(\xi)-\omega(\xi+\eta)|^2\, d\xi \frac{d\eta}{|\eta|^{1+2\epsilon}}\right]^{1/2}\right\}$$

$$\leq C\left\{\sum_{j=1}^{n} 2^{j(1-2\epsilon)} + 2^{n(1-2\epsilon)}\right\} \leq \frac{2C}{1-2^{1-2\epsilon}} < +\infty.$$

• for $\epsilon = k$, $k\in\mathbb{N}^*$, we define m_ℓ as $(\frac{d}{d\xi})^\ell m_0$ and S_ℓ as $S_\ell f = m_\ell(\frac{\xi}{2})f(\frac{\xi}{2})$: for $\ell<k$, S_ℓ is bounded from H to H and from L^2 to L^2, for $\ell = k$, S_k is bounded from H to L^2. We write moreover for $\ell\in\mathbb{N}^k$ and $n\in\mathbb{N}$, $\alpha_{n,\ell}=\sum_{i=1}^{k}\delta_{n,\ell_i}$; then we have

$$\left(\frac{d}{d\xi}\right)^k S^n\omega = \sum_{\ell\in\{0,1,\cdots,n\}^k} (\frac{1}{2})^{\sum_{j=1}^{n} j\alpha_{j,\ell}+n\alpha_{0,\ell}} S_{\alpha_{1,\ell}}\cdots S_{\alpha_{n,\ell}}\left((\frac{d}{d\xi})^{\alpha_{0,\ell}}\omega\right)$$
$$=\sum_{L=0}^{n}\sum_{\ell\in\mathbb{N}^k,\sup\ell_i=L} (\frac{1}{2})^{\sum_{j=1}^{n} j\alpha_{j,\ell}+n\alpha_{0,\ell}} S_{\alpha_{1,\ell}}\cdots S_{\alpha_{L,\ell}}S_0^{n-L}\left((\frac{d}{d\xi})^{\alpha_{0,\ell}}\omega\right)\ ;$$

for $j<L$, $\sum_i \delta_{j,\ell_i}<k$, hence $\| m_{\sum\delta_{j,\ell_i}}\|_\infty\leq C\| m_0\|_{H^k}$; using this estimate, inequality (7) and $\| m_{\sum\delta_{L,\ell_i}}\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})}\leq\| m_0\|_{H^k}$, we get :

$\| S_{\alpha_{1,\ell}}\cdots S_{\alpha_{L,\ell}}S_0^{n-L}\left((\frac{d}{d\xi})^{\alpha_{0,\ell}}\omega\right)\|_2\leq C\,2^{L/2}\| m_0\|_{H^k}^2 \sup_{p\in\mathbb{N}}\| T^p(1)\|_\infty^{\frac{k+1}{2}} \sup_{0\leq p\leq k}\| (\frac{d}{d\xi})^p\omega\|_H$

and

$$\| (\frac{d}{d\xi})^k S^n\omega\|_2\leq C\sum_{L=0}^{n}\sum_{\ell\in\mathbb{N}^k,\sup\ell_i=L}(\frac{1}{2})^{\sum_{i=1}^{n}\ell_i-\frac{1}{2}L+n\sum_{i=1}^{n}\delta_{0,\ell_i}}\leq C\left(\frac{1}{1-\frac{1}{\sqrt{2}}}\right)^k.$$

Now $(\frac{d}{d\xi})^k S^n\omega\to(\frac{d}{d\xi})^k\hat\varphi$ in $\mathcal{D}'$, thus $(\frac{d}{d\xi})^k\hat\varphi$ is square-integrable and $\hat\varphi\in H^k$. We obtain moreover :

$$(8)\qquad (\frac{d}{d\xi})^k\hat\varphi=\sum_{L=1}^{\infty}\sum_{\ell\in(\mathbb{N}^*)^k,\sup\ell_i=L}(\frac{1}{2})^{\sum_{i=1}^{k}\ell_i}\prod_{j=1}^{L} m_{\sum_{i=1}^{k}\delta_{j,\ell_i}}(\frac{\xi}{2^j})\hat\varphi(\frac{\xi}{2^L}).$$

• for $\epsilon = k+\alpha$, $k \geq 1$, $0<\alpha<1$, we have to show that

$$\int\int \left|(\frac{d}{d\xi})^k\hat{\varphi}(\xi) - (\frac{d}{d\xi})^k\hat{\varphi}(\xi+\eta)\right|^2 \frac{d\xi\, d\eta}{|\eta|^{1+2\alpha}}$$

is finite. This is done in the very same way as for the case $\frac{1}{2}<\epsilon<1$, using the fact that for $j<L$ we have $\epsilon_j = \sum_1^k \delta_{j,\ell_i} < k$, hence

$$\int |m_{\epsilon_j}(\frac{\xi}{2^j}) - m_{\epsilon_j}(\frac{\xi+\eta}{2^j})|^2 \frac{d\eta}{|\eta|^{1+2\alpha}} \leq C \| m_0 \|^2_{H^\epsilon} 2^{-2\alpha j},$$

and that if $\epsilon_L = \sum \delta_{L,\ell_i} < k$ then m_{ϵ_L} is bounded ; the only term to discuss is then :

$$\int\int (\frac{1}{2})^{2Lk} \prod_{j=1}^{L-1} | m_0(\frac{\xi+\eta}{2^j}) |^2 | m_k(\frac{\xi+\eta}{2^L}) |^2 | \hat{\varphi}(\frac{\xi+\eta}{2^L}) - \hat{\varphi}(\frac{\xi}{2^L}) |^2 d\xi \frac{d\eta}{|\eta|^{1+2\alpha}}$$

$$\leq \| T^{L-1}(1) \|_\infty (\frac{1}{2})^{2Lk-L+2\alpha L} \int\int | m_k(\xi+\eta) |^2 | \hat{\varphi}(\xi+\eta) - \hat{\varphi}(\xi) |^2 d\xi \frac{d\eta}{|\eta|^{1+2\alpha}}$$

$$\leq \| T^{L-1}(1) \|_\infty (\frac{1}{2})^{L(2k+2\alpha-1)} \| m_k \|_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \int \| V(\hat{\varphi}(\cdot+\eta) - \hat{\varphi}(\cdot)) \|^2_\infty \frac{d\eta}{|\eta|^{1+2\alpha}} ;$$

but for $0<\theta$ and $\alpha<\gamma<1$ we have

$$\begin{aligned} \| V(\hat{\varphi}(\cdot+\eta) - \hat{\varphi}(\cdot)) \|_\infty &\leq C_\theta \| \hat{\varphi}(\cdot+\eta) - \hat{\varphi}(\cdot) \|_{H^{\theta+1/2}} \\ &\leq C_{\theta,\gamma} \| \hat{\varphi}(\cdot) \|_{H^{\theta+\gamma+1/2}} \inf(1, |\eta|^\gamma) ; \end{aligned}$$

if $\alpha < \frac{1}{2}$, we may choose θ and γ such that $\theta+\gamma+1/2 = 1$ and we already know that $\hat{\varphi} \in H^1$, thus we have $\hat{\varphi} \in H^{k+\alpha}$; if $\frac{1}{2} \leq \alpha < 1$, we may choose θ and γ such that $\theta+\gamma+\frac{1}{2} < \frac{3}{2}$ and we have just proved that $\hat{\varphi} \in H^{\theta+\gamma+1/2}$, so $\hat{\varphi} \in H^{k+\alpha}$.

We thus have proved that $\hat{\varphi} \in H^\epsilon$, $\inf_{\xi\in\mathbb{R}} \sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2 > 0$ and $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$. By lemma 1, φ is an ϵ-localized scaling function and theorem 1 is proved. ∎

Theorem 1 shows that for $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ ($\epsilon > 1/2$) to be a scaling filter, a necessary and sufficient condition is that $m_0(0) = 1$, m_0 satisfies the Cohen criterion and the operator T satisfies $\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty < +\infty$. But we can describe more properties on T and hence on $\hat{\varphi}$. The following theorem is a simplified version of the analysis of the operator T by L. Hervé [HER].

THEOREM 2 (The first regularity theorem). - *Let $\epsilon > \frac{1}{2}$ and $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ be a scaling filter. Let φ be the scaling function (defined by (5.1)) and T the transition operator (defined by (5.2)) associated to m_0. Let $\alpha \in (0, \inf(\epsilon-1/2, 1))$ and $C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ be the space of 2π-periodical continuous functions which are Hölderian of exponant α (with norm $\| f \|_{C^\alpha} = \| f \|_\infty + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$). Then :*

(i) *$m_0(\pi) = 0$ and T leaves invariant the space $E_\alpha = \{f \in C^\alpha / f(0) = 0\}$.*

(ii) *The spectral radius ρ_α of T on E_α is less than 1.*

(iii) *φ belongs to the Sobolev space H^σ for all $\sigma < \frac{\ell n \frac{1}{\rho_\alpha}}{2\ell n 2}$.*

(iv) (The convergence lemma) : *Let θ_n be defined by $\hat{\theta}_n(\xi) = \prod_{j=1}^n m_0(\frac{\xi}{2^j})\chi_{[-\pi,\pi]}(\frac{\xi}{2^n})$. Then $\theta_n \to \varphi$ in H^σ as $n \to +\infty$ for all $\sigma < \frac{\ell n \frac{1}{\rho_\alpha}}{2\ell n 2}$.*

Proof. We first prove that for $g \in C^0(\mathbb{R}/2\pi\mathbb{Z})$ the functions

$$\gamma_n = g(\frac{\xi}{2^n}) \prod_{j=1}^n m_0(\frac{\xi}{2^j})\chi_{[-\pi,\pi]}(\frac{\xi}{2^n})$$

converge to $g(0)\hat{\varphi}$ in L^2 as $n \to +\infty$. We choose a compact set K satisfying Cohen's criterion (5.1) and (5.2) and such that 0 belongs to the interior set of K (which can always be done since $m_0(0) = 1$). Then

$$\tilde{\gamma}_n(\xi) = g(\frac{\xi}{2^n}) \prod_{j=1}^n m_0(\frac{\xi}{2^j})\chi_K(\frac{\xi}{2^n})$$

converges pointwise to $g(0)\hat{\varphi}$ as $n \to +\infty$ and is dominated by $\| g \|_\infty \frac{\hat{\varphi}(\xi)}{\inf_{\eta\in K} |\hat{\varphi}(\eta)|}$; thus $\tilde{\gamma}_n \to g(0)\hat{\varphi}$ in L^2 as $n \to +\infty$. Now fix η such that $[-\eta, \eta] \subset K$, $\omega \in C_c^\infty$ with $\operatorname{supp}\omega \subset [-\eta,\eta]$ and $\omega(0) = 1$ and define Ω as $\Omega(\xi) = \sum_{k\in\mathbb{Z}} \omega(\xi + 2k\pi)$. Then $\eta_n = g(\frac{\xi}{2^n}) \prod_{j=1}^n m_0(\frac{\xi}{2^j})\omega(\frac{\xi}{2^n}) \to g(0)\hat{\varphi}$ in L^2 as $n \to +\infty$ (by dominated convergence again), while

$$\| \eta_n - \gamma_n \|_2 = \| (\Omega(\frac{\xi}{2^n}) - 1)\gamma_n \|_2 = \| (\Omega(\frac{\xi}{2^n}) - 1)\tilde{\gamma}_n \|_2$$

(by (5.1)) and thus

$$\lim_{n\to+\infty} \| \eta_n - \gamma_n \|_2 = \| (\Omega(0) - 1)\hat{\varphi} \|_2 = 0.$$

The equality $m_0(\pi) = 0$ is obvious : we know that $\sum_{k\in\mathbb{Z}} | \hat{\varphi}(\pi + 2k\pi) |^2 > 0$, hence $\hat{\varphi}(\pi + 2k_0\pi) \neq$ for some $k_0 \in \mathbb{Z}$, and

$$\hat{\varphi}(2^N(\pi + 2k_0\pi)) = \prod_{j=0}^{N-1} m_0(2^j(\pi + 2k_0\pi))\hat{\varphi}(\pi + 2k_0\pi)) = m_0(\pi)\hat{\varphi}(\pi + 2k_0\pi)$$

(since $m_0(0) = 1$). Now, $\varphi \in L^1$, hence $\hat{\varphi}(\xi) \to 0$ as $\xi \to +\infty$ and we get $m_0(\pi) = 0$. Since $\alpha < \epsilon - \frac{1}{2}$, we know that $m_0 \in C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$, hence T is a bounded operator on $C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$. Moreover

$$Tf(0) = | m_0(0) |^2 f(0) + | m_0(\pi) |^2 f(\pi) = f(0),$$

hence T keeps E_α invariant.

We will now prove that the operators T^n, $n \in \mathbb{N}$, are equicontinuous on $C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$. We already know that $\| T^n f \|_\infty \leq \| T^n(1) \|_\infty \| f \|_\infty$. Moreover if $\Delta_h f = f(\xi+h) - f(\xi)$,

$$S_m f = m(\frac{\xi}{2}) f(\xi) + m(\frac{\xi}{2}+\pi) f(\frac{\xi}{2}) \ \ and \ \ \tau_h f = f(\xi+h),$$

we have the identity :

$$\Delta_h T f = T \Delta_{h/2} f + S_{\Delta_{h/2} |m_0|^2 \tau_{h/2}} f,$$

hence

$$\Delta_h T^n f = T^n \Delta_{h/2^n} f + \sum_{j=1}^{n} T^{j-1} S_{\Delta_{h/2^j} |m_0|^2 \tau_{h/2^j}} f,$$

and, writing $||| f |||_\alpha$ for $\sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$,

$$||| T^n f |||_\alpha \leq \| T^n(1) \|_\infty \frac{||| f |||_\alpha}{2^{n\alpha}} + \sum_{j=1}^{n} \| T^{j-1}(1) \|_\infty \frac{2 \, ||| \, | m_0 |^2 |||_\alpha}{2^{j\alpha}} \| f \|_\infty$$

and thus, for a constant C_0,

$$||| T^n f |||_\alpha \leq C_0 \frac{||| f |||_\alpha}{2^{n\alpha}} + C_0 \| f \|_\infty \, . \tag{9}$$

Now let ω_α be a non-negative function in $C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ such that ω_α vanishes only at $0 (\mathrm{mod}\, 2\pi)$ and such that $\omega(\xi) \sim | \xi |^\alpha$ in the neighborhood of 0. Then for any $f \in E_\alpha$, we have (for some constant C_α, which doesn't depend on f) :

$$| f(\xi) | \leq C_\alpha (||| f |||_\alpha + \| f \|_\infty) \omega_\alpha(\xi). \tag{10}$$

Moreover, the set $(T^n \omega_\alpha)_{n \in \mathbb{N}}$ is bounded in $C^\alpha(\mathbb{R}/2\pi\mathbb{Z})$, hence is relatively compact in $C^0(\mathbb{R}/2\pi\mathbb{Z})$ by the Ascoli theorem. Thus we may find a subsequence $(T^{n_k} \omega_\alpha)$ converging in $C^0(\mathbb{R}/2\pi\mathbb{Z})$ to a non-negative function ω. But we have

$$\int_0^{2\pi} \omega(\xi) d\xi = \lim_{n_k \to +\infty} \int_{-\pi 2^{n_k}}^{\pi 2^{n_k}} \prod_{j=1}^{n_k} | m_0(\frac{\xi}{2^j}) |^2 \omega_\alpha(\frac{\xi}{2^{n_k}}) d\xi = \omega_\alpha(0) \| \hat{\varphi} \|_2^2 = 0,$$

hence $\omega = 0$. Hence for all positive η, we may find an integer n_0 such that $\| T^{n_0} \omega_\alpha \|_\infty < \eta$. Then we obtain

$$\| T^{n_0+p} \omega_\alpha \|_\infty \leq \eta \sup_{q \in \mathbb{N}} \| T^q(1) \|_\infty$$

while

$$||| T^{n_0+p} \omega_\alpha |||_\alpha \leq C_0 \frac{1}{2^{p\alpha}} ||| T^{n_0} \omega_\alpha |||_\alpha + C_0 \eta,$$

so that if p_0 is big enough we have :

$$T^{n_0+p_0} \omega_\alpha(\xi) \leq C_\alpha \left(2C_0 + \sup_{q \in \mathbb{N}} \| T^q(1) \|_\infty \right) \eta \, \omega_\alpha(\xi).$$

We choose η such that

$$C_\alpha(2C_0 + \sup_{q\in\mathbb{N}} \| T^q(1) \|_\infty)\eta < 1/2 \;;$$

we then have proved that we may find an integer N such that

$$T^N \omega_\alpha(\xi) \leq \frac{1}{2}\omega_\alpha(\xi).$$

But then we obtain (since by (9) $||| T^n f |||_\alpha \leq C_0(||| f |||_\alpha + \| f \|_\infty)$) :

$$| T^{qN} f(\xi) | \leq C_\alpha(||| f |||_\alpha + \| f \|_\infty)\frac{1}{2^q}\omega_\alpha(\xi),$$

$$\| T^{2qN} f \|_\infty \leq C_\alpha(||| f |||_\alpha + \| f \|_\infty) \| \omega_\alpha \|_\infty \frac{1}{2^{2q}},$$

while

$$\begin{aligned} ||| T^{2qN} f |||_\alpha &\leq C_0 \frac{||| T^{qN} f |||_\alpha}{2^{qN\alpha}} + C_0 \| T^{qN} f \|_\infty \\ &\leq C_0(||| f |||_\alpha + \| f \|_\infty)\left(\frac{C_0}{2^{qN\alpha}} + C_\alpha \frac{1}{2^q} \| \omega_\alpha \|_\infty\right), \end{aligned}$$

hence

$$\rho_\alpha \leq \sup\left(\frac{1}{2^{\alpha/2}}, \frac{1}{2^{1/2N}}\right) < 1.$$

It is then easy to show that φ belongs to H^σ for $0 < \sigma < \frac{\ell n \frac{1}{\rho_\alpha}}{2\ell n 2}$:

$$\begin{aligned} \| \varphi \|^2_{H^\sigma} &= \int (1+ | \xi |^2)^\sigma | \hat\varphi(\xi) |^2 \, d\xi \\ &\leq \int_{-\pi}^{\pi} | \hat\varphi(\xi |^2 (1+ | \xi |^2)^\sigma d\xi + \sum_{N=1}^{\infty} (2\pi^2)^\sigma 4^{\sigma N} \int_{2^{N-1}\pi\leq|\xi|\leq 2^N\pi} | \hat\varphi(\xi) |^2 \, d\xi \;; \end{aligned}$$

but

$$| \hat\varphi(\xi) |^2 \leq C \prod_{j=1}^{N} | m_0(\frac{\xi}{2^j}) |^2 \text{ for all } N,$$

where

$$C = \sup_{q\in\mathbb{N}} \| \prod_{j=0}^{q} m_0(2^j\xi) \|^2_\infty$$

(which, we know, is finite) while for $2^{N-1}\pi \leq | \xi | \leq 2^N\pi$,

$$1 \leq \frac{\omega_\alpha(\xi/2^N)}{\inf_{\frac{\pi}{2}\leq|\eta|\leq\pi} \omega_\alpha(\eta)},$$

hence

$$\| \varphi \|_{H^\sigma}^2 \leq C'(1 + \sum_{N=1}^{\infty} 4^{\sigma N} \| T^N(\omega_\alpha) \|_{L^1(\mathbb{R}/2\pi\mathbb{Z})}) \ ;$$

now for any $\rho \in (\rho_\alpha, 1)$ we have

$$\| T^N(\omega_\alpha) \|_{L^1(\mathbb{R}/2\pi\mathbb{Z})} \leq 2\pi \| T^N(\omega_\alpha) \|_\infty \leq 2\pi C_\rho \rho^N$$

and the series $\sum_{N=1}^{\infty} 4^{\sigma N} \rho^N$ is finite if and only if $4^\sigma < \frac{1}{\rho}$. Thus $\varphi \in H^\sigma$ if $4^\sigma < \frac{1}{\rho_\alpha}$, or $\sigma < \frac{\ell n \frac{1}{\rho_\alpha}}{2\, \ell n\, 2}$.

The convergence of θ_n to φ in H^σ is then obvious. We fix again $\eta > 0$ such that $\hat{\varphi}$ doesn't vanish on $[-\eta, \eta]$. Then $| \hat{\theta}_n(\xi) | \leq \frac{|\hat{\varphi}(\xi)|}{\inf_{|\ |\leq \eta} |\hat{\varphi}(\tau)|}$ on $[-2^n\eta, 2^n\eta]$, and thus $\chi_{[-\eta,\eta]}(\frac{\xi}{2^n})\widehat{\theta_n} \to \hat{\varphi}$ in $L^2((1+ | \xi |^2)^\sigma d\xi)$ as $n \to +\infty$. Moreover

$$(1+ | \xi |^2)^\sigma \chi_{[-\pi,-\eta]\cup[\eta,\pi]}(\frac{\xi}{2^n}) \leq (2\pi^2)^\sigma 4^{n\sigma} \frac{\omega_\alpha(\frac{\xi}{2^n})}{\inf_{\eta \leq |\tau| \leq \pi} \omega_\alpha(\tau)},$$

hence

$$\| (1 - \chi_{[-\eta,\eta]}(\frac{\xi}{2^n}))\widehat{\theta_n} \|_{L^2((1+|\xi|^2)^\sigma d\xi)} \leq C\, 2^{n\sigma} \| T^n(\omega_\alpha) \|_{L^1(\mathbb{R}/2\pi\mathbb{Z})}^{1/2}$$

and this last estimate goes to 0 as $n \to +\infty$ whenever $\sigma < \frac{\ell n \frac{1}{\rho_\alpha}}{2\, \ell n\, 2}$. Theorem 2 is proved. ∎

COROLLARY 1. - *Let $m_0 \in H^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ ($\epsilon > \frac{1}{2}$) satisfy $m_0(0) = 1$ and Cohen's criterion (5.1) and (5.2) and let $\alpha \in (0, \min(\epsilon - 1/2, 1))$. Then m_0 is a scaling filter if and only if $m_0(\pi) = 0$ and the spectral radius ρ_α of the transition operator T on E_α is less than 1.*

Proof. Just write $T(1) = 1 + \omega$, where $\omega(0) = 0$. Hence

$$T^n(1) = 1 + \sum_{j=0}^{n-1} T^j(\omega) \quad \text{and} \quad \| T^n(1) \|_\infty \leq 1 + \sum_{j=0}^{n-1} \| T^j(\omega) \|_\infty \ .$$

But if $\rho_\alpha < 1$ and $\rho \in (\rho_\alpha, 1)$, we have

$$\sum_{j=0}^{n-1} \| T^j(\omega) \|_\infty \leq C_\rho \sum_{j=0}^{n+1} \rho^j \leq \frac{C_\rho}{1-\rho}$$

and thus

$$\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty$$

if finite. ∎

3. Derivatives and primitives of a regular scaling function.

By definition, a regular scaling function is a scaling function with rapid decay ($\forall k \in \mathbb{N}$, $x^k\varphi \in L^2$). A necessary and sufficient condition for a scaling function φ to be regular is then to have some decay ($\varphi \in L^2(|\ x\ |^{2\epsilon}\ dx)$ for some $\epsilon > 1/2$) and to have a smooth associated scaling filter ($m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$). The class of regular scaling functions is stable under integration or derivation, as shown by the following theorem :

THEOREM 3. - *Let φ be a regular scaling function (with $\hat{\varphi}(0) = 1$) and m_0 its scaling filter. Then :*

(i) *the function $I\varphi$ defined by :*

$$I\varphi = \int_x^{x+1} \varphi(t)dt \tag{11}$$

is a regular scaling function, with scaling filter $I_{m_0}(\xi) = \left(\frac{1+e^{+i\xi}}{2}\right) m_0(\xi)$.

(ii) *Conversely, if for some $\epsilon > 0$, φ belongs to $H^{1+\epsilon}$ or if φ belongs to H^1 and φ' belongs to $L^2(|\ x\ |^{3+\epsilon}\ dx)$, then φ' can be written as :*

$$\varphi'(x) = D\varphi(x) - D\varphi(x-1) \tag{12}$$

where $D\varphi$ is a regular scaling function associated to the filter

$$Dm_0 = \frac{2}{1+e^{-i\xi}} m_0(\xi).$$

Remark. - We don't have $DI\varphi = \varphi$ but $= \varphi(x+1)$; this choice for the definition of D and I is to ensure $< \varphi_1 \mid \varphi_2 >=< I\varphi_1 \mid D\varphi_2 >$ which we will use as an integration-by-part formula. ∎

Proof. Formulas (11) and (12) were given by G. Malgouyres in 1990 and are related to the well-known infinite product :

$$\prod_{j=1}^{\infty} \left(\frac{1+e^{-i\frac{\xi}{2^j}}}{2}\right) = \frac{1-e^{-i\xi}}{i\xi}. \tag{13}$$

The proof of (i) is straightforward. $I\varphi$ has rapid decay since

$$\int |\ x\ |^{2k} |\ I\varphi(x)\ |^2\ dx \le \int (1+|\ x\ |)^{2k}\ |\ \varphi(x)\ |^2\ dx.$$

Moreover, $(I\varphi)' = \varphi(x+1) - \varphi(x)$, hence $\widehat{I\varphi} = \frac{e^{i\xi}-1}{i\xi}\hat{\varphi}$, and thus

$$\widehat{I\varphi}(2\xi) = \left(\frac{e^{i\xi}+1}{2}\right) m_0(\xi)\widehat{I\varphi}(\xi).$$

Finally, $(I\varphi(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis for $\text{Span}(I\varphi(x-k))$, since if we choose K a compact set satisfying Cohen's criterion for $\hat{\varphi}$ (and such that $0 \in \text{Int}\, K$) then $\widehat{I\varphi}$ doesn't vanish on K, hence $\sum_{k\in\mathbb{Z}} | \widehat{I\varphi}(\xi + 2k\pi) |^2$ never vanishes. Thus (i) is proved.

Conversely let's assume that $\varphi \in H^1$ and that φ' has enough decay ($x^{3/2+\epsilon}\varphi' \in L^2$ for some positive ϵ). [If φ belongs to $H^{1+\epsilon}$, then by interpolation φ' has rapid decay.] We know that $m_0(\pi) = 0$, hence that

$$\hat{\varphi}(2^N(2k_0+1)2\pi) = m_0(\pi)\hat{\varphi}((2k_0+1)\pi) = 0,$$

so that :

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = \hat{\varphi}(0), \tag{14}$$

hence $\sum_{k\in\mathbb{Z}} \varphi'(x-k) = 0$. We define $D\varphi$ as $D\varphi(x) = \sum_{k=0}^{+\infty} \varphi'(x-k)$; then $D\varphi$ has enough decay : let $p \in \left(\frac{1}{2}, \frac{1}{2}+\epsilon\right)$ and $q = \frac{1}{2}+\epsilon$, then

$$\begin{aligned}
&\int_{-\infty}^{0} | x |^{2p} | D\varphi(x) |^2 \, dx \\
&= \sum_{\ell=1}^{+\infty} \int_0^1 | x-\ell |^{2p} | D\varphi(x-\ell) |^2 \, dx \le \sum_{\ell=1}^{\infty} \ell^{2p} \int_0^1 | D\varphi(x-\ell) |^2 \, dx \\
&\le \sum_{\ell=1}^{\infty} \ell^{2p} \int_0^1 \sum_{k=0}^{+\infty} | \varphi'(x-k-\ell) |^2 \, (1+ | x-k-\ell |)^{2q+2} dx \sum_{k=0}^{+\infty} \frac{1}{(k+\ell)^{2q+2}} \\
&\le \sum_{\ell=1}^{\infty} \ell^{2p} \parallel x^{q+1}\varphi' \parallel_2^2 \frac{C}{\ell^{2q+1}} \le C' \parallel x^{q+1}\varphi' \parallel_2^2 \; ;
\end{aligned}$$

we control $\int_0^{+\infty} | x |^{2p} | D\varphi(x) |^2 \, dx$ in the same way since $D\varphi(x) = -\sum_{k=-\infty}^{-1} \varphi'(x-k)$. Now, $\varphi' = D\varphi(x) - D\varphi(x-1)$, hence

$$\widehat{D\varphi}(\xi) = \frac{i\xi}{1-e^{-i\xi}}\hat{\varphi}(\xi) \quad \text{and} \quad \widehat{D\varphi}(2\xi) = \frac{2}{1+e^{-i\xi}} m_0(\xi)\widehat{D\varphi}(\xi).$$

Moreover, $(D\varphi(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis of $\text{Span}(D\varphi(x-k))$, since $\widehat{D\varphi}$ doesn't vanish on any compact set K satisfying Cohen's criterion for m_0. Thus $D\varphi$ is a p-localized scaling function ; since its scaling filter belongs to C^∞, $D\varphi$ has rapid decay. Theorem 3 is proved.

COROLLARY 2 (The second regularity theorem). - *Let φ be a regular scaling function and m_0 its scaling filter and let $N \in \mathbb{N}$. Then the following four assertions are equivalent :*
(i) *For some positive ϵ, $\varphi \in H^{N+\epsilon}$;*
(ii) *$\varphi \in H^N$ and $\varphi^{(N)}$ has rapid decay ;*
(iii) *There exists a regular scaling function $\tilde{\varphi}$ such that $\varphi = I^N\tilde{\varphi}$;*

(iv) m_0 *has a zero of order* $N+1$ *at* $\pi (m_0(\pi) = \cdots = m_0^{(N)}(\pi) = 0)$ *and*

$$\limsup_{q\to+\infty} \| T^q \left((\sin\frac{\xi}{2})^{2N+2} \right) \|_\infty^{1/q} < \left(\frac{1}{4}\right)^N .$$

This theorem has been proved by many authors ([HER], [VIM1], [COD1], [EIR]) in 1991-92.

Proof. (i)⇒(ii) by interpolation, (ii)⇒(iii) by theorem 3 and (iii)⇒(i) by the first regularity theorem (theorem 2). (iii)⇒(iv) is obvious : let $\tilde{T}$ be the transition operator associated to $\tilde{\varphi}$ and $\tilde{m}_0$ its scaling filter ; $\tilde{m}_0(\xi)\left(\frac{1+e^{i\xi}}{2}\right)^N = m_0(\xi)$ and since $\tilde{m}_0(\pi) = 0$ and $1+e^{i\pi} = 0$, m_0 has a zero of orderN+1 at π ; moreover

$$\begin{aligned} T(\sin^{2N}\frac{\xi}{2}\cdot f) &= (\cos\frac{\xi}{4})^{2N} \mid \tilde{m}_0(\frac{\xi}{2}) \mid^2 (\sin\frac{\xi}{4})^{2N} f(\frac{\xi}{2}) \\ &\quad + (\cos(\frac{\xi}{4}+\frac{\pi}{2}))^{2N} \mid \tilde{m}_0(\frac{\xi}{2}+\pi) \mid^2 (\sin(\frac{\xi}{4}+\frac{\pi}{2}))^{2N} f(\frac{\xi}{2}+\pi) \\ &= (\frac{1}{4})^N \sin^{2N}(\frac{\xi}{2})\tilde{T}(f) \ ; \end{aligned}$$

hence

$$\| T^q(\sin^{2N+2}\frac{\xi}{2}) \|_\infty = (\frac{1}{4})^{Nq} \| \sin^{2N}(\frac{\xi}{2})\tilde{T}^q(\sin^2\frac{\xi}{2}) \|_\infty$$

and we already know that

$$\limsup_{q\to+\infty} \| \tilde{T}^q(\sin^2\frac{\xi}{2}) \|_\infty^{1/q} < 1$$

(by theorem 2). Conversely, let $p \in \mathbb{N}^*$ and

$$\rho = \limsup_{q\to+\infty} \| T^q \left((\sin\frac{\xi}{2})^{2p} \right) \|_\infty^{1/q} \ ;$$

then $\varphi \in H^\sigma$ for all $\sigma < \frac{\ln\frac{1}{\rho}}{2\ln 2}$ (we just have to notice that

$$\int_{2^{q-1}\pi\le|\xi|\le 2^q\pi} \mid \hat{\varphi}(\xi) \mid^2 d\xi \le C_p \| T^q \left((\sin\frac{\xi}{2})^{2p} \right) \|_\infty$$

as in the proof of theorem 2). ∎

The end of this proof can be stated in a more precise way :

PROPOSITION 1. - *Let* φ *be a regular scaling function,* m_0 *its scaling filter and* T *its transition operator. For* $p \in \mathbb{N}$*, define* ρ_p *as* $\rho_p = \limsup_{q\to+\infty} \| T^q \left((\sin\frac{\xi}{2})^{2p} \right) \|_{L^1(\mathbb{R}/2\pi\mathbb{Z})}^{1/q}$. *Then* ρ_p *is non-decreasing* $(\rho_{p+1} \le \rho_p)$ *and we have :*

(i) $\rho_0 = 1$ *and* $\rho_p < 1$ *for* $p \geq 1$;
(ii) $(\frac{1}{4})^p \leq \rho_p$ *for all* p ;
(iii) *if for some* p_0, $\rho_{p_0} > (\frac{1}{4})^{p_0}$ *then* $\rho_p = \rho_{p_0}$ *for all* $p \geq p_0$;
(iv) *let* $\rho = \lim_{p \to +\infty} \rho_p$ *and* $\sigma = \frac{\ell n \frac{1}{\rho}}{2 \ell n 2}$ *then* $\varphi \in H^s$ *for* $s < \sigma$ *and* $\varphi \notin H^s$ *for* $s > \sigma$;
(v) *if* m_0 *is a trigonometric polynomial then* $\varphi \notin H^\sigma$; *moreover if*

$$m_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^N \left(\sum_{k=0}^{M} a_k e^{-ik\xi}\right)$$

with $a_0 \neq 0$ *and* $a_M \neq 0$, *where* $\tilde{m}_0(\xi) = \sum_{k=0}^{M} a_k e^{-ik\xi}$ *satisfies* $\tilde{m}_0(\pi) \neq 0$, *and if we note* $\tilde{T}$ *the transition operator*

$$\tilde{T} f(\xi) = \mid \tilde{m}_0(\frac{\xi}{2}) \mid^2 f(\frac{\xi}{2}) + \mid \tilde{m}_0(\frac{\xi}{2} + \pi) \mid^2 f(\frac{\xi}{2} + \pi),$$

then $\tilde{T}$ *keeps invariant the space* E_M *of trigonometric polynomials of degree* $\leq M$ $\left(f \in E_M \Leftrightarrow f = \sum_{k=-M}^{M} f_k e^{-ik\xi}\right)$, *the spectral radius* $\tilde{\rho}$ *of* $\tilde{T}$ *on* E_M *is no less than 1* ($\tilde{\rho} \geq 1$) *and we have :* $\rho = \frac{\tilde{\rho}}{4^N}$ *and* $\sigma = N - \frac{\ell n \tilde{\rho}}{2 \ell n 2}$.

Proof. We already know that $\rho_p < 1$ for $p \geq 1$ by theorem 2. Moreover,

$$\lim_{q \to +\infty} \| T^q(1) \|_1 = \| \hat{\varphi} \|_2^2,$$

hence $\rho_0 = 1$. T is a non-negative operator ($f \leq g \Rightarrow Tf \leq Tg$) and, since

$$\sin^{2p+2} \frac{\xi}{2} \leq \sin^{2p} \frac{\xi}{2},$$

we have

$$T^q(\sin^{2p+2} \frac{\xi}{2}) \leq T^q(\sin^{2p} \frac{\xi}{2}),$$

hence $\rho_{p+1} \leq \rho_p$. Moreover

$$\| T^q(\sin^{2p} \frac{\xi}{2}) \|_1 = \int_{2^N K} \prod_{j=1}^{N} \mid m_0(\frac{\xi}{2^j}) \mid^2 \mid \sin^{2p} \frac{\xi}{2^{N+1}} \mid d\xi$$

$$\approx \int_{2^N K} \mid \hat{\varphi}(\xi) \mid^2 \mid \sin^{2p} \frac{\xi}{2^{N+1}} \mid d\xi$$

where K is a compact set satisfying Cohen's criterion (5.1), (5.2) and $0 \in \operatorname{Int} K$. But we have, for $0 < s < p$:

(15.1) $$\sum_{k \in \mathbf{N}} 4^{Ns} \sin^{2p} \frac{\xi}{2^{N+1}} \leq C_{s,p}(1 + \mid \xi \mid^2)^s$$

(just write $4^{Ns}\sin^{2p}\frac{\xi}{2^{N+1}} \leq 4^{Ns}$ if $\xi > 2^{N+1}$ and $\leq 4^{N(s-p)}(\frac{\xi}{2})^{2p}$ if $\xi < 2^{N+1}$) while

$$\sum_{k\in\mathbb{N}} 4^{Ns}\sin^{2p}\frac{\xi}{2^{N+1}}\chi_K(\frac{\xi}{2^N}) \geq \frac{1}{C_{s,p}}\inf(|\xi|^{2s}, |\xi|^{2p}) \tag{15.2}$$

(let $\epsilon \in (0, \frac{\pi}{2})$ be such that $(-\epsilon, \epsilon) \subset K$; if $|\xi| \geq \epsilon$, choose N_0 such that

$$2^{N_0}\epsilon \leq |\xi| < 2^{N_0+1}\epsilon$$

and write

$$4^{N_0 s}\sin^{2p}\frac{\xi}{2^{N_0+1}}\chi_K(\frac{\xi}{2^{N_0}}) \geq 4^{N_0 s}\sin^{2p}(\frac{\epsilon}{2}) \geq (\frac{\xi}{2\epsilon})^s\sin^{2p}(\frac{\epsilon}{2}) \ ;$$

if $|\xi| \leq \epsilon$ then $\sin^{2p}(\frac{\xi}{2})\chi_K(\xi) \geq (\frac{2\xi}{\pi})^{2p}$). (15.1) and (15.2) give then that for $s \in (0,p)$, $\varphi \in H^s$ if and only if $\sum_{k\in\mathbb{N}} 4^{sk} \parallel T^k(\sin^{2p}\frac{\xi}{2}) \parallel_{L^1} < +\infty$. In particular if $4^s < \frac{1}{\rho_p}$ then $\varphi \in H^s$ and if $4^s > \frac{1}{\rho_p}$, $\varphi \notin H^s$. Points (iii) and (iv) are then straightforward : if $\rho_{p_0} > \frac{1}{4^{p_0}}$ then $\varphi \notin H^s$ for $s \in \left(\frac{\ell n \frac{1}{\rho_{p_0}}}{2\,\ell n\, 2}, p_0\right)$, thus we must have $\rho_p \geq \rho_{p_0}$ for $p \geq p_0$ and thus $\rho_p = \rho_{p_0}$.

Now, let's suppose that m_0 is a trigonometric polynomial, $m_0 = \left(\frac{1+e^{-i\xi}}{2}\right)^N \tilde{m}_0(\xi)$ with $\tilde{m}_0(\pi) \neq 0$ and $\tilde{m}_0(\xi) = \sum_{k=0}^M a_k e^{-ik\xi}$ with $a_0 \neq 0$, $a_M \neq 0$. (We may always assume that $\tilde{m}_0$ has no positive frequencies and that $a_0 \neq 0$, because replacing $m_0(\xi)$ by $e^{iN\xi}m_0(\xi)$ is equivalent to shifting $\varphi(x)$ into $\varphi(x+N)$). The invariance of E_M under $\tilde{T}$ is obvious. Moreover $\tilde{T}$ is a non-negative operator on E_M, and we have therefore $\parallel \tilde{T}^N(f) \parallel_\infty \leq \parallel f \parallel_\infty \parallel \tilde{T}^N(1) \parallel_\infty$, so that the operator norm of $\tilde{T}^N$ on $(E_M, \parallel\cdot\parallel_\infty)$ is exactly $\parallel \tilde{T}^N(1) \parallel_\infty$, and $\tilde{\rho} = \lim_{N\to+\infty} \parallel \tilde{T}^N(1) \parallel_\infty^{1/N}$. We want now to compute

$$\rho_N = \limsup_{q\to+\infty} \parallel T^q(\sin^{2N}\frac{\xi}{2}) \parallel_1^{1/q} .$$

But we have for $f \in E_M$,

$$T(\sin^{2N}\frac{\xi}{2}f) = (\frac{1}{4})^N\sin^{2N}\frac{\xi}{2}\tilde{T}(f),$$

hence

$$T^q(\sin^{2N}\frac{\xi}{2}) = (\frac{1}{4})^{Nq}\sin^{2N}\frac{\xi}{2}\tilde{T}^q(1) \ ;$$

on E_M all the norms are equivalent (E_M if finite dimensional), so that the norms $\parallel \sin^{2N}\frac{\xi}{2}f \parallel_1$ and $\parallel f \parallel_\infty$ are equivalent, and thus

$$\parallel T^q(\sin^{2N}\frac{\xi}{2}) \parallel_L \approx (\frac{1}{4})^{Nq} \parallel \tilde{T}^q(1) \parallel_\infty$$

and therefore $\rho_N = \frac{\tilde{\rho}}{4^N}$. If $\tilde{\rho} = 1$, we know that $\varphi \notin H^N$ (if φ belongs to H^N, then $\varphi^{(N)}$ has compact support so by corollary 2, $m_0^{(N)}(\pi) = 0$; but $m_0^{(N)}(\pi) = \frac{N!}{2^N} i^N \tilde{m}_0(\pi) \neq 0$) so $\sigma = N$, $\varphi \notin H^\sigma$ and $\rho = \frac{1}{4^N} = \frac{\tilde{\rho}}{4^N}$. If $\tilde{\rho} > 1$, then $\rho_N > \frac{1}{4^N}$ hence $\rho = \rho_N = \frac{\tilde{\rho}}{4^N}$; moreover, to prove that $\varphi \notin H^\sigma$, we just have to prove the divergence of

$$\sum_{k \in \mathbb{N}} 4^{k\sigma} \parallel T^k(\sin^{2N}\frac{\xi}{2}) \parallel_1 \approx \sum_{k \in \mathbb{N}} 4^{k(\sigma-N)} \parallel \tilde{T}^k(1) \parallel_\infty,$$

but $\parallel \tilde{T}^k(1) \parallel_\infty$ is the operator norm of $\tilde{T}^k$ and $\tilde{\rho}$ is the greatest eigenvalue of $\tilde{T}$, so $\parallel \tilde{T}^k(1) \parallel_\infty \geq \tilde{\rho}^k$ and we obtain as a minorant the series

$$\sum_{k \in \mathbb{N}} \left(\frac{4^{\sigma-N}}{\tilde{\rho}}\right)^k = +\infty. \blacksquare$$

4. Compactly supported scaling functions.

We now describe the scaling functions which are compactly supported. Let $(V_j)_{j\in\mathbb{Z}}$ be a multi-resolution analysis such that V_0 contains non-trivial compactly supported functions (i.e. functions which are of compact support but don't vanish identically). Then we know (by lemma 5 of the preceding chapter) that we may find compactly supported functions φ and ω such that $\varphi \in V_0$, $\omega \in L^2$ and

$$< \varphi(x-k) \mid \omega(x) >= \delta_{k,0} \text{ for all } k \in \mathbb{Z}.$$

It follows readily that the family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is a Riesz family, hence a Riesz basis of V_0 and therefore φ is a scaling function for (V_j). If h is any compactly supported function in V_0, we have

$$h = \sum_{k\in\mathbb{Z}} < h \mid \omega(x-k) > \varphi(x-k)$$

and all but a finite number of the coefficients $< h \mid \omega(x-k) >$ are 0 ; if

$$k_0 = \inf\{k/ < h \mid \omega(x-k) > \neq 0\} \quad \text{and} \quad k_1 = \sup\{k/ < h \mid < \omega(x-k) > \neq 0\}$$

then

$$\text{Inf Supp } h = k_0 + \text{Inf Supp } \varphi \quad \text{and} \quad \text{Sup Supp } h = k_1 + \text{Sup Supp } \varphi,$$

hence the diameter of the support of h is greater than the diameter of the support of φ, unless $h = \lambda\, \varphi(x-k)$ for some $\lambda \in \mathbf{C}$ and $k \in \mathbb{Z}$.

DEFINITION 3. - *A local multi-resolution analysis is a multi-resolution analysis (V_j) such that V_0 contains non-trivial compactly supported functions. A fundamental scaling function is a compactly supported function φ which belongs to the space V_0 of a (local) multi-resolution analysis (V_j) and which has a support of minimal diameter among the* *non-trivial compactly supported functions of V_0. A fundamental scaling filter is a scaling filter associated to a fundamental scaling function.*

If φ is a fundamental scaling function for (V_j), then any compactly supported function h in V_0 can be written as a finite linear combination of the $\varphi(x-k)$, $k \in \mathbb{Z}$; equivalently $\hat{h}$ can be written as the product of $\hat{\varphi}$ by a trigonometric polynomial. If we take $h = \varphi(\frac{x}{2})$, we see that the fundamental scaling filter m_0 associated to φ is a trigonometric polynomial. Now, if θ is another compactly supported scaling function for (V_j) and μ_0 its scaling filter, we must have

$$\hat{\theta}(\xi) = M(\xi)\hat{\varphi}(\xi)$$

for a trigonometric polynomial which doesn't vanish on $[0, 2\pi]$, and thus

$$\hat{\theta}(2\xi) = \frac{M(2\xi)m_0(\xi)}{M(\xi)}\hat{\theta}(\xi) \ ;$$

the scaling filter $\mu(\xi) = \frac{M(2\xi)m_0(\xi)}{M(\xi)}$ can be a trigonometric polynomial (if $M(\xi)$ divides $M(2\xi)m_0(\xi)$) or not. Conversely if M and m_0 are two trigonometric polynomials such that $M(0) = m_0(0) = 1$ and M doesn't vanish on $[0, 2\pi]$, then the infinite product $\prod_{j=1}^{\infty} \mu\left(\frac{\xi}{2^j}\right)$ where $\mu(\xi) = \frac{M(2\xi)m_0(\xi)}{M(\xi)}$ is the Fourier transform of a compactly supported distribution: we have :

$$\prod_{j=1}^{\infty} \mu\left(\frac{\xi}{2^j}\right) = M(\xi)\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right),$$

so we have only to prove it for $\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$; writing $m_0(\xi) = e^{-iN\xi}P(e^{-i\xi})$ for some $N \in \mathbb{Z}$ and $P \in \mathbb{C}[X]$, $P(0) \neq 0$, we may define the function

$$F(z) = \prod_{j=1}^{\infty} P\left(e^{-i\frac{z}{2^j}}\right)$$

(so that $m_0(\xi) = e^{-iN\xi}F(\xi)$) ; the infinite product converges uniformly on the compacts of $\mathbb{C}$, so that F is holomorphic ; moreover

$$| P(e^{-i\frac{z}{2^j}}) | \leq C_0 e^{|Im\, z|\frac{1}{2^j}d^0 P},$$

hence we have for $2^{j_0} \leq | z | < 2^{j_0+1}$ $(j_0 \geq 0)$

$$\begin{aligned} | F(z) | &\leq | F(\frac{z}{2^{j_0+1}}) | \prod_{j=1}^{j_0} | P(e^{-i\frac{z}{2^j}}) | \\ &\leq C_0^{j_0} e^{d^0 P|Im\, z|} \sup_{|\eta|\leq 1} | F(\eta) | \\ &\leq | z |^{\frac{\ln C_0}{\ln 2}} e^{d^0 P|Im\, z|} \sup_{|\eta|\leq 1} | F(\eta) | \end{aligned}$$

and the theorem of Paley-Wiener-Schwartz ensures us that the inverse Fourier transform of $\prod_{j=1}^{\infty} P(e^{-i\frac{\xi}{2^j}})$ has a compact support.

We begin by describing the trigonometric polynomials which are scaling filters ; then we will characterize among them the fundamental scaling filters. We may always assume that the polynomials we deal with are of the type $m_0(\xi) = P(e^{-i\xi})$ with $P(0) \neq 0$, $P(1) = 1$, $P \in \mathbb{C}[X]$; if k is the lowest index such that $a_k \neq 0$ (where $m_0(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}$), we multiply $m_0(\xi)$ by $e^{ik\xi}$; it doesn't change the multi-resolution analysis and shifts the scaling function $\varphi(x)$ into $\varphi(x+k)$.

PROPOSITION 2. - *Let $m_0(\xi) = P(e^{-i\xi})$ ($P \in \mathbb{C}[X]$, $P(0) \neq 0$, $P(1) = 1$) be a trigonometric polynomial, T its transition operator (defined by*

$$Tf = | m_0(\frac{\xi}{2}) |^2 f(\frac{\xi}{2}) + | m_0(\frac{\xi}{2}+\pi) |^2 f(\frac{\xi}{2}+\pi))$$

and φ the compactly-supported distribution defined by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}).$$

Then m_0 is a scaling filter if and only if the three following assertions are satisfied :

(i) *the characteristic polynomial of the operator T operating on the space E_P of trigonometric polynomials of degree no greater than d^0P $\left(f \in E_p \Leftrightarrow f = \sum_{k=-d^0P}^{d^0P} f_k e^{-ik\xi}\right)$ has 1 as a root of multiplicity 1 and all its other roots are of modulus less than 1 ;*

(ii) *if $P(z_0) = 0$ and $| z_0 | = 1$, then $P(-z_0) \neq 0$ ($P(z)$ and $P(-z)$ have no common root on the unit circle) ;*

(iii) *there is no $z_0 \in \mathbb{C}$ such that $| z_0 | = 1$, $z_0 \neq 1$, and $\forall N \in \mathbb{N}$, $P(-z^{2^N}) = 0$.*

Proof. We will show that (ii) and (iii) are equivalent to Cohen's criterion (5.1), (5.2). Then (i) will be equivalent to $\sup_{q\in\mathbb{N}} \| T^q(1) \|_\infty < +\infty$: if (i) is satisfied, then E_P can be decomposed into $E_P = A \oplus B$ where $\dim A = 1$, $T|_A = Id$ and where $T(B) \subset B$ and the spectral radius $\rho(T|_B)$ is less than 1 ; if $1 = a + b$, $a \in A$, $b \in B$, then $T^q(1) = a + T^q(b)$, and $T^q(b) \to 0$ for any norm on E_P ; conversely if m_0 satisfies Cohen's criterion and $\sup_{q\in\mathbb{N}} \| T^q(1) \|_\infty < +\infty$, then we know that m_0 is a scaling filter and we may decompose E_p as

$$E_p = \mathbb{C}\, C(\varphi,\varphi) \oplus \{f \in E_p / f(0) = 0\}$$

and we know that $T(C(\varphi,\varphi)) = C(\varphi,\varphi)$ while the spectral radius of T on $\{f \in E_p/f(0) = 0\}$ is less than 1 (see theorem 2) ; we may notice that

$$\varphi(\frac{x}{2}) = \sum_{k=0}^{d^0P} a_k \varphi(x-k),$$

which implies

$$2\text{Inf Supp } \varphi = \text{Inf Supp } \varphi$$

and

$$2\text{Sup Supp } \varphi = d^0 P + \text{Sup Supp } \varphi,$$

hence

$$\text{Inf Supp } \varphi = 0 \quad \text{and} \quad \text{Sup Supp } \varphi = d^0 P,$$

hence

$$C(\varphi,\varphi) = \sum_{k\in\mathbb{Z}} < \varphi(x) \mid \varphi(x-k) > e^{-ik\xi} \in E_P.$$

We now prove the equivalence between Cohen's criterion and (ii)-(iii). If for some ξ_0 we had $m_0(\xi_0) = m_0(\xi_0+\pi) = 0$ (i.e. if (ii) was false) then we would have

$$\hat{\varphi}(2\xi_0 + 2k\pi) = \hat{\varphi}(\xi_0 + k\pi)m_0(\xi_0 + k\pi) = 0 \text{ for all } k,$$

and thus Cohen's criterion would be false. Similarly, if (iii) was false, we could find $\xi_0 \notin 2\pi\mathbb{Z}$ such that for all $N \in \mathbb{Z}$, $m_0(2^N\xi_0+\pi) = 0$; but P can only have a finite number of roots and we must have $2^{N_0}\xi = 2^{M_0}\xi \bmod(2\pi)$ for at least two numbers $N_0, M_0 \in \mathbb{N}$ with $N_0 > M_0$; we then write $2^{N_0} = 2^{P_0}2^{M_0}$ and $2^{M_0}\xi_0 = 2^{N_0}\xi_0 + 2k_0\pi$; notice that we cannot have $\frac{2^{P_0}k_0}{2^{P_0}-1} \in \mathbb{Z}$ since $\xi_0 = 2\pi\frac{k_0}{2^{M_0}-2^{N_0}} = \frac{1}{2^{N_0}}\frac{2\pi 2^{P_0}k_0}{1-2^{P_0}}$: if $\frac{2^{P_0}k_0}{1-2^{P_0}} = 2^{A_0}(2B_0+1)$ then we would have $\xi_0 \in 2\pi\mathbb{Z}$ if $A_0 \geq N_0$ or $m_0(2^{N_0-A_0-1}\xi_0+\pi) = 1 \neq 0$ if $A_0 < N_0$, which contradicts the choice of ξ_0 ; we now prove that $\hat{\varphi}(2^{N_0}\xi_0 + 2k\pi) = 0$ for all $k \in \mathbb{Z}$ (so that Cohen's criterion is not satisfied) : just write

$$\hat{\varphi}(2^{N_0}\xi_0 + 2k\pi) = \hat{\varphi}\left(2^{N_0}\xi_0 + 2\pi(\frac{k}{2^{P_0}} + k_0)\right)\prod_{j=1}^{P_0} m_0\left(\frac{2^{N_0}\xi_0 + 2k\pi}{2^j}\right) \ ;$$

if k is not a multiple of 2^{P_0}, one term $m_0\left(\frac{2^{N_0}\xi_0+2k\pi}{2^j}\right)$ $(1 \leq j \leq P_0)$ is 0 ; if k is a multiple of 2^{P_0}, we change it into $\frac{k}{2^{P_0}} + k_0$ because $\hat{\varphi}(2^{N_0}\xi_0 + 2k\pi)$ vanishes as soon as $\hat{\varphi}(2^{N_0}\xi_0+2\pi(\frac{k}{2^{P_0}}+k_0))$ does ; now the sequence (a_j) defined by $a_0 = k$ and $a_{j+1} = \frac{a_j}{2^{P_0}}+k_0$ converges to $\frac{2^{P_0}k_0}{2^{P_0}-1}$ which is not an integer, and thus we may conclude that (a_j) is not integer-valued and $\hat{\varphi}(2^{N_0}\xi_0 + 2k\pi) = 0$.

Conversely, let's assume that Cohen's criterion is not satisfied by m_0 and that (ii) is satisfied. We then have to prove that (iii) is false, i.e. to exhibit $\xi_0 \notin 2\pi\mathbb{Z}$ such that $m_0(2^N\xi_0 + \pi)$ is 0 for all $N \in \mathbb{N}$. We may assume that φ is square-integrable : we have seen that for some $M \in \mathbb{Z}$ we have

$$\mid \hat{\varphi}(\xi) \mid \leq C(1+ \mid \xi \mid)^M \ ;$$

if $M \leq -2$, $\varphi \in L^2$; if $M \geq -1$, then

$$\left(\frac{1-e^{-i\xi}}{i\xi}\right)^{M+2} \hat{\varphi} \in L^2 \ ;$$

but multiplying $\hat{\varphi}$ by $\left(\frac{1-e^{-i\xi}}{i\xi}\right)^{M+2}$ is equivalent to multiplying m_0 by $\left(\frac{1+e^{-i\xi}}{2}\right)^{M+2}$ and this doesn't change neither Cohen's criterion (5.1), (5.2) nor the properties (ii) and (iii). Now, write $Q(\xi)$ for

$$C(\varphi,\varphi)(\xi) = \sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2 = \sum_{k\in\mathbb{Z}} < \varphi(x) \mid \varphi(x-k) > e^{-ik\xi} \ ;$$

if Cohen's criterion is not satisfied, then $Q(\xi_0) = 0$ for at least one $\xi_0 \in \mathbb{R}$; moreover if $Q(\xi_0) = 0$, then write

$$Q(\xi_0) = Q(\frac{\xi_0}{2}) \mid m_0(\frac{\xi_0}{2}) \mid^2 + Q(\frac{\xi_0}{2}+\pi) \mid m_0(\frac{\xi_0}{2}+\pi) \mid^2$$

to conclude (since (ii) is satisfied) that $Q(\frac{\xi_0}{2})$ or $Q(\frac{\xi_0}{2}+\pi)$ vanishes ; we may change ξ_0 into $\xi_0+2\pi$ and assume that $Q(\frac{\xi_0}{2}) = 0$; then (changing ξ_0 into $\xi_0+4\pi$ if necessary) we obtain $Q(\frac{\xi_0}{4}) = 0$, and so on. Since φ is compactly supported, Q is a trigonometric polynomial and has only a finite number of roots, which mean that we have for some $N \in \mathbb{N}^*$,

$$Q(\xi_0) = Q(\frac{\xi_0}{2}) = \cdots = Q(\frac{\xi_0}{2^N}) = 0$$

and

$$\frac{\xi_0}{2^N} - \xi_0 \in 2\pi\mathbb{Z}.$$

Moreover all the numbers $Q(\frac{\xi_0}{2}+\pi), \cdots, Q(\frac{\xi_0}{2^N}+\pi)$ are non-zero : we have seen that if $Q(\eta) = 0$ for some $M \geq 1$, $\frac{2^M-1}{2^M}\eta \in 2\pi\mathbb{Z}$; we would then have, if $Q(\frac{\xi_0}{2^j}+\pi) = 0$, for some M_j, $\frac{2^{M_j}-1}{2^{M_j}}\left(\frac{\xi_0}{2^j}+\pi\right) \in 2\pi\mathbb{Z}$; define $Z = e^{i(\frac{\xi_0}{2^j}+\pi)}$; then writing $\frac{\xi_0}{2^j}+\pi = \frac{\frac{\xi_0}{2^j}+\pi}{2^{M_j}} \text{mod}(2\pi)$ gives $Z^{2^{M_j}} = Z$ and writing $\frac{\xi_0}{2^j} = 2^{N-j}\frac{\xi_0}{2^N} = 2^{N-j}\frac{\xi_0}{2^{2N}}\text{mod}(\frac{2\pi}{2^N})$ gives $Z^{2^N} = e^{i\frac{\xi_0}{2^j}} = -Z$, and thus $-Z = Z^{2^N} = Z^{2^{N+M_j}} = (-Z)^{2^{M_j}} = Z$ which is absurd. But we have $Q\left(\frac{\xi_0}{2^{j-1}}\right) = 0$ for $1 \leq j \leq N$, while

$$Q(\frac{\xi_0}{2^{j-1}}) = Q(\frac{\xi_0}{2^j}) \mid m_0(\frac{\xi_0}{2^j}) \mid^2 + Q(\frac{\xi_0}{2^j}+\pi) \mid m_0(\frac{\xi_0}{2^j}+\pi) \mid^2 \ ;$$

since $Q(\frac{\xi_0}{2^j}+\pi) \neq 0$, we have $m_0(\frac{\xi_0}{2^j}+\pi) = 0$ for $1 \leq j \leq N$; since $\xi_0 - \frac{\xi_0}{2^N} \in 2\pi \in \mathbb{Z}$, we obtain $m_0(2^k\xi_0+\pi) = 0$ for all $k \geq 0$, while $\xi_0 \notin 2\pi\mathbb{Z}$: $Q(0) \geq | \hat{\varphi}(0) |^2 = 1$ and thus $Q(0) \neq 0$. Thus, proposition 2 is proved. ∎

We may now easily characterize the fundamental scaling filters :

PROPOSITION 3. - *Let $m_0(\xi) = P(e^{-i\xi})$ ($P \in \mathbf{C}[X]$, $P(0) \neq 0$, $P(1) = 1$) be a trigonometric polynomial. Then m_0 is a fundamental scaling filter if and only if it is a scaling filter and the polynomials $P(z)$ and $P(-z)$ have no common root in $\mathbf{C}$.*

Proof. Let's suppose that $P(z) \wedge P(-z) = R(z^2)$ and define h by $\hat{h} = \frac{\hat{\varphi}(\xi)}{R(e^{-i\xi})}$; R have no root on the unit circle so that h is well-defined ; moreover
$\hat{h}(2\xi) = R(e^{-i\xi})\frac{P(e^{-i\xi})}{R(e^{-2i\xi})}\hat{h}(\xi) = m_1(\xi)\hat{h}(\xi)$ where m_1 is a trigonometric polynomial of degree less than the degree of P (unless $R = 1$ and $h = \varphi$) ; then the support of h has a smaller diameter than the support of φ and m_0 cannot be a fundamental scaling filter. Thus if m_0 is fundamental, then $P(z)$ and $P(-z)$ are prime together.

Conversely, let's assume that $P(z)$ and $P(-z)$ are prime together. Let $m_1 = P_1(e^{-i\xi})$ be the fundamental scaling filter associated to the multi-resolution analysis generated by m_0, and φ_1 be its scaling function. Then $\varphi(x)$ is a finite linear combination of the $\varphi_1(x-k)$, and $\hat{\varphi}(\xi) = Q(e^{-i\xi})\hat{\varphi}_1(\xi)$ for a polynomial $Q \in \mathbb{C}[X]$. Then we obtain $Q(z^2)P_1(z) = Q(z)P(z)$. Since $P(z) \wedge P(-z) = 1$, we must have that $Q(z)$ divides $Q(z^2)$: $Q(z^2) = A(z)Q(z)$ for some $A \in \mathbb{C}[X]$. But now if $Q(z_0) = 0$, then $Q(z_0^2) = 0$ and $z_0, z_0^2, \cdots, z_0^{2^N}, \cdots$ are roots of Q. But $Q(0) \neq 0$, and Q has only a finite number of roots ; thus $| z_0 |= 1$; but this cannot be since if φ is a scaling function $C(\varphi,\varphi) = Q(e^{-i\xi})C(\varphi_1,\varphi_1)$ doesn't vanish.
■

DEFINITION 4. - *Let $P \in \mathbb{C}[X]$, $P(0) \neq 0$, $P(1) = 1$ be such that $P(z) \wedge P(-z) = 1$. Then the dual polynomial P^* of P is the unique polynomial $P^* \in \mathbb{C}[X]$ such that : $d^0 P^* < d^0 P$ and $P(z)P^*(z) + P(-z)P^*(-z) = 1$.*

We may now prove some prooperties of the fundamental scaling functions :

THEOREM 4. - *Let $m_0 = P(e^{-i\xi})$ ($P \in \mathbb{C}[X]$, $P(0) \neq 0$, $P(1) = 1$) be a fundamental scaling filter, and φ (defined by $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$) its scaling function. Then :*

i) *there exists a compactly supported function $h \in L^2$ such that :*

$$(16.1) \qquad \forall k \in \mathbb{Z}, \quad < h(x-k) \mid \varphi >= \delta_{k,0}.$$

If (V_j) is the multiresolution analysis generated by m_0, we thus have :

$$(16.2) \qquad \forall f \in V_0, \quad f = \sum_{k\in\mathbb{Z}} < f \mid h(x-k) > \varphi(x-k).$$

ii) *Let $f \in V_0$, $\hat{f}(\xi) = F(\xi)\hat{\varphi}(\xi)$ with $F \in L^2(\mathbb{R}/2\pi\mathbb{Z})$. Then,*

$$(17) \qquad f \in V_{-1} \Leftrightarrow \begin{vmatrix} F(\xi) & P(e^{-i\xi}) \\ F(\xi+\pi) & P(-e^{-i\xi}) \end{vmatrix} = 0.$$

iii) *(Malgouyres' separation lemma). If $f \in V_0$ vanishes identically on an interval (α,β) then $\chi_{[-\infty,\alpha]}f \in V_0$ and $\chi_{[\beta,+\infty]}f \in V_0$.*
iv) *The support of φ is the full interval $[0, d^0 P]$.*
v) *The restrictions $(\varphi(x-k) \mid_{[0,1]})_{1-d^0P\leq k\leq 0}$ are linearly independent in $L^2([0,1])$.*

Remarks.

j) The existence of a compactly supported dual function h (satisfying (16.1)) or the linear independence of the restrictions $(\varphi(x-k)\ |_{[0,1]})(1-\text{Sup Supp}\,\varphi \leq k \leq -\text{Inf Supp}\,\varphi)$ characterize the fundamental scaling functions among the compactly supported functions in V_0.

jj) Points iii) and iv) were proved in 1991 by G. Malgouyres [MAG], [LEMA2] and point v) by Y. Meyer [MEY5]. ■

Proof. Point i) has already been discussed in the beginning of this section. Point ii) is almost obvious : if $f \in V_{-1}$, then $F(\xi) = G(2\xi)P(e^{-i\xi})$ for some $G \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ and

$$F(\xi+\pi) = G(2\xi)P(-e^{-i\xi}),$$

hence

$$F(\xi)P(-e^{-i\xi}) - F(\xi+\pi)P(e^{-i\xi}) = 0\ ;$$

conversely, let P^* be the dual polynomial of P ; then we have

$$F(\xi) = (F(\xi)P^*(e^{-i\xi}) + F(\xi+\pi)P^*(-e^{-i\xi}))P(e^{-i\xi})$$
$$+(F(\xi)P(-e^{-i\xi}) - F(\xi+\pi)P(e^{-i\xi}))P^*(-e^{-i\xi})$$

which proves that if

$$F(\xi)P(-e^{-i\xi}) - F(\xi+\pi)P(e^{-i\xi}) = 0$$

then

$$F(\xi) = G(2\xi)P(e^{-i\xi})$$

for some $G \in L^2(\mathbb{R}/2\pi\mathbb{Z})$, hence $f \in V_{-1}$.

We now prove Malgouyres' separation lemma.

We know that $\text{Inf Supp}\,\varphi = 0$ and $\text{Sup Supp}\,\varphi = d^0P$. Let $A = \text{Inf Supp}\,h$ and $B = \text{Sup Supp}\,h$; $B > 0$ and $A < d^0P$. We choose now $j \geq 0$ such that : $2^j(\beta-\alpha) \geq B-A+d^0P$. Then it is easy to see that $f\chi_{(-\infty,\alpha]}$ and $f\chi_{[\beta,+\infty)}$ belong to V_j. Indeed, we know that f belongs to V_0, hence to V_j (since $V_0 \subset V_j$ for $j \geq 0$) and therefore we may write

$$f = \sum_{k\in\mathbb{Z}} 2^j < f \mid h(2^jx-k) > \varphi(2^jx-k).$$

But, since $f \equiv 0$ on $[\alpha,\beta]$,

$$< f \mid h(2^jx-k) >= 0$$

for

$$2^j\alpha - A \leq k \leq 2^j\beta - B.$$

Define now f_+ and f_- as

$$f_+ = \sum_{k\geq 2^j\alpha-A} 2^j < f \mid h(2^jx-k) > \varphi(2^jx-k)$$

and

$$f_- = \sum_{k \leq 2^j\beta - B} 2^j < f \mid h(2^j x - k) > \varphi(2^j x - k) \ ;$$

the sum for f_+ runs only on $k > 2^j\beta - B$, hence $f_+ \equiv 0$ on $(-\infty, \beta - \frac{B}{2^j})$, and similarly $f_- \equiv 0$ on $(\alpha - \frac{A}{2^j} + \frac{d^0 P}{2^j}, +\infty)$; moreover $f = f_+ + f_-$, and we obtain, since $\beta - \frac{B}{2^j} \geq \alpha - \frac{A}{2^j} + \frac{d^0 P}{2^j}$, that f_+ is 0 on $(-\infty, \beta - \frac{B}{2^j})$ and is equal to f on $(\beta - \frac{B}{2^j}, +\infty)$, hence that $f_+ = f\chi_{[\beta,+\infty)}$ and similarly $f_- = f\chi_{(-\infty,\alpha]}$.

The second step in the proof of the separation lemma is to show that if $\alpha < \beta$, if $f_- = f\chi_{(-\infty,\alpha]}$ and $f_+ = f\chi_{[\beta,+\infty)}$ belong to V_j and f belongs to V_{j-1} and vanishes on (α, β) then f_- and f_+ belong to V_{j-1}. We may replace f by $f(2^{-j}x)$ and α and β by $2^j\alpha$ and $2^j\beta$ and thus assume f_-, f_+ in V_0 and f in V_{-1}. We then write

$$f_- = \sum_{k \in \mathbb{Z}} f_{-,k}\varphi(x - k) \quad , \quad \hat{f}_-(\xi) = F_-(\xi)\hat{\varphi}(\xi)$$

with

$$F_-(\xi) = \sum_{k \in \mathbb{Z}} f_{-,k}e^{-ik\xi},$$

and similarly

$$\hat{f}_+(\xi) = F_+(\xi)\hat{\varphi}(\xi)$$

with

$$F_+(\xi) = \sum_{k \in \mathbb{Z}} f_{+,k}e^{-ik\xi}.$$

Since

$$\mathrm{Supp}\, f_- \subset (-\infty, \alpha] \quad \text{and} \quad f_{-,k} = < f_- \mid h(x - k) >,$$

we have $f_{-,k} = 0$ for $k \geq \alpha - A$; but if k_0 is the greatest index k such that $f_{-,k} \neq 0$, we have

$$\mathrm{Sup}\,\mathrm{Supp}\, f_- = \mathrm{Sup}\,\mathrm{Supp}\,\varphi(x - k_0) = d^0 P + k_0,$$

hence

$$k_0 \leq \alpha - d^0 P \ ;$$

similarly if k_1 is the lowest index k such that $f_{+,k} \neq 0$, we have $k_1 \geq \beta$, and thus $k_0 + d^0 P < k_1$. Now, if $P(z) = \sum_{k=0}^{d^0 P} P_k z^k$, we have to prove, in order to prove $f_- \in V_{-1}$, that

$$F_-(\xi)P(-e^{-i\xi}) - F_-(\xi + \pi)P(e^{-i\xi}) = 0,$$

or equivalently that :

$$\forall k \in \mathbb{Z}, \quad \sum_{q=0}^{d^0 P} f_{-,2k-1-q}(-1)^{q+1}p_q = 0 \tag{18.1}$$

but, since $f \in V_{-1}$, we already know that :

$$\text{(18.2)} \qquad \forall k \in \mathbb{Z}, \quad \sum_{q=0}^{d^0 P} (f_{+,2k-1-q} + f_{-,2k-1-q})(-1)^{q+1} p_q = 0 \ ;$$

if $2k-1 < k_1$, $f_{+,2k-1-q}$ is always 0 and (18.2) gives (18.1) ; if $2k-1 \geq k_1$ then $2k-1-q$ is always $\geq k_1 - d^0 P > k_0$ and $f_{-,2k-1-q}$ is always 0, hence (18.1) is obviously true. Thus f_- and f_+ belong to V_{-1}. The separation lemma is then proved.

Point iv) is then straightforward. If φ vanishes on $(\alpha, \beta) \subset [0, d^0 P]$, with $\alpha < \beta$ then $\chi_{(-\infty,\alpha]}\varphi$ belongs to V_0 ; but the diameter of the support of $\chi_{(-\infty,\alpha]}\varphi$ would be at most α (since φ vanishes on $(-\infty, 0]$) and thus φ wouldn't have a support with minimal diameter ; thus φ cannot vanish identically on a subinterval of $[0, d^0 P]$.

Point v) is easy as well. Assume that $\sum_{k=1-d^0 P}^{0} \lambda_k \varphi(x-k)$ vanishes identically on $[0,1]$. Then $\chi_{(-\infty,0]} \sum_{k=1-d^0 P}^{0} \lambda_k \varphi(x-k)$ would belong to V_0 and its support would be contained in $[1 - d^0 P, 0]$, hence would have a diameter less than the diameter of $\operatorname{Supp} \varphi$; thus

$$\chi_{(-\infty,0]} \sum_{k=1-d^0 P}^{0} \lambda_k \varphi(x-k)$$

has to be identically 0, and similarly

$$\chi_{[1,+\infty)} \sum_{k=1-d^0 P}^{0} \lambda_k \varphi(x-k)$$

is identically 0. We thus obtain

$$\sum_{k=1-d^0 P}^{0} \lambda_k \varphi(x-k) = 0$$

on all $\mathbb{R}$, and thus $\lambda_k = 0$ for $k = 1 - d^0 P, \cdots, 0$. Theorem 4 is proved. ∎

Chapter 5

DAUBECHIES' FUNCTIONS AND OTHER EXAMPLES OF SCALING FUNCTIONS

In this chapter, we proceed to the construction of the "classical" scaling functions, which are by now widely used in scientific applications, including the celebrated compactly supported orthonormal scaling functions of I. Daubechies.

As we shall see, the classical scaling functions are deeply related to interpolating functions, i.e. to functions Φ such that $\Phi(0) = 1$ and $\Phi(k) = 0$ for $k \in \mathbb{Z}^*$. By instance, saying that $(\varphi(x-k))_{k\in\mathbb{Z}}$ is orthonormal is equivalent to say that $\varphi * \bar{\varphi}(-x)$ is interpolating.

We thus begin with interpolating scaling functions (including the iterative dyadic interpolation scheme of Deslauriers and Dubuc), then describe orthonormal scaling functions. In a third section, we describe orthogonal spline multi-resolution analyses (including Battle-Lemarié's or Strömberg's orthonormal wavelets, Battle's pre-wavelets, Chui-Wang's pre-wavelets, Schoenberg's interpolating functions and so on) and lastly we end with bi-orthogonal multi-resolution analyses and dual compactly supported scaling functions associated to the B-spline functions.

This chapter is thus mainly a list of examples (which includes all classical scaling functions), together with a list of some easy results on interpolating or orthonormal scaling functions which justify the constructions.

1. Interpolating scaling functions.

We begin with the construction of scaling functions φ which are interpolating :

(1) $$\varphi(0) = 1, \quad \varphi(k) = 0 \ \text{ for } \ k \in \mathbb{Z}^*.$$

Of course, φ has to be continuous if we want (1) to be meaningful. With help of the Poisson formula, (1) can be written as

(2) $$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2k\pi) = 1 \text{ a.e.}$$

provided that $\hat{\varphi} \in L^1$. We therefore introduce the Banach space $E^0 = \{\varphi \in C^0(\mathbb{R}) / \hat{\varphi}$ is continuous and $\sum_{k\in\mathbb{Z}} \mid \hat{\varphi}(\xi + 2k\pi) \mid$ converges uniformly on $[-\pi, \pi]\}$ equipped with the norm

$$\| \varphi \|_{E^0} = \| \sum_{k\in\mathbb{Z}} \mid \hat{\varphi}(\xi + 2k\pi) \mid \|_\infty .$$

It is very easy to check that E^0 is complete, as well as the space E^k defined by

$$E^k = \{\varphi \in C^0 / \forall p \in \{0, 1, \cdots, k\} \ x^p \varphi \in E^0\}$$

with norm $\| \varphi \|_{E^k} = \sum_{p=0}^{k} \| x^p \varphi \|_{E^0}$. Then theorems 1 and 2 of the preceding chapter can easily be adapted to establish the following result :

PROPOSITION 1. - *Let $m_0 \in C^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ for some $\epsilon > 0$ be such that $m_0(0) = 1$. Define φ and T by :*

(3) $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$

(4) $\forall f \in C^0(\mathbb{R}/2\pi\mathbb{Z}), \quad Tf(\xi) = | m_0(\frac{\xi}{2}) | f(\frac{\xi}{2}) + | m_0(\frac{\xi}{2} + \pi) | f(\frac{\xi}{2} + \pi).$

Moreover let's assume that m_0 satisfies Cohen's criterion :

(5) *there exists a compact set K which is a finite union of closed intervals such that :*

(i) $\sum_{k \in \mathbb{Z}} \chi_K(\xi + 2k\pi) = 1$ *a.e.*

(ii) $\forall \xi \in K, \quad \hat{\varphi}(\xi) \neq 0.$

Then $\varphi \in E^0$ if and only if $\sup_{n \in \mathbb{N}} \| T^n(1) \|_\infty < +\infty$. Moreover in that case there exists a positive α such that $| \xi |^\alpha \hat{\varphi} \in L^1$. If $\varphi \in E^0$, then $\varphi \in E^k$ if and only if m_0 is C^k.

Proof. If $\varphi \in E^0$, define γ as $\gamma = \sum_{k \in \mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) |$; then $\gamma \in C^0(\mathbb{R}/2\pi\mathbb{Z})$, $T(\gamma) = \gamma$ and $\| \frac{1}{\gamma} \|_\infty \leq \frac{1}{\inf_{\xi \in K} |\hat{\varphi}(\xi)|} < +\infty$; since T is a positive operator and $1 \leq \gamma \| \frac{1}{\gamma} \|_\infty$, we obtain $\| T^n(1) \|_\infty \leq \| \gamma \|_\infty \| \frac{1}{\gamma} \|_\infty$. Conversely, let's assume that $\sup_{n \in \mathbb{N}} \| T^n(1) \|_\infty$ is finite. Define an operator S on E^0 by $\widehat{S\omega} = m_0(\frac{\xi}{2})\omega(\frac{\xi}{2})$. We will approximate $\hat{\varphi}$ by $\widehat{S^n\omega}$ where $\hat{\omega} \in C_c^\infty$ satisfies $\hat{\omega}(0) = 1$. We easily check that $\| S^n\omega \|_{E^0} \leq \| T^n(1) \|_\infty \| \omega \|_{E^0}$, and thus by Fatou's lemma we get that $\sup_{\xi \in \mathbb{R}} \sum_{k \in \mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) | < +\infty$ (we take the supremum instead of the essential supremum because $\sum_{k \in \mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) |$ is semi-continuous). To prove the uniform convergence of $\sum_{k \in \mathbb{Z}} | \hat{\varphi}(\xi + 2k\pi) |$ on $[-\pi, \pi]$ is much more difficult. This will be done by mimicking the proof of the first regularity theorem (theorem 2 of the preceding chapter) :

- for $g \in C^0(\mathbb{R}/2\pi\mathbb{Z})$, the functions $\gamma_n = g(\frac{\xi}{2^n}) \prod_{j=1}^{n} m_0(\frac{\xi}{2^j}) \chi_{[-\pi,\pi]}(\frac{\xi}{2^n})$ converge to $g(0)\hat{\varphi}$ in L^1 ;
- $m_0(\pi) = 0$ (since $\sum_{k \in \mathbb{Z}} | \hat{\varphi}(2k\pi) | < +\infty$ and $\hat{\varphi}(2^n(\pi + 2k_0\pi)) = \hat{\varphi}(\pi + 2k_0\pi)m_0(\pi)$)
- choosing $\epsilon \in (0,1)$ with $m_0 \in C^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$, we see that T keeps invariant the space $F^\epsilon = \{f \in C^\epsilon(\mathbb{R}/2\pi\mathbb{Z}) / f(0) = 0\}$ and that its spectral radius ρ_ϵ on F^ϵ is less than 1.
- Now, we choose $\Omega_\epsilon \in E_0$ such that $\hat{\Omega}_\epsilon$ is non-negative, $\hat{\Omega}_\epsilon \in C^\epsilon$, $\operatorname{Supp} \hat{\Omega}_\epsilon \subset [-\frac{3\pi}{2}, \frac{3\pi}{2}]$, $0 < \hat{\Omega}_\epsilon(\xi)$ for $0 < | \xi | \leq \pi$ and $\lim_{\xi \to 0} \frac{\hat{\Omega}_\epsilon(\xi)}{|\xi|^\epsilon} = 1$.

We remark that $C_0 = \sup_{n \in \mathbb{N}} \| \prod_{j=1}^{n} m_0(2^j\xi) \|_\infty$ is bounded by $\sup_{n \in \mathbb{N}} \| T^n(1) \|_\infty$, and that $| \hat{\varphi}(\xi) | \leq C_0 \prod_{j=1}^{N} | m_0(\frac{\xi}{2^j}) |$ for all N. Hence we have :

$$\sum_{2^{N-1}\pi \leq |\xi + 2k\pi| \leq 2^N\pi} | \hat{\varphi}(\xi + 2k\pi) | \leq$$

$$\leq \frac{C_0}{\inf_{\frac{\pi}{2}\leq|\eta|\leq\pi} \hat{\Omega}_\epsilon(\eta)} \sum_{2^{N-1}\pi\leq|\xi+2k\pi|\leq 2^N\pi} \prod_{j=1}^{N} | m_0(\frac{\xi+2k\pi}{2^j}) | \hat{\Omega}_\epsilon(\frac{\xi+2k\pi}{2^N})$$
$$\leq C' \| S^N\Omega_\epsilon \|_{E^0} .$$

Now define V as $Vf = \sum_{k\in\mathbb{Z}} | \hat{f}(\xi+2k\pi) |$; we have $VSf \leq TVf$, hence $VS^nf \leq T^nVf$; but $V\Omega_\epsilon \in F^\epsilon$ and therefore we get $\| S^N\Omega_\epsilon \|_{E^0}\leq C_\rho\rho^N$ for any $\rho \in (\rho_\epsilon, 1)$, which gives that $\varphi \in E^0$ and that $| \xi |^\alpha \hat{\varphi} \in L^1$ for any α such that $2^\alpha\rho_\epsilon < 1$. Moreover, φ is a scaling function : we have

$$\sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2 \geq \inf_{\eta\in K} | \hat{\varphi}(\eta) |^2 > 0$$

while

$$\sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2 \leq \| \hat{\varphi} \|_\infty \sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) | \leq \| \hat{\varphi} \|_{E^0}^2 ;$$

define then

$$V_j = \mathrm{Span}(2^{j/2}\varphi(2^jx-k)/k\in\mathbb{Z})$$

and apply the proof of lemma 1 in the preceding chapter to show that $\overline{\cup V_j} = L^2$ and $\cap V_j = \{0\}$. Finally, the equivalence between $\varphi \in E^k$ and $m_0 \in C^k$ is easy : if $\varphi \in E^k$, $\hat{\varphi} \in C^k$ and whenever $\hat{\varphi}(\xi_0+2k_0\pi) \neq 0$ we have $m_0(\xi) = \frac{\hat{\varphi}(2\xi+4k_0\pi)}{\hat{\varphi}(\xi+2k_0\pi)}$ in a neighborhood of ξ_0, thus m_0 is C^k. We prove the converse when $m_0 \in C^1$: we have seen that in that case

$$\varphi' = \sum_{k=1}^{\infty}\prod_{j=1}^{k-1} m_0(\frac{\xi}{2^j})m_0'(\frac{\xi}{2^k})\frac{1}{2^k}\prod_{j=k+1}^{\infty} m_0(\frac{\xi}{2^j})$$

so that $\| \hat{\varphi}' \|_\infty \leq C_0^2 \| m_0' \|_\infty$ and thus for all $N \in \mathbb{N}$ we have :

$$| \hat{\varphi}'(\xi) | \leq$$

$$C_0\sum_{p=1}^{N}(\frac{1}{2})^p | \prod_{j=1}^{p-1} m_0(\frac{\xi}{2^j}) || m_0'(\frac{\xi}{2^p}) | \prod_{j=p+1}^{N} | m_0(\frac{\xi}{2^p}) | + \frac{\| m_0' \|_\infty}{2^N} C_0^2 \prod_{j=p+1}^{N} | m_0(\frac{\xi}{2^p}) |$$

and thus defining $\widehat{S_{m_0'}f} = m_0'(\frac{\xi}{2})\hat{f}(\frac{\xi}{2}))$:

$$\sum_{2^{N-1}\pi\leq|\xi+2k\pi|\leq 2^N\pi} | \hat{\varphi}'(\xi+2p\pi) |$$

$$\leq C\left(\sum_{p=1}^{N}(\frac{1}{2})^p \| S^{p-1}S_{m_0'}S^{N-p}\Omega_\epsilon \|_{E^0} + (\frac{1}{2})^N \| S^N\Omega_\epsilon \|_{E^0}\right)$$
$$\leq C'\left(\sup_{q>N/2} \| S^q\Omega_\epsilon \|_{E^0} + (\frac{1}{2})^{N/2} \| \Omega_\epsilon \|_{E^0}\right).$$

We may in the same way mimick Proposition 1 of the preceding chapter and obtain :

PROPOSITION 2. - *Let m_0 be an even trigonometric polynomial*

$$m_0 = \left(\frac{1+\cos\xi}{2}\right)^N P(\cos\xi)$$

with $P(1) = 1$ and $P(-1) \neq 0$ such that :
(i) *it satisfies Cohen's criterion*
(ii) *m_0 is non-negative : $\forall x \in [-1,1]$ $P(x) \geq 0$*
(iii) $\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty < +\infty$

(where $Tf = m_0(\frac{\xi}{2})f(\frac{\xi}{2}) + m_0(\frac{\xi}{2}+\pi)f(\frac{\xi}{2}+\pi)$). Define $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ and $\tilde{T}$ by $\tilde{T}f = P(\cos\frac{\xi}{2})f(\frac{\xi}{2}) + P(-\cos\frac{\xi}{2})f(\frac{\xi}{2}+\pi)$. Then :

(j) *$\tilde{T}$ keeps invariant the space $\tilde{E}_M$ of even trigonometric polynomials of degree $\leq M =$ $\deg P$.*
(jj) *the spectral radius $\tilde{\rho}$ of $\tilde{T}\mid_{\tilde{E}_M}$ is no less than 1.*
(jjj) *let $\sigma = 2N + \frac{\ell n 1/\tilde{\rho}}{\ell n 2}$; then $\mid \xi \mid^s \hat{\varphi} \in L^1$ for all $s \in [0,\sigma)$ and $\mid \xi \mid^\sigma \hat{\varphi} \in L^1$; moreover φ belongs to the Hölder space C^s for all $s \in [0,\sigma)$ and to no C^s with $s > \sigma$.*

Proof. Since $\hat{\varphi}$ is non-negative, we have $\varphi \in C^{s_0}$ (or $C_*^{s_0}$ if $s_0 \in \mathbb{N}^*$) if and only if

$$\sup_{k\in\mathbb{N}} 2^{ks_0} \int_{2^k\pi\leq|\xi|\leq 2^{k+1}\pi} \hat{\varphi}(\xi)d\xi < +\infty ;$$

this gives $(1+\mid \xi \mid^s)\hat{\varphi} \in L^1$ for any $s < s_0$. Conversely if $(1+\mid \xi \mid^s)\hat{\varphi} \in L^1$ then $\varphi \in C^s$. Moreover, the proof we want to mimick gives us that for $s \in (0,N)$:

$$\int (1+\mid \xi \mid)^s \mid \hat{\varphi}(\xi) \mid d\xi \approx \sum_{k\in\mathbb{Z}} 2^{k(s-2N)} \| \tilde{T}^k(1) \|_\infty$$

and we know, since $\| \tilde{T}^k(1) \|_\infty$ is the operator norm of $\tilde{T}$ on $(\tilde{E}_M, \| \cdot \|_\infty)$, that for some positives constants C and M we have

$$\frac{1}{C}\tilde{\rho}^k \leq \| \tilde{T}^k(1) \|_\infty \leq C(1+k)^M \tilde{\rho}^k.$$

This gives the proposition if $\tilde{\rho} > 1$, if $\tilde{\rho} = 1$, we have to see that $\xi^{2N}\hat{\varphi} \notin L^1$: if $\xi^{2N}\hat{\varphi} \in L^1$, then φ is C^{2N}, hence H^{2N} because φ has compact support and then we have that $m_0^{(2N)}(\pi) = 0$; but this is not possible since $P(-1) \neq 0$ and thus $\mid \xi \mid^{2N} \hat{\varphi} \notin L^1$. ∎

We may now describe the interpolating scaling functions :

THEOREM 1. - *Let* $m_0 \in C^\epsilon(\mathbb{R}/2\pi\mathbb{Z})$ *with* $m_0(0) = 1$, $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ *and* T *the operator defined by* $Tf = | m_0(\frac{\xi}{2}) | f(\frac{\xi}{2}) + | m_0(\frac{\xi}{2} + \pi) | f(\frac{\xi}{2} + \pi)$. *Then :*

(A) $\varphi \in E_0$ *and is interpolating* ($\varphi(0) = 1$, $\varphi(k) = 0$ *for* $k \in \mathbb{Z}^*$) *if and only if* m_0 *satisfies Cohen's criterion (5),* $\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty < +\infty$ *and equality :*

$$m_0(\xi) + m_0(\xi + \pi) = 1. \tag{6}$$

(B) *Let* m_0 *be non-negative and satisfy equality (6). Then the following assertions are equivalent :*
(i) $\varphi \in E_0$ *and is interpolating*
(ii) m_0 *satisfies Cohen's criterion*
(iii) $\varphi \in E_0$ *and* $\varphi(0) = 1$.

Proof. If $\varphi \in E_0$ and is interpolating, the Poisson formula gives

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2k\pi) = 1$$

hence φ is a scaling function and m_0 satisfies Cohen's criterion ; we then obtain $\sup_n \| T^n(1) \|_\infty < +\infty$ by Proposition 1 and $m_0(\xi) + m_0(\xi + \pi) = 1$ as a direct consequence of $\sum \hat{\varphi}(\xi + 2k\pi) = 1$ and $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$. Conversely, if m_0 satisfies (5), (6) and $\sup_n \| T^n(1) \|_\infty < +\infty$, we know that γ_n, defined by $\hat{\gamma}_n = \prod_{j=1}^{n} m_0(\frac{\xi}{2^j})\chi_{[-\pi,\pi]}(\frac{\xi}{2^n})$, converges uniformly to φ as $n \to +\infty$; but it is easy to see that $\gamma_n(0) = 0$ and $\gamma_n(k) = 0$ if $k \neq 0$. **(A)** is proved.

If m_0 is non-negative and satisfies (6), then $T(1) = 1$ and thus $\sup_{n\in\mathbb{N}} \| T^n(1) \|_\infty < +\infty$. The equivalence between (i) and (ii) is then obvious. We just have to prove (iii)⇒(i) : we know that $\hat{\varphi}$ is non-negative, that $\hat{\varphi}$ is the pointwise limit of $\hat{\gamma}_N$, hence by Fatou's lemma $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2k\pi) \leq 1$ a.e.; if $\varphi(0) = 1$, then $\frac{1}{2\pi}\int \hat{\varphi} d\xi = 2\pi$, which implies $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi + 2k\pi) = 1$ a.e. and thus φ is interpolating. ■

Counter-example n° 1 : A very simple counter-example to theorem 1 when Cohen's criterion is not satisfied is the following one : take $m_0(\xi) = \frac{1+\cos 3\xi}{2}$. Then m_0 is non-negative and $m_0(\xi) + m_0(\xi + \pi) = 1$. But φ can be explicitly computed : for $| x | \leq 3$, $\varphi(x) = \frac{1}{3}\left(1 - \frac{|x|}{3}\right)$ and $\varphi = 0$ elsewhere. Thus $\varphi(0) \neq 1$.

Example n° 1 : The dyadic interpolation scheme of Deslauriers and Dubuc [DES].

The problem studied by Deslauriers and Dubuc is the following one : knowing a sequence $(x_n)_{n\in\mathbb{N}}$ defined on the integers, to describe an interpolation $(x_d)_{d\in D}$ of the sequence (x_n) into a sequence defined on all dyadic numbers by the following self-similar process :

Step 1 : interpolate (x_n) to the half-integers by :

$$x_{n+1/2} = \sum_{k\in\mathbb{Z}} \alpha_k x_{n-k} \;; \tag{7}$$

this is a convolution, which means that the interpolation process has to be linear and shift-invariant ;we require moreover that the constant sequence 1 is interpolated by 1 :

$$1 = \sum_{k\in\mathbb{Z}} \alpha_k$$

and that the (α_k) are rapidly decreasing (so that we may interpolate data with polynomial growth).

Step 2 : rescale the indexes to apply formula (7) recursively :

$$x_{\frac{n}{2^j}+\frac{1}{2^{j+1}}} = \sum_{k\in\mathbb{Z}} \alpha_k x_{\frac{n-k}{2^j}} \tag{8}$$

using the same sequence (α_k).

We would like to know whether we may further interpolate $(x_d)_{d\in\mathcal{D}}$ to a continuous function $(x(t))_{t\in\mathbb{R}}$ with polynomial growth. If it is possible to do such an interpolation, we must have (writing Riemann sums instead of integrals) that the distributions

$$X_j = \sum_{k\in\mathbb{Z}} \frac{1}{2^j} x_{k/2^j} \delta(t - k/2^j)$$

converge to x in$\mathcal{S}$' ; but

$$\begin{aligned} X_{j+1} &= \frac{1}{2}X_j + \sum_{k\in\mathbb{Z}} \frac{1}{2^{j+1}} x_{\frac{k}{2^j}+\frac{1}{2^{j+1}}} \delta\left(t - \frac{k}{2^j} - \frac{1}{2^{j+1}}\right) \\ &= X_j * \left(\frac{1}{2}\delta + \sum_{k\in\mathbb{Z}} \frac{1}{2}\alpha_k \delta(t - \frac{k}{2^j} - \frac{1}{2^{j+1}})\right), \end{aligned}$$

hence, writing $m_0(\xi) = \frac{1}{2} + \sum_{k\in\mathbb{Z}} \frac{1}{2} e^{-i(2k+1)\xi}\alpha_k$,

$$\hat{X}_j = \left(\sum_{n\in\mathbb{Z}} x_n e^{-in\xi}\right) \prod_{\ell=1}^{j} m_0(\frac{\xi}{2^j}). \tag{9}$$

We have $m_0 \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$, $m_0(0) = 1$ and $m_0(\xi) + m_0(\xi+\pi) = 1$. Thus, provided m_0 is non-negative and satisfies Cohen's criterion, we may conclude from Theorem 1 that $\hat{\varphi} = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$ is the Fourier transform of a regular interpolating scaling function and that the interpolation scheme converges to the function $\sum_{n\in\mathbb{Z}} x_n\varphi(x-n)$. ■

Example n° 2 : Interpolating scaling functions with minimal support.

THEOREM 2. - *Let (E_N) be the following problem : to construct a scaling function φ such that :*

(i) *φ has compact support and for some positive ϵ, $\varphi \in C^\epsilon$*
(ii) *φ is even and real-valued*
(iii) *φ is interpolating : $\varphi(0) = 1$ and $\varphi(k) = 0$ for $k \in \mathbb{Z}^*$*
(iv) *φ reconstructs polynomials up to degree $2N+1$*

$$\forall p \in \mathbf{C}_{2N+1}[X], \quad p = \sum_{k \in \mathbb{Z}} p(k)\varphi(x-k). \tag{10}$$

Then (E_N) has a unique solution ${}_N\varphi$ with support $[-2N-1, 2N+1]$ and all other solutions have greater support. Moreover ${}_N\varphi$ has the following properties :

(j) ${}_N\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} {}_Nm_0(\frac{\xi}{2^j})$ *with* ${}_Nm_0(\xi) = \left(\frac{1+\cos\xi}{2}\right)^{N+1} \sum_{k=0}^{n} \binom{N+k}{k} \left(\frac{1-\cos\xi}{2}\right)^k$
(jj) *${}_N\hat{\varphi}$ is non-negative*
(jjj) *if α_N is defined as $\alpha_N = \sup\{\alpha > 0 / {}_N\varphi \in C^\alpha\}$ then*

$$\alpha_N \sim N\left(2 - \frac{\ell n 3}{\ell n 2}\right) \quad \text{as } N \to +\infty. \tag{11}$$

Proof. We first notice that (10) is equivalent to $m_0^{(k)}(0) = 0$ for $1 \leq k \leq 2N+1$, hence (since $m_0(\xi+\pi) = 1 - m_0(\xi)$) to $m_0^{(k)}(\pi) = 0$ for $0 \leq k \leq 2N+1$ (where m_0 is the scaling filter associated to φ : $m_0(\xi) = \sum_{k\in\mathbb{Z}} \frac{1}{2}\varphi(\frac{k}{2})e^{-ik\xi}$). Thus we may write m_0 as $m_0(\xi) = \left(\frac{1+\cos\xi}{2}\right)^{N+1} P(\cos\xi)$, where P must satisfy :

$$\left(\frac{1+X}{2}\right)^{N+1} P(X) + \left(\frac{1-X}{2}\right)^{N+1} P(-X) = 1. \tag{12}$$

Let A_N and $B_N \in \mathbf{C}_N[N]$ be the unique solutions of the Bezout identity

$$\left(\frac{1+X}{2}\right)^{N+1} A_N(X) + \left(\frac{1-X}{2}\right)^{N+1} B_N(X) = 1 \ ;$$

by uniqueness of the solution, we have $B_N(X) = A_N(-X)$, so that A_N is a solution of (12), and any other solution P of (12) can be written as $P = A_N + X\left(\frac{1-X}{2}\right)^{N+1} Q(X^2)$ (with $Q \in \mathbf{C}[X]$); thus $\deg P \geq N+2$ and $\operatorname{Supp}\varphi \supset [-2N-2, 2N+2]$, unless $P = A_N$. We thus have to show that $P = A_N$ leads to a solution ${}_N\varphi$.

Define $Y = \frac{1-X}{2}$; then equation (12) for A_N reads as :

$$A_N(1-2Y) = \frac{1}{(1-Y)^{N+1}} + Y^{N+1}\frac{A_N(2Y-1)}{(1-Y)^{N+1}}$$

and thus $A_N(1-2Y)$ is the Taylor development of $\frac{1}{(1-Y)^{N+1}}$ at 0 up to order N, hence $A_N(X)=\sum_{k=0}^{N}\binom{N+k}{k}\left(\frac{1-X}{2}\right)^k$. Thus $\deg A_N = N$ and $\mathrm{Supp}_N\varphi = [-2N-1, 2N+1]$. We see that $A_N(X)$ is non-negative on $[-1,1]$, hence ${}_Nm_0(\xi)$ is non-negative ; moreover A_N never vanishes on $[-1,1]$, hence ${}_Nm_0$ satisfies Cohen's criterion ; Theorem 1 states then that ${}_N\varphi$ is a solution of (E_N). Now, to estimate α_N, we will estimate $\beta_N = \sup\{\beta / \mid \xi \mid^{\beta_n} \hat{\varphi} \in L^\infty\}$: we have $\beta_N - 1 \leq \alpha_N \leq \beta_N$ (since $\mid \xi \mid^\beta \hat{\varphi} \in L^\infty \Rightarrow \mid \xi \mid^\alpha \hat{\varphi} \in L^1$ for $\alpha < \beta - 1 \Rightarrow \varphi \in C^\alpha$ for $\alpha < \beta - 1$, while $\varphi \in C^\alpha$ and $\mathrm{Supp}\,\varphi$ is compact implies that $\mid \xi \mid^\alpha \hat{\varphi} \in L^\infty$). We will show that :

$$2N+2-\frac{\ell n[9\binom{2N}{N}(\frac{3}{4})^N]}{\ell n 2} \leq \beta_N \leq 2N+2-\frac{\ell n[\binom{2N}{N}(\frac{3}{4})^N]}{\ell n 2} \tag{13}$$

which gives (11) by Stirling's formula : $\binom{2N}{N} = \frac{(2N)!}{(N!)^2} \sim \frac{4^N}{\sqrt{\pi N}}$. To prove (13), let's observe that for $\ell \in \mathbb{N}$, $\cos(2^\ell \frac{2\pi}{3}) = -1/2$, so that

$${}_N\hat{\varphi}(2^\ell\frac{2\pi}{3}) = (\frac{1}{4^\ell})^{N+1} A_N(-\frac{1}{2})^\ell {}_N\hat{\varphi}(\frac{2\pi}{3})$$

and thus

$$\beta_N \leq 2N+2-\frac{\ell n A_N(-1/2)}{\ell n 2} \leq 2N+2-\frac{\ell n[\binom{2N}{N}(\frac{3}{4})^N]}{\ell n 2}.$$

Conversely let $q_N = \max(A_N(-1/2); \sup_{-1\leq x\leq -1/2}\sqrt{A_N(x)A_N(2x^2-1)})$; then we have $q_N \leq A_N(-1)$ and $\prod_{j=1}^{\ell} A_N(\cos\frac{\xi}{2^j}) \leq q_N^{\ell-1}A_N(-1)$ (by induction on ℓ, since we have $A_N(\cos\frac{\xi}{2^\ell}) \leq q_N$ if $\cos\frac{\xi}{2^\ell} \geq -\frac{1}{2}$, whereas $A_N(\cos\frac{\xi}{2^{\ell-1}})A_N(\cos\frac{\xi}{2^\ell}) \leq q_N^2$ if $\cos(\frac{\xi}{2^\ell}) \leq -\frac{1}{2}$); moreover $\prod_{j=1}^{\infty}\frac{1+\cos\frac{\xi}{2^j}}{2} = \frac{4\sin^2\frac{\xi}{2}}{\xi^2}$, hence we obtain for $2^\ell\pi \leq \mid \xi \mid \leq 2^\ell 2\pi$,

$$|_N\,\hat{\varphi}(\xi)\,| \leq (\frac{4}{\xi^2})^{N+1} q_N^{\ell-1} A_N(-1) \sup_{|\eta|\leq 2\pi}\prod_{j=1}^{\infty} A_N(\cos\frac{\eta}{2^j})$$

which gives $\beta_N \geq 2N+2-\frac{\ell n q_N}{\ell n 2}$. Thus it is enough to prove that $q_N \leq 9\binom{2N}{N}(\frac{3}{4})^N$. This is easy to do : for $-1 \leq x \leq -\frac{1}{8}$, we have

$$\begin{aligned} A_N(x) = \sum_{k=0}^{N}\binom{N+k}{k}\left(\frac{1-x}{2}\right)^k &\leq \sum_{k=0}^{N}(\frac{1}{2})^{N-k}\binom{2N}{N}\left(\frac{1-x}{2}\right)^k \\ &\leq \binom{2N}{N}\left(\frac{1-x}{2}\right)^N \frac{1-x}{-x} \\ &\leq 9\binom{2N}{N}\left(\frac{1-x}{2}\right)^N \end{aligned}$$

and thus :

$$A_N(-1) \le 9\binom{2N}{N},\ A_N(-\tfrac{1}{2}) \le 9\binom{2N}{N}(\tfrac{3}{4})^N \text{ and } A_N(-\tfrac{1}{8}) \le 9\binom{2N}{N}(\tfrac{9}{16})^N;$$

- for $-1 \le x \le -\frac{\sqrt{7}}{4}$ we have $2x^2-1 \ge -\frac{1}{8}$ and thus

$$A_N(x)A_N(2x^2-1) \le A_N(-1)A_N(-\frac{1}{8}) \le 81(\binom{2N}{N})^2(\frac{3}{4})^{2N}$$

- for $-\frac{\sqrt{7}}{4} \le x \le -\frac{1}{2}$ we have $2x^2-1 \le -\frac{1}{8}$ and thus :

$$A_N(x)A_N(2x^2-1) \le 81(\binom{2N}{N})^2\left(\frac{1-x}{2}\right)^N(1-x^2)^N \le 81(\binom{2N}{N})^2(\frac{3}{4})^{2N}$$

since $(1-x)(1-x^2)$ increases on $[-1,-\frac{1}{3}]$. Thus (13) is proved. ■

REMARK. - The polynomial A_N has been introduced by I. Daubechies for her construction of compactly supported orthonormal wavelets (see below). Estimate (11) has been given independently by Volkner [VOL] and by A. Cohen and J. P. Conze [COHC]. Cohen and Conze proved more precisely that $q_N = A_N(-\frac{1}{2})$ and hence that $\beta_N = 2N+2-\frac{\ell n A_n(-1/2)}{\ell n 2}$. ■

Of course, estimate (11) is an asymptotic one, and for the small values of N one uses the spectral analysis of the transition operator T to get the regularity exponent of ${}_N\varphi$ (see Proposition 2).

N	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N \to +\infty$
α_n	1	2	2.83	3.55	4.19	4.77	5.31	$\alpha_N \sim N(2-\frac{\ell n3}{\ell n2})$

Table n° 1 : Value of $\alpha_n = \sup\{\alpha / {}_N\varphi \in C^\alpha\}$

Example n° 3 : Littlewood-Paley multi-resolution analysis and interpolation.

Following Y. Meyer [MEY2], we will call a multi-resolution analysis a *Littlewood-Paley multi-resolution analysis* if it is generated from a scaling function φ such that $\hat{\varphi}$ is C^∞, $\hat{\varphi}$ has compact support, $\text{Supp}\,\hat{\varphi} \subset [-\frac{2\pi}{3},\frac{2\pi}{3}]$ and $\hat{\varphi}$ doesn't vanish on $[-\pi,\pi]$. This is equivalent to require that the associated scaling filter $m_0 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ doesn't vanish on $[-\frac{\pi}{2},\frac{\pi}{2}]$ and vanishes identically on $[\frac{2\pi}{3},\frac{4\pi}{3}]$. Such a filter always provides a scaling function (provided $m_0(0)=1$ of course). If we require moreover m_0 to satisfy $m_0(\xi)+m_0(\xi+\pi)=1$ identically, then we obtain an interpolating scaling function satisfying $P(x)=\sum_{k\in\mathbb{Z}} P(k)\varphi(x-k)$ for all polynomial $P \in \mathbb{C}[X]$.

REMARK. - *Existence of interpolating functions in a multi-resolution analysis.*

If (V_j) is a multi-resolution analysis with a scaling function $\varphi \in E^1$, then V_0 contains an interpolating function μ (with $\mu(0)=1, \mu(k)=0$ for $k \in \mathbb{Z}^*$) if and only if $\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+2k\pi)$ doesn't vanish on $[-\pi,\pi]$; μ is then given by $\hat{\mu}(\xi)=\frac{\hat{\varphi}(\xi)}{\sum_{k\in\mathbb{Z}}\hat{\varphi}(\xi+2k\pi)}$. This condition is fulfilled whenever $\hat{\varphi}$ is non-negative, since then

$$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+2k\pi) \geq \frac{1}{\| \hat{\varphi} \|_\infty} \sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2 . \blacksquare$$

2. Orthogonal multi-resolution analyses.

The first wavelet bases which were constructed in the years 85-87 were orthonormal bases [LEME], [BAT1], [LEM1], [DAU1]. Any multi-resolution analysis is related to an orthonormal wavelet basis through the following scheme :

- choose a scaling function φ in V_0
- orthonormalize the basis $(\varphi(x-k))$ into an orthonormal basis $(\varphi_\perp(x-k))$ of V_0 by the formula

$$\hat{\varphi}_\perp = \frac{\hat{\varphi}(\xi)}{\sqrt{\sum_{k\in\mathbb{Z}} | \hat{\varphi}(\xi+2k\pi) |^2}} \tag{14}$$

- define W_0 as the orthogonal complement of V_0 in V_1
- obtain an orthonormal basis $(\psi(x-k))_{k\in\mathbb{Z}}$ of W_0 by choosing ψ as :

$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}} \bar{m}_{0,\perp}(\frac{\xi}{2}+\pi)\hat{\varphi}_\perp(\frac{\xi}{2})$$

  where $m_{0,\perp}$ is the scaling filter associated to $\varphi_\perp$.
- obtain finally an orthonormal basis of $L^2(\mathbb{R})$ as $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k)_{j\in\mathbb{Z},k\in\mathbb{Z}}$ (or $(\varphi_\perp(x-k))_{k\in\mathbb{Z}} \cup (\psi_{j,k})_{j\geq 0,k\in\mathbb{Z}})$.

We have then got an *orthogonal multi-resolution analysis* of $L^2(\mathbb{R})$; every function f of L^2 is approximated by its orthogonal projection $P_j f$ on V_j, given by :

$$P_j f = \sum_{k\in\mathbb{Z}} 2^j < f \mid \varphi_\perp(2^j x - k) > \varphi_\perp(2^j x - k) \; ; \tag{15}$$

the projectors P_j commute : $P_j \circ P_\ell = P_\ell \circ P_j = P_{\inf(j,\ell)}$ and the operator $Q_j = P_{j+1} - P_j$ is the projection operator on $W_j = V_j^\perp \cap V_{j+1}$ given by :

$$Q_j f = \sum_{k\in\mathbb{Z}} < f \mid \psi_{j,k} > \psi_{j,k}. \tag{16}$$

REMARKS. - (i) The fact that $(\psi(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0 is easy to check : let U be the operator $f \in V_1 \rightarrow Uf \in L^2(0,2\pi)$ defined by $\hat{f}(\xi) = Uf(\frac{\xi}{2})\hat{\varphi}_\perp(\frac{\xi}{2})$; then we have $\| f \|_2 = \| Uf \|_2 \sqrt{2}$, and thus

$$< \varphi_\perp \mid \psi(x-k) >= 2\int_0^{2\pi} m_0(\xi)e^{i\xi}m_0(\xi+\pi)e^{2ik\xi}d\xi = 0$$

(by antisymmetry around π) ; moreover every $f \in V_1$ can be decomposed as

$$\hat{f} = \left(Uf(\frac{\xi}{2})\bar{m}_0(\frac{\xi}{2}) + Uf(\frac{\xi}{2}+\pi)\bar{m}_0(\frac{\xi}{2}+\pi) \right) \hat{\varphi}_\perp +$$

$$\left(e^{i\frac{\xi}{2}} Uf(\frac{\xi}{2}) m_0(\frac{\xi}{2}+\pi) + e^{i(\frac{\xi}{2}+\pi)} Uf(\frac{\xi}{2}+\pi) m_0(\frac{\xi}{2}) \right) \hat{\psi}.$$

(ii) As we have already seen it in chapter 3, the orthonormalization formula (14) preserves size properties of φ : if φ belongs to $L^2(|\, x\,|^{1+\epsilon}\, dx)$ for some $\epsilon > 0$ so does $\varphi_\perp$, and the same holds for rapid or exponential decay (changing the rate of the decay in the last case, however). But this formula doesn't preserve compactness of support, and the construction of a compactly supported orthonormal scaling function needs a direct approach. ∎

THEOREM 3. - *Let $m_0 \in H^{1/2+\epsilon}(\mathbb{R}/2\pi\mathbb{Z})$ for some positive ϵ, with $m_0(0) = 1$, and assume that m_0 satisfies Cohen's criterion (5). Then m_0 is a scaling filter associated to an orthonormal scaling function φ (i.e. the scaling function φ generates an orthonormal family $(\varphi(x-k))_{k\in\mathbb{Z}}$) if and only if :*

(17) $$\forall \xi \in [0, 2\pi], \qquad |\, m_0(\xi)\,|^2 + |\, m_0(\xi+\pi)\,|^2 = 1.$$

Proof. If the family $(\varphi(x-k))_{k\in\mathbb{Z}}$ is orthonormal, then $\sum_{k\in\mathbb{Z}} |\, \hat{\varphi}(\xi + 2k\pi)\,|^2 = 1$ a.e., this gives (17), since $\hat{\varphi}(\xi) = m_0(\frac{\xi}{2})\hat{\varphi}(\frac{\xi}{2})$. Conversely, if (17) is satisfied, then the filter $M_0(\xi) = |\, m_0(\xi)\,|^2$ is C^ϵ, non-negative, satisfies Cohen's criterion and $M_0(\xi) + M_0(\xi + \pi) = 1$; hence we know by theorem 1 that the associated scaling function F, defined by $\hat{\Phi}(\xi) = \prod_{j=0}^{+\infty} M_0(\frac{\xi}{2^j})$, is interpolating; we now conclude that $(\varphi(x-k))$ is orthonormal since $\Phi(k) = < \varphi \mid \varphi(x-k) >$. ∎

Theorem 3 gives us another approach to construct orthonormal bases : we begin with an interpolating scaling function Φ and try to write it as $\Phi(x) = \int \varphi(y)\bar{\varphi}(y-x)dy$. Of course, the filter M_0 associated to Φ has to be non-negative ; the problem is then to find m_0 such that $|\, m_0(\xi)\,|^2 = M_0(\xi)$. If M_0 is an even non-negative trigonometric polynomial, we would like to choose m_0 polynomial as well ; this can be done, as expressed by the well-known Riesz theorem (of common use in signal analysis).

PROPOSITION 3 (Riesz theorem). - *Let $P \in \mathbb{R}[X]$ be a polynomial such that $P(x) \geq 0$ for all $x \in [-1, 1]$. Then there exists $Q \in \mathbb{R}[X]$ such that :*

(18) $$P(\cos\xi) = |\, Q(e^{-i\xi})\,|^2 \quad \textit{for all} \quad \xi \in \mathbb{R}.$$

Proof. We will show that if $\deg P > 0$ we may find two polynomials Q_1, $P_1 \in \mathbb{R}[X]$ with $P(\cos\xi) = |\, Q_1(e^{-i\xi})\,|^2\, P_1(\cos\xi)$ and $\deg P_1 < \deg P$; iterating the construction, we will get a solution Q to (18). Let $N = \deg P$ and write $P(\cos\xi) = e^{iN\xi}\tilde{P}(e^{-i\xi})$; then if $z_0 \in \mathbf{C}$ is a root of $\tilde{P}$, we have $\tilde{P}(\overline{z_0}) = 0$ as well (since $\tilde{P}$ has real coefficients) and $\tilde{P}(\frac{1}{z_0}) = \tilde{P}(\frac{1}{\overline{z_0}}) = 0$ (since $z_0 \neq 0$ and $z^{2N}\tilde{P}(\frac{1}{z}) = \tilde{P}(z)$). Moreover, if $| z_0 |= 1$, then z_0 (and $\overline{z_0}$) is a zero of $\tilde{P}$ with an even multiplicity, since, writing $z_0 = e^{-i\xi_0}$, we know that $P(\cos\xi)$ doesn't change its sign when ξ goes through the value ξ_0. Thus, if $z_0 \notin \mathbb{R}$, we may write $\tilde{P} = (z - z_0)(z - \overline{z_0})(1 - zz_0)(1 - \overline{zz_0})\tilde{R}(z)$ and thus

$$P(\cos\xi) =| Q_1(e^{-i\xi}) |^2 P_1(\cos\xi)$$

where

$$P_1(\cos\xi) = e^{i(N-2)\xi}\tilde{R}(e^{-i\xi})$$

and $Q_1 = (z-z_0)(z-\overline{z_0})$ or $(1-zz_0)(1-z\overline{z_0})$; if $z_0 \in \mathbb{R}$, then we may write $\tilde{P}(z) = (z-z_0)(1-zz_0)\tilde{R}(z)$ and $P(\cos\xi) =| Q_1(e^{-i\xi}) |^2 P_1(\cos\xi)$ with $P_1(\cos\xi) = e^{i(N-1)\xi}\tilde{R}(e^{-i\xi})$ and $Q_1(z) = z - z_0$ or $1 - zz_0$. Thus Riesz theorem is proved.

REMARK. - The proof shows a way to construct Q : first compute the roots of $\tilde{P}$ and write $\tilde{P}$ as :

$$\tilde{P}(z) = C_0 \prod_{i=1}^{N_1}(z - z_i)(z - \overline{z_i})(1 - zz_i)(1 - z\overline{z_i}) \prod_{j=1}^{N_2}(z - \alpha_j)(1 - z\alpha_j)$$

where $z_i \notin \mathbb{R}$ and $\alpha_j \in \mathbb{R}$. Then a solution Q is given by

$$Q(z) = \sqrt{C_0}\prod_{i=1}^{N_1}(z - z_i)(z - \overline{z_i}) \prod_{j=1}^{N_2}(z - \alpha_j).$$

If M_1 is the number of z_i such that $| z_i |\neq 1$ and M_2 the number of α_j such that $\alpha_j \notin \{1, -1\}$, we have $2^{M_1+M_2}$ choices for Q (choosing z_i or $\frac{1}{z_i}$ and α_j or $\frac{1}{\alpha_j}$). Thus Q is not unique. ∎

Example n° 4 : The Haar basis (1909).

In 1909, A. Haar [HAA] introduced an orthonormal basis for $L^2(0,1)(e_n)_{n\in\mathbf{N}}$ such that the partial sums $\sum_1^N < f \mid e_n > e_n$ of a continuous function f converge uniformly to f as N goes to $+\infty$ (in opposition to the Fourier system $(e^{2\pi inx})_{n\in\mathbb{Z}}$). The basis is simply described by $e_0 = \varphi(x) = \chi_{[0,1]}$ and for $n = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$, $e_n = \psi(2^j x - k)$ where $\psi(x) = \chi_{[0,1/2]} - \chi_{[1/2,1]}$; this basis can be extended obviously to a wavelet basis $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with orthonormal scaling function $\varphi = \chi_{[0,1]}$ and orthonormal wavelet $\psi(x) = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]}$; the associated scaling filter is then $m_0(\xi) = \frac{1+e^{-i\xi}}{2}$. The scaling function φ is not continuous; its Sobolev regularity exponent $\sigma_0 = \sup\{s/\varphi \in H^s\}$ is $\sigma_0 = \frac{1}{2}$.

Example n° 5 : Spline bases : Franklin (1927), Strömberg (1981), Battle and Lemarié (1986).

In spite of its simplicity, the Haar system was not satisfying, since it approximated continuous functions by discontinuous ones. In order to overpass this difficulty, Faber [FAB] and Schauder [SCA] proposed to integrate the functions of the Haar system and to obtain a system $(\epsilon_n)_{n\geq -1}$, with $\epsilon_{-1} = 1$, $\epsilon_0 = x$ and for $n = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$, $\epsilon_n = \tilde{\Delta}(2^j x - k)$, where $\tilde{\Delta}(x) = (1 - \mid 2x - 1 \mid)^+$. We then obtain the Schauder basis which allows one to approximate uniformly on $[0,1]$ a continuous function f by $\sum_{n=-1}^{N} C_n(f)\epsilon_n$, where $C_{-1}(f) = f(0)$, $C_0(f) = f(1) - f(0)$ and $C_{2^j+k}(f) = f\left(\frac{2k+1}{2^{j+1}}\right) - \frac{1}{2} f\left(\frac{k}{2^j}\right) - \frac{1}{2} f\left(\frac{k+1}{2^j}\right)$. In particular, $\sum_{n=-1}^{2^j-1} C_n(f)\epsilon_n$ is the function f_j which is continuous on $[0,1]$, linear on each interval $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ $(0 \leq k < 2^j)$ and has the same value as f on each node $\frac{k}{2^j}$. This basis, however, is not fitted to the analysis of discontinuous functions.

In 1927, Franklin [FRA] orthonormalized the Faber-Schauder basis and got an Hilbertian basis of $L^2([0,1])$ composed of continuous functions and such that every continuous functions was uniformly approximated on $[0,1]$ by its partial sums. But the Gram-Schmidt orthonormalization destroys the simplicity of the basis (ϵ_n) and the Franklin system cannot be expressed by simple formulas. The Franklin system $(f_n)_{n\geq -1}$ can be expressed in the simplest way by : $f_{-1} = 1$, $f_0 = \sqrt{3}(2x - 1)$ and f_{2^j+k} is defined (up to a multiplicative constant $C_{j,k}$) by the following properties : it is continuous, piece-wise linear (linear on each interval $\left[\frac{p}{2^{j+1}}, \frac{p+1}{2^{j+1}}\right]$ $0 \leq p \leq 2k+1$ and each interval $\left[\frac{q}{2^j}, \frac{q+1}{2^j}\right]$ $k+1 \leq q \leq 2^j - 1$) and is orthogonal to each continuous function which is linear on each interval $\left[\frac{p}{2^{j+1}}, \frac{p+1}{2^{j+1}}\right]$ $0 \leq p \leq 2k-1$ and each interval $\left[\frac{q}{2^j}, \frac{q+1}{2^j}\right]$ $k \leq q \leq 2^j - 1$. This property led J. O. Strömberg [STR] to define a function ψ such that it is square-integrable on $\mathbb{R}$ with $\| \psi \|_2 = 1$, continuous, piece-wise linear (linear on each interval $\left[\frac{p}{2}, \frac{p+1}{2}\right]$ $-\infty < p \leq 1$ and on each interval $[q, q+1]$ $1 \leq q < +\infty$) and orthogonal to each continuous square-integrable function which is linear on each interval $\left[\frac{p}{2}, \frac{p+1}{2}\right]$ $-\infty < p \leq -1$ and on each interval $[q, q+1]$ $0 \leq q < +\infty$. Then by definition ψ is orthogonal to $\psi(2^j x - k)$ as soon as j is negative or as soon as $j = 0$ and k is negative ; thus the system $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is orthonormal; moreover it is complete and we thus have a wavelet basis for $L^2(\mathbb{R})$. The wavelet ψ has Sobolev regularity exponent $\frac{3}{2}$ and Hölder regularity exponent 1, and it has exponential decay.

Strömberg's basis was created in the context of functional analysis (the study of Hardy spaces $H^p(\mathbb{R})$ for $p < 1$) and was not known in the "wavelet community" until 1988. In 1986, Battle [BAT1] and Lemarié [LEM1] introduced independently another spline orthonormal wavelet. Spline bases have played a key rôle in the development of wavelet theory and have turned to be a focus of interest for many searchers. We devote below a special section to spline multi-resolution analyses.

Example n° 6 : Littlewood-Paley analysis and wavelets : the Meyer-Lemarié basis (1985).

The Littlewood-Paley multi-resoluton analysis described in example n° 3 and the orthonormalization process described on the beginning of this section gives us directly an orthonormal scaling function φ such that $\hat{\varphi}$ is C^∞, compactly supported and takes identically the value 1 in a neighborhood of 0 ; the associated orthonormal wavelet ψ is such that $\hat{\psi}$ is C^∞, compactly supported and $0 \notin \operatorname{Supp} \hat{\psi}$. This analysis has already been introduced in example b) of chapter 3.

The construction of the first Littlewood-Paley wavelet ψ by Y. Meyer (and of the associated scaling function φ by P. G. Lemarié) [LEME] was not as simple as it might seem now : in 1985, the notion of multi-resolution analysis had not yet been introduced and Meyer had to construct directly the function ψ (φ was constructed *after* ψ). The notion of multi-resolution analysis was introduced by S. Mallat and Y. Meyer in november 1986, after the discovery by G. Battle and P. G. Lemarié of the spline orthonormal wavelets ψ and of their associated scaling functions φ : the existence of various scaling functions associated to wavelet bases lead Mallat and Meyer to think of a general functional scheme in order to describe the fortuitous calculations of Meyer-Lemarié or Battle-Lemarié.

The most attractive feature of the Littlewood-Paley wavelet analysis is the fact that every tempered distribution can be expressed with help of this basis :

PROPOSITION 4. - *Let $\varphi, \psi \in \mathcal{S}$ be such that :*

(i) $(\varphi(x-k))_{k\in\mathbb{Z}}$ *is orthonormal*
(ii) $\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} < \varphi(\frac{y}{2}) \mid \varphi(y-k) > \varphi(x-k)$
(iii) $\psi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} (-1)^k < \varphi(y+k+1) \mid \varphi(\frac{y}{2}) > \varphi(x-k)$
(iv) $\operatorname{Supp} \hat{\varphi} \subset \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$.

Then $(\varphi(x-k))_{k\in\mathbb{Z}} \cup \left(2^{j/2}\psi(2^j x - k) = \psi_{j,k}\right)_{j\geq 0, k\in\mathbb{Z}}$ *is an Hilbertian basis of* $L^2(\mathbb{R})$. *Moreover for every* $f \in \mathcal{S}$ *we have*
(19)

$$f = \lim_{N\to+\infty} \left(\sum_k < f \mid \varphi(x-k) > \varphi(x-k) + \sum_{j=0}^{N} \sum_k 2^j < f \mid \psi(2^j x - k) > \psi(2^j x - k) \right)$$

where the limit (19) is taken in $\mathcal{S}$; *and similarly if* $f \in \mathcal{S}'$ *is an arbitrary tempered distribution, the convergence (19) is true in* $\mathcal{S}'$.

Proof. We already know that we have an Hilbertian basis of $L^2(\mathbb{R})$, and that the right-hand side of equality (19) can be written as well

$$\lim_{N\to+\infty} \sum_{k\in\mathbb{Z}} 2^j < f \mid \varphi(2^j x - k) > \varphi(2^j x - k),$$

which gives the Proof. ∎

The Littlewood-Paley basis is an unconditional basis for most spaces useful in harmonic analysis (Besov spaces, Lebesgue or Hardy spaces,...), as we shall see later.

Example n° 7 : The Daubechies orthonormal wavelets.

In 1987, I. Daubechies constructed orthonormal scaling functions φ_N with compact support ($\operatorname{Supp}\varphi_N = [0, 2N-1]$) and arbitrarily great regularity (φ_N is of class C^{α_N}, where $\alpha_N \sim N\left(1 - \frac{\log 3}{2\log 2}\right)$). In order to get Daubechies' functions, we just have to use theorem 2 (on interpolating scaling functions) and proposition 3 (Riesz theorem) :

THEOREM 4. - *Let $Q_N \in \mathbb{R}[X]$ be such that $Q_N(0) \neq 0$, $Q_N(1) = 1$ and $|Q_N(e^{-i\xi})|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \left(\frac{1-\cos\xi}{2}\right)^k$, and define $m_N \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ by*

$$m_N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q_N(e^{-i\xi}).$$

Then the distribution φ_N defined $\hat{\varphi}_N(\xi) = \prod_{j=1}^{\infty} m_N\left(\frac{\xi}{2^j}\right)$ satisfies :

(i) *$\varphi_N \in L^2$, $(\varphi_N(x-k))_{k\in\mathbb{Z}}$ is orthonormal*

(ii) $\operatorname{Supp}\varphi_N = [0, 2N-1]$

(iii) *φ_N is continuous for $N \geq 2$ and if $\alpha_N = \sup\{\alpha/\varphi_N \in C^\alpha\}$, we have $\alpha_N \sim N\left(1 - \frac{\log 3}{2\log 2}\right)$ as N goes to $+\infty$.*

(iv) *Every polynomial $P \in \mathbb{C}[X]$ with $\deg P \leq N-1$ satisfies*

$$P = \sum_{k\in\mathbb{N}} < P \mid \varphi_N(x-k) > \varphi_N(x-k).$$

Point (iv) says that φ_N is of approximation order $N-1$: if P_j is the orthogonal projection operator onto $\operatorname{Span}(2^j\varphi_N(2^jx-k))$, then we have for every $g \in H^{N-1}$, $\| g - P_j(g) \|_2 = o(2^{-j(N-1)})$. The link between approximation order $N-1$ and decomposition of polynomials on $\varphi_N(x-k)$ up to order $N-1$ is expressed by the well-known Strang-Fix condition [FIX]. We thus are bound with finite orders of approximation when dealing with compactly supported wavelets : approximation order $N-1$ is equivalent to the vanishing of m_N and its derivatives (up to order $N-1$) at $\xi = \pi$, and a polynomial has only finite-order zeros. Example n° 10 will show a compactly supported basis with infinite approximation order.

Daubechies's bases are now widely used, because the compactness of the support of $\varphi_N$allows the use of finite filters in the fast wavelet algorithm (see chapter 8). The most commonly used φ_N are φ_3 (which is C^1) and φ_7 (which is C^2) corresponding to the filters $m_N(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2N-1} a_k e^{-ik\xi}$ with :

$N = 3$

$a_0 =$	0.3326705529500825	$a_3 =$	−0.1350110200102546
$a_1 =$	0.8068915093110924	$a_4 =$	−0.0854412738820267
$a_2 =$	0.4598775021184914	$a_5 =$	0.0352262918857095

$N = 7$

$a_0 =$	0.0778520540850037	$a_7 =$	0.0806126091510774
$a_1 =$	0.3965393194818912	$a_8 =$	$-\ 0.0380299369350104$
$a_2 =$	0.7291320908461957	$a_9 =$	$-\ 0.0165745416306655$
$a_3 =$	0.4697822874051889	$a_{10} =$	0.0125509985560986
$a_4 =$	$-\ 0.1439060039285212$	$a_{11} =$	0.0004295779729214
$a_5 =$	$-\ 0.2240361849938412$	$a_{12} =$	$-\ 0.0018016407040473$
$a_6 =$	0.0713092192668272	$a_{13} =$	0.0003537137999745

(Those values are borrowed from [DAU3], where the reader may find the values of the coefficients of m_N for $N = 2$ to 10).

The choice of the normalization $\sum a_k = \sqrt{2}$ corresponds to the numerical filters in the fast wavelet algorithm : we may write the three following relationships between φ_N and $\varphi_N(\frac{x}{2})$:

- $\hat{\varphi}_N(2\xi) = m_N(\xi)\hat{\varphi}_N(\xi) = \frac{1}{\sqrt{2}}\left(\sum_0^{2N-1} a_k e^{-ik\xi}\right)\hat{\varphi}_N(\xi)$ which corresponds to the calculus of $\hat{\varphi}_N = \prod_{j=1}^{\infty} m_N(\frac{\xi}{2^j})$ (the normalization of the coefficients is then

$$\sum\left(\frac{a_k}{\sqrt{2}}\right) = 1)\ ;$$

- $\varphi_N(\frac{x}{2}) = \sqrt{2}\sum_0^{2N-1} a_k\varphi_N(x-k)$, the two-scale difference equation which corresponds to the calculus of $\varphi_N(x)$ (see chapter 8) through the cascade algorithm (the normalization is then $\sum(\sqrt{2}a_k) = 2$)

- $\frac{1}{\sqrt{2}}\varphi_N(\frac{x}{2}) = \sum_0^{2N-1} a_k\varphi_N(x-k)$, a formula which allows the computation of the coefficients $s_{j,k} = < f \mid 2^{j/2}\varphi_N(2^j x - k) >$ of a function

$$f = \sum_{k\in\mathbb{Z}} s_{j-1,k}2^{(j-1)/2}\varphi_N(2^{j-1}x - k)$$

in V_{j-1} as $s_{j,k} = \sum_q a_{k-2q}s_{j-1,q}$; the good normalization is then $\sum a_k = \sqrt{2}$.

(The beginners in wavelet theory are often puzzled by the lack of uniformity in the literature about the normalization of the coefficients a_k. The point to be understood is that each normalization has its own interest which depends strongly on the point of view of the user. In practice, however, the three points of view are useful and cannot be easily dissociated, as we shall see in chapter 8).

Figure 1: Daubechies functions for N = 3

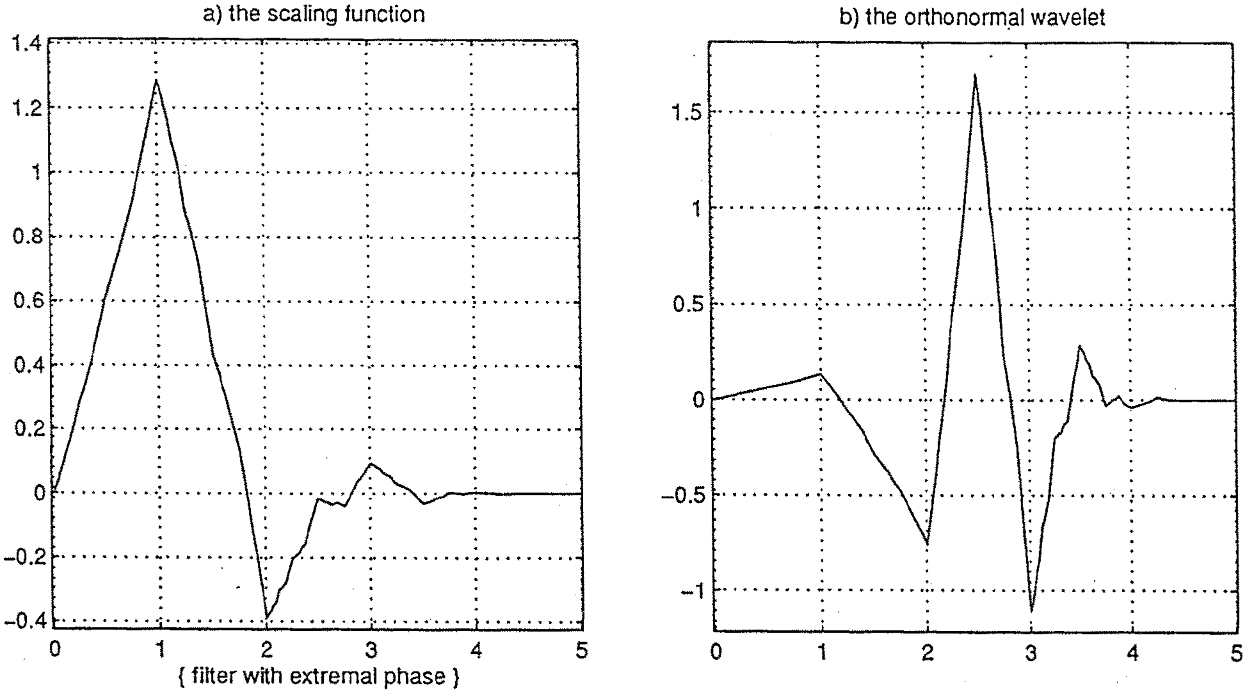
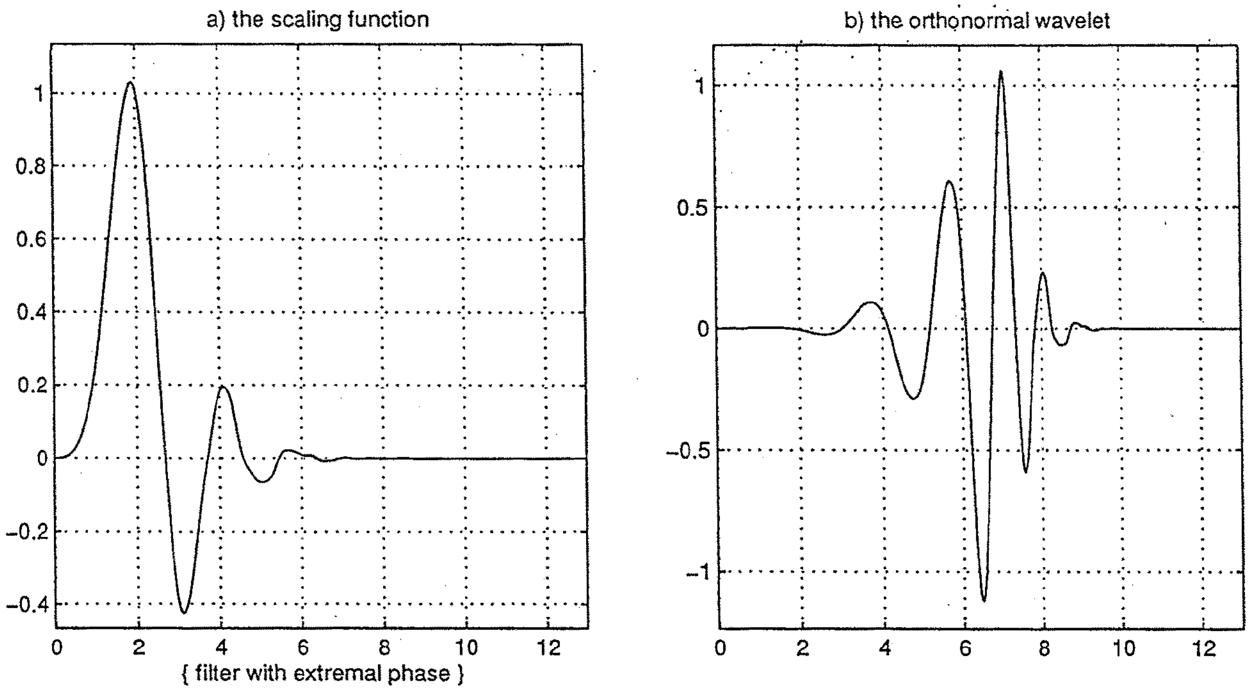


Figure 2: Daubechies functions for N = 7



Of course, the choice of Q_N in theorem 4 is not unique, since the description of Q_N gives only $Q_N(1) = 1$ and $| Q_N(e^{-i\xi}) |^2$. We already noticed that these are (exactly) $2^{[N/2]}$ choices for Q_N. The coefficients we gave for m_3 and m_7 corresponds to choices of Q_N with extremal phase [it was the choice of Daubechies [DAU1] in 1987]. Other choices are discussed in [DAU4] and [RIO] : we may improve the phase of φ_N (going closer to a linear phase) or its Hölder regularity (all the choices of Q_N give the same Sobolev regularity $\sigma_N = \sup\{s/\varphi_N \in H^s\}$, so that the Hölder regularity α_N is fixed up to 1/2 : $\alpha_N = \sup\{\alpha/\varphi_N \in C^\alpha\} \in [\sigma_N - 1/2, \sigma_N])$. Of course, we ought to be able to compute the Hölder regularity of φ_N (we gave exact formulae only for σ_N) : the interested reader will find such results in the works of Daubechies [DAUL] and Rioul [RIO].

N	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N \to +\infty$
σ_N	0.5	1	1.415	1.775	2.096	2.388	2.658	$\sigma_N \sim N(1 - \frac{\ell n\, 3}{2\, \ell n\, 2})$
α_N (extremal − phased m_N)	0	0.5500	1.0878	1.6179				$\alpha_N \sim \sigma_N$

Table n° 2 : Values of σ_N and α_N for the first values of N.

Example n° 8 : The Coiflets.

If φ is a compactly-supported orthonormal scaling function (with scaling filter m_0) and if m_0 has a zero of order N at π ($m_0(\pi) = \frac{d}{d\xi} m_0(\pi) = \cdots = \frac{d^{N-1}}{d\xi^{N-1}} m_0(\pi) = 0$), then each polynomial $P \in \mathbb{C}_{N-1}[X]$ of degree $\leq N-1$ can be written as

$$P(x) = \sum_{k \in \mathbb{Z}} < P \mid \varphi(x-k) > \varphi(x-k).$$

R. Coifman asked to I. Daubechies to construct a minimally supported φ so that $< P \mid \varphi(x-k) >$ was obtained by the sampling $P(k)$:

$$P(x) = \sum_{k \in \mathbb{Z}} P(k) \varphi(x-k).$$

We have then $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N A_N(e^{-i\xi})$ where $A_N(e^{-i\xi}) = \sum_{k=K_0}^{K_1} a_k e^{-ik\xi}$ should satisfy :

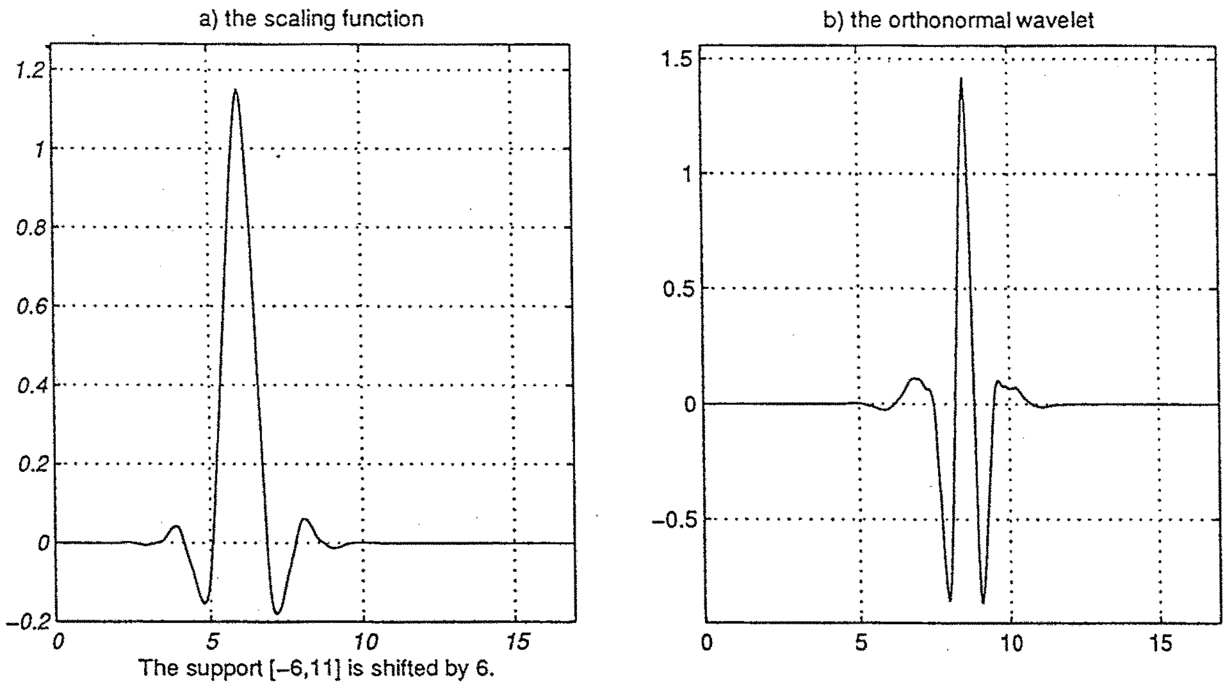
$$\left(\frac{1+\cos\xi}{2}\right)^N \mid A_N(e^{-i\xi}) \mid^2 + \left(\frac{1+\cos\xi}{2}\right)^N \mid A_N(e^{-i\xi}) \mid^2 = 1$$

and

$$\left(\frac{d}{d\xi}\right)^k A_N(e^{-i\xi}) \mid_{\xi=0} = 0 \quad \text{for} \quad 0 < k \leq N-1.$$

We thus added $N-1$ requirements to the previous ones (bading to orthonormality) Daubechies pointed reasons why a solution m_0 should exist with length $3N$ (instead of $2N$ for the minimally-supported scaling functions of theorem 4) and gave solutions for the first values of N in [DAU4].

Figure 3: The coiflet with five vanishing moments



Example n° 9 : Rational filters.

In 1989, G. Malgouyres [LEMA1] pointed out that the construction of rational orthonormal scaling filters $m_0(\xi) = \frac{P(e^{-i\xi})}{Q(e^{-i\xi})}$ was very easy. Indeed, if $P, Q \in \mathbb{R}[X]$ and $| P(e^{-i\xi}) |^2 = A(\cos\xi)$ and $| Q(e^{-i\xi}) |^2 = B(\cos\xi)$ with $A \wedge B = 1$ (which means that we exclude factors $\frac{\lambda - e^{-i\xi}}{\lambda e^{-i\xi} - 1}$ ($\lambda \in \mathbb{C}$) from $m_0(\xi)$), $| m_0(\xi) |^2 + | m_0(\xi + \pi) |^2 = 1$ becomes $A(X)B(-X) + B(X)A(-X) = B(X)B(-X)$ hence $A(X) + A(-X) = B(X)$: thus, in order to get m_0, it suffices to choose $P \in \mathbb{R}[X]$ such that $P(1) = 1$, $P(-1) = 0$, $P(z)$ and $P(-z)$ have no common root on the unit circle and $P(z)$ has no factors $\prod_{k=1}^{N}(z + z_0^{2^k})$ with $z_0^{2^N} = z_0$ and $z_0 \neq 1$ (Cohen's criterion); then constructing $Q \in \mathbb{R}[X]$ such that $| Q(e^{-i\xi}) |^2 = | P(e^{-i\xi}) |^2 + | P(-e^{-i\xi}) |^2$ gives an orthonormal scaling filter $m_0(\xi) = \frac{P(e^{-i\xi})}{Q(e^{-i\xi})}$.

A very interesting example is the *Butterworth scaling filter* where

$$P(e^{-i\xi}) = \left(\frac{1 + e^{-i\xi}}{2}\right)^N .$$

Then we have to construct Q such that

$$| Q(e^{-i\xi}) |^2 = \left|\frac{1 + \cos\xi}{2}\right|^N + \left|\frac{1 - \cos\xi}{2}\right|^N .$$

The polynomial

$$R_N(X) = \left(\frac{1+X}{2}\right)^N + \left(\frac{1-X}{2}\right)^N$$

has explicit roots, so that Q can be computed exactly. For every N, one good choice for Q is to select the roots so that $Q(z)$ doesn't vanish inside the unit disk ; then $\frac{P(e^{-i\xi})}{Q(e^{-i\xi})}$ is holomorphic on $\Im\mathrm{m}\, \xi < 0$, so that the associated φ has its support contained in $[0, +\infty)$. When N is odd, another choice is possible :

$$Q(e^{-i\xi}) = \left(\frac{1+e^{-i\xi}}{2}\right)^N + \left(\frac{1-e^{-i\xi}}{2}\right)^N = e^{-i\xi\frac{N}{2}}\left((\cos\frac{\xi}{2})^N + i^N(\sin\frac{\xi}{2})^N\right).$$

In that case, m_0 satisfies moreover $m_0(\xi) + m_0(\xi + \pi) = 1$, hence we have $\varphi(k) = \delta_{k,0}$: this choice of Q leads to an *orthonormal and interpolating* scaling function !

Rational filters have been also introduced by Evangelista [EVG] and more recently by Herley and Vetterli [HEL].

Example n° 10 : The basis of Berkolaiko and Novikov.

In 1992, V. Berkolaiko and I. Novikov introduced a basis of compactly supported "almost" wavelets which were C^∞ and had an infinite approximation order [BER]. This basis was also introduced by A. Cohen and N. Dyn in 1993 [COHD] as an example of *non-stationary multi-resolution analysis.*

If μ is a compactly supported square-integrable function ($\mu \neq 0$), the non-stationary multi-resolution analysis generated by μ is the sequence of spaces $(V_j(\mu))_{j\geq 0}$ defined by: $V_j(\mu)$ is the smallest closed sub-space of L^2 such that $\mu \in V_j(\mu)$ and for any $f \in V_j(\mu)$ and $k \in \mathbb{Z}$, $f\left(x - \frac{k}{2^j}\right) \in V_j(\mu)$. Then, following our results on shift-invariant subspaces of L^2 in chapter 3, it is easy to see that $(V_j(\mu))$ has the following properties :

(i) $V_j(\mu) \subset V_{j+1}(\mu)$ and $\bigcup_{j=0}^{+\infty} V_j(\mu)$ is dense in $L^2(\mathbb{R})$;
(ii) For every $j \geq 0$, there exists a function N_j with compact support such that

$$(2^{j/2}N_j(2^jx - k))_{k\in\mathbb{Z}}$$

is a Riesz basis of V_j.

When N_j doesn't depend on j, the $(V_j(\mu))$ are a multi-resolution analysis of $L^2(\mathbb{R})$ with scaling function N_0. A necessary and sufficient condition to have a multi-resolution analysis is $\mu(\frac{x}{2}) \in V_0(\mu)$.

We write P_j for the orthogonal projection operator onto $V_j(\mu)$. Then μ will be said to have approximation order k if for all $y \in H^k(\mathbb{R})$ we have $\| P_j(y) - y \|_2 = o(2^{-jk})$.

PROPOSITION 5. - *Let $(m_N)_{N\geq 1}$ be Daubechies' filter (where*

$$m_N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q_N(e^{-i\xi})$$

with $Q_N \in \mathbb{R}[X]$, $\deg Q_N = N-1$, $Q_N(1) = 1$ *and* $| m_N(\xi) |^2 + | m_N(\xi+\pi) |^2 = 1$). *Define for* $N \geq 0$, Φ_N *and* Ψ_N *by*

$$\hat{\Phi}_N(\xi) = \prod_{j=1}^{\infty} m_{N+j}\left(\frac{\xi}{2^j}\right) \tag{20}$$

$$\hat{\Psi}_N(\xi) = e^{-i\frac{\xi}{2}} \bar{m}_{N+1}(\frac{\xi}{2}+\pi)\hat{\Phi}_{N+1}(\frac{\xi}{2}). \tag{21}$$

Then :

- (j) Φ_N *and* Ψ_N *are compactly-supported* C^∞ *functions, with* $\operatorname{Supp}\Phi_N \subset [0, 2N+3]$ *and* $\operatorname{Supp}\Psi_N \subset [-N, N+2]$.
- (jj) *The family* $(2^{N/2}\Phi_N(2^N x - k))_{k\in\mathbb{Z}}$ *is an Hilbertian basis for* $V_N(\Phi_0)$ *and the family* $(2^{N/2}\Psi_N(2^N x - k))_{k\in\mathbb{Z}}$ *is an Hilbertian basis for* $V_{N+1}(\Phi_0) \cap V_N(\Phi_0)^\perp$.
- (jjj) *The family* $(\Phi_0(x-k))_{k\in\mathbb{Z}} \cup (2^{N/2}\Psi_N(2^N x - k))_{N\geq 0, k\in\mathbb{Z}}$ *is an Hilbertian basis for* $L^2(\mathbb{R})$.
- (jv) Φ_0 *has an infinite approximation order. More precisely, for every* $s \in \mathbb{R}$ *and* $f \in \mathcal{D}'(\mathbb{R})$, f *belongs to the Sobolev space* H^s *if and only if* $N_s(f) < +\infty$, *where*

$$N_s(f) = \left\{ \sum_{k\in\mathbb{Z}} |< f \mid \Phi_0(x-k) >|^2 + \sum_{N=0}^{+\infty} \sum_{k\in\mathbb{Z}} 4^{Ns} |< f \mid 2^{N/2}\Psi_N(2^N x - k) >|^2 \right\}^{1/2},$$

and the norms $\|\cdot\|_{H^s}$ *and* $N_s(\cdot)$ *are equivalent.*

Proof. The pointwise convergence of the infinite products (20) and (21) is obvious : we have $\| m_N \|_\infty \leq 1$ and $\deg m_N = 2N-1$, hence by Bernstein's inequality we get $\| \frac{d}{d\xi} m_N \|_\infty \leq CN$, so that $| m_{N+j}(\frac{\xi}{2^j}) - 1 | \leq C(N+j)\frac{|\xi|}{2^j}$. Since $\sum_{j=1}^{\infty}(N+j)\frac{1}{2^j} = N+2 < +\infty$, the products converge. Moreover, each finite product is bounded by 1, hence the products converge in $\mathcal{S}'$ (by the bounded convergence theorem). Thus Φ_N is an infinite convolution product of sums of Dirac masses, where the $j-th$ sum has its support contained in $[0, \frac{2N+2j-1}{2^j}]$, so that $\operatorname{Supp}\Phi_N \subset [0, 2N+3]$.

Moreover, since $| m_N(\xi) |^2 + | m_N(\xi+\pi) |^2 = 1$ for all N and ξ, the functions $\theta_{N,p}$ defined by

$$\hat{\theta}_{N,p}(\xi) = \chi_{[-\pi,\pi]}\left(\frac{\xi}{2^p}\right) \prod_{j=1}^{p} m_{N+j}\left(\frac{\xi}{2^j}\right)$$

generate orthonormal families $(\theta_{N,p}(x-k))_{k\in\mathbb{Z}}$, while $\hat{\theta}_{N,p} \to \hat{\Phi}_N$ as $p \to +\infty$ pointwise (and in $\mathcal{S}'$). The proof of proposition 5 then reduces to the following estimates :

$$| \hat{\theta}_{N,p}(\xi) | \leq C_1 \mid \xi \mid^{-\alpha|\xi|} \tag{22}$$

$$\text{For } k \in \mathbb{N},\ k \leq N+1, \quad | \hat{\Psi}_N(\xi) | \leq D_k \mid \xi \mid^k \tag{23}$$

where C_1 and α are positive constants which don't depend on N nor p, and the D_k's are positive constants which don't depend on N.

(22) is obvious for $|\xi| \leq 2^{30}\pi$: we just write $|\hat{\theta}_{N,p}(\xi)| \leq 1$. Now, if $|\xi| \geq 2^{30}\pi$ we write $\ell \geq 30$ the integer such that $2^\ell \pi \leq |\xi| < 2^{\ell+1}\pi$. If $p \leq \ell$, $\hat{\theta}_{N,p}(\xi) = 0$; if $p \geq \ell+1$, we have

$$|\hat{\theta}_{N,p}(\xi) \leq \prod_{j=1}^{\ell+1} | m_{N+j}(\frac{\xi}{2^j}) |= A_{N,\ell}(\xi) B_{N,\ell}(\xi),$$

where

$$A_{N,\ell}(\xi) = \prod_{j=1}^{\ell+1} \left| \frac{1+e^{-i\frac{\xi}{2^j}}}{2} \right|^{N+j} \quad \text{and} \quad B_{N,\ell}(\xi) = \prod_{j=1}^{\ell+1} | Q_{N+j}(e^{-i\frac{\xi}{2^j}}) | .$$

$A_{N,\ell}$ is easily estimated (since $|\sin\frac{\xi}{2^{\ell+2}}| \geq \frac{2}{\pi}\frac{|\xi|}{2^{\ell+2}}$ for $|\xi| \leq 2^{\ell+1}\pi$) :

$$\begin{aligned} A_{N,\ell}(\xi) &= \prod_{j=1}^{\ell+1} \left| \frac{1+e^{-i\frac{\xi}{2^j}}}{2} \right|^N \prod_{k=0}^{\ell} \left| \prod_{j=k+1}^{\ell+1} | \frac{1+e^{-i\frac{\xi}{2^j}}}{2} | \right| \\ &= \left(\frac{|\sin\frac{\xi}{2}|}{2^{\ell+1}|\sin\frac{\xi}{2^{\ell+2}}|} \right)^N \prod_{k=0}^{\ell} \frac{|\sin\frac{\xi}{2^{k+1}}|}{2^{\ell+1-k}|\sin\frac{\xi}{2^{\ell+2}}|} \\ &\leq (\frac{\pi}{|\xi|})^N \prod_{k=0}^{\ell} \frac{2^k\pi}{|\xi|} \leq 2^{-\ell N} 2^{-\frac{1}{2}\ell(\ell+1)}. \end{aligned}$$

Now, we recall that $|Q_N(e^{-i\xi})|^2 = \sum_{k=0}^{N-1} \binom{N+k-1}{k} (\frac{1-\cos\xi}{2})^k$; Berkolaiko and Novikov noticed that $\binom{N+k-1}{k} \leq \binom{N+k}{k}$ hence $|Q_N(e^{-i\xi})| \leq |Q_{N+1}(e^{-i\xi})|$ for all N and ξ. Now, we see that

$$|Q_N(e^{-i\xi})Q_N(e^{-2i\xi})| \leq |Q_N(-1)| |Q_N(e^{-2i\pi/3})| :$$

the function

$$A_N(X) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{1-X}{2}\right)^k$$

is clearly decreasing on $[-1,1]$ and we always have $\cos\xi \geq -\frac{1}{2}$ or $\cos 2\xi \geq -\frac{1}{2}$. Moreover, if $X < 0$,

$$A_N(x) \leq 2^{2N-2} \left(\frac{1-X}{2}\right)^{N-1} \left(\frac{1-X}{-X}\right)$$

(just write $\binom{N+k-1}{k} \leq 2^{k-N+1} \binom{2N-2}{N-1} \leq 2^{2N-2} 2^{k-N+1}$), thus $|Q_N(-1)| \leq 2^{N-1/2}$, $|Q_N(e^{-2i\pi/3})| \leq 3^{N/2}$ and for $\frac{\pi}{2} < |\xi| \leq \pi$, $|m_N(\xi)| \leq \frac{|\sin\xi|^N}{\sqrt{2|\cos\xi|}}$.

We may estimate $B_{N,\ell}(\xi)$: we distinguish odd ℓ's and even ℓ's : if $\ell = 2\ell' + 1$, then

$$\begin{aligned} B_{N,\ell}(\xi) \leq & \mid Q_{N+2}(e^{-i\frac{\xi}{2}}) \mid\mid Q_{N+2}(e^{-i\frac{\xi}{4}}) \mid \cdot \mid Q_{N+4}(e^{-i\frac{\xi}{8}}) \mid\mid Q_{N+4} \mid e^{-i\frac{\xi}{16}}) \mid \cdots \\ & \mid Q_{N+2\ell'+2}(e^{-i\frac{\xi}{2^{2\ell'+1}}}) \mid\mid Q_{N+2\ell'+2}(e^{-i\frac{\xi}{2^{2\ell'+2}}}) \mid \leq \\ & \leq \prod_{j=1}^{\ell'+1} 2^{N+2j-1/2} 3^{\frac{N}{2}+j} \\ & = 2^{-1/2(\ell'+1)} (2\sqrt{3})^{N(\ell'+1)+1/2(\ell'+1)(\ell'+2)}. \end{aligned}$$

If $\ell = 2\ell'$, we write $Q_{N,\ell}(\xi) \leq Q_{N+1}(e^{-i\xi}) B_{N+1,\ell-1}(\frac{\xi}{2})$, and get

$$\mid B_{N,\ell}(\xi) \mid \leq 2^{N+1/2} 2^{-1/2\ell'} (2\sqrt{3})^{(N+1)\ell'+1/2\ell'(\ell'+1)}.$$

In any case, we've got :

$$\mid B_{N,\ell}(\xi) \mid \leq (2\sqrt{3})^{N(\frac{\ell+2}{2})+\frac{1}{8}(\ell+1)(\ell+6)},$$

so that

$$A_{N,\ell} B_{N,\ell} \leq 2^{-\ell N} (2\sqrt{3})^{N\frac{\ell+2}{2}} 2^{-1/2\ell(\ell+1)} (2\sqrt{3})^{\frac{1}{8}(\ell+1)(\ell+6)}.$$

Now, we recall that $\ell \geq 30$, hence $\frac{\ell+2}{2} \leq \frac{8}{15}\ell$, and thus :

$$A_{N,\ell} B_{N,\ell} \leq \left(\frac{(2\sqrt{3})^{8/15}}{2}\right)^{\ell N} (2\sqrt{3})^{6/8} \left(\frac{(2\sqrt{3})^{7/8}}{\sqrt{2}}\right)^{\ell} \left(\frac{(2\sqrt{3})^{1/8}}{\sqrt{2}}\right)^{\ell^2}.$$

We check easily that $(2\sqrt{3})^{8/15} < 2$ and $(2\sqrt{3})^{1/8} < \sqrt{2}$, thus $A_{N,\ell} B_{N,\ell} \leq C\gamma^{\ell N} e^{-\alpha \ell^2}$ for some $C > 0$, $\alpha > 0$ and $\gamma \in (0,1)$. Since $\ell \leq \frac{\log \frac{|\xi|}{\pi}}{\log 2} \leq \ell + 1$, (22) is proved.

(23) is easy : for $\mid \xi \mid \geq 1$, we just write $\mid \hat{\psi}_N(\xi) \mid \leq 1$; for $\mid \xi \mid \leq 1$ we have $\frac{\xi}{2} + \pi \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$, hence $\mid \hat{\psi}_N(\xi) \mid \leq \mid m_{N+1}(\frac{\xi}{2} + \pi) \mid \leq \mid \sin \frac{\xi}{2} \mid^{N+1} \leq \frac{|\xi|^{N+1}}{2^{N+1}}$.

The proposition is now easily proved. Because of the estimate (22), we may use the dominated convergence theorem to get that $(\Phi_N(x-k))_{k\in\mathbb{Z}}$ is an orthonormal family. Points (ii) and (iii) are thus proved.

Point (iv) is proved by the following easy consequences of (22) and (23) :

- $\sup_{N\geq 0} \sup_{\xi} \sum_{p\in\mathbb{Z}} \mid \xi + 2p\pi \mid^{k} \mid \hat{\psi}_N(\xi + 2p\pi) \mid^2 < +\infty$
- $\sup_{N\geq k-1} \sup_{\xi} \sum_{p\in\mathbb{Z}} \mid \xi + 2p\pi \mid^{-k} \mid \hat{\psi}_N(\xi + 2p\pi) \mid^2 < +\infty.$

Thus, if Λ^s is the operator $\widehat{\Lambda^s f} = \mid \xi \mid^s \hat{f}$, we have for every $s \in \mathbb{R}$ and $N \geq -s$, $\| \Lambda^s(Q_N f) \|_2 \approx 2^{Ns} \| Q_N f \|_2$, and $\| f \|_{H^s} \leq C N_s(f)$. We have

$$\mid < \Lambda^s Q_N f \mid \Lambda^s Q_{N'} f > \mid \leq 2^{Ns} 2^{N's} 2^{-|N-N'|} \| Q_N f \|_2 \| Q_{N'} f \|_2$$

writing

$$< \Lambda^s Q_N f \mid \Lambda^s Q_{N'} f > = < \Lambda^{s+1} Q_N f \mid \Lambda^{s-1} Q_{N'} f >,$$

and thus

$$\| \Lambda^s \sum_{N \geq -s} Q_N f \|_2 \leq C_s \left(\sum_{N \geq -s} 4^{Ns} \| Q_N f \|_2^2 \right)^{1/2} .$$

The reverse inequality is obtained by duality.

REMARKS. - (i) The basis of Berkolaiko and Novikov is a generalization of the Rvachev function $up(x)$, defined by :

$$up(x) = \chi * 2\chi(2x) * \cdots * 2^j \chi(2^j x) * \cdots$$

(where $\chi = \chi_{[0,1]}$) or $\widehat{up}(\xi) = \prod_{j=1}^{\infty} \left(\frac{1+e^{-i\frac{\xi}{2^j}}}{2} \right)^j$.

The Rvachev function is a compactly supported C^∞ function with infinite approximation order [RVA], [DYN].

(ii) The proof of proposition 5 gives us also the following estimates :

$$\lim_{N \to +\infty} \sum_{|\xi + 2k\pi| > \pi} | \hat{\Phi}_N(\xi + 2k\pi) |^2 = 0$$

$$\lim_{N \to +\infty} | \hat{\Phi}_N(\xi) | = 1 \quad \text{if} \quad | \xi | < \pi$$

$$\lim_{N \to +\infty} | \hat{\Phi}_N(\xi) | = \frac{1}{\sqrt{2}} \quad \text{if} \quad | \xi | = \pi,$$

hence $\lim_{N \to +\infty} \sum_{k \in \mathbb{Z}} < f \mid \Phi_N(x-k) > \Phi_N(x-k) = g$ with $\hat{g} = \chi_{[-\pi,\pi]} \hat{f}$ for any $f \in L^2$.

3. Spline functions : the case of orthogonal spline wavelets.

The theory of spline functions has been thoroughly developped in the last fifty years as a key tool in approximation theory or in computer-aided design. A small part of this theory, namely the theory of cardinal splines developped by I. J. Schoenberg in the early 50's [SCH], fits very well to the frame of wavelet theory.

A *spline function* of degree m with nodes in a discrete subset X of $\mathbb{R}$ is a function f of class C^{m-1} such that (writing $X = (\cdots < x_j < x_{j+1} < \cdots)_{j \in \mathbb{Z}}$) the restriction of f to each interval (x_j, x_{j+1}) is a polynomial of degree at most m. Thus f is determined exactly by the polynomial $f \mid_{(x_0, x_1)}$ and by the sequence $(f^{(m)}(x_j + 0) - f^{(m)}(x_j - 0))_{j \in \mathbb{Z}}$. Thus, we may say equivalently that a distribution f is a spline function with nodes in X if and only if $f^{(m+1)}$ can be written as a (locally finite) sum of Diract masses at X : $f^{(m+1)} = \sum_{j \in \mathbb{Z}} a_j \delta(x - x_j)$, where $a_j = f^{(m)}(x_j + 0) - f^{(m)}(x_j - 0)$; integrating $(m+1)$-times $f^{(m+1)}$ gives us f up to $(m+1)$ constants of integration (which are exactly determined by the polynomial $f \mid_{(x_0, x_1)}$).

A *cardinal spline function* is a spline function with nodes in a regular grid ($X = a\mathbb{Z}$ for some $a > 0$), and especially in $\mathbb{Z}$. If V_0^m is the space of square-integrable spline functions of degree m with nodes in $\mathbb{Z}$, then it generates a multi-resolution analysis. Clearly we have for any $f \in V_0^m$, $f(\frac{x}{2}) \in V_0^m$ (since $2\mathbb{Z} \subset \mathbb{Z}$), so that we get only to check that V_0^m has a Riesz basis of the form $N_m(x-k))_{k\in\mathbb{Z}}$. This is obvious by induction on m, since V_0^m has clearly an orthonormal basis $(N_0(x-k) = \chi(x-k))_{k\in\mathbb{Z}}$ where $\chi = \chi_{[0,1]}$, and since $f \in V_0^{m+1}$ if and only if $f \in L^2$ and $f' \in V_0^m$. Our study of scaling functions, and more precisely theorem 3 of chapter 4, allows one to conclude that V_0^{m+1} has a scaling function N_{m+1} given by

$$N_{m+1}(x) = \int_x^{x+1} N_m(t)dt = \chi * N_m = \chi^{(*)m+2}. \tag{24}$$

The function N_m is called the normalized B-spline of degree m ; it is the function of minimal support in V_0^m, and its support is $[0, m+1]$.

Formula (24) allows one to compute exactly $N_m(x)$ by induction. By instance we have:

$$N_0(x) = \chi_{[0,1]}$$
$$N_1(x) = \chi_{[0,1]} + (2-x)\chi_{[1,2]}$$
$$N_2(x) = \frac{x^2}{2}\chi_{[0,1]} + \frac{6x-3-2x^2}{2}\chi_{[1,2]} + \frac{(x-3)^2}{2}\chi_{[2,3]}.$$

Moreover, the Fourier transform of N_m is easily given by (24) :

$$\hat{N}_m(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^{m+1}. \tag{25}$$

Moreover, since χ is symmetric with symmetry center at $x = 1/2$, N_m is symmetric, with symmetry center at $x = \frac{m+1}{2}$. Thus, the auto-correlation function $C_m(\xi)$ of N_m, given by

$$C_m(\xi) = \sum_{k\in\mathbb{Z}} \mid \hat{N}_m(\xi + 2k\pi) \mid^2 = \sum_{k\in\mathbb{Z}} < N_m(x) \mid N_m(x-k) > e^{-ik\xi},$$

can be computed with help of N_{2m+1} :

$$C_m(\xi) = \sum_{k=-m}^{m} N_{2m+1}(k+m+1)e^{-ik\xi}. \tag{26}$$

The trigonometric polynomial C_m has been called by I. J. Schoenberg a *generalized Euler-Frõbenius polynomial*. Of course, the function $L_{2m+1}(x)$ defined by $\widehat{L_{2m+1}}(\xi) = \frac{|\hat{N}_m(\xi)|^2}{C_m(\xi)}$ satisfies $\sum_{k\in\mathbb{Z}} \widehat{L_{2m+1}}(\xi + 2k\pi) = 1$, hence $L_{2m+1}(0) = 1$ and $L_{2m+1}(k) = 0$ for $k \in \mathbb{Z}^*$; moreover $\mid \hat{N}_m(\xi) \mid^2 = e^{-i(m+1)\xi}\hat{N}_{2m+1}(\xi)$, hence $L_{2m+1} \in V_0^{2m+1}$: it is the so-called *Lagrangian interpolation spline* of degree $2m+1$, studied by Schoenberg in [SCH].

Spline functions are related to interpolation through Holladay's theorem. This theorem can be stated in the following way, for spline with nodes in $\mathbb{Z}$:

PROPOSITION 6. - *Let* $m \in \mathbb{N}$. *Then* :

(i) $V_0^{m\perp} = \{f \in L^2 / \exists g \in H^{m+1},\ f = g^{(m+1)}$ *and* $g(k) = 0$ for all $k \in \mathbb{Z}\}$
(ii) $f \in V_0^{2m+1}$ *if and only if* $f \in H^{m+1}$ *and*

$$\| f^{(m+1)} \|_2 = \inf\{\| F^{(m+1)} \|_2 / F \in H^{m+1} \ \ and \ \ F(k) = f(k) \ \ for\ all \ \ k \in \mathbb{Z}\}.$$

Proof. It is easy to see that on H^{m+1} the $\| \cdot \|_{H^{m+1}}$ norm is equivalent to the following $N_{m+1}(\cdot)$ norm :

$$N_{m+1}(f) = \| f^{(m+1)} \|_2 + \left(\sum_{k \in \mathbb{Z}} | f(k) |^2 \right)^{1/2}.$$

(Indeed,

$$\begin{aligned} \| f \|_{H^{m+1}}^2 &\approx \sum_{k \in \mathbb{Z}} \| f |_{[k,k+1]} \|_{H^{m+1}([k,k+1])}^2 \\ &\approx \sum_{k \in \mathbb{Z}} | f(k) |^2 + \sum_{j=1}^{m+1} \| f^{(j)} \|_2^2 \ ; \end{aligned}$$

now, to conclude, it is enough to write $\| f^{(j)} \|_2^2 \le C\{\epsilon^j \| f \|_{H^{m+1}}^2 + \epsilon^{j-m-1} \| f^{(m+1)} \|_2^2\}$ where C doesn't depend on $\epsilon > 0$). Thus $(\frac{d}{dx})^{m+1}$ is an isomorphism between the subspace K^m of H^{m+1} defined by $K^m = \{f \in H^{m+1} / \forall k \in \mathbb{Z},\ f(k) = 0\}$ and the subspace of $L^2 (\frac{d}{dx})^{m+1} K^m$, which is therefore closed in L^2. To see that $(\frac{d}{dx})^{m+1} K^m = (V_0^m)^\perp$ is then obvious, since $f \in V_0^m$ if and only if $f \in L^2$ and $f^{(m+1)} = \sum_{k \in \mathbb{Z}} a_k \delta_k$ for some $(a_k) \in \ell^2$. Point (i) is proved, and point (ii) follows easily, since $f \in V_0^{2m+1}$ if and only if $f \in H^{m+1}$ and $f^{(m+1)} \in V_0^m$. ∎

We may now construct the orthonormal multi-resolution analysis associated to V_0^m.

The orthogonal projection operator P_0^m onto V_0^m can be written as

$$P_0^m f = \sum_{k \in \mathbb{Z}} < f \mid g_m(x-k) > N_m(x-k), \tag{27}$$

where g_m is defined by

$$\hat{g}_m(\xi) = \frac{\hat{N}_m(\xi)}{C_m(\xi)}.$$

We can also introduce an orthonormal scaling function φ_m for V_0^m by choosing a 2π-periodical function $D_m(\xi)$ such that $| D_m(\xi) |^2 = C_m(\xi)$ and defining φ_m by $\hat{\varphi}_m = \frac{\hat{N}_m}{D_m}$. The choice of $D_m = \sqrt{C_m}$ leads to *the function of Battle* [BAT1] *and Lemarié* [LEM1], which is symmetrical with symmetry center at $x = \frac{m+1}{2}$. A choice of D_m as a trigonometric polynomial (through Riesz' lemma) with no roots in the lower half-plane ($D_m(\xi) = \sum a_k e^{-ik\xi}$ and $| \sum a_k e^{-ikz} | > 0$ for $\Im m\, z < 0$) leads to the *causal function of Strömberg* [STR].

In order to get a Riesz basis for $W_0^m = V_1^m \cap (V_0^m)^\perp$, it is enough to see that (because of proposition 6) we have $f \in W_0^m$ if and only if $f \in V_1^m \cap (\frac{d}{dx})^{m+1} K^m$, hence if and only if $f = (\frac{d}{dx})^{m+1} g$ for some $g \in V_1^{2m+1}$ such that $g(k) = 0$ for all $k \in \mathbb{Z}$. Thus $f = \sum_{k\in\mathbb{Z}} g(k+\frac{1}{2}) 2^{m+1} \left\{(\frac{d}{dx})^{m+1} L_{2m+1}\right\}(2x-2k-1)$, and we've got a Riesz basis $\Lambda_m(x-k)$ of W_0^m with $\Lambda_m = (\frac{d}{dx})^{m+1}\{L_{2m+1}(2x-1)\}$. This is *Battle's pre-wavelet* [BAT4], also introduced in [LEM2] and [CHW1]. We have :

$$\hat{\Lambda}_m(\xi) = \frac{1}{2}(i\xi)^{m+1} e^{-i\frac{\xi}{2}} \frac{\left| \frac{1-e^{-i\xi/1}}{i\xi/2} \right|^{2m+2}}{C_m(\xi/2)}$$
$$= \frac{e^{-i\frac{\xi}{2}}}{C_m(\frac{\xi}{2})} 2^{2m+1} (i)^{m+1} \left(\frac{1-e^{+i\xi/2}}{2}\right)^{m+1} \hat{N}_m(\frac{\xi}{2}).$$

Replacing Λ_m by ψ_m, where $\hat{\psi}_m = \frac{C_m(\frac{\xi}{2})C_m(\frac{\xi}{2}+\pi)}{2^{2m+1} i^{m+1}} \hat{\Lambda}_m$, we obtain a compactly supported function ψ_m such that $(\psi_m(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis of W_0^m : this is the *compactly supported pre-wavelet of Chui and Wang* [CHW2] ψ_m defined by

$$\hat{\psi}_m(\xi) = e^{-i\frac{\xi}{2}} C_m(\frac{\xi}{2}+\pi) \left(\frac{1-e^{+i\xi/2}}{2}\right)^{m+1} \hat{N}_m(\frac{\xi}{2}). \tag{28}$$

(More generally, if φ is a compactly supported scaling function, with scaling filter m_0 and auto-correlation function $C(\xi)$, and if (V_j) is the multi-resolution analysis generated by φ, then one may associate a compactly supported pre-wavelet ψ to φ, i.e. a compactly supported function ψ such that $(\psi(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis of $W_0 = V_0^{\perp} \cap V_1$ [CHW3] ; this function is given by the formula

$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}} C(\frac{\xi}{2}+\pi) \bar{m}_0(\frac{\xi}{2}+\pi) \hat{\varphi}(\frac{\xi}{2})).$$

We may now compute the orthogonal projection operator Q_0^m onto $W_0^m = V_1^m \cap (V_0^m)^{\perp}$ by defining $\Gamma_m(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}_m(\xi+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} < \psi_m \mid \psi_m(x-k) > e^{-i\xi}$, $\gamma_m(x)$ by $\hat{\gamma}_m = \frac{\hat{\psi}_m}{\Gamma_m}$, such that :

$$Q_0^m f = \sum_{k\in\mathbb{Z}} < f \mid \gamma_m(x-k) > \psi_m(x-k). \tag{29}$$

We may also define an orthogonal wavelet, i.e. a function θ_m such that $(\theta_m(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of W_0^m, by choosing a 2π-periodical function Δ_m such that $|\Delta_m(\xi)|^2 = \Gamma_m(\xi)$ and defining θ_m by $\hat{\theta}_m = \frac{\hat{\psi}_m}{\Delta_m}$. The choice of $\Delta_m = \sqrt{\Gamma_m}$ leads to the orthogonal wavelet of Battle [BAT1] and Lemarié [LEM1]. Another choice of θ_m is proposed by Strömberg [STR].

We may then summarize these results in the following way :

PROPOSITION 7. - *Let L_{2m+1}, $\tilde{L}_{2m+1}$ be the following Lagrangian interpolation spline functions of degree $2m+1$:*

- *L_{2m+1} has nodes in $\mathbb{Z}$, $L_{2m+1}(0) = 1$, $L_{2m+1}(k) = 0$ for $k \in \mathbb{Z}^*$*

• *$\tilde{L}_{2m+1}$ has nodes in $-\mathbb{N} \cup \frac{1}{2}\mathbb{N}^*$, $\tilde{L}_{2m+1}(\frac{1}{2}) = 1$, $\tilde{L}_{2m+1}(k) = 0$ for $k \in -\mathbb{N} \cup \frac{1}{2}\mathbb{N}^* \backslash \{\frac{1}{2}\}$.*
Let $(V_j^m)_{j\in\mathbb{Z}}$ be the multi-resolution analysis of square-integrable spline functions of degree $2m+1$ (with nodes in $\frac{1}{2^j}\mathbb{Z}$). Then :

(i) *the B-spline N_m (defined by $\hat{N}_m(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^{m+1}$) is a compactly supported scaling function (with minimal support) for (V_j^m), with associated scaling filter $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{m+1}$ and auto-correlation function*

$$C_m(\xi) = \sum_{k=-m}^{m} N_{2m+1}(k+m+1)e^{-ik\xi}.$$

(ii) *the function φ_m defined by $\hat{\varphi}_m(\xi) = \frac{\hat{N}_m}{\sqrt{C_m}}$ is an orthonormal scaling function for (V_j^m), which is symmetric with symmetry center at $x = \frac{m+1}{2}$.*
(iii) *the function $\tilde{\varphi}_m$ defined up to a multiplicative constant as a spline with nodes in $\mathbb{N}$ which is orthogonal to every spline with nodes in $\mathbb{N}^*$ is a causal orthonormal scaling function for (V_j^m),*
(iv) *the compactly supported function ψ_m defined by*

$$\hat{\psi}_m(\xi) = e^{-i\frac{\xi}{2}} C_m(\frac{\xi}{2}+\pi)\left(\frac{1-e^{i\,\xi/2}}{2}\right)^{m+1} N_m(\frac{\xi}{2})$$

is a pre-wavelet for (V_j^m) (i.e., $(\psi_m(x-k))_{k\in\mathbb{Z}}$ is a Riesz basis for $V_1^m \cap (V_0^m)^\perp$)
(v) *the function $\Lambda_m = (\frac{d}{dx})^{2m+1}(L_{2m+1}(2x-1))$ is a pre-wavelet for (V_j^m).*
(vi) *The function θ_m defined by*

$$\hat{\theta}_m(\xi) = e^{-i\frac{\xi}{2}}\left(\frac{1-e^{i\,\xi/2}}{2}\right)^{m+1} \frac{\sqrt{C_m(\frac{\xi}{2}+\pi)}}{\sqrt{C_m(\xi)}\sqrt{C_m(\xi/2)}}\hat{N}_m(\frac{\xi}{2})$$

is an orthonormal wavelet for (V_j^m), which is symmetric with symmetry center at $x = \frac{1}{2}$ if m is odd, and anti-symmetric if m is even.
(vii) *The function $(\frac{d}{dx})^{m+1}\tilde{L}_{2m+1}$ is, up to a multiplicative constant, Strömberg's orthonormal wavelet for (V_j^m).*

These results can be generalized to other dilation factors than 2. Indeed for every $A \in \mathbb{N}^*$ and $f \in V_0^m$ we have $f(\frac{x}{A}) \in V_0^m$. (One can show that if (V_j) is a multi-resolution analysis with a compactly supported scaling function and if moreover V_0 satisfies $f \in V_0 \Rightarrow f(\frac{x}{A}) \in V_0$ for $A=2$ and for some another factor A which is odd, then (V_j) is a spline multi-resolution analysis [LEM7]). Now, if we want to describe $W_{0,A}^m = V_{1,A}^m \cap (V_0^m)^\perp$ (where $V_{1,A}^m = \{f(Ax)/f \in V_0^m\}$), we may use again Proposition 6 to see that if $f \in W_{0,A}^m$ then $f = (\frac{d}{dx})^{m+1}g$, where g is a spline of degree $2m+1$ with nodes in $\frac{1}{A}\mathbb{Z}$ and such that $g(k) = 0$ for every $k \in \mathbb{Z}$, thus we obtain a Riesz basis $(\Lambda_{m,r}(x-k))_{1\leq r\leq A-1, k\in\mathbb{Z}}$ of

$W^m_{0,A}$ with $\Lambda_{m,r} = (\frac{d}{dx})^{m+1}(L_{2m+1}(Ax-r))$, where L_{2m+1} is the Lagrangian interpolation cardinal spline of degree $2m+1$.

4. Bi-orthogonal spline wavelets.

Bi-orthogonal multi-resolution analyses have been introduced in 1990 by Cohen, Daubechies and Feauveau [CODF] in order to produce linear-phased F.I.R. filters adapted to the fast wavelet transform. It is a general belief in image processing that linear-phased filters produce less visual artifacts than the other ones. But, for the wavelet transform, linear phase corresponds to a symmetric scaling function, while F.I.R. corresponds to a compactly supported scaling function ; I. Daubechies proved that the Haar function $\varphi = \chi_{[0,1]}$ is the only orthonormal compactly-supported scaling function to be symmetric, so that orthonormality has to be dropped if linear phase is to be used.

Another interesting feature of bi-orthogonal multi-resolution analyses is the fact that they fit very well the differentiation.

DEFINITION 1. - *A bi-orthogonal multi-resolution analysis with compactly supported dual scaling function is a pair of multi-resolution analyses* $(V_j)_{j\in\mathbb{Z}}$, $(V_j^*)_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R})$ *with compactly supported scaling functions* φ, φ^* *such that* $< \varphi \mid \varphi^*(x-k) >= \delta_{k,0}$.

To a bi-orthogonal multi-resolution analysis, we may associate oblique projection operators P_j from L^2 onto V_j in direction of $(V_j^*)^\perp$:

(30)
$$P_j f = \sum_{k\in\mathbb{Z}} 2^j < f \mid \varphi^*(2^j x - k) > \varphi(2^j x - k).$$

PROPOSITION 8. - *Let* (V_j), (V_j^*) *be a bi-orthogonal multi-resolution analysis with compactly supported dual scaling functions* φ, φ^*, *and let* m_0, m_0^* *be the associated scaling filters. Then :*

(i) *the functions* ψ *and* ψ^* *defined by*

(31)
$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\bar{m}_0^*(\frac{\xi}{2}+\pi)\hat{\varphi}(\frac{\xi}{2}) \quad \text{and} \quad \hat{\psi}^*(\xi) = e^{-i\frac{\xi}{2}}\bar{m}_0(\frac{\xi}{2}+\pi)\hat{\varphi}^*(\frac{\xi}{2})$$

satisfy $< \psi \mid \psi^*(x-k) >= \delta_{k,0}$, $< \psi \mid \varphi^*(x-k) >= 0$, $< \psi^* \mid \varphi(x-k) >= 0$ *for any* $k \in \mathbb{Z}$.

(ii) $Q_j = P_{j+1} - P_j$ *is an oblique projection operator from* $L^2(\mathbb{R})$ *onto* $W_j = V_{j+1} \cap (V_j^*)^\perp$ *in direction of* $(W_j^*)^\perp$, *where* $W_j^* = V_{j+1}^* \cap V_j^\perp$. *Moreover* Q_j *is given by :*

(32)
$$Q_j f = \sum_{k\in\mathbb{Z}} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j x - k).$$

The functions ψ, ψ^* are called the bi-orthogonal wavelets associated to φ, φ^*.

(iii) *The family $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$, with dual basis $(2^{j/2}\psi^*(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$, and similarly the family*

$$(\varphi(x-k))_{k\in\mathbb{Z}}\cup(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{N},k\in\mathbb{Z}}$$

is a Riesz basis for $L^2(\mathbb{R})$ with dual basis

$$(\varphi^*(x-k))_{k\in\mathbb{Z}}\cup(2^{j/2}\psi^*(2^jx-k))_{j\in\mathbb{N},k\in\mathbb{Z}}.$$

Proof. Points (i) and (ii) are straightforward, using the correlation function $C(f,g)=\sum\hat{f}(\xi+2k\pi).\bar{\hat{g}}(\xi+2k\pi)$ for φ,φ^*, ψ and ψ^*. We have $C(\varphi,\varphi^*)=1$, hence for $f\in V_1$, $\hat{f}=U(\frac{\xi}{2})\hat{\varphi}(\frac{\xi}{2})$, and $g\in V_1^*$, $\hat{g}(\xi)=V(\frac{\xi}{2})\hat{\varphi}^*(\frac{\xi}{2})$, $C(f,g)=U(\frac{\xi}{2})\bar{V}(\frac{\xi}{2})+U(\frac{\xi}{2}+\pi)\bar{V}(\frac{\xi}{2}+\pi)$, which gives

$$m_0(\frac{\xi}{2})\bar{m}_0^*(\frac{\xi}{2})+m_0(\frac{\xi}{2}+\pi)\bar{m}_0^*(\frac{\xi}{2}+\pi)=C(\varphi,\varphi^*)=1,$$

$$C(\psi,\varphi^*)=0,\qquad C(\varphi,\psi^*)=0\quad\text{and}\quad C(\psi,\psi^*)=1.$$

Point (i) is proved, as well as point (ii), since W_0 is a finitely generated shift invariant space of multiplicity 1 so that $(\psi(x-k))_{k\in\mathbb{Z}}$ generates W_0 : since $P_{j+1}\circ P_j=P_j$ (because of $V_j\subset V_{j+1}$) and $P_j\circ P_{j+1}=P_j$ (because of $V_j^*\subset V_{j+1}^*$), Q_j is a projection operator and (32) is then obvious. Point iii) is easy as well, because the families $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ and $(2^{j/2}\psi^*(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ are dual to each other and almost orthogonal (just apply the "vaguelette lemma", theorem 3 of chapter 2 : we know by the first regularity theorem (theorem 2 of chapter 4) that ψ, ψ^* belong to H^ϵ for some positive ϵ).

We may now complete Theorem 3 of chapter 4 on the differentiation of scaling functions :

PROPOSITION 9. - *Let (V_j), (V_j^*) be a bi-orthogonal multi-resolution analysis with compactly supported dual scaling functions φ, φ^* and associated scaling filters m_0, m_0^*. Assume moreover that $\varphi\in H^1$. Then :*

(i) *the derivative φ' of φ can be written as $\varphi'=\tilde{\varphi}(x)-\tilde{\varphi}(x-1)$ where $\tilde{\varphi}$ is a compactly supported scaling function, with scaling filter $\tilde{m}_0(\xi)=\frac{2}{1+e^{-i\xi}}m_0(\xi)$.*

(ii) *the primitive $\int_{-\infty}^x\varphi^*(t)dt$ satisfy $\int_x^{x+1}\varphi^*(t)dt=\tilde{\varphi}^*(x)$ where $\tilde{\varphi}^*$ is a compactly supported scaling function, with associated scaling filter $\tilde{m}_0^*(\xi)=\frac{1+e^{i\xi}}{2}m_0^*(\xi)$.*

(iii) *$\tilde{\varphi}$ and $\tilde{\varphi}^*$ are compactly supported dual scaling functions for a bi-orthogonal multi-resolution analysis $(\tilde{V}_j)$, $(\tilde{V}_j^*)$. Moroever the projection operators P_j onto V_j in direction of $(V_j^*)^\perp$ and $\tilde{P}_j$ onto $\tilde{V}_j$ in direction of $(\tilde{V}_j^*)^\perp$ satisfy :*

$$\frac{d}{dx}\circ P_j=\tilde{P}_j\circ\frac{d}{dx}. \tag{33}$$

(iv) *the bi-orthogonal wavelets $\tilde{\psi}$, $\tilde{\psi}^*$ associated to $\tilde{\varphi}$, $\tilde{\varphi}^*$ satisfy :*

$$\tilde{\psi}=\frac{1}{4}\frac{d}{dx}\psi\quad\textit{and}\quad\tilde{\psi}^*=-4\int_{-\infty}^x\psi^*(t)dt. \tag{34}$$

Proof. (i) and (ii) have already been proved in chapter 4, and point (iv) is a direct consequence of points (i) and (ii). The only thing to check is point (iii). But

$$< \tilde{\varphi}(x) \mid \tilde{\varphi}^*(x-k) >= \sum_{p=0}^{+\infty} < \varphi'(x-p) \mid \tilde{\varphi}^*(x-k) >$$

$$= - \sum_{p=0}^{+\infty} < \varphi(x-p) \mid \varphi^*(x-k+1) - \varphi^*(x-k) >= \sum_{p=0}^{+\infty} (\delta_{k,p} - \delta_{k-1,p}) = \delta_{k,0}.$$

Moreover, we have

$$\frac{d}{dx}(P_0 f) = \sum_{k \in \mathbb{Z}} < f \mid \varphi^*(x-k) > (\tilde{\varphi}(x-k) - \tilde{\varphi}(x-k-1)) =$$

$$\sum_{k \in \mathbb{Z}} < f \mid \varphi^*(x-k) - \varphi^*(x-k+1) > \tilde{\varphi}(x-k) = \tilde{P}_0 \left(\frac{df}{dx} \right).$$

Thus proposition 9 is proved. ■

Due to proposition 9, we may now easily construct dual scaling functions associated to the B-splines $N_m(x)$.

PROPOSITION 10. - *Let $m \in \mathbb{N}$, N_m be the B-spline of degree m*

$$(\hat{N}_m(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^{m+1})$$

and $\varphi \in L^2$ be compactly supported. Then the following assertions are equivalent :

(A1) *φ is a dual scaling function associated to N_m.*
(A2) *the function $\Delta^{m+1}\varphi$ (where $\Delta f = f(x+1) - f(x)$ and $\Delta^{k+1} f = \Delta(\Delta^k f)$) can be written as $\Delta^{m+1}\varphi = (\frac{d}{dx})^{m+1}\Phi$ where Φ is a compactly supported interpolating scaling function such that $\Phi \in H^{m+1}$.*

Proof. (A1) $\Rightarrow$ (A2) is obvious : we have $\hat{\Phi} = \overline{\hat{N}_m(\xi)} \hat{\varphi}$, hence $\Phi = \varphi * N_m(-x)$; thus Φ is compactly supported and interpolating ; it is easy to see moreover that $\Phi(\frac{x}{2})$ belong to the linear span of the $\Phi(x-k)$'s. (A2) $\Rightarrow$ (A1) is a consequence of Proposition 9. ■

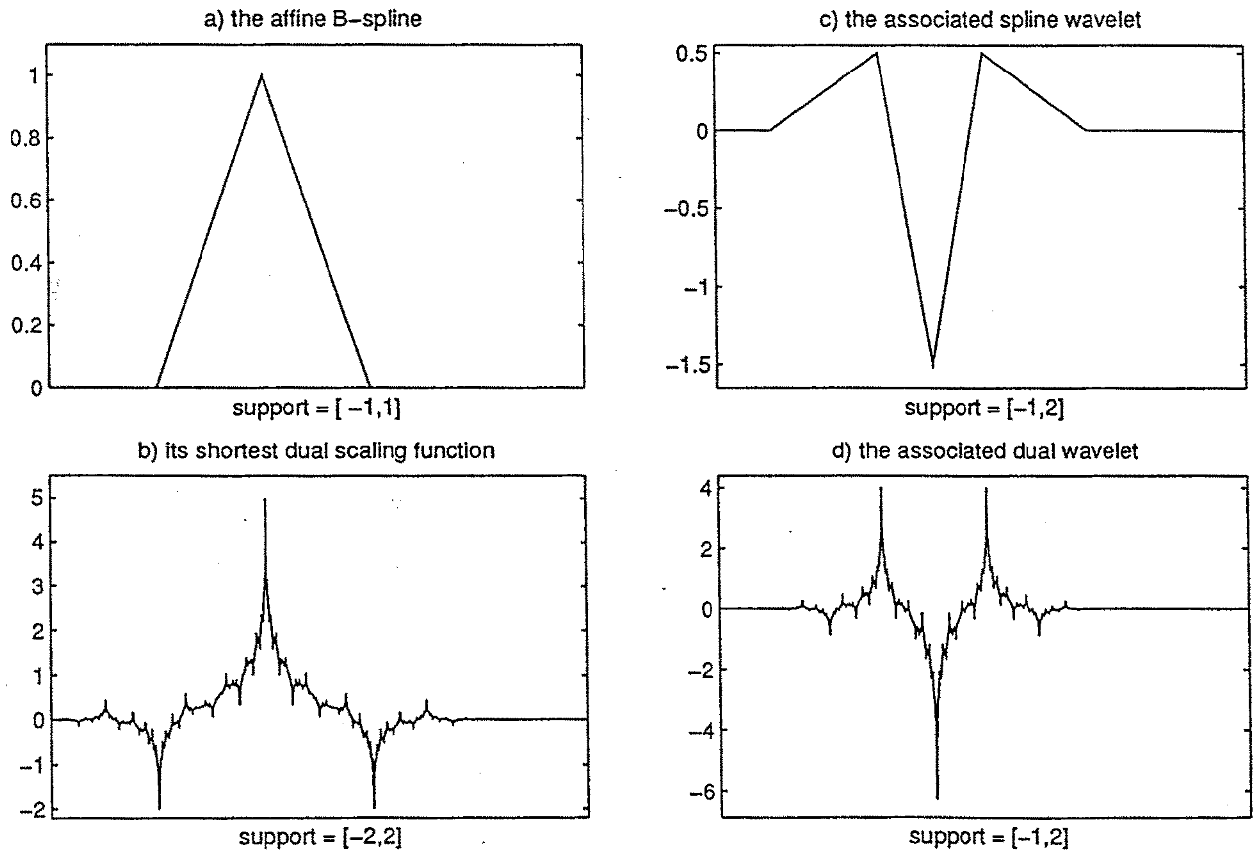
If Φ is of the form $\Phi = {}_N\varphi(x) *_N \varphi(-x)$ for a Daubechies function ${}_N\varphi$, then Φ is symmetric with a symmetry center at $x = 0$, while φ will be symmetric with a symmetry center at $x = \frac{m+1}{2}$.

Thus, we see that if φ is a dual scaling function associated to N_m, then the length of $\operatorname{Supp}\varphi$ is at least $m+2$ (since $\Phi \in H^{m+1}$ and is interpolating, we know that the filter M_0 associated to Φ contains a factor $\left(\frac{1+e^{-i\xi}}{2}\right)^{m+2}$, hence that the support of Φ has length at least $m+2+m+1=2m+3$ [see Theorem 2]), while we know that we may find a solution φ with length of support $O\left(m^{\frac{\ell n 12}{\ell n 4/3}}\right)$ as $m \to +\infty$ (see Proposition 2 again : write φ as $\Delta^{m+1}\varphi = (\frac{d}{dx})^{m+1}{}_N\Phi$ where ${}_N\Phi$ is the interpolating scaling function with minimal support associated to N : we know that $\mathrm{Supp}_N\Phi = [-2N-1, 2N+1]$ and that ${}_N\Phi \in C^{\alpha_N}$ where $\alpha_N \sim N\left(2-\frac{\ell n 3}{\ell n 2}\right)$; thus we may choose $N = O\left(\frac{m}{2-\frac{\ell n 3}{\ell n 2}}\right)$ and the length of the support of our solution is $4N+2-m-1 = O\left(m^{\frac{\ell n 12}{\ell n 4/3}}\right)$. Thus we approximately get a length of the support of φ nine times greater than the one of N_m ; it would be of course of great interest getting a much smaller support.

A very nice property of those dual scaling functions is that the associated filters are very easy to compute (if φ is such that $\Delta^{m+1}\varphi = \left(\frac{d}{dx}\right)^{m+1} {}_N\Phi$, then the associated filter m_0 is just $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N \left(\frac{1+e^{i\xi}}{2}\right)^{N-m-1} \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{1-\cos\xi}{2}\right)^k$ and there is no need of Riesz lemma or another algebraic algorithm to get the coefficients.

Example n° 11. The shortest dual scaling function associated to $N_1(x)$ is associated to the filter $\left(\frac{1+e^{-i\xi}}{2}\right)^2 \left(1+2\cdot\frac{1-\cos\xi}{2}\right) = -\frac{1}{8}e^{-3i\xi} + \frac{1}{4}e^{-2i\xi} + \frac{3}{4}e^{-i\xi} + \frac{1}{4} - \frac{1}{8}e^{i\xi}$. ■

Figure 4: bi-orthogonal wavelets



Chapter 6

WAVELETS AND FUNCTIONAL ANALYSIS

Wavelets and functional analysis is a theme which has been fully explored in the beginning of the theory. Results were established with the tools developped in real harmonic analysis in the 70's (and which are to be found in the books by E. M. Stein [STEW], [STE1], [STE2], Yves Meyer [COIM1], [MEY2], [MEY3], [MEY4] and Frazier, Jawerth and Weiss [FRJW]) ; these tools are mainly the theory of Calderón-Zygmund singular integral operators, the atomic decompositions and interpolation theorems. We develop mainly the example of Lebesgue spaces (using the Calderón-Zygmund splitting) and of Besov spaces (using a molecular approach).

1. Bi-orthogonal wavelets and functional analysis.

The functional analysis provided by wavelet bases is often given in the following presentation. We start from two conjugate filters m_0, $m_0^* \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$ satisfying $m_0(0) = m_0^*(0) = 1$ and

(1) $$m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1$$

and define the tempered distributions φ, φ^* by their Fourier transform

(2) $$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}) \quad \text{and} \quad \hat{\varphi}^*(\xi) = \prod_{j=1}^{\infty} m_0^*(\frac{\xi}{2^j})$$

and the associated wavelets ψ, ψ^* by

(3) $$\hat{\psi}(2\xi) = e^{-i\xi}\bar{m}_0^*(\xi+\pi) \quad \text{and} \quad \hat{\psi}^*(2\xi) = e^{-i\xi}\bar{m}_0(\xi+\pi)\hat{\varphi}^*(\xi).$$

In all cases we have to deal with, the function $m_0(\xi)\bar{m}_0^*(\xi)$ is non-negative-valued ($\forall \xi \in \mathbb{R}$, $m_0(\xi)\bar{m}_0^*(\xi) \geq 0$) and (1) implies that $\int \hat{\varphi}(\xi)\bar{\hat{\varphi}}^*(\xi)d\xi \leq 2\pi$ ($\hat{\varphi}\bar{\hat{\varphi}}^*$ being also non-negative-valued). If $\int \hat{\varphi}(\xi)\bar{\hat{\varphi}}^*(\xi)d\xi = 2\pi$ (which is generally ensured through Cohen's criterion), we have

(4) $$\sum_{k\in\mathbb{Z}} \hat{\varphi}(\xi+2k\pi)\bar{\hat{\varphi}}^*(\xi+2k\pi) = 1 \quad \text{a.e.}$$

and similarly

(6) $$< \psi(x) \mid \varphi^*(x-k) >=< \varphi(x) \mid \psi^*(x-k) >= 0 \quad \text{for} \quad k \in \mathbb{Z}$$

(7) $$< \psi(x) \mid \psi^*(x-k) >= \delta_{k,0} \quad \text{for } k \in \mathbb{Z}.$$

We know that $\hat{\varphi}$ and $\hat{\varphi}^*$ are smooth functions which have a slow growth at infinity ($\exists N \ \ \forall \xi, \mid \hat{\varphi}(\xi) \mid + \mid \hat{\varphi}^*(\xi) \mid \leq C(1+ \mid \xi \mid)^N$). Thus we may define operators P_j and Q_j from S to S' by :

(8) $$\widehat{P_j f}(\xi) = \left(\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi 2^j) \bar{\hat{\varphi}}^*(\frac{\xi}{2^j} + 2k\pi) \right) \hat{\varphi}(\frac{\xi}{2^j})$$

(9) $$\widehat{Q_j f}(\xi) = \left(\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi 2^j) \bar{\hat{\psi}}^*(\frac{\xi}{2^j} + 2k\pi) \right) \hat{\psi}(\frac{\xi}{2^j})$$

[which formally correspond to

(10) $$P_j f = \sum_{k\in\mathbb{Z}} 2^j < f \mid \varphi^*(2^j x - k) > \varphi(2^j x - k)$$

(11) $$Q_j f = \sum_{k\in\mathbb{Z}} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j x - k)].$$

P_j and Q_j can be extended as operators on the space $E = \{T \in S'/\hat{T}$ is a locally bounded function and $\forall j \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} \mid \hat{T}(\xi + 2k\pi 2^j) \hat{\varphi}^*(\frac{\xi}{2^j} + 2k\pi) \mid \in L^\infty\}.$$

In that case, we have $P_j \circ P_j = P_j$ and P_j is a projection operator on

$$V_j = \{f \in E / \exists \mu \in L^\infty(\mathbb{R}/2\pi\mathbb{Z}) \hat{f} = \hat{\mu}\hat{\varphi}(\frac{\xi}{2^j})\}.$$

Moreover $P_{j+1} \circ P_j = P_j$ (since $V_j \subset V_{j+1}$) and $P_j \circ P_{j+1} = P_j$ (since

$$P_{j+1} f = 0 \Leftrightarrow \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi 2^{j+1}) \bar{\hat{\varphi}}^*(\frac{\xi}{2^{j+1}} + 2k\pi) = 0 \text{ a.e.}$$
$$\Rightarrow \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi 2^j) \bar{\hat{\varphi}}^*(\frac{\xi}{2^j} + 2k\pi) = 0 \text{ a.e. } \Rightarrow P_j f = 0$$

hence $P_j(f - P_{j+1} f) = 0$). At last, $Q_j = P_{j+1} - P_j$ and $Q_j \circ Q_j = Q_j$.

Thus, we have "natural" definitions for the projection operators P_j and Q_j. Now, if B and B^* are two Banach spaces of distribution such that we have a duality bracket between B and B^* induced by the bracket between S and S', we may ask the following questions :

(i) if $P_j(E \cap B) \subset E \cap B$, does P_j operate on B as a bounded projection operator ? If so, we note V_j for the range of P_j and V_j^* for the orthogonal space in B^* to $\operatorname{Ker} P_j$. Note that $V_j \subset V_{j+1}$ and $V_j^* \subset V_{j+1}^*$.

(ii) Similarly, we note W_j for the range of Q_j in B and W_j^* for the orthogonal space to $\operatorname{Ker} Q_j$.

(iii) If the distributions $\varphi(2^j x - k)$ belong to B and $\varphi^*(2^j x - k)$ to B^*, do we have for all $f \in B$

$$P_j f = \sum_{k \in \mathbb{Z}} 2^j < f \mid \varphi^*(2^j x - k) >_{B,B^*} \varphi(2^j x - k)$$

where the series converges strongly in B ? And in that case, is the family $(\varphi(2^j x - k))_{k \in \mathbb{Z}}$ a Riesz basis for V_j ? (Equivalently, do the operators $P_{j,\eta}$, $\eta \in \ell^\infty(\mathbb{Z})$, $\| \eta \|_\infty \leq 1$, form an equicontinuous family of operators on B, where

$$P_{j,\eta} f = \sum_{k \in \mathbb{Z}} \eta_k 2^j < f \mid \varphi^*(2^j x - k) >_{B,B^*} \varphi(2^j x - k) \text{ ?}).$$

(iv) Suppose that the P_j are bounded on B. Do we have $\lim_{j \to +\infty} P_j f = f$ in B for all $f \in B$? (This is equivalent to the equicontinuity of $(P_j)_{j \geq 0}$ and the density of $\bigcup_{j \in \mathbb{Z}} V_j$. Note that B is separable in that case).

(v) Do we have $\lim_{j \to -\infty} P_j f = 0$ in B ? If B is separable and B^* is the dual space of B, this is equivalent to the equicontinuity of $(P_j)_{j \leq 0}$ and $\bigcap_{j \in \mathbb{Z}} V_j^* = \{0\}$ (and implies of course $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$). Indeed, if $(P_j)_{j \leq 0}$ is equicontinuous and $\bigcap_{j \in \mathbb{Z}} V_j^* = \{0\}$, we write for $f \in B$, $\| P_j f \|_B = < P_j f \mid g_j >_{B,B^*}$ with $\| g_j \|_{B^*} = 1$, and $< P_j f \mid g_j > = < f \mid P_j^*(g_j) >$. But the range of P_j^* is V_j^* ; moreover $\| P_j^*(g_j) \|_{B^*} \leq \| P_j \|_{B,B}$, so that $(P_j^*(g_j))_{j \leq 0}$ is pre-compact in the weak topology of B^*. If g_∞ is an accumulation point of $P_j^*(g_j)$, necessarily $g_\infty \in V_j^*$, hence $g_\infty = 0$. Thus we may conclude $\lim_{j \to -\infty} P_j f = 0$. Conversely, if $\lim_{j \to -\infty} P_j f = 0$ for all f, we may conclude by Banach-Steinhaus' theorem that the P_j are equicontinuous, while if $g \in \cap V_j^*$ and $f \in B$ is such that $< f \mid g > = \| g \|_{B^*}$, $\| f \|_B = 1$, we have $< f \mid g > = < P_j f \mid g >$ for all j, hence $g = 0$.

(vi) If $(\psi(2^j x - k))_{k \in \mathbb{Z}}$ is an unconditional basis for W_j for all j, and $(\varphi(2^j x - k))_{k \in \mathbb{Z}}$ an unconditional basis for V_j, if $P_j f \to f$ as $j \to +\infty$, then we have a Schauder basis for B :

$$\text{(12)} \qquad \forall f \in B, \ \ f = \lim_{N \to +\infty} \sum_{k \in \mathbb{Z}} < f \mid \varphi^*(x - k) >_{B,B^*} \varphi(x - k) +$$

$$\sum_{j=0}^{N} \sum_{k \in \mathbb{Z}} 2^j < f \mid \psi^*(2^j x - k) >_{B,B^*} \psi(2^j x - k).$$

The next question is then to check whether we have an unconditional basis. It is equivalent to check the equicontinuity of the operators T_ϵ, $\epsilon \in \ell^\infty(\mathbb{N}) \oplus \ell^\infty(\mathbb{N} \times \mathbb{Z})$, defined by :

$$T_\epsilon f = \sum_{k\in\mathbb{Z}} \epsilon_k < f \mid \varphi^*(x-k) >_{B,B^*} \varphi(x-k) +$$

$$\sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} 2^j \epsilon_{j,k} < f \mid \psi^*(2^j x - k) >_{B,B^*} \psi(2^j x - k).$$

(vii) We have a similar question, when $\lim\limits_{+\infty} P_j f = f$ and $\lim\limits_{-\infty} P_j f = 0$, so that $f = \lim\limits_{N,M\to+\infty} \sum_{j=-N}^{M} Q_j f$, for the family $(2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$.

A way to answer the question of the continuity is often given by the theory of Calderón-Zygmund operators, a class of operators which have been extensively studied by A. P. Calderón, E. Stein, R. Coifman and Y. Meyer [STE1], [COIM1].

DEFINITION 1. - *A Calderón-Zygmund operator of class ϵ $(0 < \epsilon \leq 1)$ is a bounded linear operator T on L^2 such that its distribution kernel $K(x,y) \in \mathcal{S}'(\mathbb{R} \times \mathbb{R})$ is given outside from the diagonal by a locally Hölderian function which satisfies for some constants $C \geq 0$ and $\epsilon \in (0,1]$:*

(13) $$| K(x,y) | \leq C \frac{1}{| x-y |}$$

(14)
$$| K(x+h,y) - K(x,y) | + | K(x,y+h) - K(x,y) | \leq C \frac{| h |^\epsilon}{| x-y |^{1+\epsilon}} \text{ for } | h | < \frac{1}{2} | x-y | .$$

The point is that the opeators T_ϵ are often Calderón-Zygmund operators :

LEMMA 1. - *Let ψ, ψ^* be continuous functions such that for some $\alpha, \beta > 0$:*

(15) $$| \psi(x) \leq \frac{1}{(1+ | x |)^{1+\alpha}} \quad \textit{and} \quad | \psi^*(x) | \leq \frac{1}{(1+ | x |)^{1+\alpha}}$$

(16) $$| \psi(x+h) - \psi(x) | \leq | h |^\beta \quad \textit{and} \quad | \psi^*(x) - \psi^*(x+h) | \leq | h |^\beta$$

(17) $$\int \psi \, dx = \int \psi^* dx = 0.$$

Then for $\eta \in \ell^\infty(\mathbb{Z} \times \mathbb{Z})$ the operator T_η defined by $T_\eta f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \eta_{j,k} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j - k)$ is a Calderón-Zygmund operator of class ϵ_0, where ϵ_0 depends *only on α and β. Moreover, the kernel $K_\eta(x,y)$ of T_η satisfies for some constant C_0 which depends only on α, β and ϵ_0 :*

$$\| T_\eta \|_{L^2,L^2} \leq C_0 \| \eta \|_\infty, \quad \sup_{x\neq y} | x-y | \, K_\eta(x,y) | \leq C_0 \| \eta \|_\infty$$

and

$$\sup_{x\neq y, |h|<\frac{1}{2}|x-y|} | h |^{-\epsilon_0} | x-y |^{1+\epsilon_0} \{| K_\eta(x,y) - K(x,y+h) | + | K_\eta(x,y) - K_\eta(x+h,y) |\}$$

$$\leq C_0 \| \eta_0 \| .$$

Proof. The L^2-boundedness is given by the vaguelettes lemma (see chapter 2). The estimates on the size of the kernel are classical. We prove for instance the estimate for $K_\eta(x+h,y) - K_\eta(x,y)$. It is enough to prove that for $2^j \mid h \mid \geq 1$

$$A_j = \left| \sum_{k\in\mathbb{Z}} \eta_{j,k} 2^j (\psi(2^j x - k) - \psi(2^j x + 2^j h - k)) \bar{\psi}^*(2^j y - k) \right|$$

$$\leq \| \eta \|_\infty C_0' \frac{2^j}{(2^j \mid x-y \mid)^{1+\epsilon_0}}$$

while for $2^j \mid h \mid \leq 1$, $A_j \leq \| \eta \|_\infty C_0' 2^j \frac{(2^j|h|)^{2\epsilon_0}}{(2^j|x-y|)^{1+\epsilon_0}}$ and then to sum over j to get the estimate.

Now, to estimate A_j, it is enough to check that :

$$| \psi(2^j x - k) - \psi(2^j x + 2^j h - k) | \leq$$

$$C \inf(1, 2^j \mid h \mid)^{2\epsilon_0} \left\{ \frac{1}{(1+ \mid 2^j x - k \mid)^{1+\epsilon_0}} + \frac{1}{(1+ \mid 2^j x + 2^j h - k \mid)^{1+\epsilon_0}} \right\}$$

and $| \psi^*(2^j y - k) | \leq C \frac{1}{(1+|2^j y - k|)^{1+\epsilon_0}}$, and to sum over k. If $| 2^j h | \geq 1$, it is a direct consequence of (15) and is valid for all $\epsilon_0 \in (0,\alpha)$. If $| 2^j h | \leq 1$, we have by (15),

$$| \psi(2^j x - k) - \psi(2^j x + 2^j h - k) | \leq \frac{C}{(1+ \mid 2^j x - k \mid)^{1+\alpha}}$$

and by (16), $| \psi(2^j x - k) - \psi(2^j x + 2^j h - k) | \leq C \mid 2^j h \mid^\beta$, hence

$$\psi(2^j x - k) - \psi(2^j x + 2^j h - k) | \leq \left(C \frac{1}{(1+ \mid 2^j x - k \mid)^{1+\alpha}} \right)^\lambda (\mid 2^j h \mid)^{(1-\lambda)\beta}$$

for all $\lambda \in [0,1]$, and thus

$$| \psi(2^j x - k) - \psi(2^j x + 2^j h - k) | \leq C(\epsilon_0) \frac{(2^j \mid h \mid)^{2\epsilon_0}}{(1+ \mid 2^j x - k \mid)^{1+\epsilon_0}}$$

for any $\epsilon_0 \in \left(0, \frac{\alpha\beta}{2(1+\alpha)+\beta}\right)$.

2. Wavelets and Lebesgue spaces.

A striking feature of wavelets is the fact that they provide unconditional bases for Lebesgue spaces L^p $(1 < p < +\infty)$, while Fourier series provide an unconditional basis only for L^2.

The case of L^2 is very clear from the results of the preceding chapters :

PROPOSITION 1. - *Let m_0, and m_0^* be two scaling filters in $H^{1/2+\epsilon}(\mathbb{R}/2\pi\mathbb{Z})$ $(\epsilon > 0)$ such that $m_0(\xi)\bar{m}_0^*(\xi)+m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1$ and let φ, φ^* be the associated ϵ-localized scaling functions. Let $\hat{\psi}(\xi) = e^{-i\xi k}\bar{m}_0^*(\frac{\xi}{2}+\pi)\hat{\varphi}(\frac{\xi}{2})$ and $\hat{\psi}^*(\xi) = e^{-i\frac{\xi}{2}}\bar{m}_0(\frac{\xi}{2}+\pi)\hat{\varphi}^*(\frac{\xi}{2})$. Then $< \varphi(x-k) \mid \varphi^*(x-\ell) >= \delta_{k,\ell}$, and the families $(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^jx-k))_{j\geq 0,k\in\mathbb{Z}}$ and $(\varphi^*(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi^*(2^jx-k))_{j\geq 0,k\in\mathbb{Z}}$ are dual unconditional bases of $L^2(\mathbb{R})$. Similarly, the families $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ and $(2^{j/2}\psi^*(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ are dual unconditional bases of L^2. Moreover for $f \in L^2$ the three following norms are equivalent :*

(i) $\left(\int \mid f \mid^2 dx\right)^{1/2}$

(ii) $\left(\sum_{k\in\mathbb{Z}} |< f \mid \varphi^*(x-k) >|^2 + \sum_{j=0}^{+\infty}\sum_{k\in\mathbb{Z}} 2^j \, |< f \mid \psi^*(2^jx-k) >|^2\right)^{1/2}$

(iii) $\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |< f \mid \psi^*(2^jx-k) >|^2\right)^{1/2}$.

Proof. See chapter 4. Define θ_p and θ_p^* by $(\hat{\theta}_p\xi) = \prod_{j=1}^p m_0(\frac{\xi}{2^j})\chi_{[-\pi,\pi]}(\frac{\xi}{2^p})$ and $\hat{\theta}_p^*(\xi) = \prod_{j=1}^p m_0^*(\frac{\xi}{2^j})\chi_{[-\pi,\pi]}(\frac{\xi}{2^p})$. From (1), we have $< \theta_p(x-k) \mid \theta_p^*(x-\ell) >= \delta_{k,\ell}$. We know that $\theta_p \to \varphi$ in L^2 and $\theta_p^* \to \varphi^*$ in L^2 as $p \to +\infty$, so that $< \varphi(x-k) \mid \varphi^*(x-\ell) >= \delta_{k,\ell}$. Moroever, we know that φ, φ^* belong to $L^2((1+\mid x \mid)^{1+2\epsilon}dx)$, so that $P_jf = \sum_{k\in\mathbb{Z}} 2^j < f \mid \varphi^*(2^jx-k) > \varphi(2^jx-k)$ is well defined on L^2. We have seen that UV_j is dense in L^2 and that $\cap V_j^* = \{0\}$; the P_j are obviously an equicontinuous family since $P_j = D_jP_0D_j^{-1}$ where D_j is the isometry $D_jf = 2^{j/2}f(2^jx-k)$. Thus we already have Schauder bases $(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^jx-k))_{j\geq 0,k\in\mathbb{Z}}$ and $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$. Moreover $\psi, \psi^* \in H^\alpha$ for some positive α and $\int \psi \, dx = 0$ and we may apply the vaguelettes lemma in order to prove that

$$\| \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \lambda_{j,k}2^{j/2}\theta(2^jx-k) \|_2^2 \leq C \sum\sum \mid \lambda_{j,k} \mid^2$$

with $\theta = \psi$ or ψ^*. ∎

We now turn to the case of $p \neq 2$. For sake of simplicity, we consider regular scaling functions :

THEOREM 1. - *Let m_0, $m_0^* \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})$ be regular scaling filters satisfying*

$$m_0(\xi)\bar{m}_0^*(\xi) + m_0(\xi+\pi)\bar{m}_0^*(\xi+\pi) = 1$$

and φ, φ^* the associated scaling functions (given by (2)) and ψ, ψ^* the associated wavelets (given by (5)).

Let $p \in (1,2)$ and $q = \frac{p}{p-1}$ the conjugate Lebesgue exponent of p. If for some $q' \in (q, +\infty]$ we have $\varphi^* \in L^{q'}$, then the family $(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^j x - k))_{j\geq 0, k\in\mathbb{Z}}$ is an unconditional basis for L^p, and its dual basis $(\varphi^*(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi^*(2^j x - k))_{j\geq 0, k\in\mathbb{Z}}$ is an unconditional basis for L^q. The same results holds for $(2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z}, k\in\mathbb{Z}}$ and $(2^{j/2}\psi^*(2^j x - k))_{j\in\mathbb{Z}, k\in\mathbb{Z}}$. Moreover we have equivalent norms N_1, N_2, N_3 on L^p and N_1^*, N_2^*, N_3^* on L^q given by :

$$N_1(f) = \left(\int |f|^p \, dx\right)^{1/p}$$

$$N_2(f) = \left(\sum_{k\in\mathbb{Z}} |< f \mid \varphi^*(x-k) >|^p\right)^{\frac{1}{p}} + \left(\int (\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} 2^j \, |< f \mid \psi^*(2^j x - k) >|^2 \, \chi_{[0,1]}(2^j x - k))^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$

$$N_3(f) = \left(\int \left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^j \, |< f \mid \psi^*(2^j x - k) >|^2 \, \chi_{[0,1]}(2^j x - k)\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$

and similarly

$$N_1^*(f) = \left(\int |f|^q \, dx\right)^{1/q}$$

$$N_2^*(f) = \left(\sum_{k\in\mathbb{Z}} |< f \mid \varphi^*(x-k) >|^q\right)^{\frac{1}{q}} + \left(\int (\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} 2^j \, |< f \mid \psi^*(2^j x - k) >|^2 \, \chi_{[0,1]}(2^j x - k))^{\frac{q}{2}} dx\right)^{\frac{1}{q}}$$

$$N_3^*(f) = \left(\int \left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^j \, |< f \mid \psi^*(2^j x - k) >|^2 \, \chi_{[0,1]}(2^j x - k)\right)^{\frac{q}{2}} dx\right)^{\frac{1}{q}}.$$

REMARKS. - (i) If $p \in (2, +\infty)$, just interchange φ and φ^*, and p and q.

(ii) For $p = 1$ or $+\infty$, we cannot have a similar result since L^1 and L^∞ don't have unconditional bases. See below for the discussion of the case $p = 1$ for the norms N_2 and N_3. ■

Proof. By assumption, we have φ, φ^* rapidly decaying at infinity in L^2 : $\forall k \in \mathbb{N}$, $x^k\varphi$ and $x^k\varphi^* \in L^2$. We get easily that $x^k\varphi \in L^{p''}$ for all $p'' \in [1,2]$, while $x^k\varphi^* \in L^{q''}$ for all $q'' \in [1, q')$. Indeed, we have for $q'' < 2$

$$\int | x^k \varphi^* |^{q''} x < \left(\int x^{\frac{2k}{q''}} (1+ | x |)^{\frac{2N}{q''}} | \varphi^* |^2 \, dx \right)^{q''/2} \left(\int \frac{dx}{(1+ | x |^{\frac{2}{2-q''} N})} \right)^{1-\frac{q''}{2}} ,$$

and we conclude by choosing $N > 1 - \frac{q''}{2}$; for $2 < q'' < q'$ we have

$$\int | x^k \varphi^* |^{q''} x < \left(\int | \varphi^* |^{q'} \, dx \right)^{\frac{q''-2}{q'-2}} \left(\int | x |^{kq'' \frac{q'-1}{q'-q''}} | \varphi^* |^2 \, dx \right)^{\frac{q'-q''}{q'-2}} .$$

Thus we get that P_j is well defined as a projection operator from L^p onto V_j :

$$\begin{aligned} &\| \sum_k c_k \varphi(x-k) \|_p^p \\ &\leq \int \left(\sum_k | c_k |^p (1+ | x-k |)^{Np} | \varphi(x-k |^p \right) \left(\sum_k (1+ | x-k |)^{-Nq} \right)^{\frac{p}{q}} dx \\ &\leq \| \sum_k (1+ | x-k |)^{-Nq} \|_\infty^{\frac{p}{q}} \| (1+ | x |)^N \varphi \|_p \sum_k | c_k |^p \end{aligned}$$

and we conclude by choosing $N > \frac{1}{q}$; similarly we get

$$\| \sum_k c_k^* \varphi^*(x-k) \|_q^q \leq C \sum_k | c_k^* |^q .$$

The family $(P_j)_{j\in\mathbb{Z}}$ is equicontinuous on L^p since $P_j = D_j^* \circ P_0 \circ D_j^{-1}$ with $D_j f = 2^{\frac{j}{p}} f(2^j x)$ (so that D_j is an isometry of L^p). The density of $\cup V_j$ in L^p and $\cup V_j^*$ in L^q is easy : if $\theta \in \overline{\cup V_j}^{\perp}$, choose $\omega \in \mathcal{C}_c^\infty$ such that $\| \theta - \omega \|_q \leq \epsilon \| \theta \|_q$ (hence $\leq \frac{\epsilon}{1-\epsilon} \| \omega \|_q$) and write

$$|< \omega \mid P_j((\operatorname{sgn} \omega) \mid \omega |^{q-1}) >| \leq \frac{\epsilon}{1-\epsilon} \| P_j \|_{L^p, L^p} \| \omega \|_q^q ;$$

since ω and $(\operatorname{sgn} \omega) \mid \omega |^{q-1} \in L^2$, we get $\lim_{j\to+\infty} < \omega \mid P_j(\operatorname{sgn} \omega \mid \omega |^{q-1}) > = \| \omega \|_q^q$ and thus, if ϵ is small enough, $\omega = 0$ and $\theta = 0$. Similarly, $\bigcap_{j\in\mathbb{Z}} V_j = 0$ in L^p and $\bigcap_{j\in\mathbb{Z}} V_j^* = 0$ in L^q is easy : since we have $\ell^p \subset \ell^2$, we know that $V_j \subset L^2$ and thus $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ is a consequence of the same property in L^2 ; then we know that for all $f \in L^q$, $\lim_{j\to-\infty} P_j^* f = 0$ in L^q, and thus $\bigcap_{j\in\mathbb{Z}} V_j^* = \{0\}$.

We thus know that we have Schauder bases for L^p $(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^j x - k))_{j\geq 0, k\in\mathbb{Z}}$ or $(2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z}, k\in\mathbb{Z}}$. In order to prove that they are unconditional, we will study the operators

$$U_\eta f = \sum_j \sum_k \eta_{j,k} 2^j < f \mid \psi^*(2^j x - k) > \theta(2^j x - k), \quad \eta \in \ell^\infty(\mathbb{Z} \times \mathbb{Z})$$

$$V_\eta f = \sum_j \sum_k \eta_{j,k} 2^j < f \mid \psi(2^j x - k) > \theta(2^j x - k)$$

where $\theta = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. We will show that for some constant $C \geq 0$

$$\forall \eta \in \ell^\infty(\mathbb{Z} \times \mathbb{Z}),\ \forall f \in L^p, \quad \| U_\eta f \|_p \leq C \| f \|_p \| \eta \|_\infty \tag{18}$$

$$\forall \eta \in \ell^\infty(\mathbb{Z} \times \mathbb{Z}),\ \forall f \in L^q, \quad \| V_\eta f \|_q \leq C \| f \|_q \| \eta \|_\infty . \tag{19}$$

From (18) and (19) we conclude easily that we have unconditional bases for L^p or L^q. First, we sum estimate (18) for all $r_{j,k} = \pm 1$:

$$\int_{\{-1,1\}^{\mathbb{Z}}} \| U_\eta f \|_p^p \, d\eta \leq C^p \| f \|_p^p$$

and thus

$$\int \left[\int_{\{-1,1\}^{\mathbb{Z}}} | U_\eta f(x) |_p^p \, d\eta \right] dx \leq C^p \| f \|_p^p .$$

We then use Khintchin's inequalities to get :

$$\left(\sum_j \sum_k 4^j |< f \mid \psi^*(2^j x - k) >|^2 \chi_{[0,1]}(2^j x - k) \right)^{p/2} \leq \frac{1}{C_p} \int_{\{-1,1\}^{\mathbb{Z}}} | U_\eta f(x) |_p^p \, d\eta$$

and thus $N_3(f) \leq C \| f \|_p$. Similarly we have $N_3^*(f) \leq C \| f \|_q$.

Moreover we have for $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^q$:

$$\begin{aligned} < f \mid g > &= \sum_j \sum_k 2^j < f \mid \psi^*(2^j x - k) >< \psi(2^j x - k) \mid g > \\ &= \int \sum_j \sum_k 4^j < f \mid \psi^*(2^j x - k) >< \psi(2^j x - k) \mid g > \chi_{[0,1]}(2^j x - k) dx, \end{aligned}$$

hence

$$\begin{aligned} |< f, g >| \leq & \int \left(\sum_j \sum_k 4^j |< f \mid \psi^*(2^j x - k) >|^2 \chi_{[0,1]}(2^j x - k) \right)^{1/2} \\ & \qquad \left(\sum_j \sum_k 4^j |< g \mid \psi(2^j x - k) >|^2 \chi_{[0,1]}(2^j x - k) \right)^{\frac{1}{2}} dx \\ \leq & \; N_3(f) N_3^*(g) \end{aligned}$$

Thus $\| f \|_p \leq C N_3(f)$ and $\| g \|_q \leq C N_3^*(g)$. The equivalence between $\| f \|_p$ and $N_3(f)$ gives the unconditionality of the basis $(2^{j/2}\psi(2^j x - k))_{j \in \mathbb{Z}, k \in \mathbb{Z}}$.

We now prove (18) and (19). This will be done by a tool commonly used for proving L^p continuity of Calderón-Zygmund operators, namely the Calderón-Zygmund splitting of a function f into $f = g + b$ where the "good" function g has small L^2-norm and the "bad" function b has small support and oscillates.

We first prove (18) and (19) in the case where ψ and ψ^* are compactly supported : $\operatorname{Supp}\psi \cup \operatorname{Supp}\psi^* \subset [-M, M]$ $(M \geq 1)$. For proving the continuity of V_η on L^q, we shall prove that V_η^* (the adjoint of V_η) is of weak type $(1,1)$, hence due to the Marcinkiewicz interpolation theorem and since V_η^* is bounded on L^2 [due to the vaguelettes lemma] we get that V_η^* is bounded on L^r for all $r \in (1, 2]$, hence V_η is bounded on L^r for all $r \in [2, +\infty)$.

We suppose $\| \eta \|_\infty \leq 1$. In order to show that V_η^* is of weak type $(1,1)$, we have to show that for some constant C we have for all $\epsilon > 0$ and all $f \in L^1$

$$| \{ x \in \mathbb{R} / \mid V_\eta^* f(x) \mid > \epsilon \} | \leq \frac{C}{\epsilon} \| f \|_1 . \tag{20}$$

This is done by using the Calderón-Zygmund splitting of f : define Ω as the union of those dyadic intervals $I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j}]$ such that $\frac{1}{|I_{j,k}|} \int_{I_{j,k}} \mid f \mid dx > \epsilon$ and decompose Ω in maximal dyadic intervals $\Omega = \bigcup_{(j,k)\in\Lambda} I_{j,k}$; then b is defined as

$$b = \sum_{(j,\ell)\in\Lambda} f\chi_{I_{j,k}} - \left(\frac{1}{\mid I_{j,k} \mid} \int_{J_{j,k}} f \, dx \right) \chi_{I_{j,k}} .$$

Since $I_{j,k}$ is maximal, the dyadic interval $I_{j,k}^*$ of length 2^{-j+1} containing $I_{j,k}$ is not contained in Ω, and thus $\int_{I_{j,k}} \mid f \mid dx \leq 2\epsilon \mid I_{j,k} \mid$. Thus $g = f - b$ satisfies $\| g \|_\infty \leq 2\epsilon$ (since $\mid f \mid \leq \epsilon$ on $\mathbb{R} \backslash \Omega$) and

$$\int \mid g \mid dx = \int_{\mathbb{R}\backslash\Omega} \mid f \mid dx + \sum_{(j,k)\in\Lambda} \mid \int_{I_{j,k}} f \, dx \mid \leq \| f \|_1 .$$

The $I_{j,k}$, $(j,k) \in \Lambda$, are disjoint and we have

$$\mid \Omega \mid = \sum_{(j,k)\in\Lambda} \mid I_{j,k} \mid \leq \frac{1}{\epsilon} \int_\Omega \mid f \mid dx.$$

Write $V_\eta^* f = V_\eta^* g + V_\eta^* b$. We have obviously :

$$V_\eta^* \left(f\chi_{I_{j,k}} - \left(\frac{1}{\mid I_{j,k} \mid} \int_{I_{j,k}} f \, dx\right)\chi_{I_{j,k}} \right)$$

$$= \sum_{I_{\ell,r} \subset I_{j,k}} \eta_{\ell,r} 2^\ell < f\chi_{I_{j,k}} \mid \theta(2^\ell x - r) > \psi(2^\ell x - r)$$

and thus

$$\operatorname{Supp} V_\eta^* \left(f\chi_{I_{j,k}} - \frac{1}{\mid I_{j,k} \mid} \left(\int_{I_{j,k}} f \, dx\right)\chi_{I_{j,k}} \right) \subset \left[\frac{k - M}{2^j}, \frac{M + 1 + k}{2^j} \right] = \tilde{I}_{j,k}$$

and

$$| \operatorname{Supp} V_\eta^*(b) | \leq \sum_{(j,k)\in\Lambda} | \tilde{I}_{j,k} | \leq (2M+1)\frac{1}{\epsilon}\int_\Omega | f | \, dx.$$

We thus get :

$$| \{x \in \mathbb{R} / \mid V_\eta^*(f)(x) \mid > \epsilon\} | \leq (2M+1)\frac{1}{\epsilon} \parallel f \parallel_1 + | \{x \in \mathbb{R} / \mid V_\eta^* g(x) \mid > \epsilon\} |$$

$$\leq (2M+1)\frac{1}{\epsilon} \parallel f \parallel_1 + \frac{1}{\epsilon^2} \parallel V_\eta^* g \parallel_2^2$$

and, since V_η^* is bounded on L^2 (with $\parallel V_\eta^* \parallel_{2,2}$ bounded by $C \parallel \eta \parallel_\infty$) and $\parallel g \parallel_2^2 \leq \parallel g \parallel_\infty \parallel g \parallel_1 \leq 2\epsilon \parallel f \parallel_1$, (20) is proved.

Similarly we prove that U_η is of weak-type (p'', p'') for any $p'' \in (p', p)$:

$$| \{x \in \mathbb{R} / \mid U_\eta f(x) \mid > \epsilon\} | \leq C_{p''} \frac{\parallel f \parallel_{p''}^{p''}}{\epsilon^{p''}}. \tag{21}$$

In order to prove (21), we follow an idea of Gripenberg [GRI2] and use a modified Calderón-Zygmund splitting of f, adapted to the wavelet basis. We define Ω as the union of those $I_{j,k}$ such that $\int_{I_{j,k}} | f |^{p''} dx > \epsilon \mid I_{j,k} \mid$ and decompose Ω as the union of the maximal dyadic intervals contained in Ω : $\Omega = \bigcup_{(j,k)\in\Lambda} I_{j,k}$. Then b is defined *with help of the projection operators* P_j as :

$$b = \sum_{(j,k)\in\Lambda} f\chi_{I_{j,k}} - P_j(f\chi_{I_{j,k}}).$$

The point is that for this choice of b we have :

$$U_\eta(b) = \sum_{(j,k)\in\Lambda} \left\{ \sum_{\ell \geq j} \sum_{[\frac{r-M}{2^\ell}, \frac{r+M}{2^\ell}] \cap I_{j,k} \neq \emptyset} \eta_{\ell,r} 2^\ell < f\chi_{I_{j,k}} \mid \psi^*(2^\ell x - r) > \theta(2^\ell x - r) \right\}$$

so that, once again, $\operatorname{Supp} U_\eta(b) \subset \bigcup_{(j,k)\in\Lambda} \left[\frac{k-M}{2^j}, \frac{k+M+1}{2^j}\right]$ and $| \operatorname{Supp} U_\eta(b) | \leq (2M+1)\frac{\|f\|_{p''}^{p''}}{\epsilon^{p''}}$.

Thus (21) will be proved when we have proved that $\parallel g \parallel_2^2 \leq C \parallel f \parallel_{p''}^{p''} \epsilon^{2-p''}$. This is quite obvious. We have :

$$P_j(f\chi_{j,k}) = \sum_{r\in\mathbb{Z}} 2^j < f\chi_{I_{j,k}} \mid \varphi^*(2^j x - r) > \varphi(2^j x - r)$$

where we know (chapter 3) that φ and φ^* can be chosen with compact support (say that $\operatorname{Supp}\varphi \cup \operatorname{Supp}\varphi^* \subset [-M, M]$). Since $\varphi^* \in L^{q''}$, we get

$$|< f\chi_{I_{j,k}} \mid \varphi^*(2^j x - r) >| \leq \parallel f\chi_{I_{j,k}} \parallel_{p''} \parallel \varphi^* \parallel_{q''} 2^{-j/q''} \leq \epsilon 2^{-j} \parallel \varphi^* \parallel_{q''} 2^{1/p''},$$

and since $< f\chi_{I_{j,k}} \mid \varphi^*(2^j x - r) >= 0$ if $\mid k - r \mid \geq M + 1$, we have :

$$\mid P_j(f\chi_{j,k}) \mid \leq C\epsilon \sum_{|r|\leq M} \mid \varphi \mid (2^j x - k - r) = C\epsilon\Phi(2^j x - k).$$

Thus we have

$$\mid g(x) \mid \leq \mid f \mid \chi_{\mathbb{R}\setminus\bigcup_\Lambda I_{j,k}} + \sum_{(j,k)\in\Lambda} C\,\epsilon\,\Phi(2^j x - k).$$

On $\mathbb{R}\setminus\Omega$, $\mid f(x) \mid \leq \epsilon$ and thus $\int_{\mathbb{R}\setminus\Omega} \mid f \mid^2 dx \leq \epsilon^{2-p''} \parallel f \parallel_{p''}^{p''}$. For estimating $\parallel \sum_\Lambda \epsilon\Phi(2^j x - k) \parallel_2$, we note that Φ is compactly supported and belongs to H^α for some $\alpha > 0$ (because φ does); now we write $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1 \in C_c^\infty$, $\operatorname{Supp}\Phi_1 \subset [0,1]$ and $\int \Phi_2 = 0$; the functions $\Phi_1(2^j x - k)$, $(j,k) \in \Lambda$, have disjoint supports, while we may apply the vaguelettes lemma (chapter 2) to Φ_2, and thus we get :

$$\begin{aligned}\parallel \sum_\Lambda \epsilon\Phi(2^j x - k) \parallel_2^2 &\leq C\,\epsilon^2 \sum_\Lambda 2^{-j} = C\,\epsilon^2 \sum_\Lambda \mid I_{j,\ell} \mid = C\,\epsilon^2\Omega \\ &\leq C\,\epsilon^2 \frac{\parallel f \parallel_{p''}^{p''}}{\epsilon^{p''}},\end{aligned}$$

and (21) is proved.

The proof of (18) and (19) for rapidly decaying ψ and ψ^* in L^2 (instead of compactly supported wavelets) is quite the same. The point is that we cannot have $\mid \operatorname{Supp} V_\eta^*(b) \mid < +\infty$ or $\mid \operatorname{Supp} U_\eta(b) \mid < +\infty$ any more. This will be replaced by estimates of the type :

$$\text{(22)} \qquad \int_{\mathbb{R}\setminus\tilde\Omega} \mid V_\eta^*(b) \mid dx \leq C \parallel f \parallel_1 \quad \text{and} \quad \parallel g \parallel_2^2 \leq C \parallel f \parallel_1 \epsilon$$

$$\text{(23)} \qquad \int_{\mathbb{R}\setminus\tilde\Omega} \mid U_\eta(b) \mid^{p''} dx \leq C \parallel f \parallel_{p''}^{p''} \quad \text{and} \quad \parallel g \parallel_2^2 \leq C \parallel f \parallel_{p''}^{p''} \epsilon^{2-p''}$$

and (20) and (21) will then be proved by :

$$\begin{aligned}\mid \{x/ \mid V_\eta^*(f) \mid \geq \epsilon\} \mid &\leq \mid \tilde\Omega \mid + \mid \{x \notin \tilde\Omega/ \mid V_\eta^*(b) \mid > \frac{\epsilon}{2}\} \mid + \mid \{x \notin \tilde\Omega/ \mid V_\eta^*(g) \mid > \frac{\epsilon}{2}\} \mid \\ &\leq \mid \tilde\Omega \mid + \frac{2}{\epsilon} \int_{\mathbb{R}\setminus\tilde\Omega} \mid V_\eta^*(b) \mid dx + \frac{4}{\epsilon^2} \parallel V_\eta^* g \parallel_2^2\end{aligned}$$

and similarly

$$\mid \{x/ \mid U_\eta f(x) \mid \geq \epsilon\} \mid \leq \mid \tilde\Omega \mid + (\frac{2}{\epsilon})^{p''} \int_{\mathbb{R}\setminus\tilde\Omega} \mid U_\eta(b) \mid^{p''} dx + \frac{4}{\epsilon^2} \parallel U_\eta g \parallel_2^2 .$$

Estimate (22) is easy. We use the classical Calderón-Zygmund splitting $f = g + b$ for $f \in L^1$. We have seen that $\| g \|_\infty \leq 2\epsilon$ and $\| g \|_1 \leq \| f \|_1$ so that $\| g \|_2^2 \leq 2\epsilon \| f \|_1$. Now we write :

$$| V_\eta^*(f\chi_{I_{j,k}} - \frac{1}{| I_{j,k} |}(\int_{I_{j,k}} f\, dx)\chi_{I_{j,k}}) | =$$

$$= \left| \sum_{I_{\ell,r} \subset I_{j,k}} \eta_{\ell,r} 2^\ell < f\chi_{I_{j,k}} \mid \theta(2^\ell x - r) > \psi(2^\ell x - r) \right| \leq$$

$$\leq \int_{I_{j,k}} | f(y) | \sum_{I_{\ell,r} \subset I_{j,k}} 2^\ell \chi_{[0,1]}(2^\ell y - r) \mid \psi(2^\ell x - r) \mid dy.$$

Now ψ has rapid decay in L^1, hence $\int_{\mathbb{R}\setminus\tilde{\Omega}} 2^\ell \mid \psi(2^\ell x - r) \mid dx \leq \| x^N \psi \|_1 \sup_{\tilde{\Omega}} \frac{1}{|2^\ell x - r|^N}$ and we have of course $\left| x - \frac{r}{2^\ell} \right| \geq \frac{1}{2^j}$, hence $| 2^\ell x - r | \geq 2^{\ell - j}$, so that

$$\int_{\mathbb{R}\setminus\tilde{\Omega}} | V_\eta^*(b) | \, dx \leq C \sum_{(j,k)\in\Lambda} \int_{I_{j,k}} \sum_{\ell\geq j} (2^{j-\ell})^N \mid f(y) \mid dy = 2C \int_\Omega | f | \, dx.$$

We now turn to estimate (23). We use the modified Calderón-Zygmund splitting $f = g + b$ for $f \in L^{p''}$. The estimate on $\| g \|_2$ remains quite easy. On $\mathbb{R}\setminus\Omega$, we have $| f | \leq \epsilon$ and thus $\int_{\mathbb{R}\setminus\Omega} | f |^2 \, dx \leq \epsilon^{2-p''} \| f \|_{p''}^{p''}$. Thus we have only to deal with $g_1 = \sum_{(j,k)\in\Lambda} P_j(f\chi_{I_{j,k}})$. But we know that φ^* has rapid decay in $L^{q''}$ ($q'' = \frac{p''}{p''-1}$) and that $\| f\chi_{I_{j,k}} \|_{p''}^{p''} \leq 2\epsilon^{p''} | I_{j,k} |$, hence :

$$| g_1(x) | \leq C_N \epsilon \sum_{(j,k)} \Phi(2^j x - k)$$

where $\Phi(x) = \sum_{r\in\mathbb{Z}} (1 + | r |)^{-N} | \varphi(x - r) |$. Φ satisfies $| x |^{N-1} \Phi \in L^2$ and $\Phi \in H^\alpha$ for some $\alpha > 0$ (since $\varphi \in H^\alpha$ and has rapid decay in L^2); thus writing $\Phi = \Phi_1 + \Phi_2$ where $\operatorname{Supp} \Phi_1 \subset [0,1]$ and $\int \Phi_2 dx = 0$, we obtain again by the vaguelettes lemma $\| g_1 \|_2^2 \leq C\epsilon^2 \sum_{(j,k)\in\Lambda} | I_{j,k} | \leq C\epsilon^{2-p''} \| f \|_{p''}^{p''}$.

We now turn our attention to $\int_{\mathbb{R}\setminus\tilde{\Omega}} | U_\eta(b) |^{p''} dx$. We write

$$U_\eta(b) = \sum_{(j,k)\in\Lambda} \sum_{\ell\geq j} \sum_{r\in\mathbb{Z}} \eta_{\ell,r} 2^\ell \int_{I_{j,k}} f(y) \bar{\psi}^*(2^\ell y - m) dy \, \theta(2^\ell x - m),$$

and thus

$$| U_\eta(b) | = \sum_{(j,k)\in\Lambda} \sum_{\ell\geq j} \sum_{\Lambda\in\mathbb{Z}} 2^\ell (\int_{I_{j,k}} | f(y) || \bar{\psi}^*(2^\ell y - r) | \, dy) \chi_{[0,1]}(2^\ell x - r).$$

Now we fix $\tilde{p} \in (p', p'')$ and $\tilde{q} = \frac{\tilde{p}}{\tilde{p}-1}$; ψ^* has rapid decay in $L^{\tilde{q}}$ so that for all $N \in \mathbb{N}$:

$$
\begin{aligned}
&| U_\eta(b) | \\
&\le \sum_{(j,k)\in\Lambda} \sum_{\ell\ge j} \sum_{r\in\mathbb{Z}} \sum_{I_{\ell,r+m}\subset I_{j,k}} 2^\ell \int_{I_{\ell,r+m}} | f(y) | \bar{\psi}^*(2^\ell y - r) | dy \chi_{[0,1]}(2^\ell x - r) \\
&\le \sum_{(j,k)\in\Lambda} \sum_{\ell\ge j} \sum_{r\in\mathbb{Z}} \sum_{I_{\ell,r+m}\subset I_{j,k}} \left(\frac{1}{| I_{\ell,r+m} |} \int_{I_{\ell,r+m}} | f(y) |^{\tilde{p}} dy \right)^{\frac{1}{\tilde{p}}} \frac{1}{(1+ | m |)^N} \chi_{[0,1]}(2^\ell x - r).
\end{aligned}
$$

We introduce the maximal function

$$
M_{\tilde{p}} f(x) = \sup_{x\in I} \left(\frac{1}{| I |} \int_I | f(y) |^{\tilde{p}} dy \right)^{\frac{1}{\tilde{p}}}.
$$

Then

$$
\frac{1}{(1+ | m |)^N} \left(\frac{1}{| I_{\ell,r+m} |} \int_{I_{\ell,r+m}} | f(y) |^{\tilde{p}} dy \right)^{\frac{1}{\tilde{p}}} \le C \frac{1}{(1+ | 2^\ell z - r |)^N} M_{\tilde{p}} f(z)
$$

for all $z \in I_{\ell,r+m}$, hence :

$$
\begin{aligned}
| U_\eta(b) | &\le C \sum_{j,k)\in\Lambda} \sum_{r} \sum_{I_{\ell,r+m}\subset I_{j,k}} \int_{I_{\ell,r+m}} 2^\ell M_{\tilde{p}} f(y) \frac{1}{(1+ | 2^\ell y - r |)^N} \chi_{[0,1]}(2^\ell x - r) dy \\
&\le C' \sum_{(j,k)\in\Lambda} \sum_{I_{\ell,m}\subset I_{j,k}} \int_{I_{\ell,m}} M_{\tilde{p}} f(y) \frac{2^\ell}{(1+ | 2^\ell (y-x) |)^N} dy \\
&= C' \sum_{(j,k)\in\Lambda} \sum_{\ell\ge j} \int_{I_{j,k}} M_{\tilde{p}} f(y) \frac{2^\ell}{(1+ | 2^\ell (y-x) |)^N} dy \\
&\le C'' \sum_{(j,k)} \int_{I_{j,k}} M_{\tilde{p}} f(y) \frac{2^j}{| 2^j (y-x) |^N} dy.
\end{aligned}
$$

But if we look at the operator

$$
F \to \left(\sum_{(j,k)\in\Lambda} \int_{I_{j,k}} F(y) \frac{2^j dy}{| 2^j (y-x) |^N} \right) = \tilde{F},
$$

we have clearly

$$
\int_{\mathbb{R}\setminus\tilde{\Omega}} | \tilde{F}(x) | dx \le C \int_\Omega | F(y) | dy
$$

(since $\int_{\mathbb{R}|\widetilde{I_{j,k}}} \frac{2^j dx}{|2^j(y-x)|^N} \le C$ for $y \in I_{j,k}$), while $\int_{\mathbb{R}\setminus\tilde{\Omega}} | \tilde{F}(x) |^2 dx \le C \int_\Omega | F(y) |^2 dy$: we have for $x \notin \tilde{\Omega}$,

$$
| \int_{I_{j,k}} F(y) \frac{2^j dy}{| 2^j (y-x) |^N} | \le C \| F\chi_{I_{j,k}} \|_2 \frac{2^{j/2}}{(1+ | 2^j x - k |)^{N/2}}
$$

and one more time we may use the vaguelettes lemma. Thus, we may conclude by interpolation that

$$\int_{\mathbb{R}\setminus\tilde{\Omega}} |\tilde{F}(x)|^{p''}\,dx \le C\int_{\Omega} |F(y)|^{p''}\,dy,$$

which gives

$$\int_{\mathbb{R}\setminus\Omega} |U_\eta(b)|^{p''}\,dx \le C\int_{\Omega} |M_{\tilde{p}}f(y)|^{p''}\,dy \le C\int |f|^{p''}\,dy$$

since $M_{\tilde{p}}$ is bounded on L^s for all $s \in (\tilde{p}, +\infty]$.

Theorem 1 is proved. ∎

3. H¹ and BMO.

We now turn to the case $p = 1$.

THEOREME 2. - *If φ and φ^* are regular dual scaling functions such that they have rapid decay in L^∞ and belong to C^α for some $\alpha > 0$, then the following inequality holds for any $f \in H^1$ (real Hardy space)*

$$(24)\quad \frac{1}{C}\| f \|_{H^1} \le N(f) = \int \left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 4^j \,|< f \mid \psi^*(2^j x-k)>|^2\, \chi_{[0,1]}(2^j x-k)\right)^{1/2} dx$$

$$\le C\| f \|_{H^1}$$

and similarly for any $f \in$ BMO

(25)

$$\frac{1}{C}\| f \|_{\mathrm{BMO}} \le N^*(f) = \sup_I \left(\frac{1}{|I|}\sum_{I_{j,k}\subset I} 2^j \,|< f \mid \psi(2^j x-k)>|^2\right)^{1/2} \le C\| f \|_{\mathrm{BMO}}$$

(where the supremum runs over the set of all intervals I, and $I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j}]$).

REMARKS. - i) For $\varphi = \varphi^* = \chi_{[0,1]}$ (the Haar system), theorem 2 is not true. We have to replace H^1 by H^1_d, the dyadic Hardy space which is associated to dyadic martingale, and BMO by BMO_d.

ii) If we look at the basis $(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^j x-k))_{j\ge 0,k\in\mathbb{Z}}$ instead of $(2^{j/2}\psi(2^j x-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$, we cannot obtain a basis for H^1 (indeed, $\varphi \notin H^1$ since $\int \varphi\,dx \ne 0$). We obtain a basis for the local Hardy space h^1 of Goldberg, and its dual the local bmo space [GOL].

Proof. We recall first some properties of H^1 and BMO (which are to be found in the celebrated paper of Fefferman and Stein [FEF]).

H^1 is the space of the functions $f \in L^1$ such that the Hilbert transform Hf of f is integrable (where $Hf = f * V.P.\frac{1}{\pi x}$), equipped with the norm $\| f \|_{H^1} = \| f \|_1 + \| Hf \|_1$.

The *atomic characterization* of H^1 is the following one : f belongs to H^1 if and only if there exists a family of intervals $(I_n)_{n\in\mathbb{N}}$, a family of atoms $(a_n)_{n\in\mathbb{N}}$ associated to (I_n) (i.e. $\operatorname{Supp} a_n \subset I_n$, $\| a_n \|_2 \leq | I_n |^{-1/2}$, $\int a_n dx = 0$) and a family of scalars $(\lambda_n) \in \ell^1$ such that : $f = \sum_{n=0}^{+\infty} \lambda_n a_n$, and moreover the norm $\| f \|_{H^1}$ is equivalent to the infimum of $\sum_0^{+\infty} | \lambda_n |$ over all atomic decompositions $f = \sum \lambda_n a_n$.

BMO is the dual space of H^1 and is the space of locally integrable functions such that $\| f \|_{\mathrm{BMO}} = \sup_I \left\{ \frac{1}{|I|} \int_I | f - \frac{1}{|I|} \int f\, dx |^2 \, dy \right\}^{1/2}$ is finite. This is a space of functions modulo the constants ($\| 1 \|_{\mathrm{BMO}} = 0$) . [Notice that the duality between BMO and the atomic H^1 space is obvious : for any intervall and $a \in L^2(I)$ with $\int a\, dx = 0$, we have $|< a \mid f >| \leq \| a \|_2 | I |^{1/2} \| f \|_{\mathrm{BMO}}$, hence $|< \sum \lambda_n a_n \mid f >| \leq \left(\sum_{n=0}^{+\infty} | \lambda_n | \right) \| f \|_{\mathrm{BMO}}$ if (a_n) is a family of atoms].

An useful notion is the notion of a molecule, an elementary function which can be easily decomposed into atoms [COIWE], [FRJW].

LEMMA 2. - *For $\epsilon > 0$, an ϵ-molecule is a function m such that for some $x_0 \in \mathbb{R}$ and some $\lambda > 0$, $\int (1 + \lambda \mid x - x_0 \mid)^{1+\epsilon} \mid m(x) \mid^2 dx \leq \lambda$ and $\int m\, dx = 0$.*

(i) *If m is an ϵ-molecule, then $m \in H^1$ (Hardy space) and $\| m \|_{H^1} \leq C_\epsilon$.*
(ii) *If T is a Calderón-Zygmund operator of class ϵ, then for any atom a associated to an interval $I = [x_0 - \lambda, x_0 + \lambda]$ (i.e.* $\operatorname{Supp} a \subset I$, $\| a \|_2 \leq | I |^{-1/2}$, $\int a\, dx = 0$) *and for any $\alpha < \epsilon$:*

$$\int (1+ \mid \lambda(x - x_0) \mid)^{1+2\alpha} \mid Ta(x) \mid^2 dx \leq C(T, \alpha, \epsilon)\lambda.$$

Thus if every atom satisfies $\int Ta\, dx = 0$ (which is usually written as "$T^*(1) = 0$"), then T is bounded on H^1.

Proof of the lemma. m is a molecule associated to $\lambda = \lambda_0$ and $x_0 = X_0$ if and only if $m = \lambda_0 M(\lambda_0(x - X_0))$, where M is a molecule associated to $\lambda = 1$ and $x_0 = 0$. Thus in (i) and (ii) (due to the invariance through dilations and translations of the atomic H^1-norm or of the estimation on the kernel of T) we may assume $\lambda = 1$, $x_0 = 0$.

Point (i) is easy. We write $m = \sum_{k\in\mathbb{N}} m\chi_{\Gamma_k}$ where $\Gamma_0 = [-1, 1]$ and $\Gamma_k = [-2^k, 2^k] \setminus [-2^{k-1}, 2^{k-1}]$ for $k \geq 1$. Then we have

$$\left(\int_{\Gamma_k} \mid m \mid^2 dx \right)^{1/2} \leq C \parallel (1+ \mid x \mid)^{\frac{1+\epsilon}{2}} m \parallel_2 2^{-k\frac{\epsilon}{2}} 2^{-\frac{k}{2}},$$

thus

$$\parallel (m - \frac{1}{\mid \Gamma_k \mid} \int_{\Gamma_k} \mid m \mid dx)\chi_{\Gamma_k} \parallel_2 \leq C\, 2^{-k\frac{\epsilon}{2}} 2^{-\frac{k}{2}}$$

and $m_0 = \sum_k (m - \frac{1}{|\Gamma_k|} \int_{\Gamma_k} m\, dx)\chi_{\Gamma_k}$ belongs to H^1. Now $m - m_0 = \sum_{k=0}^{+\infty} \lambda_k \chi_{\Gamma_k}$ with

$| \lambda_k | \leq C 2^{-k\frac{\epsilon}{2}} | \Gamma_k |^{-1}$ and $\sum_0^{+\infty} | \Gamma_k | \lambda_k = 0$. Rewrite $m - m_0$ as

$$m - m_0 = \sum_0^{+\infty} | \Gamma_k | \lambda_k \frac{1}{| \Gamma_k |} \chi_{\Gamma_k}$$
$$= \sum_0^{+\infty} \left\{ \sum_{p=0}^{k} | \Gamma_p | \lambda_p \right\} \left(\frac{1}{| \Gamma_k |} \chi_{\Gamma_k} - \frac{1}{| \Gamma_{k+1} |} \chi_{\Gamma_{k+1}} \right).$$

The atomic H^1-norm of $m - m_0$ is thus controlled by

$$\sum_{k=0}^{+\infty} | \sum_{p=0}^{k} | \Gamma_p | \lambda_p | = \sum_{k=0}^{+\infty} | \sum_{p=k+1}^{+\infty} | \Gamma_p | \lambda_p | \leq C \sum_{k=0}^{+\infty} 2^{-k\frac{\epsilon}{2}} < +\infty.$$

Point (ii) is easy as well. Suppose $\operatorname{Supp} a \subset [-1,1]$, $\| a \|_2 \leq \frac{1}{\sqrt{2}}$ and $\int a\, dx = 0$. Then $\int_{-3}^{3} | Ta |^2\, dx \leq \| T \|_{2,2}^2 \| a \|_2^2 \leq \frac{1}{2} \| T \|_{2,2}^2$. Moreover, if $x \notin [-3,3]$, we have

$$Ta(x) = \int_{-1}^{1} K(x,y) a(y) dy = \int_{-1}^{1} (K(x,y) - K(x,0)) a(y) dy,$$

hence

$$| Ta(x) | \leq \frac{1}{\sqrt{2}} \left(\int_{-1}^{1} | K(x,y) - K(x,0) |^2\, dy \right)^{1/2} \leq$$
$$\leq \frac{1}{\sqrt{2}} \frac{1}{| x |^{1+\epsilon}} \left(\int_{-1}^{1} | y |^{2\epsilon}\, dy \right)^{1/2} C(T)$$

and thus

$$\int_{\mathbb{R} \setminus [-3,3]} | x |^{1+2\alpha} | Ta(x) |^2\, dx \leq C'(T, \epsilon, \alpha)$$

provided that $\alpha < \epsilon$. Thus lemma 2 is proved. ■

Proof of theorem 2 (continued). Using lemma 2, we see that it is enough to prove theorem 2 for compactly supported wavelets ψ, ψ^*. Indeed, if we may prove it for a pair of compactly supported wavelets ψ_0, ψ_0^*, we look (for the wavelets ψ, ψ^* of theorem 2) at the operators

$$Tf = \sum_j \sum_k 2^j < f \mid \psi_0^*(2^j x - k) > \psi(2^j x - k)$$

and

$$Uf = \sum_j \sum_k 2^j < f \mid \psi^*(2^j x - k) > \psi_0(2^j x - k).$$

Lemma 1 guarantees that T and U are Calderón-Zygmund operators, and lemma 2 guarantees that they operate boundedly on H^1 and on BMO. (For BMO, prove that T^* and U^* operate boundedly on H^1); moreover $T = U^{-1}$. Thus the sequence space

$$H = \left\{ (\lambda_{j,k}) / \int (\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\lambda_{j,k}|^2 \, 2^j \chi_{[0,1]}(2^j x - k))^{1/2} dx < +\infty \right\}$$

(which is the space of coefficients of functions of H^1 in the basis $(2^{j/2}\psi_0(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$) is as well the space of coefficients of H^1 functions in the basis $(2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$, and similarly for the space $B = \{(\mu_{j,k}) / \sup_I \left(\frac{1}{|I|}\sum_{I_{j,k}\subset I} |\mu_{j,k}|^2\right)^{1/2} < +\infty\}$ which is the space of coefficients of BMO functions in the basis $(2^{j/2}\psi_0^*(2^j x - k))$ or $(2^{j/2}\psi^*(2^j x - k))$.

Now for proving (24) and (25) it is enough to prove :

i) $\int \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 4^j \, |< f \mid \psi^*(2^j x - k) >|^2 \, \chi_{[0,1]}(2^j x - k)\right)^2 dx \leq C \parallel f \parallel_{H^1}$

ii) $\sup_I \left(\frac{1}{|I|}\sum_{I_{j,k}\subset I} 2^j \, |< f \mid \psi(2^j x - k) >\right)^{1/2} \leq C \parallel f \parallel_{\mathrm{BMO}}$

iii) for $(\lambda_{j,k}) \in H$ and $(\mu_{j,k}) \in B$, $\sum_j \sum_k |\lambda_{j,k}| |\mu_{j,k}| \leq C \parallel \lambda_{j,k} \parallel_H \parallel \mu_{j,k} \parallel_B$.

i) is obvious via Khinchin's inequality and lemma 2. Choose any $\omega \in C_c^\infty$ such that $\int \omega \, dx = 0$ and $\omega = 1$ on $[0,1]$. Then define T_η for $\eta \in \{-1,1\}^{\mathbb{Z}\times\mathbb{Z}}$ by

$$T_\eta f = \sum_i \sum_k \eta_{j,k} 2^j < f \mid \psi^*(2^j x - k) > \omega(2^j x - k).$$

Then $\parallel T_\eta f \parallel_1 \leq C \parallel f \parallel_{H^1}$ where C doesn't depend on η. Thus we have :

$$\int_{\{-1,1\}^{\mathbb{Z}\times\mathbb{Z}}} \parallel T_\eta f \parallel_1 d\eta \leq C \parallel f \parallel_{H^1},$$

and Khinchin's inequality gives :

$$\int \left(\sum_j \sum_k 4^j \, |< f \mid \psi^*(2^j x - k) >|^2 | \omega(2^j x - k)|^2\right)^{1/2} dx \leq C' \parallel f \parallel_{H^1} .$$

Thus, i) is proved (since $\chi(x) \leq |\omega(x)|$).

ii) is easy to prove when ψ is compactly supported. Indeed if $I_{j,k} \subset I = [x_I - R_I, x_I + R_I]$ then

$$\mathrm{Supp}\, \psi(2^j x - k) \subset \tilde{I} = [x_I - (M+1)R_I, x_I + (M+1)R_I]$$

(if $\operatorname{Supp}\psi \subset [-M,M]$) and thus

$$< f \mid \psi(2^j x-k) > = < f\chi_{\tilde I} \mid \psi(2^j x-k) >,$$

so that (since $\int \psi\, dx = 0$) :

$$\sum_{I_{j,k}\subset I} 2^j \mid < f \mid \psi(2^j x-k) > \mid^2 = \sum_{I_{j,k}\subset I} 2^j \mid < f\chi_{\tilde I} - \frac{1}{\mid \tilde I \mid}\int_{\tilde I} f\, dx \mid \psi(2^j x-k) > \mid^2$$

$$\leq C \parallel f\chi_{\tilde I} - \frac{1}{\tilde I}\int_{\tilde I} f\, dx \parallel_2^2 \leq C \mid \tilde I \mid \parallel f \parallel_{\mathrm{BMO}}^2 \leq C(M+1) \mid I \mid \parallel f \parallel_{\mathrm{BMO}}^2 .$$

iii) is a classical lemma in harmonic analysis, related to the so-called Carleson measures [COIM1]. We define E_N as

$$E_N = \{x / \sum_j \sum_k \mid \lambda_{j,k} \mid^2 \chi_{I_{j,k}}(x) 2^j > 4^N\},$$

$$F_N = \{I_{j,k} / \mid I_{j,k} \cap E_N \mid \geq \frac{1}{2} \mid I_{j,k} \mid\}$$

and

$$G_N = F_N - F_{N+1}.$$

Then we have

$$\sum_j \sum_k \mid \lambda_{j,k} \mid\mid \mu_{j,k} \mid = \sum_{N\in\mathbb{Z}} \left\{ \sum_{I_{j,k}\in G_N} \mid \lambda_{j,k} \mid\mid \mu_{j,k} \mid \right\}$$

$$\leq \sum_{N\in\mathbb{Z}} \left(\sum_{I_{j,k}\in G_N} \mid \lambda_{j,k} \mid^2 \right)^{1/2} \left(\sum_{I_{j,k}\in G_N} \mid \mu_{j,k} \mid^2 \right)^{1/2} .$$

We write
$H_N = \bigcup_{I_{j,k}\in G_N} I_{j,k} = \bigcup_{I_{j,k}\in G_N^*} I_{j,k}$, where G_N^* is the collection of maximal dyadic intervals (for inclusion) among the elements of G_N. Then

$$\sum_{I_{j,k}\in G_N} \mid \mu_{j,k} \mid^2 = \sum_{I_{\ell,q}\in G_N^*} \left(\sum_{I_{j,k}\in G_N, I_{j,k}\subset I_{\ell,q}} \mid \mu_{j,k} \mid^2 \right)$$

$$\leq \parallel \mu \parallel_B^2 \sum_{I_{\ell,q}\in G_N^*} \mid I_{\ell,q} \mid = \parallel \mu \parallel_B^2 \mid H_N \mid .$$

Similarly

$$\sum_{I_{j,k}\in G_N} \mid \lambda_{j,k} \mid^2 = \sum_{I_{\ell,q}\in G_N^*} \left(\sum_{I_{j,k}\in G_N, I_{j,k}\subset I_{\ell,q}} \mid \lambda_{j,k} \mid^2 \right).$$

Since $I_{j,k} \in G_N$, we have $| I_{j,k} \cap E_{N+1} |< \frac{1}{2} | I_{j,k} |$, hence

$$\sum_{I_{j,k} \in G_N, I_{j,k} \subset I_{\ell,q}} | \lambda_{j,k} |^2 \leq 2 \int_{I_{\ell,q} \setminus E_{N+1}} \sum | \lambda_{j,k} |^2 \frac{1}{| I_{j,k} |} \chi_{I_{j,k}}(x) dx \leq$$

$$2 \cdot 4^{N+1} | I_{\ell,q} |$$

and we have proved :

$$\sum_j \sum_k | \lambda_{j,k} || \mu_{j,k} | \leq \sqrt{2} \| \mu \|_B \sum_N 2^{N+1} | H_N | .$$

But $H_N = \bigcup_{I_{\ell,q} \in G_N^*} I_{\ell,q}$, hence $| H_N \cap E_N | \geq \frac{1}{2} | H_N |$ and thus

$$\sum_j \sum_k | \lambda_{j,k} || \mu_{j,k} | \leq 4\sqrt{2} \| \mu \|_B \sum_N 2^N | E_N |$$

$$\leq 8\sqrt{2} \| \mu \|_B \sum_N 2^N | E_N \setminus E_{N+1} | \leq 8\sqrt{2} \| \mu \|_B \| \lambda \|_H .$$

Thus, theorem 2 is proved. ∎

REMARK. - This proof gives a way to exhibit an atomic decomposition for $f \in H^1$. Write

$$f = \sum_N \sum_{I_{\ell,q} \in G_N^*} \sum_{I_{j,k} \in G_N, I_{j,k} \subset I_{\ell,q}} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j x - k)$$

where G_N and G_N^* are defined as before. Then if ψ and ψ^* are compactly supported, we get that

$$a_{\ell,q} = \sum_{I_{j,k} \in G_N, I_{j,k} \subset I_{\ell,q}} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j x - k)$$

has its support contained in $\tilde{I}_{\ell,q} = \left[\frac{q-M}{2^\ell}, \frac{q+M+1}{2^\ell}\right]$, while $\int a_{\ell,q} dx = 0$ and

$$\| a_{\ell,q} \|_2 \approx \left(\sum_{I_{j,k} \in G_N, I_{j,k} \subset I_{\ell,q}} 2^j |< f \mid \psi^*(2^j x - k) >|^2 \right)^{1/2} \leq \sqrt{2} 2^{N+1} | I_{\ell,q} |^{1/2},$$

so that $\frac{1}{C\, 2^N | I_{\ell,q} |} a_{\ell,q}$ is an atom, whereas

$$\sum_N \sum_{I_{\ell,q} \in G_N^*} 2^\ell | I_{N,q} | \leq 2 \int \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 4^j |< f \mid \psi^*(2^j x - k) >|^2 \chi(2^j x - k) \right)^{1/2} dx.$$

If ψ and ψ^* are no more compactly supported, but have rapid decay in L^2, then

$$\frac{1}{C\,2^N \mid I_{\ell,q} \mid} a_{\ell,q}$$

is a molecule. ∎

4. Weighted Lebesgue spaces.

Theorem 1 can be extended to more general spaces of integrable functions. However, one must introduce restrictions on the weight of the Lebesgue space in order to get an unconditional wavelet basis for $L^p(w\,dx)$.

DEFINITION 2. - *For $p \in (1, +\infty)$, a positive function $w(x)$ belongs to the Muckenhoupt class A_p if for any interval I we have*

$$\frac{1}{\mid I \mid}\left(\int_I w\,dx\right)^{1/p}\left(\int_I w^{-\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \leq C$$

for some constant C. Similarly, $w \in A_1$ if :

$$\sup_I \frac{1}{\mid I \mid}\left(\int_I w\,dx\right) \text{ess. supp.}_{x\in I} \frac{1}{w} \leq +\infty.$$

THEOREM 3. - *If $p \in (1, +\infty)$ and $w \in A_p$, then if φ and φ^* are regular dual scaling functions with rapid decay in L^∞ and belonging to C^α for some $\alpha > 0$, the associated wavelet basis $(2^{j/2}\psi(2^j x-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is an unconditional basis for $L^p(w\,dx)$ and the norms $\| f \|_{L^p(w\,dx)}$ and*

$$\left(\int (\sum_j \sum_k 4^j \mid< f \mid \psi^*(2^j x - k) >\mid^2 \chi_{[0,1]}(2^j x - k))^{p/2} w(x)dx\right)^{1/p}$$

are equivalent on $L^p(w\,dx)$.

Proof. The proof is essentially given by the continuity of the Calderón-Zygmund operator on $L^p(w\,dx)$ if $w \in A_p$ (see [MEY3], chapter 7). Thus we have for $\eta \in \{-1,1\}^{\mathbb{Z}}$ and $w \in C_c^\infty$, $w \equiv 1$ on $[0,1]$ and $\int w\,dx = 0$:

$$\int \left(\sum_j \sum_k 4^j \eta_{j,k} < f \mid \psi^*(2^j x - k) > w(2^j x - k)\right)^p dw(x)dx \leq C(\omega) \int \mid f \mid^p w\,dx$$

where $C(\omega)$ doesn't depend on η. One more time, Khinchin's inequality gives

$$\int \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 4^j \mid< f \mid \psi^*(2^j x - k) >\mid^2 \mid \omega(2^j x - k) \mid^2\right)^{p/2} w(x)dx \leq C \int \mid f \mid^p w(x)dx,$$

and we may use $\chi_{[0,1]} \leq |\omega|^2$. Similarly $w^{-\frac{p}{p-1}}$ belongs to $A_{\frac{p}{p-1}}$, and thus

$$\int \left(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 4^j \,|< f \mid \psi(2^jx-k) >|^2\, \chi_{[0,1]}(2^jx-k) \right)^{\frac{p}{2(p-1)}} w(x)^{-\frac{p}{p-1}}\,dx \leq$$

$$C \int |f|^{\frac{p}{p-1}}\, w(x)^{-\frac{p}{p-1}}\,dx.$$

This gives the equivalence of norms on $L^p(w\,dx)$ (using the duality between $L^p(w\,dx)$ and $L^{\frac{p}{p-1}}(w^{-\frac{p}{p-1}}\,dx)$). ∎

The restriction "$w \in A_p$" is unavoidable, even for the far less restrictive condition: "the P_j, $j \in \mathbb{Z}$, are equicontinuous on $L^p(w\,dx)$" instead of the unconditionality of the wavelet basis. More precisely, let's define for a positive function $w(x)$ the function $\alpha_w(R)$ by :

$$\alpha_w(R) = \sup_{|I|=R} \left(\int_I w\,dx\right)^{1/p} \left(\int_I w^{-\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}.$$

Then we have :

THEOREM 4. - *Let $p \in (1,+\infty)$ and let φ, φ^* be compactly supported dual scaling functions which belongs to some C^α, $\alpha > 0$. Then :*

(i) *For each $j \in \mathbb{Z}$, P_j is bounded on $L^p(w\,dx)$ if and only if $\alpha_w(2^{-j}) < +\infty$. (Moreover, if $\alpha_w(R_0) < +\infty$), then $\alpha_w(R) < +\infty$ for every $R > 0$, so that if P_{j_0} is bounded, then any P_j is bounded).*

(ii) *$(P_j)_{j\geq 0}$ is equicontinuous on $L^p(w\,dx)$ (or equivalently P_0 is bounded on $L^p(w\,dx)$ and $P_jf \to f$ in $L^p(w\,dx)$ for any $f \in L^p(w\,dx)$ as $j \to +\infty$) if and only if $\sup_{|R|\leq 1} \alpha_w(R) < +\infty$. Moreover, this condition is equivalent to the unconditionality of the basis*

$$(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^jx-k))_{j\geq 0, k\in\mathbb{Z}} \ \text{ in } \ L^p(w\,dx),$$

and in that case $\| f \|_{L^p(w\,dx)}$ is equivalent to :

$$\left(\int (\sum_{j\geq 0} \sum_{k\in\mathbb{Z}} 4^j \,|< f \mid \psi^*(2^jx-k) >|^2\, \chi_{[0,1]}(2^jx-k))^{p/2} w(x)dx \right)^{1/p}$$

$$+ \left(\sum_{k\in\mathbb{Z}} |< f \mid \varphi^*(x-k) >|^p \int_k^{k+1} w(x)dx \right)^{1/p}.$$

(iii) *$(P_j)_{j\in\mathbb{Z}}$ is equicontinuous on $L^p(w\,dx)$ (or equivalently P_0 is bounded on $L^p(w\,dx)$, $P_jf \to f$ as $j \to +\infty$, $P_jf \to 0$ as $j \to -\infty$) if and only if $\sup_R \alpha_w(R) < +\infty$ (i.e. $w \in A_p$). This condition is equivalent to the unconditionality of $(2^{j/2}\psi(2^jx-k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$.*

This theorem is proved (for $\varphi = \varphi^*$) in [LEM8].

5. Besov spaces.

The case of Besov spaces has been recently treated in an enlightening paper of G. Bourdaud [BOR]. Our approach will be slightly different. We want to include the Dirac mass δ in the class of admissible scaling distributions, so that we drop the usual Calderón-Zygmund operator analysis, and use instead a "molecular" analysis, where the molecule is adapted to the Besov space.

DEFINITION 3. - *i) A bump element of a Besov space* $B_q^{s,p}$ ($s \in \mathbb{R}$, $p, q \in [1, +\infty]$) *is a distribution* ω *such that* ω *has rapid decay in* $B_q^{s,p}$ ($\forall k \in \mathbb{N}$, $x^k \omega \in B_q^{s,p}$) *and* ω *belongs to* $B_q^{s+\epsilon,p}$ *for some* $\epsilon > 0$.

ii) A regular molecule in $B_q^{s,p}$ *is a distribution* ω *such that* ω *is a bump element of* $B_q^{s,p}$ *and* $\int x^k \omega \, dx = 0$ *for* $0 \leq k \leq -s$.

LEMMA 3. *i) If* ω *is a bump element of* $B_q^{s,p}$ *then for any* $(\lambda_k) \in \ell^p$, $\sum_{k \in \mathbb{Z}} \lambda_k \omega(x-k)$ *(which converges in* S'*) belongs to* $B_q^{s,p}$ *and*

$$\| \sum_{k \in \mathbb{Z}} \lambda_k \omega(x-k) \|_{B_q^{s,p}} \leq C \left(\sum_{k \in \mathbb{Z}} | \lambda_k |^p \right)^{1/p} \tag{26}$$

(where C *depends only on* ω, s, p *and* q*).*

ii) If ω *is a regular molecule in* $B_q^{s,p}$ *then for any sequence* $(\lambda_{j,k})_{j \geq 0, k \in \mathbb{Z}}$ *such that*

$$\left(\sum_{j \geq 0} 2^{j(s - \frac{1}{p} + \frac{1}{2})q} \left(\sum_{k \in \mathbb{Z}} | \lambda_{j,k} |^p \right)^{q/p} \right)^{\frac{1}{q}} < +\infty$$

the sequence

$$\sum_{j \geq 0} \sum_{k \in \mathbb{Z}} 2^{j/2} \lambda_{j,k} \omega(2^j x - k)$$

converges in S' *to an element of* $B_q^{s,p}$ *and :*

$$\| \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} 2^{j/2} \lambda_{j,k} \omega(2^j x - k) \|_{B_q^{s,p}} \leq C \left(\sum_{j \geq 0} 2^{j(s - \frac{1}{2} + \frac{1}{p})q} \left(\sum_{k \in \mathbb{Z}} | \lambda_{j,k} |^p \right)^{q/p} \right)^{1/q} . \tag{27}$$

Before proving lemma 3, it will be convenient to prove some preliminary results on bumps and molecules.

LEMMA 4. - *Let $s \in \mathbb{R}$, $p,q \in [1,+\infty]$ and $\omega \in B_q^{s+\epsilon,p}$ for some $\epsilon > 0$.*

i) Let $\gamma \in C_c^\infty(\mathbb{R})$ such that $\sum_{k\in\mathbb{Z}} \gamma(x-k) = 1$. Then ω is a bump element of $B_q^{s,p}$ if and only if the sequence $(\| \gamma(x-k)\omega(x) \|_{B_q^{s,p}})$ is rapidly decaying :

$$\forall N \in \mathbb{N}, \quad \sup_{k\in\mathbb{Z}} | k |^N \| \gamma(x-k)\omega(x) \|_{B_q^{s,p}} < +\infty.$$

Moreover in that case $x^N\omega$ is a bump element of $B_q^{s,p}$ for all $N \in \mathbb{N}$.

ii) The Fourier transform of a bump element ω is a C^∞ function and can be computed as $\hat{\omega}(\xi) = \sum_{k\in\mathbb{Z}} < \omega\gamma(x-k) \mid e^{ix\xi} >_{\mathcal{E},C^\infty}$ where $\sum_{k\in\mathbb{Z}} \gamma(x-k) = 1$ as in part i)).

Thus ω is a regular molecule if and only if $\hat{\omega}(0) = \frac{\partial}{\partial\xi}\hat{\omega}(0) = \cdots \frac{\partial^N}{\delta\xi^N}\hat{\omega}(0) = 0$ with $N = [-s]$.

iii) The derivative of a bump element in $B_q^{s,p}$ is a bump element in $B_q^{s-1,p}$. Conversely, a bump element ω in $B_q^{s,p}$ is the derivative of some bump element in $B_q^{s+1,p}$ if and only if $\hat{\omega}(0) = 0$.

Proof of lemma 4. i) is obvious. Recall that a function $\mu \in C^K$ where $K > | s |$ is a pointwise multiplier of $B_q^{s,p}$:

$$\| \mu f \|_{B_q^{s,p}} \leq C \| f \|_{B_q^{s,p}} \sum_{j=0}^{K} \| \frac{\partial^j}{\partial x^j}\mu \|_\infty . \tag{28}$$

Thus

$$\begin{aligned}\| \gamma(x-k)\omega(x) \|_{B_q^{s,p}} &\leq C \| (1+x^2)^N\omega \|_{B_q^{s,p}} \sum_{j=0}^{K} \| \frac{\partial^j}{\partial x^j}\{\frac{\gamma(x-k)}{(1+x^2)^N}\} \|_\infty \\ &\leq C'_N \| (1+x^2)^N\omega \|_{B_q^{s,p}} (1+k^2)^{-N}\end{aligned}$$

while

$$\begin{aligned}\| x^N\omega \|_{B_q^{s,p}} &\leq \sum_{k\in\mathbb{Z}} \| x^N\gamma(x-k)\omega \|_{B_q^{s,p}} \\ &\leq C \sum_{k\in\mathbb{Z}} \| \gamma(x-k)\omega \|_{B_q^{s,p}} \sum_{j=0}^{K} \| \frac{\partial^j}{\partial x^j}(x^N\Gamma(x-k)) \|_\infty \\ &\leq C'_N \sum_{k\in\mathbb{Z}} (1+ | k |)^N \| \gamma(x-k)\omega \|_{B_q^{s,p}}\end{aligned}$$

where $\Gamma \in C_c^\infty$ is such that $\Gamma \equiv 1$ on $\operatorname{Supp}\gamma$.

ii) From point i), we see that $\sum_k \omega\,\gamma(x-k)$ converges to ω in $B^{s,p}_q$ (and similarly $\sum_k x^N \omega\gamma(x-k)$ to $x^N\omega$), so that $\hat\omega$ is the limit of $\sum_k \omega\,\gamma\widehat{(x-k)}$ in S'. We have only to prove that we have uniform convergence on any compact of $\mathbb{R}$ of $\sum_k \omega\,\gamma\widehat{(x-k)}(\xi)$ (for the uniform convergence of derivatives, replace ω by $x^N\omega$).

Choose again $\Gamma \in C_c^\infty$ such that $\bar\Gamma\gamma = 1$, and write :

$$\begin{aligned}|\,\omega\,\gamma\widehat{(x-k)}(\xi)\,| &= |<\omega\,\gamma(x-k)\mid \Gamma(x-k)e^{ix\xi}>| \le \\ &\|\,\omega\,\gamma(x-k)\,\|_{B^{s,p}_q}\|\,\Gamma(x-k)e^{ix\xi}\,\|_{B^{-s,p'}_{q'}}\end{aligned}$$

where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$. Then notice that we have (using (28) again) :

$$\|\,\Gamma(x-k)e^{ix\xi}\,\|_{B^{-s,p'}_{q'}} = \|\,\Gamma(x)e^{ix\xi}\,\|_{B^{-s,p'}_{q'}} \le C\,\|\,\Gamma\,\|_{B^{-s,p'}_{q'}}\,(1+|\,\xi\,|)^K.$$

iii) We notice first that for a bump element ω of $B^{s,p}_q$ and a C^∞ function θ, the sequence $\|\,\theta(x-k)\omega\,\|_{B^{s,p}_q}$ has rapid decay (same proof as for i), hence, writing

$$\begin{aligned}\|\,\gamma(x-k)\frac{d}{dx}\omega\,\|_{B^{s-1,p}_q} &\le \|\,\frac{d}{dx}(\gamma(x-k)\omega)\,\|_{B^{s-1,p}_q} + \|\,\omega\frac{d}{dx}\gamma(x-k)\,\|_{B^{s-1,p}_q}\\ &\le C\left(\|\,\gamma(x-k)\omega\,\|_{B^{s,p}_q} + \|\,\omega\frac{d}{dx}\gamma(x-k)\,\|_{B^{s,p}_q}\right),\end{aligned}$$

we get that $\|\,\gamma(x-k)\frac{d\omega}{dx}\,\|_{B^{s-1,p}_q}$ has rapid decay if ω is a bump element of $B^{s,p}_q$. Conversely, assume that ω is a bump element of $B^{s,p}_q$ such that $\hat\omega(0) = 0$. Write $\omega\gamma(x-k) = \alpha_k + \left(\int \omega\gamma(x-k)dx\right)\gamma(x-k) = \alpha_k + \epsilon_k\gamma(x-k)$. Then (ϵ_k) is a rapidly decaying sequence, so that $\Omega_1 = \sum_k \epsilon_k\gamma(x-k) \in S(\mathbb{R})$; moreover $\int \Omega_1 dx = \sum_k \epsilon_k = \int \omega\,dx = 0$, thus $\widehat{\Omega_1}(0) = 0$ and $\frac{\widehat{\Omega_1}(\xi)}{i\xi} = \widehat{\Omega_2} \in S'(\mathbb{R})$: $\Omega_1 = \frac{d}{dx}\Omega_2$ with $\Omega_2 \in S(\mathbb{R})$, so that Ω_2 is a bump element of $B^{s+1,p}_q$. We now turn our attention to $\omega_1 = \sum_k \alpha_k$. If we look at $\alpha_k(x+k) = \beta_k$, we see that $\operatorname{Supp}\beta_k \subset \operatorname{Supp}\gamma$ and $<\beta_k \mid 1> = 0$; hence $\beta_k = \frac{d}{dx}B_k$ where $\operatorname{Supp} B_k \subset <\operatorname{Supp}\gamma>$ (the convex closure of $\operatorname{Supp}\gamma$) ;moreover $\|\,B_k\,\|_{B^{s+1,p}_q} \le C_{p,q}\,\|\,\beta_k\,\|_{B^{s,p}_q}$ (since β_k is supported in a fixed compact set), and thus $\|\,B_k\,\|_{B^{s+1,p}_q} \le C\,\|\,\gamma(x-k)\omega\,\|_{B^{s,p}_q}$; then we get $\omega_1 = \sum_k \frac{d}{dx}B_k(x-k) = \frac{d}{dx}\omega_2$ where $\omega_2 = \sum_{k\in\mathbb{Z}} B_k(x-k)$ is a bump element of $B^{s+1,p}_q$. ■

Proof of lemma 3. i) is easy. If ω is a bump element of $B^{s,p}_q$ and $\omega \in B^{s+\epsilon,p}_q$, then ω is a bump element of $B_p^{s+\frac{\epsilon}{2},p}$: using the Littlewood-Paley-Stein characterization of $B^{s,p}_q$ (i.e. choosing θ_0 and $\theta_1 \in C^\infty$ such that $\operatorname{Supp}\theta_0 \subset [-2,2]$, $\operatorname{Supp}\theta_1 \subset [-4,-1]\cup[1,4]$ and $|\,\theta_0\,|^2 + \sum_{j=0}^{+\infty}|\,\theta_1(\frac{\xi}{2^j})\,|^2 = 1$, we define $\widehat{S_0 f} = |\,\theta_0\,|^2\,\hat f$ and $\widehat{\Delta_j f} = |\,\theta_1(\frac{\xi}{2^j})\,|^2\,\hat f$; then $f \in B^{s,p}_q$ if and only if $f \in S'$, $S_0 f \in L^p$ and $(2^{js}\,\|\,\Delta_j f\,\|_p)_{j\ge 0} \in \ell^q(\mathbb{N})$ and $\|\,f\,\|_{B^{s,p}_q}$ is equivalent to $\|\,S_0 f\,\|_p + \|\,2^{js}\,\|\,\Delta_j f\,\|_p\|_{\ell^q}$), we see easily that for $\alpha \in (0,\epsilon)$,

$$\|\,f\,\|_{B^{s+\alpha,p}_q} \le C\,\|\,f\,\|_{B^{s,p}_q}^{\frac{\epsilon-\alpha}{\epsilon}}\|\,f\,\|_{B^{s+\epsilon,p}_q}^{\frac{\alpha}{\epsilon}},$$

while $\| f \|_{B_p^{s+\frac{\epsilon}{2},p}} \leq C_\alpha \| f \|_{B_q^{s+\alpha,p}}$ for $\alpha \in (\frac{\epsilon}{2}, \epsilon)$; putting $f = \gamma(x-k)\omega$, we see that ω is a bump element of $B_p^{s+\frac{\epsilon}{2},p}$.

Now, in order to prove (26), it is enough to prove (writing $\sigma = s + \frac{\epsilon}{2}$)

$$(29) \qquad \| \sum_{k\in\mathbb{Z}} \lambda_k \omega(x-k) \|_{B_p^{\sigma,p}} \leq C_\omega \| \lambda_k \|_{\ell^p} \text{ for } \omega \text{ a bump element of } B_p^{\sigma,p}.$$

But when $p = q$ we may localize the calculus of the norm $B_q^{s,p}$: if $\gamma \in \mathcal{C}_c^\infty$ is such that $\sum_{\mathbb{Z}} \gamma(x-k) = 1$ then $\| f \|_{B_p^{\sigma,p}}$ is equivalent to $(\sum_{k\in\mathbb{Z}} \| f\gamma(x-k) \|_{B_p^{\sigma,p}}^p)^{\frac{1}{p}}$. Hence we have :

$$\begin{aligned}
&\| \sum_{k\in\mathbb{Z}} \lambda_k \omega(x-k) \|_{B_p^{\sigma,p}}^p \\
&\leq C \sum_{p\in\mathbb{Z}} \| \sum_{k\in\mathbb{Z}} \gamma(x-p)\lambda_k\omega(x-k) \|_{B_p^{\sigma,p}}^p \\
&\leq C \sum_{p\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} | \lambda_k | \| \omega\gamma(x-p+k) \|_{B_p^{\sigma,p}})^p \\
&\leq C \sum_{p\in\mathbb{Z}} (\sum_{k\in\mathbb{Z}} | \lambda_k |^p \| \omega\gamma(x-p+k) \|_{B_p^{\sigma,p}})(\sum_{k\in\mathbb{Z}} \| \omega\gamma(x-p+k) \|_{B_p^{\sigma,p}})^{p-1} \\
&= C(\sum_k \| \omega\gamma(x-k) \|_{B_p^{\sigma,p}})^p (\sum_{k\in\mathbb{Z}} | \lambda_k |^p).
\end{aligned}$$

Point ii) is more delicate. We first notice that we may assume $s > 0$. Indeed if $s < 0$, we may write ω as $\omega = \frac{d^N\Omega}{dx^N}$ with $N = [-s]+1$ and Ω a regular molecule in $B_q^{s+N,p}$; moreover we have obviously

$$\sum^{+\infty} \sum_{\ell} \lambda_{j,k} 2^{\frac{j}{2}} \omega(2^j x - k) = \frac{d^N}{dx^N} \left(\sum_{j=0}^{+\infty} \sum_{\ell} \{2^{-jN}\lambda_{j,k}\} 2^{\frac{j}{2}} \Omega(2^j x - k) \right),$$

and the estimate (27) for Ω and $s + N > 0$ gives (27) for ω and s.

Now if $\sigma > 0$ and $\lambda \geq 1$ we have

$$\| f(\lambda x) \|_{B_q^{\sigma,p}} \leq C\lambda^{\sigma-\frac{1}{p}} \| f \|_{B_q^{\sigma,p}} .$$

Hence, for ω a bump element in $B_q^{s,p}$ ($s > 0$) which belongs to $B_q^{s+\epsilon,p}$, we choose α such that $\alpha \in (0, \min(s,\epsilon))$ and define

$$\omega_j = \sum_{k\in\mathbb{Z}} 2^{\frac{j}{2}} \lambda_{j,k} \omega(2^j x - k) ;$$

then :

$$\begin{aligned}
\| \omega_j \|_{B_q^{s+\alpha,p}} &\leq C\, 2^{j(\frac{1}{2}-\frac{1}{p}+s+\alpha)} \| \sum_k \lambda_{j,k}\omega(x-k) \|_{B_q^{s+\alpha,p}} \\
&\leq C' 2^{j(\frac{1}{2}-\frac{1}{p}+s+\alpha)} (\sum_k | \lambda_{j,k} |^p)^{\frac{1}{p}}
\end{aligned}$$

(since ω is a bump element of $B_q^{s+\alpha,p}$), and similarly

$$\| \omega_j \|_{B_q^{s-\alpha,p}} \leq C\, 2^{j(\frac{1}{2}-\frac{1}{p}+s-\alpha)} (\sum_k | \lambda_{j,k} |^p)^{\frac{1}{p}}.$$

We may now estimate

$$\| \sum_0^{+\infty} \omega_\ell \|_{B_q^{s,p}} = \| S_0(\sum_0^{+\infty} \omega_\ell) \|_p + \left(\sum_{j=0}^{+\infty} 2^{jsq} \| \Delta_j(\sum_0^{\infty} \omega_\ell) \|_p^q \right) .$$

We have :

$$\| S_0(\omega_\ell) \|_p \leq C 2^{-\ell\alpha} 2^{\ell(\frac{1}{2}-\frac{1}{p}+s)} (\sum_k | \lambda_{\ell,k} |^p)^{\frac{1}{p}} = C 2^{-\ell\alpha} \epsilon_\ell \; ;$$

$(\epsilon_\ell)_{\ell \geq 0} \in \ell^q$ and $(2^{-\ell\alpha})_{\ell \geq 0} \in \ell^{\frac{q}{q-1}}$, hence $\sum \| S_0(\omega_\ell) \|_p < +\infty$. Similarly, for $\sigma = s \pm \alpha$

$$\begin{aligned} 2^{js} \| \Delta_j(\omega_\ell) \|_p &\leq C\, 2^{js} 2^{-j\sigma} \| \omega_\ell \|_{B_q^{\sigma,p}} \\ &\leq C 2^{js} 2^{-js} 2^{\ell(\frac{1}{2}-\frac{1}{p}+\sigma)} (\sum_k | \lambda_{\ell,k} |^p)^{\frac{1}{p}} \\ &\leq C 2^{(j-\ell)(s-\sigma)} 2^{\ell(\frac{1}{2}-\frac{1}{p}+s)} (\sum_k | \lambda_{\ell,k} |^p)^{\frac{1}{p}} \end{aligned}$$

and thus $2^{js} \| \Delta_j(\omega_\ell) \|_p \leq C 2^{-\alpha|j-\ell|} \epsilon_\ell$, and (27) is proved. ∎

A direct consequence of lemma 3 is the following one :

THEOREM 5. - *Let φ, φ^* be distributions such that for some $s \in \mathbb{R}$, $p, q \in [1, +\infty)$ φ is a bump element in $B_q^{s,p}$ and φ^* a bump element in $B_{q'}^{-s,p'}$ ($p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$). Assume moreover that :*

(*i*) $\quad$ *for* $k \in \mathbb{Z}, \quad < \varphi \mid \varphi^*(x-k) > = \delta_{k,0}$

(*ii*) $\quad \varphi(\frac{x}{2}) \in \{\sum \lambda_k \varphi(x-k) / (\lambda_k) \in \ell^p\}$

(*iii*) $\quad \varphi^*(\frac{x}{2}) \in \{\sum \lambda_k \varphi^*(x-k) / (\lambda_k) \in \ell^q\}$

and define m_0, m_0^* as $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$, $\hat{\varphi}^*(2\xi) = m_0^*(\xi)\hat{\varphi}^*(\xi)$ and the wavelets ψ, ψ^* as

$$\hat{\psi}(2\xi) = e^{-i\xi} \bar{m}_0^*(\xi+\pi)\hat{\varphi}(\xi), \hat{\psi}^*(2\xi) = e^{-i\xi} \bar{m}_0(\xi+\pi)\hat{\varphi}(\xi).$$

Then :

$$(\varphi(x-k))_{k\in\mathbb{Z}} \cup (2^{j/2}\psi(2^j x - k))_{j\geq 0, k\in\mathbb{Z}}$$

is an unconditional basis for $B_q^{s,p}$*, and the norm* $\| f \|_{B_q^{s,p}}$ *is equivalent to :*

$$(30) \qquad \left(\sum_{k \in \mathbb{Z}} |< f \mid \varphi^*(x-k) >|^p \right)^{\frac{1}{p}} +$$

$$\left(\sum_{j \geq 0} 2^{j(\frac{1}{2}-\frac{1}{p}+s)q} (\sum_{k \in \mathbb{Z}} 2^{j\frac{p}{2}} |< f \mid \psi^*(2^j x - k) >|^p)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

REMARK. - If $p = +\infty$ or $q = +\infty$, we cannot have a basis ($B_q^{s,p}$ is no more separable). But we have still equivalence between $\| f \|_{B_q^{s,p}}$ and (30), and we have weak convergence of

$$\sum_k < f \mid \varphi^*(x-k) > \varphi(x-k) + \sum_{j=0}^{+\infty} \sum_{k \in \mathbb{Z}} 2^j < f \mid \psi^*(2^j x - k) > \psi(2^j x - k)$$

to f (in the topology $\sigma(B_q^{s,p}, B_{q'}^{-s,p'})$).

Proof. We prove that ψ and ψ^* are regular molecules (respectively in $B_q^{s,p}$ and $B_{q'}^{-s,p'}$). Indeed, we have $\varphi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} < \varphi(\frac{x}{2}) \mid \varphi^*(x-k) > \varphi(x-k)$, and $(\lambda_k =< \varphi(\frac{x}{2}) \mid \varphi^*(x-k) >)_{k \in \mathbb{Z}}$ is a rapidly decaying sequence. Thus, we see at once that m_0 and m_0^* are $\mathcal{C}^\infty$ periodical functions and that ψ and ψ^* are bump elements. The only thing to check is that (in case $s \geq 0$) $\int \psi^* x^k dx = 0$ for $0 \leq k \leq [s]$. We may of course assume $s \notin \mathbb{N}$ (because if φ is a bump in $B_q^{s,p}$, it is a bump in $B_q^{s+\alpha,p}$ for every $\alpha \in (0, \epsilon)$ where $\varphi \in B_q^{s+\epsilon,p}$, while φ^* is obviously a bump in $B_{q'}^{-s-\alpha,p'}$).

We will prove that if $s > 0$ then necessarily $m_0(\pi) = 0$ (so that $\hat{\psi}^*(0) = 0$). Assume this is true, and define $\tilde{\varphi}$ and $\tilde{\varphi}^*$ by $\hat{\tilde{\varphi}}(\xi) = \frac{i\xi}{1-e^{-i\xi}} \hat{\varphi}(\xi)$ and $\hat{\tilde{\varphi}}^*(\xi) = \frac{e^{i\xi}-1}{i\xi} \hat{\varphi}^*(\xi)$. Then $\tilde{\varphi}$ and $\tilde{\varphi}^*$ satisfy the hypotheses of theorem 5 for $B_q^{s-1,p}$ and $B_{q'}^{-s+1,p'}$: $\tilde{\varphi}^*$ is clearly a bump element in $B_{q'}^{-s+1,p'}$, because $\varphi^*(x+1) - \varphi^*(x)$ is a bump element in $B_{q'}^{-s,p'}$ and $\int (\varphi^*(x+1) - \varphi^*(x)) dx = 0$; since $s > 0$, we know that φ is integrable (since $\| \varphi\gamma(x-k) \|_1 \leq | \operatorname{Supp} \gamma |^{\frac{p-1}{p}} \| \varphi\gamma(x-k) \|_p \leq C \mid \operatorname{Supp} \gamma |^{1-\frac{1}{p}} \| \varphi\gamma(x-k) \|_{B_q^{s,p}}$) so that in $\mathcal{S}'$, $\sum_{k \in \mathbb{Z}} \varphi(x-k) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(2k\pi) e^{2ik\pi\xi} = \hat{\varphi}(0)$ (since $\hat{\varphi}(2k\pi) = 0$ for $k \neq 0$, as a consequence of $m_0(\pi) = 0$); then $\tilde{\varphi}$ can be defined as

$$\tilde{\varphi} = \sum_{k=0}^{+\infty} \frac{d}{dx} \varphi(x-k) = - \sum_{k=-\infty}^{-1} \frac{d}{dx} \varphi(x-k) \quad \text{(remember that } \sum_{k \in \mathbb{Z}} \frac{d}{dx} \varphi(x-k) = 0) \text{ ;}$$

then using a partition of unity $1 = \sum_{k \in \mathbb{Z}} \gamma(x-k)$, we see that

$$(\sum_{k<0} \gamma(x-k))(\sum_{k \geq 0} \frac{d}{dx} \varphi(x-k)) \quad \text{and} \quad (\sum_{k \geq 0} \gamma(x-k))(\sum_{k<0} \frac{d}{dx} \varphi(x-k))$$

have rapid decay in $B_q^{s-1,p}$. We have moreover

$$\tilde{\varphi}(2\xi) = \frac{2m_0(\xi)}{1+e^{-i\xi}}\tilde{\varphi}(\xi) \quad \text{and} \quad \tilde{\varphi}^*(2\xi) = \frac{e^{i\xi}+1}{2}\tilde{\varphi}^*(\xi).$$

Finally, the duality between $\tilde{\varphi}$ and $\tilde{\varphi}^*$ is almost obvious : we write $< \tilde{\varphi} \mid \tilde{\varphi}^*(x-m) >= \sum_{k\geq 0} < \frac{d}{dx}\varphi(x-k) \mid \tilde{\varphi}^*(x-m)$ (we have no problem for x growing to $+\infty$ since $\tilde{\varphi}^*$ decays rapidly while $\sum \frac{d}{dx}\varphi(x-k)$ grows slowly : $\| \gamma(x-k)\sum_{k\geq 0}\frac{d}{dx}\varphi(x-\ell) \|_{B_q^{s-1,p}} \leq C(1+k)$ for $k \geq 0$), and thus

$$\begin{aligned} < \tilde{\varphi} \mid \tilde{\varphi}^*(x-m) > &= \sum_{k\geq 0} - < \varphi(x-k) \mid \varphi^*(x+1-m) > + < \varphi(x-k) \mid \varphi^*(x-m) > \\ &= \sum_{k\geq 0}(-\delta_{k,m-1} + \delta_{k,m}) = \delta_{k,0}. \end{aligned}$$

Now, iterating the proof that $m_0(\pi) = 0$ if $s > 0$, we get that if $s > N$ then $m_0(\pi) = \frac{d}{d\xi}m_0(\pi) = \cdots = \frac{d^N}{d\xi^N}m_0(\pi) = 0$, hence $\hat{\psi}^*(0) = \cdots = \frac{\partial^N}{\partial\xi^N}\hat{\psi}^*(0) = 0$.

Thus we have reduced the proof to the proof that $m_0(\pi) = 0$.

We begin by the following easy lemma :

LEMMA 5. - *Let $\Omega \in L^\infty$ be such that for some $\epsilon > 0$, $| x |^{1+\epsilon}\, \Omega \in L^\infty$ and $\Omega \in C^\epsilon$. Then the following Poisson formula holds for all ξ :*

$$\sum_{k\in\mathbb{Z}} \Omega(k)e^{-2ik\pi\xi} = \sum_{k\in\mathbb{Z}} \hat{\Omega}(\xi + 2k\pi) \tag{31}$$

(i.e. for all ξ both series converge to the same limit).

We postpone the proof of this lemma at the end of this proof.

We apply lemma 5 to $\Omega = \varphi(x) * \bar{\varphi}^*(-x)$. It is easy to see that convolution is well defined on $B_q^{s,p} \times B_{q'}^{\sigma,p'}$, with values in $B_1^{s+\sigma,\infty}$: if θ_0 defines S_0 and θ_1 defines the dyadic blocks Δ_j and if $\gamma_0, \gamma_1 \in C_c^\infty$, $\gamma_0\theta_0 = 1$, $\gamma_1\theta_1 = 1$ and $0 \notin \text{Supp}\,\gamma_1$, define $\widehat{\sum_0 f} =| \gamma_0(\xi) |^2 \hat{f}$ and $\widehat{\Gamma_j f} =| \gamma_1(\frac{\xi}{2^j}) |^2 \hat{f}$, then : $S_0(f*g) = S_0 f * \sum_0 g$ and $\Delta_j(f*g) = \Delta_j f * \Gamma_j g$, so that

$$\| S_0 f * \sum_0 g \|_\infty \leq \| S_0 f \|_p \| \sum_0 g \|_{p'}$$

and

$$\begin{aligned} \| \Delta_j(f*g) \|_\infty 2^{j(s+\sigma)} &\leq \sum_{j\geq 0} \| \Delta_j f \|_p \| \Gamma_j g \|_{p'} 2^{j(s+\sigma)} \\ &\leq \left(\sum_{j\geq 0} \| \Delta_j f \|_p^q 2^{jsq}\right)^{\frac{1}{q}} \left(\sum_{j\geq 0} \| \Gamma_j g \|_{p'}^{q'} 2^{j\sigma q'}\right)^{\frac{1}{q'}}. \end{aligned}$$

Proof of the lemma. If g is such that $g \in C^\epsilon$ and $|x|^{1+\epsilon} g \in L^\infty$, then $\sum_{k\in\mathbb{Z}} g(x+k)$ is Hölderian, hence its Fourier series converges pointwise :

$$\sum_{k\in\mathbb{Z}} g(x+k) = \sum_{k\in\mathbb{Z}} \hat{g}(2k\pi)e^{-2ik\pi x}.$$

Apply this to $g_\xi = \Omega(x)e^{-2i\pi\xi x}$ and $x = 0$. ∎

REMARKS. - i) We may as well characterize homogeneous Besov spaces by using the Lemarié-Meyer wavelet basis (where the orthonormal scaling function φ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$). We find that a temperate distribution f belongs to $\dot{B}_q^{s,p}$ if and only if $\left(\sum_{k\in\mathbb{Z}} 2^{j\frac{p}{2}} |< f \mid \psi(2^j x - k) >|^p\right)^{\frac{1}{p}}$ belongs to $\ell^q(\mathbb{Z})$.

We cannot easily replace the Lemarié-Meyer wavelet basis by another basis, because in general the null set of the wavelet transform $f \to (< f \mid \psi^*(2^j x - k) >|)_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is not included in the set of polynomials [LEM10].

ii) We may encounter non-zero coefficients for a wavelet series converging in some Besov spaces to zero. For instance, let ψ be the Haar function $\chi_{[0,1/2]} - \chi_{[1/2,1]}$ and write δ, the Dirac mass at 0, like $\delta(x-0^+)$ or $\delta(x-0^-)$:

$$\delta(x-0^+) = \chi_{[0,1]} + \sum_{j=0}^{+\infty} 2^j \psi(2^j x)$$

$$\delta(x-0^-) = \chi_{[-1,0]} - \sum_{j=0}^{+\infty} 2^j \psi(2^j x + 1).$$

Both series converge to δ in $B_2^{s,2}$ for $s < -\frac{1}{2}$. Thus, δ seems to have no unique expansion in the Haar basis. This is not a paradox, since in fact the Haar basis cannot analyze Besov spaces with too negative indexes : the analyzing wavelet ψ doesn't belong to $B_2^{\sigma,2}$ for $\sigma \geq \frac{1}{2}$ so that $< \cdot \mid \psi_{j,k} >$ is not a bounded linear form on $B_2^{s,2}$ for $s \leq -\frac{1}{2}$. If we look at lemma 3, we see that ψ is a regular molecule in $B_2^{s,2}$ for $-1 < s < \frac{1}{2}$ so that for any sequence $(\lambda_{j,k})_{j\geq 0,k\in\mathbb{Z}}$ so that $\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\lambda_{j,k}|^2 4^{js} < +\infty$ and $(\mu_k) \in \ell^2(\mathbb{Z})$, the series $\sum_k \mu_k \varphi(x-k) + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} \lambda_{j,k} 2^{\frac{j}{2}} \psi(2^j x - k)$ converges in $B_2^{s,2}$ $(-1 < s < \frac{1}{2})$: in the range $-1 \leq s \leq -\frac{1}{2}$, where the oscillation of ψ is stronger than its regularity, one cannot recover $\lambda_{j,k}$ from the sum of the series. ∎

6. Local analysis.

Of course, as well as the continuous wavelet transform, the wavelet bases are a good tool for the investigation of local regularity. For example, Jaffard's theorem (theorem 3 of chapter 1) becomes :

THEOREM 6. - *Let φ, φ^* be compactly supported scaling functions, $\alpha \in (0,1)$ and f a measurable function on $\mathbb{R}$ such that $|f(t)| \leq C(1+|t|)^\alpha$, a.e. Then :*

i) if for some $x_0 \in \mathbb{R}$ we have $\sup\limits_{x \neq x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|^\alpha} < +\infty$, then for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$|< f \mid \psi^*_{j,k} >| \leq C' 2^{-j(\frac{1}{2}+\alpha)} (1+ \mid k - 2^j x_0 \mid^\alpha)$$

(where $\psi^*_{j,k} = 2^{\frac{j}{2}} \psi^*(2^j x - k)$ and ψ, ψ^* are the compactly supported dual wavelets associated to φ, φ^*).

ii) Conversely, if $|< f \mid \psi^*_{j,k} >| \leq C'' 2^{-j(\frac{1}{2}+\alpha)} \left(1 + \frac{|k-2^j x_0|^\alpha}{1+\log_+ |x_0 - \frac{k}{2^j}|}\right)$, if $\varphi \in C^{\alpha+\epsilon}$ for some $\epsilon > 0$ and $f \in C^\epsilon$ for some $\epsilon > 0$, then $\mid f(x) - f(x_0) \mid \leq C''' \mid x - x_0 \mid^\alpha$.

If the assumption $f \in C^\epsilon$ is dropped, we can only conclude that f belongs to the microlocal space $C^{\alpha,-\alpha}_{x_0}$ of J. M. Bony (see [JAM] for a discussion of wavelets and microlocalization).

Chapter 7

MULTIVARIATE WAVELETS

In this chapter, we present the theory of multivariate wavelets. Besides the separable wavelet bases, which were introduced in the very beginning of the theory, we describe briefly the theory of generalized wavelets associated to dilation matrices. We focus mainly on the existence of localized wavelets, and give only bibliographical indications for the other questions on multivariate wavelets. The reason why we choose this presentation is simple : separable wavelets present no difficulty, non-separable wavelets are still not clearly understood, while the existence problem has been thoroughly investigated by K. Gröchenig, R. Q. Jia and Ch. Micchelli. Moreover, the algebraic computation, involved in the existence theorem (theorem 4) are a good example of multivariate wavelets techniques.

1. Multivariate wavelets : a general description.

Morlet wavelets can easily be generalized to the multivariate case. Indeed, the affine group $ax+b$ (which describes the action of translations and positive dilations) can be defined on $\mathbb{R}^d$ as well as on $\mathbb{R}$. Moreover, in contrast with the univariate case where the space H^2 of analytical signals plays a significant rôle as an irreducible subspace of $L^2(\mathbb{R})$ invariant through the action of $ax+b$, there is no such canonical subspace of $L^2(\mathbb{R}^d)$. The *admissibility condition* on $\psi \in L^2(\mathbb{R}^d)$ to define a wavelet transform in $L^2(\mathbb{R}^d)$ is :

$$(1) \qquad \exists C_\psi > 0 \text{ such that } \forall \xi_0 \neq 0, \quad \int_0^{+\infty} |\hat{\psi}(\lambda\xi_0)|^2 \frac{d\lambda}{\lambda} = C_\psi.$$

(For $d=1$, and for ψ real-valued, (1) is just equivalent to $\int_0^{+\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < +\infty$ and $\psi \neq 0$). If (1) is satisfied, then we have the same formulas as in dimension 1 :

(2.1) the $\psi_{a,b}(x) = \frac{1}{a^{d/2}}\psi\left(\frac{x-b}{a}\right)$ are the wavelets $(a>0, b \in \mathbb{R}^d)$

(2.2) the $C(a,b)(f) = \langle f \mid \psi_{a,b} \rangle$ are the wavelet coefficients

$$(2.3) \qquad f = \frac{1}{C_\psi}\int_0^{+\infty}\int_{\mathbb{R}^d} \langle f \mid \psi_{a,b} \rangle \psi_{a,b} \frac{da}{a^{1+d}} db.$$

Formula (2.3) expresses that the family $(\psi_{a,b})_{a>0, b\in\mathbb{R}^d}$ is a tight frame on $L^2(\mathbb{R}^d)$ (for the measure $\frac{da}{a^{1+d}}db$ on $(0,+\infty)\times\mathbb{R}^d$).

A convenient way to impose (1) is to choose a radial function ψ. In that case, (1) is equivalent to a mean-zero condition : if ψ is radial and $|x|^{\frac{d}{2}+\epsilon}\psi$ belongs to $L^2(\mathbb{R}^d)$, then (1) is equivalent to $\int \psi\, dx = 0$. A popular choice of ψ is the so-called Mexican hat

$$(3) \qquad \psi = (-\Delta)(e^{-\|x\|^2}).$$

This function was by instance recommended by D. Marr in his book *Vision* [MAR] for the multi-scale processing of images. The image f (considered as a signal $f \in L^2(\mathbb{R}^2)$) is represented by its approximates $f_\sigma = P_\sigma f$, where $\widehat{P_\sigma f} = e^{-\sigma\|x\|^2}\hat{f}$; the advantages one may advocate for the use of P_σ is the optimality of space and frequency localization of the impulse response $\frac{1}{(2\pi\sigma)^{d/2}}e^{-\frac{\|x\|^2}{\sigma}}$ and its isotropy (invariance under rotations); moreover one have the relationship $P_\delta(P_\tau f) = P_{\sigma+\tau}(f)$, which says that the approximate $P_{\sigma+\tau}(f)$ can be computed from the approximate $P_\tau f$. Then, in order to compress the information contained in $(P_\sigma f)_{\sigma>0}$, D. Marr suggests to sample the scales σ as $\sigma_j = \sigma_1^j$, and to replace $(P_{\sigma_j}(f))$ by the information on the *zero-crossings* of $(-\Delta)(P_{\sigma_j}(f))$: x_j is a zero-crossing of $(-\Delta)P_{\sigma_j}(f)$ if $\vec{\nabla}(P_{\sigma_j}(f))(x_j) \neq 0$ and $(-\Delta)P_{\sigma_j}(f)(x_j) = 0$; the choice of $-\Delta$ was advocated by D. Marr because of the localness of $-\Delta$ (hence the choice of a differential operator), its invariance under translations (constant coefficients), dilations (homogeneity) and rotations (isotropy); $-\Delta$ is the operator with the lowest degree satisfying these properties. Roughly speaking, the zero-crossings correspond to edges in the image, and Marr's conjecture was that one should be able to reconstruct f from the zero-crossings of $(-\Delta)(P_\sigma(f))$. This conjecture was proved to be false by Y. Meyer [MEY7].

Another way to introduce isotropy in the Morlet wavelet representation is to take a bigger group G than the affine group, by including in G *the rotations* besides the dilations and the translations. This generalization has been developped by R. Murenzi [MUR] and is generally used in physics when applying the continuous wavelet transform.

The *discretization* of the (affine) Morlet wavelet representation is similar to the univariate case. Indeed, one first discretizes the scale parameter a, introducing thus filter banks with constant quality factor: $f \to (f * \check{\psi}_{a_j})_{j\in\mathbb{Z}}$, where $\check{\psi}_{a_j}(x) = a_j^{-d/2}\bar{\psi}(-\frac{x}{a_j})$, $a_j = a_1^j$, and then one samples the filtered outputs on an uniform grid whose mesh is proportionnal to the scale:

$$(f * \check{\psi}_{a_j})_{j\in\mathbb{Z}} \to (f * \check{\psi}_{a_j}(nb_0a_j))_{j\in\mathbb{Z},n\in\mathbb{Z}^d}.$$

The d-dimensional analogue of the vaguelettes lemma holds (for functions $f_{j,k}$ such that $|x|^{\frac{d}{2}+\epsilon} f_{j,k} \in L^2$ and $|\xi|^\epsilon \hat{f}_{j,k} \in L^2$ uniformly in j and k and $\int f_{j,k}dx = 0$, we have:

$$\left\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\lambda_{j,k}a^{-j\frac{d}{2}}f_{j,k}(a^{-j}(x-kb_0a^j))\right\|_{L^2}^2 \leq C(a,b_0)\sum_j\sum_k|\lambda_{j,k}|^2).$$

Thus Daubechies' theorem (theorem 1 of chapter 2) can be easily adapted to the d-dimensional case :

THEOREM 1. - *Let $\psi \in L^2(\mathbb{R}^d)$ be a Morlet wavelet :*

$$\forall\xi_0 \neq 0, \quad \int_0^{+\infty} |\hat{\psi}(\lambda\xi_0)|^2 \frac{d\lambda}{\lambda} = 1 \tag{4}$$

and suppose that for some $\epsilon > 0$, $\int |x|^{d+2\epsilon}|\psi(x)|^2\, dx < +\infty$ and $\int |\xi|^{2\epsilon}|\hat{\psi}(\xi)|^2\, d\xi < +\infty$. Then if a is close enough to 1 $(1 < a < a_0)$ and if b is close enough to 0

$(0 < b < b_0(a))$ *the family*

$$(\psi_{j,k}(x) = a^{j\frac{d}{2}}\psi(a^j x - kb))_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$$

is a frame in $L^2(\mathbb{R}^d)$ *: there exist constants* $A, B > 0$ *such that for every* $f \in L^2(\mathbb{R}^d)$

$$A \parallel f \parallel_2^2 \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |< f \mid \psi_{j,k} >|^2 \leq B \parallel f \parallel_2^2 .$$

The next step is to introduce wavelet bases. In case one uses dyadic dilations, one has to use $2^d - 1$ wavelets to generate the basis, instead of a single one. Indeed, the theorem of P. Auscher and P.-G. Lemarié-Rieusset described in chapter 2 (theorem 2) can be partly generalized to the case of dimension d :

THEOREM 2. - *Let E be a finite set, $(\psi_\epsilon)_{\epsilon\in E}$ be $\mid E \mid$ functions in $L^2(\mathbb{R}^d)$ and assume that:*

(i) *the family* $(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}^d,\epsilon\in E}$ *is an Hilbertian basis of* $L^2(\mathbb{R}^d)$;

(ii) *for some positive* α, $\mid x \mid^{\frac{d}{2}+\alpha} \psi_\epsilon \in L^2$ *and* $\mid \xi \mid^\alpha \hat{\psi}_\epsilon \in L^2$ *(and* $\int \psi_\epsilon dx = 0$) *then the space* V_0, *the closed linear span of the functions* $2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k)$ *for* $j < 0$, $k \in \mathbb{Z}^d$, $\epsilon \in E$, *has a Riesz basis of the form* $(\varphi_\delta(x-k))_{k\in\mathbb{Z}^d,\delta\in D}$ and we have : $\mid E \mid = (2^d - 1) \mid D \mid$.

We will comment below theorem 2 in a more general setting and make the comparison with dimension 1.

Before this, we recall the brief history of multivariate wavelet bases. The first bases to be constructed were *separable bases* and they were constructed in analogy with the Haar basis.

In dimension 1, the decomposition of a function on the Haar system may be viewed as a dyadic martingale : if $f \in L^2$ has its support included in $[0,1]$, then $P_j f$ $(j \geq 0)$, the orthogonal projection of f on the space V_j associated to the Haar basis, is measurable with respect to the σ-algebra$\mathcal{F}$j generated by the dyadic intervals $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ $(0 \leq k < 2^j)$ and the conditional expectation $E(P_{j+1}f \mid \mathcal{F}_j)$ satisfies $E(P_{j+1}f \mid \mathcal{F}_j) = P_j f$. In dimension d, one replaces the dyadic intervals by dyadic cubes $\left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right] \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right]$ $(k_1, \cdots, k_d \in \{0, 1, \cdots, 2^j - 1\})$ and one obtains martingales on $[0,1]^d$. Of course, the σ-algebra $\mathcal{F}_j^{(d)}$ generated by dyadic cubes is the product σ-algebra of d copies of the σ-algebra $\mathcal{F}_j^{(1)}$ generated by dyadic intervals.

Similarly, starting from a multiresolution analysis $(V_j^{(1)})_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with associated orthonormal scaling function φ, orthonormal wavelet ψ and orthogonal projection operators $P_j^{(1)}$, we may define an orthogonal projection operator $P_j^{(d)}$ in $L^2(\mathbb{R}^d)$ by taking the (tensorial) product of d copies of $P_j^{(1)}$:

$$P_j^{(d)} = P_j^{(1)} \otimes \cdots \otimes P_j^{(1)}. \tag{5}$$

The range of $P_j^{(d)}$ is then $V_j^{(d)} = V_j^{(1)}\hat{\otimes}\cdots\hat{\otimes}V_j^{(1)}$, the closure in $L^2(\mathbb{R}^d)$ of $V_j^{(1)}\otimes\cdots\otimes V_j^{(1)}$. $V_j^{(d)}$ has an orthonormal basis $\varphi_{j,k}^{(d)} = 2^{j\frac{d}{2}}\varphi^{(d)}(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ where $\varphi^{(d)} = \varphi\otimes\cdots\otimes\varphi$ (so that $\varphi_{j,k}^{(d)} = \varphi_{j,k_1}\otimes\cdots\otimes\varphi_{j,k_d}$). The description of $W_j^{(d)}$, the orthogonal complement of $V_j^{(d)}$ in $V_{j+1}^{(d)}$, is then very easy to get : writing $V_{j+1}^{(1)} = V_j^{[0]}\oplus V_j^{[1]}$ where $V_j^{[0]} = V_j^{(1)}$ and $V_j^{[1]} = W_j^{(1)}$ ($W_j^{(1)}$ being the orthogonal complement of $V_j^{(1)}$ in $V_{j+1}^{(1)}$), we obtain :

$$W_j^{(d)} = \bigoplus_{\epsilon\neq(0,\cdots,0)} V_j^{[\epsilon_1]}\hat{\otimes}\cdots\hat{\otimes}V_j^{[\epsilon_d]} \tag{6}$$

and thus we have an associated orthonormal basis of $L^2(\mathbb{R}^d)$

$$\begin{cases}\psi_{\epsilon,j,k} = 2^{j\frac{d}{2}}\psi^{[\epsilon_1]}(2^j x - k_1)\cdots\psi^{[\epsilon_d]}(2^j x - k_d)\\ \epsilon\neq(0,\cdots,0),\ \psi^{[0]} = \varphi,\ \psi^{[1]} = \psi,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d.\end{cases} \tag{7}$$

For most applications, formula (7) is better than taking directly the basis of $L^2(\mathbb{R}^d)$ obtained by tensor products of the basis of $L^2(\mathbb{R})$: $\psi_{j_1,k_1}\otimes\psi_{j_2,k_2}\cdots\otimes\psi_{j_d,k_d}$. See however [MADA] where the tensor product of bases is preferred.

Of course, the formalism of (5), (6), (7) can be used for d different multi-resolution analysis $V_{j,1}^{(1)},\cdots,V_{j,d}^{(1)}$ (to define $V_j^{(d)} = V_{j,1}^{(1)}\hat{\otimes}\cdots\hat{\otimes}V_{j,d}^{(d)}$) and oblique projection operators $P_{j,i}$ (associated to a bi-orthogonal multi-resolution analysis $(V_{j,i}),(V_{j,i}^*)$). For instance, if P_j^1 is associated to V_j^1, V_j^{*1} and P_j^0 is associated to V_j^0, V_j^{*0} and satisfies $\frac{d}{dx}P_j^1 = P_j^0\frac{d}{dx}$ (see chapter 5 proposition 9), then the vector projection operators $\vec{P}_j = (P_j^{\{1\}}, P_j^{\{2\}},\cdots,P_j^{\{d\}})$ with $P_j^{\{k\}} = P_j^{\delta_{1,k}}\otimes\cdots\otimes P_j^{\delta_{d,k}}$ provides a multiresolution analysis of the square integrable vector fields on $\mathbb{R}^d$ i.e. of $L^2(\mathbb{R}^d)^d$) such that :

$$\vec{\nabla}\cdot\vec{P}_j(\vec{f}) = P_j^0\otimes\cdots\otimes P_j^0(\vec{\nabla}\cdot\vec{f})$$

(so that divergence free vector fields are projected on divergence-free vector fields [LEM9]).

We then obtain a notion of multi-resolution analysis for $L^2(\mathbb{R}^d)$:

DEFINITION 1. - *A multi-resolution analysis of $L^2(\mathbb{R}^d)$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ such that :*

$$V_j\subset V_{j+1},\quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\},\quad \bigcup_{j\in\mathbb{Z}} V_j \textit{ is dense in } L^2. \tag{8.1}$$

$$f\in V_j \ \ iff \ \ f(2x)\in V_{j+1} \tag{8.2}$$

(8.3) V_0 *has a Riesz basis of the form* $(\varphi(x-k))_{k\in\mathbb{Z}^d}$.

The next step was to describe multi-resolution analyses which were not obtained by tensor products of univariate multi-resolution analyses. Such examples have been given by S. Jaffard (using box splines [JAF1], see also E. Riemenschneider and Shen [RIE]) and G. Battle (using polyharmonic splines [BAT2]; see also P.-G. Lemarié [LEM2] and W. Madych [MADY]).

Of course, the use of tensor products gives a special rôle to the direction of the axes. In order to define a more isotropic discrete wavelet transform in $L^2(\mathbb{R}^2)$, J. C. Feauveau [FEA] introduced an intermediate space between V_j and V_{j+1}, called $V_{j+1/2}$, by the following axiom : let A be the operator $(x,y) \to (x+y, x-y)$ (so that $A^2(x,y) = (2x, 2y)$) and assume that for all $f \in V_0$, $f(A^{-1}(x,y)) \in V_0$; then $f \in V_{j+1/2}$ iff and only if $f(A^{-2j-1}(x,y)) \in V_0$. Then if $(\psi(x-k))_{k\in\mathbb{Z}^2}$ is an Hilbertian basis of the orthogonal complement of V_0 in $V_{1/2}$, the family $(2^{j/2}\psi(A^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}^2}$ is an Hilbertian basis of $L^2(\mathbb{R}^2)$.

This construction has been extended to other linear transformations than A :

DEFINITION 2. - *A dilation matrix is a matrix $A \in \mathcal{M}_d(\mathbb{Z})$ such that all eigenvalues of A have a modulus greater than 1.*

A generalized multi-resolution analysis associated to a dilation matrix A is a family of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^d)$ such that :

(9.1) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R}^d)$

(9.2) $f \in V_j \Leftrightarrow f(Ax) \in V_{j+1}$

(9.3) *V_0 has a Riesz basis of the form $(\varphi(x-k))_{k\in\mathbb{Z}^d}$.*

If W_0 is the orthogonal complement of V_0 in V_1, then W_0 has an orthonormal basis of the form $(\psi_\epsilon(x-k))_{k\in\mathbb{Z}^d}$, $1 \le \epsilon \le |\det A| - 1$. However, these wavelets ψ_ϵ are generally not easy to compute, except for the case $\det A = 2$ where there is one wavelet and where there is an explicit way to compute it. This is a very good reason for which dilation matrices with determinant equal to 2 have been frequently used in generalized wavelet transforms.

2. Existence of multivariate wavelets.

As in dimension 1, wavelet bases are provided by multiresolution analyses. In [LEM9], the following result is proved :

THEOREM 3. - *Let A be a dilation matrix on $\mathbb{R}^d$ and let $\rho, \tilde\rho \in \mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$ be two positive-valued functions such that $\rho(Ax) = |\det A|\,\rho(x)$ and $\tilde\rho({}^tAx) = |\det A|\,\tilde\rho(x)$. If*

$$(\psi_{\epsilon,j,k} = |\det A|^{j/2}\,\psi_\epsilon(A^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}^d,1\le\epsilon\le E}$$

and

$$(\psi^*_{\epsilon,j,k} = |\det A|^{j/2}\,\psi^*_\epsilon(A^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}^d,1\le\epsilon\in E}$$

are bi-orthogonal Riesz bases of $L^2(\mathbb{R}^d)$ such that :

(i) *for some $\alpha > 0$, $\rho(x)^{1/2+\alpha}\psi_\epsilon$ and $\rho(x)^{1/2+\alpha}\psi^*_\epsilon$ are in L^2*

(ii) *for some* $\alpha > 0$, $\tilde{\rho}(x)^\alpha \hat{\psi}_\epsilon$ *and* $\tilde{\rho}(x)^\alpha \hat{\psi}_\epsilon^*$ *are in* L^2
(iii) $\int \psi_\epsilon dx = \int \psi_\epsilon^* dx = 0$

then the oblique projection operator P_0 *defined by*

$$P_0 f = \sum_{\epsilon=1}^{E} \sum_{j<0} \sum_{k \in \mathbb{Z}^d} < f \mid \psi_{\epsilon,j,k}^* > \psi_{\epsilon,j,k}$$

can be rewritten as

$$P_0 f = \sum_{\delta=1}^{D} \sum_{k \in \mathbb{Z}^d} < f \mid \varphi_\delta^*(x-k) > \varphi_\delta(x-k)$$

where $(\varphi_\delta(x-k))_{1\leq\delta\leq D, k\in\mathbb{Z}^d}$ *is a Riesz basis of* $\operatorname{Ran} P_0$ *and* $(\varphi_\delta^*(x-k))_{1\leq\delta\leq D, k\in\mathbb{Z}^d}$ *a Riesz basis of* $(\operatorname{Ker} P_0)^\perp$, *and* $D = \frac{E}{|\det A|-1}$.

The main contrast with dimension 1 is that we may loose decay properties of the basic functions. Indeed, Y. Meyer has given a very nice example : take the wavelet basis $(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z}, k\in\mathbb{Z}}$ described by Meyer and Lemarié (example n° 6 of chapter 5), and the associated separable basis in $L^2(\mathbb{R}^2)$ (with wavelet, $\psi_1 = \psi \otimes \psi$, $\psi_2 = \psi \otimes \varphi$, $\psi_3 = \varphi \otimes \psi$ which belong to $S(\mathbb{R}^2)$ and whose Fourier transforms are identically 0 on a neighbourhood of 0) ; then apply the operator U defined by $\widehat{Uf}(\xi,\eta) = \frac{\xi+i\eta}{|\xi+i\eta|}\hat{f}(\xi,\eta)$; U is an isometry on L^2 and transforms the wavelet basis $(\psi_{\epsilon,j,k} = 2^j \psi_\epsilon(2^j x - k))_{1\leq\epsilon\leq 3, j\in\mathbb{Z}, k\in\mathbb{Z}^2}$ into another orthonormal wavelet basis $U(\psi_{\epsilon,j,k}) = (U\psi_\epsilon)_{j,k} = 2^j (U\psi_\epsilon)(2^j x - k)$; the wavelets $U\psi_\epsilon$ belong to S but we cannot find a Riesz basis $(\tilde{\varphi}(x-k))_{k\in\mathbb{Z}^2}$ of the associated shift-invariant space V_0 such that $\tilde{\varphi}$ be integrable [LEM9], [AUS2].

We now discuss the reverse problem. Provided that we have a multi-resolution analysis of $L^2(\mathbb{R}^d)$, do we have an associated wavelet basis with the same decay properties as the properties of the scaling function ?

This problem has been answered positively by K. Gröchenig in 1987 provided that $|\det A| > \frac{d+1}{2}$ [GROE1]. In case of a compactly supported scaling function with globally independent translates, a positive answer was given by Jia and Micchelli in 1991 [JIA]. Finally, the case where $\det A = 2$ is obvious.

THEOREM 4. - *Let* (V_j), (V_j^*) *be two generalized multiresolution analyses associated to the same dilation matrix* A. *Assume that* V_0 *has a Riesz basis* $(\varphi(x-k))_{k\in\mathbb{Z}^d}$ *and* V_0^* *a Riesz basis* $(\varphi^*(x-k))_{k\in\mathbb{Z}^d}$ *such that*

$$< \varphi(x-k) \mid \varphi^*(x-\ell) >= \delta_{k,\ell}(k,\ell \in \mathbb{Z}^d).$$

Let finally W_0 *be the space* $W_0 = V_1 \cap (V_0^*)^\perp$ *and* $W_0^* = V_1^* \cap V_0^\perp$. *Then :*

(i) if $\det A = 2$, W_0 *has a Riesz basis* $(\psi(x-k))_{k\in\mathbb{Z}^d}$ *and* W_0^* *has a Riesz basis* $(\psi^*(x-k))_{k\in\mathbb{Z}^d}$ *such that* $< \psi(x-k) \mid \psi^*(x-\ell) >= \delta_{k,\ell}$ *and moreover one can choose* ψ *and* ψ^*

with the same decay properties as φ and φ^* (i.e. ψ and $\psi^* \in L^2(\rho(x)^{1+\alpha}dx)$ if φ and φ^* do [for any positive α], ψ and ψ^* have rapid decay in L^2 if φ and φ^* do (i.e. $\forall\alpha \in \mathbb{N}^d$, $x^\alpha\psi$ and $x^\alpha\psi^*$ belong to L^2), ψ and ψ^* have compact support if φ and φ^* do).

(ii) if $\mid \det A \mid > \frac{d+1}{2}$ and φ, φ^* have rapid decay, then W_0 has a Riesz basis $(\psi_\epsilon(x-k))_{k\in\mathbb{Z}^d, 1\leq\epsilon\leq|\det A|-1}$ and W_0^* has a Riesz basis $(\psi_\epsilon^*(x-k))_{k\in\mathbb{Z}^d,1\leq\epsilon\leq|\det A-1|}$ such that $< \psi_\epsilon(x-k) \mid \psi_\eta^*(x-\ell) >= \delta_{\epsilon,\eta}\delta_{k,\ell}$ and moreover one can choose ψ_ϵ, ψ_ϵ^* with the same decay properties as φ and φ^*.

(iii) if φ and φ^* have compact support, then W_0 has a Riesz basis

$$(\psi_\epsilon(x-k))_{k\in\mathbb{Z}^d,1\leq\epsilon\leq|\det A|-1}$$

and W_0^* a Riesz basis

$$(\psi_\epsilon^*(x-k))_{k\in\mathbb{Z}^d,1\leq\epsilon\leq|\det A|-1}$$

such that $< \psi_\epsilon(x-k) \mid \psi_\eta^*(x-\ell) >= d_{\epsilon,\eta}\delta_{k,\ell}$ and $\psi_\epsilon, \psi_\epsilon^*$ have compact support.

REMARK. - If $\varphi = \varphi^*$, one can take $\psi = \psi^*$ in (i). Similarly, in (ii), one can make a Gram orthonormalization of $(\psi_\epsilon(x-k))_{1\leq\epsilon\leq|\det A|-1,k\in\mathbb{Z}^d}$ to obtain an orthonormal basis $\mid \tilde{\psi}_\epsilon(x-k))_{1\leq\epsilon\leq|\det A|-1,k\in\mathbb{Z}^d}$ of W_0 with rapid decay (if the ψ_ϵ's are rapidly decaying). The case of compactly supported wavelets, however, seems to remain open. ■

Proof. The idea of the proof for (i) and (ii) is very easy (except for the case of compactly supported wavelets in (ii)). If we write $\psi_0 = \varphi$, we have two shift invariant bases for V_1 : namely the family $(\psi_\epsilon(x-k))_{0\leq\epsilon\leq|\det A|-1,k\in\mathbb{Z}^d}$ and the family $((\det A)^{1/2}\varphi(Ax-k))_{k\in\mathbb{Z}^d}$. We take $(k_\epsilon)_{0\leq\epsilon\leq|\det A|-1}$ $\mid \det A \mid$ representants of the classes in $\mathbb{Z}^d/A\mathbb{Z}^d$, and rewrite the second family as $(\varphi_\epsilon(x-k))_{0\leq\epsilon\leq|\det A|-1,k\in\mathbb{Z}^d}$ where $\varphi_\epsilon = (\det A)^{1/2}\varphi(Ax-k_\epsilon)$. Similarly, we write $\varphi_\epsilon^* = (\det A)^{1/2}\varphi^*(Ax-k_\epsilon)$.

We may now forget the multi-resolution setting of our problem. If $(\varphi_\epsilon)_{0\leq\epsilon<D}$ is such that $(\varphi_\epsilon(x-k))_{0\leq\epsilon<D}$ is a Riesz basis of a closed subspace V_0 of L^2, and if $(\psi_\epsilon)_{0\leq\epsilon<D}$ are D functions in V_0, at which condition do those functions ψ_ϵ generate by translations a Riesz basis of V_0 ? Similarly, if $(\varphi_\epsilon^*(x-k))_{0\leq\epsilon<D,k\in\mathbb{Z}^d}$ is a Riesz basis of some space V_0^* and $(\psi_\epsilon^*)_{0\leq\epsilon<D}$ are functions in V_0^*, if moreover $< \varphi_\epsilon(x-k)/\varphi_\eta^*(x-\ell) >= \delta_{\epsilon,\eta}\delta_{k,\ell}$, at which condition do the functions ψ_ϵ^* generate by translations a Riesz basis of V_0^* such that $< \psi_\epsilon(x-k) \mid \psi_\eta^*(x-\ell) >= \delta_{\epsilon,\eta}\delta_{k,\ell}$?

Jia and Micchelli have provided an elegant setting for this problem. They assume that φ_ϵ and φ_ϵ^* belong to a space $\mathcal{L}_2$ defined by :

$$f \in \mathcal{L}_2 \Leftrightarrow \sum_{k\in\mathbb{Z}^d} \mid f(x-k) \mid \in L^2([0,1]^d).$$

If $\psi \in V_0$, then $\psi \in \mathcal{L}_2$ if and only if $\sum_{k\in\mathbb{Z}^d}\sum_\epsilon |< \psi \mid \varphi_\epsilon^*(x-k) >| < +\infty$. Moreover we have

$$\hat{\psi} = \sum_\epsilon \left\{ \sum_{k\in\mathbb{Z}^d} < \psi \mid \varphi_\epsilon^*(x-k) > e^{-i<k|\xi>} \right\} \hat{\varphi}_\epsilon(\xi).$$

Therefore, one introduces for $f, g \in \mathcal{L}_2$ the $2\pi\mathbb{Z}^d$-periodical function :

$$C(f,g)(\xi) = \sum_{k\in\mathbb{Z}^d} < f \mid g(x-k) > e^{-i<k|\xi>}$$
$$= \sum_{k\in\mathbb{Z}^d} \hat{f}(\xi + 2k\pi)\bar{\hat{g}}(\xi + 2k\pi) \quad \text{a. e.}$$

Then the answers to our two questions are (if $\varphi_\epsilon, \varphi_\epsilon^*$ belong to $\mathcal{L}_2$) :

j) if ψ_ϵ belong to $\mathcal{L}_2$, the family $(\psi_\epsilon(x-k))_{k\in\mathbb{Z}^d, 0\leq\epsilon<D}$ is a Riesz basis of V_0 if and only if the function $\det((C(\psi_\epsilon, \varphi_\eta^*))_{0\leq\epsilon<D, 0\leq\eta<D})$ never vanishes.

jj) if ψ_ϵ and ψ_ϵ^* belong to $\mathcal{L}_2$, they generate by translations bi-orthogonal bases of V_0 and V_0^* if and only if we have :

$$(C(\psi_\epsilon, \varphi_\eta^*))_{0\leq\epsilon<D, 0\leq\eta<D} \cdot (C(\varphi_\epsilon, \psi_\eta^*))_{0\leq\epsilon<D, 0\leq\eta<D} = Id_D.$$

Moreover, if $\varphi_\epsilon, \varphi_\epsilon^*$ and ψ_ϵ belong to $\mathcal{L}_2$ and if $(\psi_\epsilon(x-k))_{0\leq\epsilon<D, k\in\mathbb{Z}^d}$ is a Riesz basis of V_0, then its dual basis $(\psi_\epsilon^*(x-k))_{0\leq\epsilon<D, k\in\mathbb{Z}^d}$ in V_0^* exists and the functions ψ_ϵ^* belong to $\mathcal{L}_2$. This is based on the fact that the Wiener algebra $A = \{f \in C(\mathbb{R}^d/2\pi\mathbb{Z}^d)/f = \sum_{k\in\mathbb{Z}^d} f_k e^{-i<k|\xi>}$ with $\sum \mid f_k \mid < +\infty)$ satisfies that if $f \in A$ doesn't vanish then $\frac{1}{f} \in A$ (Wiener's lemma).

We may replace $\mathcal{L}_2$ by subspaces as for instance $\{f \in L^2/\rho(x)^{1/2+\alpha} f \in L^2\}$. One obtain similar properties, based on the fact that Wiener's lemma is true in the associated spaces of periodical functions [LEM9].

Thus the proof of (i), (ii) or (iii) is reduced to the following problem : given the first row of the matrix $(C(\psi_\epsilon, \varphi_\eta^*)(\xi))$ (since $\psi_0 = \varphi$), can we complete the matrix in such a way that it belongs to $GL_D(\mathbb{R})$ for every ξ ? Of course, we want the coefficients $m_{\epsilon,\eta}$ to satisfy the same estimates as $m_{0,\eta}$ (i.e. $m_{\epsilon,\eta} = \sum_{k\in\mathbb{Z}^d} m_{\epsilon,\eta,k} e^{-i<k<\xi>}$ with $\sum_{k\in\mathbb{Z}^d} \rho(k)^{1+2\alpha}(m_{\epsilon,\eta,k})^2 < +\infty$ if $\varphi_\epsilon, \varphi_\epsilon^*$ belong to $L^2(\rho(x)^{1+2\alpha}dx))$; if φ_ϵ and φ_ϵ^* have compact support, we are looking for trigonometric polynomials $m_{\epsilon,\eta}$ such that $\det((m_{\epsilon,\eta})) = c_0 e^{-i<k_0|\xi>}$ for some $c_0 \neq 0$ and $k_0 \in \mathbb{Z}^d$ (in order that $(m_{\epsilon,\eta})^{-1}$ has coefficients which are still trigonometric polynomials, hence that ψ_ϵ^* have compact support).

Now, if $\mid \det A \mid = 2$, we have to complete the matrix $\begin{pmatrix} C(\varphi, \varphi_0^*) & ? \\ C(\varphi, \varphi_1^*) & ? \end{pmatrix}$. An easy solution is given by $\begin{pmatrix} C(\varphi, \varphi_0^*) & -C(\varphi_1, \varphi^*) \\ C(\varphi, \varphi_1^*) & C(\varphi_0, \varphi^*) \end{pmatrix}$: indeed, we have its determinant equal to $C(\varphi, \varphi_0^*)C(\varphi_0, \varphi^*) + C(\varphi, \varphi_1^*)C(\varphi_1, \varphi^*) = C(\varphi, \varphi^*) = 1$.

For a more general A, we have to complete the matrix $\begin{pmatrix} C(\varphi, \varphi_0^*) & \\ \vdots & ?\ldots? \\ C(\varphi, \varphi_{D-1}^* & \end{pmatrix}$ (where $D = \mid \det A \mid$, and we know that $\sum_{k=0}^{D-1} C(\varphi, \varphi_k^*)C(\varphi_k, \varphi^*) = C(\varphi, \varphi^*) = 1$).

If φ, φ^* are rapidly decaying at ∞, then the functions

$$C(\varphi, \varphi_\ell^*)(\xi) = \sum_{\ell\in\mathbb{Z}^d} \mu_{k,\ell} e^{-i<\ell|\xi>}$$

are C^∞.

Now, if we define $\mu : \mathbb{R}^d/2\pi\mathbb{Z}^d \to S^{2D-1}$,

$$\xi \to \frac{1}{\sqrt{\sum_0^{D-1} | C(\varphi,\varphi_\ell^*) |^2}}(C(\varphi,\varphi_0^*),\cdots,C(\varphi,\varphi_{D-1}^*)),$$

we see that μ is well defined and smooth. Thus, if $d < 2D-1$, μ cannot be surjective, and one may find a point $(\alpha_0,\cdots,\alpha_{D-1})$ which doesn't belong to $\mu(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. Let $(\vec{\epsilon}_i)_{0\leq i\leq D-1}$ be an orthonormal family in $\mathbf{C}^D$ with $\vec{\epsilon}_0 = (\alpha_0,\cdots,\alpha_{D-1})$ and write

$$\frac{1}{\sqrt{\sum_{k=0}^{D-1} C(\varphi,\varphi_k^*) |^2}}\begin{pmatrix} C(\varphi,\varphi_0^*) \\ \vdots \\ C(\varphi,\varphi_{k-1}^*)\end{pmatrix} = \sum_{k=0}^{D-1} \omega_k(\xi)\vec{\epsilon}_k.$$

We claim that we may complete our matrix in the following way : in the basis $(\vec{\epsilon}_i)$, we complete the first row $\sum_{k=0}^{D-1}\omega_k(\xi)\vec{\epsilon}_k$ by the $D-1$ rows $\lambda\vec{\epsilon}_k + \bar{\omega}_k(\xi)\vec{\epsilon}_0$ $(1\leq k\leq D-1)$; if $\lambda \in (0,1)$ is small enough, the determinant never vanishes on $\mathbb{R}^d/2\pi\mathbb{Z}^d$: indeed the determinant is equal to $\lambda^{D-2}(\lambda\omega_0 - \sum_{k=1}^{D-1} | \omega_k |^2) = \lambda^{D-2}(-1+ | \omega_0 |^2 + \lambda\omega_0)$; we know that $\omega_0 \neq 1$ for all ξ, hence $| 1-\omega_0 | \geq \epsilon$ for some $\epsilon \in (0,1)$; now if $0 < \lambda < \frac{\epsilon(2-\epsilon)}{1-\epsilon}$ we have if $\omega_0 \notin (0,1]$, $\lambda\omega_0 - (1- | \omega_0 |^2) \neq 0$ and if $\omega_0 \in (0,1]$, $-1+\lambda\omega_0+\omega_0^2 \leq -1+\lambda(1-\epsilon)+(1-\epsilon)^2 < 0$.

Thus, we have proved point (ii) for rapidly decaying scaling functions φ,φ^*. The case of compactly supported scaling function is more delicate. Indeed, we have to prove directly the existence of dual compactly supported wavelets $\psi_\epsilon,\psi_\epsilon^*$: the compactness of the support would be lost in a Gram-Schmidt orthonormalization process.

As a matter of fact, the answer is given by a theorem of algebraic geometry, the *Quillen-Suslin theorem* which solved in 1976 Serre's celebrated conjecture on the freeness of finitely generated projective modules over $k[t_1,\cdots,t_n]$. A by-product of this theorem (see [LAM, p. 146) expresses that if k is a field, then any finitely generated projective (left) module on $k[t_1,t_1^{-1},\cdots,t_N,t_N^{-1}]$ is free. In our case, we want to find ψ_ϵ. If we define $\mathcal{M}$ as the class of compactly supported functions in V_1, we have a module over $C[t_1,t_1^{-1},\cdots,t_d,t_d^{-1}]$, where the scalar multiplication is given by :

$$P(t_1,t_1^{-1},\cdots,t_d,t_d^{-1})f = \mathcal{F}^{-1}\{P(e^{-i\xi_1},e^{i\xi_1},\cdots,e^{-i\xi_d},e^{i\xi_d})\hat{f}(\xi)\}$$

(where $\mathcal{F}^{-1}$ is the inverse Fourier transform). Of course, $\mathcal{M}$ is free, with $(\varphi_0,\cdots,\varphi_{D-1})$ as a basis. Now if $\mathcal{M}_0$ is the class of compactly supported functions in V_0 and $\mathcal{M}_1$ the class of compactly supported functions in $W_0 = V_1 \cap (V_0^*)^\perp$ we have still modules for our scalar multiplication ; moreover, $\mathcal{M}_0$ is free (with basis φ) and we want to prove that $\mathcal{M}_1$ is free (any basis of $\mathcal{M}_1$ would provide us with compactly supported wavelets). $\mathcal{M}_1$ is projective, since $\mathcal{M}_0 \oplus \mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}$ is free, and finitely generated : if P_0 is the projection operator on V_0 associated to φ and φ^*, then $(I-P_0)(\varphi_0),\cdots,(I-P_0)(\varphi_{D-1})$ generate $\mathcal{M}_1$. Thus the Quillen-Suslin theorem can be applied.

3. Properties of multivariate wavelets.

Separable dyadic wavelets don't present any peculiar difficulty. Their properties (functional analysis of Lebesgue, Sobolev, Hölder, Besov spaces) are the same as for the univariate case and are fully described in the book of Y. Meyer [MEY2]. Just like for the univariate dyadic wavelets, this analysis lies deeply on the Littlewood-Paley-Stein theory [STE1], [COIM1].

Non-separable wavelets are still not clearly understood. See for instance the paper by Cohen and Daubechies on the regularity of scaling functions associated to some dilation matrices on $\mathbb{R}^2$ with determinant 2 [COD2]. Another valuable reference is [VIM2].

One may also quote the investigation of scaling functions φ which are characteristic functions of sets in $\mathbb{R}^d$ (so that we obtain *self-similar tilings* in $\mathbb{R}^d$ [GROEM]).

Multivariate wavelets have begun to be an active field of investigation, but we believe that useful results are still to be found or understood before we are able to present a coherent state of the art as in dimension 1.

Chapter 8

ALGORITHMS

The numerical aspects of wavelet theory played an important part in the success of wavelets, but are still bewildering the beginners by some unusual ways of computation.

Mallat's algorithm is of course the heart of the matter. (Indeed, when I. Daubechies introduces her basis in [DAU1], she deals only with the associated scaling filters and always keeps close to the related fast wavelet transform). But, whereas it is a fast and efficient algorithm, those qualities can be spoilt by many difficult issues for the wavelet beginner ; among them, we may quote two widely discussed points : the initialization of Mallat's algorithm (how do we compute the projection on the scaling functions on the finer scale, before using Mallat's algorithm for going from fine scales to coarse ones ?) and the border effects (when dealing with a function defined on an interval instead of the whole real line).

1. The continuous wavelet transform.

The first wavelet transform, introduced by J. Morlet, is still used in many applications, and many people use an algorithm written by R. Kronland-Martinet for complex-valued wavelet transforms (using analytical wavelets).

In practice, the computation of a continuous wavelet transform may be described in the following way. For a fixed scale a, one tries to compute the coefficients

$$C(a,b) = \frac{1}{\sqrt{a}} \int f(t)\bar{\psi}(\frac{t-b}{a})dt. \tag{1}$$

The mapping $f \rightarrow C(a,b)$ is a convolution operator and (1) is performed through the fast Fourier transform. If we have a sample $(f(kb_0))_{k_0<k\leq k_0+2^M}$ and if the spectrum of $\frac{1}{\sqrt{a}}\psi(\frac{x}{a})$ is essentially contained in $\left[0, \frac{2\pi}{b_0}\right]$, then one may write :

(2.1) $\hat{f}\left(\frac{2\pi}{b_0}\frac{\ell}{2^M}\right) \approx \alpha_\ell$ for $1 \leq \ell \leq 2^M$, where $\alpha_\ell = b_0 \sum_{k=1}^{2^M} f((k_0+k)b_0)e^{-\frac{2i\pi\ell(k+k_0)}{2^M}}$

(2.2) $C(a,k+k_0)b_0) \approx \beta_k$ for $1 \leq k \leq 2^M$, where $\beta_k = \sum_{\ell=1}^{2^M} \sqrt{a}\bar{\hat{\psi}}\left(a\frac{2\pi}{2^M b_0}\right)\alpha_\ell e^{\frac{2i\pi\ell(k+k_0)}{2^M}}$

Thus we need in order to compute $C(a,b)$ to know a tabulation of $\bar{\hat{\psi}}\left(a\frac{2\pi}{2^M b_0}\ell\right)$ (or to compute these values), and then to perform two FFT : the cost of those FFT is $0(N\log N)$, where $N = 2^M$ is the number of datas. If we want to compute J voices (i.e. $C(a_j,(k+k_0)b_0)$ for $1 \leq j \leq J$ and $1 \leq k \leq N = 2^M$), the total cost is $0(JN\,\mathrm{Log}\,N)$.

In case ψ is compactly supported, one may be tempted to compute the convolution directly. Typically, one knows $(f((k+k_0)b_0))_{1\le k\le 2^M}$ and we want to compute

$$C(a_0^j,(k+k_0)b_0)=\int f(x)\frac{1}{\sqrt{a_0^j}}\bar{\psi}\left(\frac{x-(k+k_0)b_0}{a_0^j}\right)dx$$
$$\simeq\sum_{q=1}^{2^M}b_0f((q+k_0)b_0)\frac{1}{\sqrt{a_0^j}}\bar{\psi}\left(\frac{(q-k)b_0}{a_0^j}\right)$$

for $j_0<j\le j_0+J$ and $1\le k\le 2^M$. If $\operatorname{Supp}\psi\subset[-A,A]$ and $\frac{b_0}{a_0^{j_0}}\le 1$, we see that the number of operations needed to compute $C(a_0^j,(k+k_0)b_0)$ is $0\left(2A\frac{a_0^j}{b_0}\right)$ (hence $0\left(a_0^{j-j_0}\right)$ if A,a_0,b_0 are fixed), hence the total cost for computing the $J2^M$ coefficients $C(a_0^j,(k+k_0)b_0)$ will be $0(a_0^JN)$. This is a very high cost, since it is exponential in J. Therefore, M. Holschneider introduced the so-called *"algorithme à trous"* [HOL] in order to reduce the cost.

Let us explain how we may reduce the cost. If we assume that we want to compute

$$C(m,n)=\int f(x)\frac{1}{2^{m/2}}\psi\left(\frac{x-n}{2^m}\right)dx$$

for $1\le m\le J$ and $1\le n\le N=2^M$ and if we assume that $\psi(\frac{x}{2})$ belongs to the space V_0 of a multi-resolution analysis ($\psi(\frac{x}{2})=\sum_{q=q_0}^{q_1}\alpha_q\varphi(x-q)$, $\varphi(\frac{x}{2})=\sum_{q=\tilde q_0}^{\tilde q_1}\beta_q\varphi(x-q)$), then one sees that one may compute $C(m,n)$ recursively, by defining :

$$D(m,n)=\int f(x)\frac{1}{2^{m/2}}\varphi\left(\frac{x-n}{2^m}\right)dx,\quad 0\le m<J,\ n\in\mathbb{N}\ :$$

we have obviously

$$D(m+1,n)=\sum_{q=\tilde q_0}^{\tilde q_1}\frac{1}{\sqrt2}\beta_qD(m,n+q2^m)$$

$$C(m+1,n)=\sum_{q=q_0}^{q_1}\frac{1}{\sqrt2}\alpha_qD(m,n+q2^m).$$

Hence, the cost of the computation of $D(m+1,n)$ or $C(m+1,n)$ (once the $D(m,n)$ are computed) is $0(1)$, and therefore the total cost of the $J\cdot N$ coefficients $C(m,n)$ is $0(JN)$ (which is linear in J instead of exponential !).

REMARK. - We have not exactly presented the *"algorithme à trous"* , but a modified presentation given by I. Daubechies in [DAU3] which makes this algorithm very close to Mallat's algorithm.

2. Mallat's algorithm.

Mallat's algorithm, which is a fast wavelet transform adapted to orthogonal or bi-orthogonal bases, may be viewed as a mixture of the Laplacian pyramidal algorithm of Burt and Adelson [BUR] and of the sub-band coding scheme of Esteban and Galand [EST].

The algorithm of Burt and Adelson was introduced in 2 dimensions, for image processing, but we will explain it here in an 1-dimensional setting to make the presentation more clear. Given a discrete signal $(f_k)_{k\in\mathbb{Z}}$, one smoothes the signal by taking average values :

$$(f_k)_{k\in\mathbb{Z}} \to A((f_k)_{k\in\mathbb{Z}}) = \left(\tilde{f}_k = \sum_{q=q_0}^{q_1} c_q f_{k-q}\right)_{k\in\mathbb{Z}} .$$

Of course, this average signal is varying slower than the initial one, so that one may sub-sample it :

$$(\tilde{f}_k)_{k\in\mathbb{Z}} \to S((\tilde{f}_k)_{k\in\mathbb{Z}}) = (\tilde{f}_{2k})_{k\in\mathbb{Z}}.$$

If we want to make a comparison between our former signal (f_k) and our new signal $(\tilde{f}_{2k})$, we've got to interpolate $(\tilde{f}_{2k})$ to odd values of k : this is done in two steps ; the first one interpolates the missing values by 0 :

$$(\tilde{f}_{2k})_{k\in\mathbb{Z}} \to I((\tilde{f}_{2k})) = (g_k = \tilde{f}_k \text{ if } k \in 2\mathbb{Z}, = 0 \text{ if } k \in 2\mathbb{Z}+1)$$

and the second one smoothens this interpolation :

$$(g_k) \to A'(g_k) = (h_k = \sum_{q=q'_0}^{q'_1} d_q g_{k-q})_{k\in\mathbb{Z}}.$$

We may then represent our signal as :

$$(f_k) = A'ISA(f_k) + (\epsilon_k) = A'I((\tilde{f}_{2k})) + (\epsilon_k).$$

The signal $(\tilde{f}_{2k})$ is defined on a coarser grid than (f_k) ($2\mathbb{Z}$ instead of $\mathbb{Z}$) and the residual signal $(\epsilon_k = f_k - h_k)$ is small in the regions where (f_k) is regular. Of course, one iterates the transform on $(\tilde{f}_{2k})$ in order to have a signal $(\tilde{\tilde{f}}_{4k})$ defined on a still coarser grid and another residual error, and so on.

The difference between this algorithm and the fast wavelet transform lies in the analysis of the residual signal $(\epsilon_k)_{k\in\mathbb{Z}}$. In Mallat's formalism, the $(\epsilon_k)_{k\in\mathbb{Z}}$ are over-sampled whatever regular or not the function f is and one may decimate the ϵ_k and keep only the ϵ_{2k}, $k \in \mathbb{Z}$. Wherever f is regular, the remaining ϵ_{2k} will be small and one may neglect them as well as in Burt and Adelson's scheme.

In the subband coding scheme, one splits a discrete signal (f_k) in two signals (g_{2k}), (h_{2k}) with the following requirements : (g_{2k}), (h_{2k}) are sub-samples of convoluted signals:

(3) $$g_{2k} = \sum_{q=q_0}^{q_1} c_q f_{2k-q} \quad \text{and} \quad h_{2k} = \sum_{q=q'_0}^{q'_1} d_q f_{2k-q}$$

and the transformation is isometric :

$$\sum_{k\in\mathbb{Z}} |f_k|^2 = 2\sum_{k\in\mathbb{Z}} |g_{2k}|^2 + 2\sum_{k\in\mathbb{Z}} |h_{2k}|^2 . \tag{4}$$

If we take the polar form of (4), we obtain :

$$\sum_{k\in\mathbb{Z}} f_k\bar{\varphi}_k = 2\sum_{k\in\mathbb{Z}} \left\{\sum_{q=q_0}^{q_1} c_q f_{2k-q}\right\}\left\{\sum_{q=q_0}^{q_1} c_q\bar{\varphi}_{2k-q}\right\} + \left\{\sum_{q=q'_0}^{q'_1} d_q f_{2k-q}\right\}\left\{\sum_{q=q'_0}^{q'_1} d_q\bar{\varphi}_{2k-q}\right\} \tag{5}$$

which gives (taking $f_k = \delta_{k,k_0}$, $\varphi_k = \delta_{k,k_1}$) :

$$\text{For } k_0, k_1 \in \mathbb{Z},\quad 2\sum_{k\in\mathbb{Z}} c_{2k-k_0}c_{2k-k_1} + 2\sum_{k\in\mathbb{Z}} d_{2k-k_0}d_{2k-k_1} = \delta_{k_0,k_1}. \tag{6}$$

(6) is equivalent to (4) and may be rewritten (defining

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi} \quad\text{and}\quad m_1(\xi) = \sum_{k\in\mathbb{Z}} d_k e^{-ik\xi}) \quad \text{as}$$

$$|m_0(\xi) + m_0(\xi+\pi)|^2 + |m_1(\xi) + m_1(\xi+\pi)|^2 = 2 \tag{7.1}$$

$$|m_0(\xi) - m_0(\xi+\pi)|^2 + |m_1(\xi) - m_1(\xi+\pi)|^2 = 2 \tag{7.2}$$

$$(m_0(\xi)+m_0(\xi+\pi))(\bar{m}_0(\xi)-\bar{m}_0(\xi+\pi))+(m_1(\xi)+m_1(\xi+\pi))(\bar{m}_1(\xi)-\bar{m}_1(\xi+\pi)) = 0. \tag{7.3}$$

(Just take for (7.1) k_0 and k_1 even, for (7.2) k_0 and k_1 odd, for (7.3) k_0 even and k_1 odd). Thus, if we write

$$m_0(\xi) = \frac{1}{\sqrt{2}}(u_0(2\xi) + e^{i\xi}v_0(2\xi))$$

and

$$m_1(\xi) = \frac{1}{\sqrt{2}}(u_1(2\xi) + e^{i\xi}v_1(2\xi)),$$

where u_0, v_0, u_1 and v_1 are 2π-periodical trigonometric polynomials, we obtain that (4) is equivalent to the fact that $\begin{pmatrix} u_0(\xi) & v_0(\xi) \\ u_1(\xi) & v_1(\xi) \end{pmatrix}$ is unitary for all ξ. Notice that v_0, v_1 are almost uniquely determined by u_0, u_1 : indeed we have necessarily

$$v_0(\xi) = -e^{iN\xi}\gamma_0\bar{u}_1(\xi) \quad\text{and}\quad v_1(\xi) = e^{iN\xi}\gamma_0\bar{u}_0(\xi)$$

for some unitary constant γ_0 (and since we have implicitly assumed the coefficients c_k and d_k to be real-valued, this gives $\gamma_0 = \pm 1$) and some constant $N \in \mathbb{Z}$. (This unicity may be proved easily by algebraic reasoning ; it is also a by-product of theorem 1 of chapter 3 : if ω is any compactly supported function such that $(\sqrt{2}\omega(2x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of a space $V_1 \subset L^2$, if $\hat{\alpha}(\xi) = m_0(\frac{\xi}{2})\hat{\omega}(\frac{\xi}{2})$ and $\hat{\beta}(\xi) = m_1(\frac{\xi}{2})\hat{\omega}(\frac{\xi}{2})$, then $(\alpha(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of a space $V_0 \subset V_1$ and $(\beta(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_0 = V_0^{\perp} \cap V_1$; β has automatically a minimal support and thus is unique up to a shift or a multiplication by an unitary scalar). Another useful remark is that $| u_0(\xi) |^2 + | v_0(\xi) |^2 = 1$ is equivalent to $| m_0(\xi) |^2 + | m_0(\xi+\pi) |^2 = 1$.

Thus the basic equalities of orthonormal scaling filters

$$| m_0(\xi) |^2 + | m_0(\xi+\pi) |^2 = 1$$

(expressing that the scaling function φ generates an orthonormal family $(\varphi(x-k))_{k\in\mathbb{Z}}$) and $m_1(\xi) = e^{-i\xi}\bar{m}_0(\xi+\pi)$ (where $\hat{\psi} = m_1(\frac{\xi}{2})\hat{\varphi}(\frac{\xi}{2})$ is the Fourier transform of an orthogonal wavelet ψ) are equivalent to the isometric relationship (4).

Moreover the polar formula (5) gives us a reconstruction formula :

$$f_{k_0} = 2\sum_{k\in\mathbb{Z}} c_{2k-k_0} g_{2k} + 2\sum_{k\in\mathbb{Z}} d_{2k-k_0} h_{2k}. \tag{8}$$

The link between wavelets and *quadrature mirror filters* (i.e. functions $m_0(\xi)$, $m_1(\xi)$ satisfying (7.1) to (7.3)) is easily described. Indeed if (V_j) is a multiresolution analysis associated to a compactly supported orthonormal scaling function φ and a compactly supported orthonormal wavelet ψ, and if P_j is the orthogonal projection operator onto V_j, and $Q_j = P_{j+1} - P_j$, the decomposition $P_{j+1}f = P_j(P_{j+1}f) + Q_j(Q_{j+1}f)$ can be read as:

$$\sum_k f_{j+1,k} 2^{(j+1)/2}\varphi(2^{j+1}x-k) = \sum_k f_{j,k} 2^{j/2}\varphi(2^j x - k) + \sum_k \tilde{f}_{j,k} 2^{jk}\psi(2^j x - k)$$

where :

$$\begin{aligned} f_{j,k} &= \sum_q f_{j+1,q} < 2^{(j+1)/2}\varphi(2^{j+1}x - q) \mid 2^{j/2}\varphi(2^j x - k) > \\ &= \sum_q < \sqrt{2}\varphi(2x - q + 2k) \mid \varphi(x) > f_{j+1,q} \\ &= \sum_q < \sqrt{2}\varphi(2x + q) \mid \varphi(x) > f_{j+1,2k-q} \\ \text{and } \tilde{f}_{j,k} &= \sum_q < \sqrt{2}\varphi(2x + q) \mid \psi(x) > f_{j+1,2k-q} \end{aligned}$$

so that $f_{j,k}$ and $\tilde{f}_{j,k}$ are given by formulas similar to (3). Moreover

$$\| P_{j+1}f \|_2^2 = \| P_j f \|_2^2 + \| Q_j f \|_2^2$$

becomes

$$\sum | f_{j+1,k} |^2 = \sum | f_{j,k} |^2 + \sum | \tilde{f}_{j,k} |^2,$$

hence we see that we have a quadrature mirror filter

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_q < \sqrt{2}\varphi(x+q) \mid \varphi(x) > e^{-iq\xi} = \frac{1}{2} \sum_q < \varphi(\frac{x}{2}) \mid \varphi(x+q) > e^{-iq\xi} ;$$

thus our orthonormal scaling filter of chapter 5 is a quadrature mirror filter.

We may now present the Fast Wavelet Transform of S. Mallat. Given two bi-orthogonal compactly supported scaling functions φ, φ^*, and associated wavelets ψ, ψ^*, we are looking for a fast computation of the wavelet coefficients

$$< f \mid \psi^*_{j,k} > = < f \mid 2^{j/2}\psi^*(2^j x - k) > .$$

This computation is organized in the following way.

Define $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ as $a_k =< \varphi(\frac{x}{2}) \mid \varphi^*(x-k) >$ and $b_k =< \varphi(x-k) \mid \varphi^*(\frac{x}{2}) >$: we have $\hat{\varphi}(2\xi) = \frac{1}{2}(\sum a_k e^{-ik\xi})\hat{\varphi}(\xi)$ and $\hat{\varphi}^*(2\xi) = \frac{1}{2}(\sum b_k e^{-ik\xi})\hat{\varphi}^*(\xi)$. We suppose that φ, φ^* are compactly supported (and real-valued), so that a_k and b_k equal 0 for all but finitely many k_0. Now, remember that ψ, ψ^* are defined by

$$\hat{\psi}(2\xi) = \frac{1}{2} \sum (-1)^k b_k e^{i(k-1)\xi} \hat{\varphi}(\xi)$$

and

$$\hat{\psi}^*(2\xi) = \frac{1}{2} \sum (-1)^k a_k e^{i(k-1)\xi} \hat{\varphi}^*(\xi).$$

The wavelet coefficients $< f \mid \psi^*_{j,k} >$, $k \in \mathbb{Z}$, correspond to the determination of

$$Q_j f = \sum_k < f \mid \psi^*_{j,k} > \psi_{j,k} = P_{j+1}f - P_j f = P_{j+1}f - P_j(P_{j+1}f).$$

Hence we have :

$$\sum_{j=j_0}^{j_1} \sum_k < f \mid \psi^*_{j,k} > \psi_{j,k} = P_{j_1+1}f - P_{j_1}(P_{j_1+1}f) + P_{j_1}f - P_{j_1-1}(P_{j_1}f) +$$

$$\cdots + P_{j_0+1}f - P_{j_0}(P_{j_0+1}f) + P_{j_0}f$$

and the computations are organized as follows :

Step 1. - Begin with the determination of $P_{j_1+1}f$, i.e. of

$$s_{j_1+1,k} =< f \mid \varphi^*_{j_1+1,k} > .$$

Step 2. - Compute recursively $s_{j,k} =< f \mid \varphi^*_{j,k} >$ and $d_{j,k} =< f \mid \psi^*_{j,k} >$ by the relationships :

(9.1)
$$s_{j,k} = \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} b_{-\ell} s_{j+1,2k-\ell}$$

(9.2)
$$d_{j,k} = \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} (-1)^{\ell-1} a_{\ell-1} s_{j+1,2k-\ell}$$

(9.1) and (9.2) are easy to prove. For instance, write

$$\begin{aligned} s_{j,k} =< P_{j+1} f \mid \varphi^*_{j,k} > &= \sum_{\ell\in\mathbb{Z}} < f \mid \varphi^*_{j+1,\ell} >< \varphi_{j+1,\ell} \mid \varphi^*_{j,k} > \\ &= \sum_{\ell\in\mathbb{Z}} s_{j+1,\ell} \frac{1}{\sqrt{2}} b_{\ell-2k}. \end{aligned}$$

As we have already noticed it, (9.1) and (9.2) can be viewed as convolutions followed by sub-sampling.

Finally, the *Fast Wavelet Transform* of S. Mallat is the algorithm which transforms the sequence $(s_{j_1+1,k})_{k\in\mathbb{Z}}$ into the sequences

$$(s_{j_0,k})_{k\in\mathbb{Z}} \cup ((d_{j,k})_{k\in\mathbb{Z}})_{j_0\leq j\leq j_1}.$$

The inverse transform is given by the formula :

(10)
$$s_{j+1,k} = \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} a_{k-2\ell} s_{j,\ell} + \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} (-1)^{k+1} b_{2\ell-k-1} d_{j,\ell}$$

(10) is as easy as (9.1) : just write

$$s_{j+1,k} =< P_{j+1} f \mid \varphi^*_{j+1,k} >=< P_j f \mid \varphi^*_{j+1,k} > + < Q_j f \mid \varphi^*_{j+1,k} > .$$

This formula can be viewed as interpolation by 0 ($(s_{j,k}) \to (\tilde{s}_{j,k})$ with $\tilde{s}_{j,k} = 0$ if k is odd, $= s_{j,k/2}$ if k is even and similarly $(d_{j,k}) \to (\tilde{d}_{j,k})$) followed by convolution:

$$s_{j+1,k} = \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} a_\ell \tilde{s}_{j,k-\ell} + \sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{2}} (-1)^\ell b_{-\ell-1} \tilde{d}_{j,k-\ell}.$$

The complexity of this algorithm is easy to evaluate. Assume (for the sake of simplicity) that f is periodical with period $2^N \geq (\frac{1}{2})^{j_0}$ and that we want to compute

$$(d_{j,k})_{j_0\leq j\leq j_1, 1\leq k\leq 2^{N+j}} \quad \text{and} \quad (s_{j,k})_{1\leq k\leq 2^{N+j_0}}$$

knowing $(s_{j_1+1,k})_{1\le k\le 2^{N+j_1+1}}$. (Indeed, if f is 2^N-periodical and $j \ge N$ then

$$s_{j,k+2^{N+j}} = s_{j,k} \quad \text{and} \quad d_{j,k+2^{N+j}} = d_{j,k}).$$

We know 2^{N+j_1+1} coefficients $s_{j_1+1,k}$ and we want to compute

$$2^{N+j_0} + \sum_{j=j_0}^{j_1} 2^{N+j} = 2^{N+j_1+1}$$

coefficients $s_{j,k}$ or $d_{j,k}$: this is the same number, and indeed Mallat's algorithm is a mere change of basis. Now, the complexity for computing $s_{j,k}$ or $d_{j,k}$ from $(s_{j+1,k})$ is $0(1)$ because of formula (9.1) and (9.2). Thus we find that the total complexity is

$$2^{N+j_0}O(1) + 2^{N+j_0+1}O(1) + \cdots + 2^{N+j_1}O(1) = O(2^{N+j_1+1}),$$

and it remains proportional to the number of data whatever the number of scales of decomposition is.

3. Wavelets on the interval.

One of the main problem in the wavelet formalism was the fact that it is defined on the whole line, while in numerical applications one could deal only with functions defined on a bounded interval. Thus, one had to investigate the problem of truncation in the wavelet approximation.

Among the solutions that were used in the beginning, we may quote the following ones (we will always assume that we deal with compactly supported bi-orthogonal scaling functions φ, φ^* and that we deal with a truncated function $f\,|_{[0,2^N]}$ for which we want to compute the coefficients $d_{j,k}$, $j_0 \le j \le j_1$) :

• the first solution is to extrapolate f outside $[0,2^N]$ by 0. Thus we have a function $\tilde{f}$ defined on the whole line. But we encounter two problems. The main one is that, unless f vanishes at the boundary of $[0,2^N]$, we have introduced discontinuities at 0 and 2^N, and this will be reflected in high-valued wavelet coefficients near 0 and 2^N and thus in a non-negligible border effect in the reconstruction of f. The second one is the fact that we destroyed the independency of our analyzing functions : indeed if ℓ is the length of the support of φ^* and L the length of the support of the associated wavelet ψ^*, in order to compute $P_{j+1}(\tilde{f})$ we need $2^{N+j+1} + \ell - 1$ coefficients (corresponding to the scaling functions $\varphi^*_{j+1,k}$ whose support encounters $(0,2^N)$), and $2^{N+j} + \ell - 1$ coefficients for $P_j(\tilde{f})$ and $2^{N+j} + L - 1$ coefficients for $Q_j(\tilde{f})$: thus we need more coefficients for $P_j(\tilde{f})$ and $Q_j(\tilde{f})$ than for $P_{j+1}(\tilde{f})$.

• a second solution, which is frequently used, is to extend f into a 2^N-periodical function $\tilde{f}$. Besides the possibility of using FFT in the computations (which is mainly interesting when dealing with non-compactly supported scaling functions), the advantage of this solution is that we recover the independency of our analyzing functions. Indeed, the coefficients $<\tilde{f} \mid \varphi^*_{j,k}>$ or $<\tilde{f} \mid \psi^*_{j,k}>$ (for $j \geq -N$) are clearly periodical: $<\tilde{f} \mid \varphi^*_{j,k+2^{N+j}}> = <\tilde{f} \mid \varphi^*_{j,k}>$. Thus we need 2^{N+j} coefficients for computing $P_j(\tilde{f})$, 2^{N+j} coefficients for $Q_j(\tilde{f})$ and 2^{N+j+1} coefficients for computing $P_{j+1}(\tilde{f})$.

As a matter of fact, we have thus constructed a *periodical wavelet basis* for $L^2([0,2^N])$: we have

$$\int_{-\infty}^{+\infty} \tilde{f}\varphi^*_{j,k}dx = \int_0^{2^N} f \sum_{p\in\mathbb{Z}} \varphi^*_{j,k}(x+p2^N)dx$$

(and so on...); it is then easy to see that

$$\left(\frac{1}{\sqrt{2^N}} \sum_{p\in\mathbb{Z}} \varphi_{-N,0}(x+p2^N)\right) \cup \bigcup_{j=-N}^{\infty} \left(\sqrt{2^j} \sum_{p\in\mathbb{Z}} \psi_{j,k}(x+p2^N)\right)_{0\leq k<2^{N+j}}$$

and

$$\left(\frac{1}{\sqrt{2^N}} = \sum_{p\in\mathbb{Z}} \varphi_{-N,0}(x+p2^N)\right) \cup \sum_{j=-N}^{\infty} \left(\sqrt{2^j} \sum_{p\in\mathbb{Z}} \psi^*_{j,k}(x+p2^N)\right)_{0\leq k<2^{N+j}}$$

are bi-orthogonal bases of $L^2([0,2^N])$ and provide bases for spaces of regular periodical functions (if $\varphi \in H^\sigma$, then we have a basis for $H^s(\mathbb{R}/2^N\mathbb{Z})$ for $0 < s < \sigma$).

• a third solution, introduced in order to alleviate the discontinuities at 0 or 2^N, is to extend f on $[-2^N, 2\cdot 2^N]$ by symmetries : on $[-2^N,0]$, $\tilde{f}(x) = f(-x)$; on $[2^N, 2^N+2^N]$, $\tilde{f}(x) = f(2^{N+1}-x)$. If $f \in C^\alpha([0,2^N])$ $(0<\alpha<1)$ then $\tilde{f} \in C^\alpha([-2^N, 2^{N+1}])$. Besides keeping the regularity of f up to order 1, one finds another advantage in this solution : one recovers again the independency of the analyzing functions, provided that one deals with *symmetric scaling functions* (hence with bi-orthogonal scaling functions, since there is no symmetric compactly supported orthonormal scaling function but the Haar scaling function $\varphi = \chi_{[0,1]}$). Indeed, let's assume that φ, φ^* are even functions, and thus that ψ, ψ^* are also symmetrical: $\psi(1-x) = \psi(x)$, $\psi^*(1-x) = \psi^*(x)$. Now, if $k/2^j \in [-2^N, 0]$ and $\operatorname{Supp}\varphi^*_{j,k} \subset [-2^N, 2^N]$, we have

$$<\tilde{f} \mid \varphi^*_{j,k}> = <\tilde{f}(-x) \mid \varphi^*_{j,k}(-x)> = <\tilde{f}(x) \mid \varphi^*_{j,-k}(x)>,$$

and similarly for $k/2^j \in [2^N, 2\cdot 2^N]$ and $\operatorname{Supp}\varphi^*_{j,k} \subset [0, 2\cdot 2^N]$,

$$<\tilde{f} \mid \varphi^*_{j,k}> = <\tilde{f} \mid \varphi^*_{j,2^{N+j+1}-k}> ;$$

thus we have only to know the coefficients $<\tilde{f} \mid \varphi^*_{j,k}>$ for $0 \leq k \leq 2^{N+j}$ in order to know $P_j(\tilde{f})$ on the neighbourhood of $[0,2^N]$. Similarly, for $\operatorname{Supp}\psi^*_{j,k} \subset [-2^N, 2^N]$ we have

$$<\tilde{f} \mid \psi^*_{j,k}> = <\tilde{f}(-x) \mid \psi^*_{j,k}(-x)> = <\tilde{f} \mid \psi^*_{j,-k-1}>$$

and for $\mathrm{Supp}\,\psi^*_{j,k} \subset [0, 2^{N+1}]$,

$$< \tilde{f} \mid \psi^*_{j,k} > = < \tilde{f}(2^{N+1} - x) \mid \psi^*_{j,k}(2^{N+1} - x) > = < \tilde{f} \mid \psi^*_{j,2^{N+j+1}-k-1} >$$

so that we need only to know the coefficients $< \tilde{f} \mid \psi^*_{j,k} >$, $0 \le k \le 2^{j+N} - 1$.

Once again, we have constructed a basis for $L^2([0, 2^N])$. Indeed if j_0 is big enough so that 2^{-j_0} times the diameter of $\mathrm{Supp}\,\varphi$ is less than 2^{N-1} (and the same for φ^*, ψ and ψ^*) we have a basis $(\tilde{\varphi}_{j_0,k})_{0 \le k \le 2^{N+j_0}} \cup (\tilde{\psi}_{j,k})_{j \ge j_0, 0 \le k \le 2^{j+N}-1}$ with dual basis $(\varphi^*_{j,k})_{0 \le k \le 2^{j_0+N}} \cup (\tilde{\psi}^*_{j,k})_{j \ge j_0, 0 \le k \le 2^{j_0+N}-1}$, where :

$$\tilde{\varphi}_{j_0,k} = 2^{j_0/2} \varphi(2^{j_0} x - k)$$

for $\frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\,\varphi < k < 2^{j_0+N} - \frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\,\varphi$,

$$\tilde{\varphi}_{j_0,k} = 2^{j_0/2} \varphi(2^{j_0} x - k) + \varphi(2^{j_0} x + k) \mid_{[0,2^N]}$$

for $0 \le k \le \frac{1}{2}\mathrm{diam}(\mathrm{Supp}\,\varphi)$,

$$\tilde{\varphi}_{j_0,k} = 2^{j_0/2} (\varphi(2^{j_0} x - k) + \varphi(2^{j_0} x - 2^{j_0+N+1} + k)) \mid_{[0,2^N]}$$

for $2^{j_0+N} - \frac{1}{2}\mathrm{diam}(\mathrm{Supp}\,\varphi) \le k \le 2^{j_0+N}$ and

$$\tilde{\psi}_{j,k} = 2^{j/2} \psi(2^j x - k)$$

for $\frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\,\psi < k < 2^{j+N} - \frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\psi - 1$,

$$\tilde{\psi}_{j,k} = 2^{j/2} \psi(2^j x - k) + \psi(2^j x + k + 1) \mid_{[0,2^N]}$$

for $0 \le k \le \frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\,\psi$ and

$$\tilde{\psi}_{j,k} = 2^{j/2} \psi(2^j x - k) + \psi(2^j x - 2^{j+N+1} + k + 1) \mid_{[0,2^n]}$$

for $2^{j+N} - \frac{1}{2}\mathrm{diam}\ \mathrm{Supp}\,\psi - 1 \le k \le 2^{j+N} - 1$

(and similar formulas for $\tilde{\varphi}^*_{j_0,k}$ and $\tilde{\psi}^*_{j,k}$). Indeed, the bi-orthogonality is easy to check, and we have already seen the completeness of $(\tilde{\varphi}^*_{j_0,k}) \cup (\tilde{\psi}^*_{j,k})$ (since $< \tilde{f} \mid \varphi^*_{j_0,k} >_{\mathbb{R}} = < f \mid \tilde{\varphi}^*_{j_0,k} >_{[0,2^N]}$ and $< \tilde{f} \mid \psi^*_{j_0,k} >_{\mathbb{R}} = < f \mid \tilde{\psi}^*_{j_0,k} >_{[0,2^N]}$). The stability of the bases is easy to check, as a consequence of the stability of the wavelet bases on the whole line. This basis is called the *folded basis* by Cohen, Daubechies and Vial in [CODV].

Both the periodical basis and the folded basis are not well fitted to the study of regular function spaces with regularity greater then 1 (such as $C^r([0, 2^N])$, $H^s([0, 2^N]), \cdots$). A reason that we may easily point is that they don't allow the reconstruction of polynomials on the big scales basic functions : indeed $x\ |_{[0,2^N]}$ is not periodical and cannot be written as $\sum_{k=0}^{2^{j_0+N}} c_k \tilde{\varphi}_{j_0,k}$.

In 1990, Y. Meyer proposed a solution which allowed the analysis of $H^s([0,2^N])$. He started from an orthonormal Daubechies scaling function φ (with $\operatorname{Supp}\varphi = [0, 2M-1]$) and defined $V_j^{[0,2^N]}$ as the space $\{f\ |_{[0,2^N]}\ / f \in V_j\}$ where V_j was as usually the closed linear span of the $2^{j/2}\varphi(2^j x - k) = \varphi_{j,k}$, $k \in \mathbb{Z}$. Then $V_j^{[0,2^N]}$ contains the polynomials $1, x, \cdots, x^{M-1}$ (restricted to $[0,2^N]$), and we may easily characterize $H^s[0,2^N]$ for $s < s_0$ (where s_0 is the regularity index of φ : $s_0 = \max_{\varphi \in H^s} s$) in terms of the orthogonal projections $p_j^{[0,2^N]}$ from $L^2([0,2^N])$ onto $V_j^{[0,2^N]}$.

As we have already seen it (theorem 4 of chapter 4), a basis for $V_j^{[0,2^N]}$ is exactly the family $(\varphi(2^j x - k)\ |_{[0,2^N]})_{-2M+2 \le k \le 2^{j+N}-1}$. If $j \ge j_0$ where $2M-1 \le 2^{j_0+N-1}$, then a Gram-Schmidt orthonormalization of this basis gives a basis $\tilde{\varphi}_{j,k}$ such that :

- for $-2M+2 \le k \le -1$, $\tilde{\varphi}_{j,k}(x) = 2^{(j-j_0)/2}\tilde{\varphi}_{j_0,k}(2^{j-j_0}x)$
- for $0 \le k \le 2^{j+N} - 2M + 1$, $\tilde{\varphi}_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$
- for $2^{j+N} - 2M + 2 \le k \le 2^{j+N} - 1$, $\tilde{\varphi}_{j,k}(x) = 2^{(j-j_0)/2}\tilde{\varphi}_{j_0,k-2^{j+N}}(2^{j-j_0}(x - 2^N) + 2^N)$.

Now, if we try to describe the orthogonal complement $W_j^{[0,2^N]}$ of $V_j^{[0,2^N]}$ into $V_{j+1}^{[0,2^N]}$, we may search for a supplement X_j and use again a Gram-Schmidt orthonormalization. Of course, from $V_{j+1} = V_j + W_j$, we see that $V_{j+1}^{[0,2^N]}$ is generated by the functions $2^{j/2}\varphi(2^j x - k)\ |_{[0,2^N]}$ $(-2M+2 \le k \le 2^{j+N}-1)$ supplemented by the restrictions of the wavelets $2^{j/2}\psi(2^j x - k)\ |_{[0,2^N]}$ $(-2M+2 \le k \le 2^{j+N}-1)$ (where the wavelet ψ satisfies $\operatorname{Supp}\psi = [0, 2M-1]$); but we have too many supplementary functions, as we already have noticed it when discussing the extension-by-0 solution. But as a matter of fact it is very easy to determine, when $j \ge j_0$, the $2M-2$ functions to be eliminated : for $-2M+2 \le k \le -M$ and $2^{j+N} - M + 1 \le k \le 2^{j+1} - 1$ we have indeed $\psi_{j,k}\ |_{[0,2^N]} \in V_j^{[0,2^N]}$. (Just write $\varphi_{j+1,2k}\ |_{[0,2^N]} = 0$ for $-2M+2 \le k \le -M$ and expand $\varphi_{j+1,2k}$ on $\varphi_{j,\ell}$ and $\psi_{j,\ell} \cdots$). Now, a Gram-Schmidt orthonormalization of the family

$$\varphi_{j,-2M+2}\ |_{[0,2^N]}, \cdots, \varphi_{j,2^{j+N}-1}\ |_{[0,2^N]}, \psi_{j,-M+1}\ |_{[0,2^N]}, \cdots, \psi_{j,2^{j+N}-M}\ |_{[0,2^N]}$$

gives an orthonormal basis $\tilde{\psi}_{j,k}$ of $W_j^{[0,2^N]}$, with the properties :

- $\tilde{\psi}_{j,k} = 2^{\frac{j-j_0}{2}}\tilde{\psi}_{j_0,k}(2^{j-j_0}x)$ for $-M+1 \le k \le -1$
- $\tilde{\psi}_{j,k} = 2^j\psi(2^j x - k)$ for $0 \le k \le 2^{j+N} - 2M + 1$
- $\tilde{\psi}_{j,k} = 2^{\frac{j-j_0}{2}}\tilde{\psi}_{j_0,k-2^{j+N}}(2^{j-j_0}(x - 2^N) + 2^N)$ for $2^{j+N} - 2M + 2 \le k \le 2^{j+N} - M$.

Thus we have border functions α_k^L, α_k^R, β_k^L, β_k^R such that for $j \ge j_0$:

$$p_j^{[0,2^N]}f = \sum_{k=1}^{2M-2} < f \mid 2^{j/2}\alpha_k^L(2^jx) > 2^{j/2}\alpha_k^L(2^jx) + \sum_{k=0}^{2^{j+N}-2M+1} < f \mid \varphi_{j,k} > \varphi_{j,k} +$$

$$\sum_{k=1}^{2M-2} < f \mid 2^{j/2}\alpha_k^R(2^j(x-2^N) > 2^{j/2}\alpha_k^R(2^j(x-2^N))$$

while

$$(p_{j+1}^{[0,2^N]} - p_j^{[0,2^N]})f = \sum_{k=1}^{M-1} < f \mid 2^{j/2}\beta_k^L(2^jx) > 2^{j/2}\beta_k^L(2^jx) +$$

$$\sum_{k=0}^{2^{j+N}-2M+1} < f \mid \psi_{j,k} > \psi_{j,k} + \sum_{k=1}^{M-1} < f \mid 2^{j/2}\beta_k^R(2^j(x-2^N)) > 2^{j/2}\beta_k^R(2^j(x-2^N))$$

with moreover

$$\mathrm{Supp}\,\alpha_k^L, \mathrm{Supp}\,\beta_k^L \subset [0, 2M-2]$$

and

$$\mathrm{Supp}\,\alpha_k^R, \mathrm{Supp}\,\beta_k^R \subset [-2M+2, 0].$$

Although Meyer's construction provides a nice theoretical solution for wavelets on $[0, 2^N]$, it had to be modified because the Gram-Schmidt orthonormalization was not well conditionned.

The simplest modification is to be found in [CODV]. The construction of Cohen, Daubechies and Vial is very easy ; since we want to have the polynomials in our big scale basic space, we have to include them in our basis ; but since we want to separate border 0 and border 2^N, we have to split each polynomial on two half-polynomials (a left-hand part and a right-hand part). Now remember that for $0 \le k \le M-1$, $\sum_{\ell\in\mathbb{Z}} \ell^k\varphi(x-\ell) = x^k + Q_{k-1}(x)$ where Q_{k-1} is a polynomial of degree less than k. Thus we may consider the subspace $\tilde{V}_j$ of $V_j^{[0,2^N]}$ spanned by :

- for $k = 0$ to $M-1$, $\sum_{\ell<\ell_0} \ell^k\varphi_{j,\ell}\,|_{[0,2^N]}$
- for $\ell_0 \le \ell \le 2^{j+N} - \ell_1$, $\varphi_{j,\ell}$
- for $k = 0$ to $M-1$, $\sum_{\ell>2^{j+N}-\ell_1} \ell^k\varphi_{j,\ell}\,|_{[0,2^N]}$

where $\ell_0 \ge 0$, $\ell_1 \ge 2M-1$. We have thus a basis of $\tilde{V}_j$ (which has dimension $d = 2^{j+N} - \ell_1 - \ell_0 + 1 + 2M$). Moreover it is easy to see that $\tilde{V}_j \subset \tilde{V}_{j+1}$, so that we may look at the sequence of the orthogonal projection operators $\tilde{V}_j$ from $L^2([0,2^N])$ onto $\tilde{V}_j$ as a multi-resolution approximation process. Cohen, Daubechies and Vial choose $\ell_0 = 1$, $\ell_1 = 2M$ in order to have $\dim\tilde{V}_j = 2^{j+N}$. Such a choice is motivated by the fact that in image processing the size of the images is usually a power of two and also by the fact that in that case $\dim V_{j+1} = 2\dim\tilde{V}_j$, which corresponds to the empirical meaning of the scale (and allows to develop wavelet packets on the interval by iterative division of even-dimensional spaces into two subspaces of equal dimension [see chapter 9]).

Other modifications have been proposed, more or less equivalent to the one we exposed here. In particular A. Jouini and P.-G. Lemarié-Rieusset [JOU] developped the notion of bi-orthogonal multi-resolution analysis on the interval and could extend to the interval the commutation property between differentiation and projection operators (with applications to the duality $H_0^k[0,2^N], H^{-k}[0,2^N]$).

4. Quadrature formulas.

In this section, we pay a few words to the topic of effective calculus with wavelets and scaling functions. We consider bi-orthogonal compactly supported scaling functions φ, φ^* with scaling filters m_0, m_0^*. Standard references for this section are [LEM3], [BEY1], [BEY2], [SWE], [TEN] among many others.

i) Moments : The calculus of $\int x^k \varphi\, dx$ is very easy to do. Indeed, it is equivalent to compute $\left(\frac{d}{d\xi}\right)^k \hat{\varphi}\mid_{\xi=0}$. But we have $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$, hence :

$$(11) \qquad (2^k-1)\frac{d^k}{d\xi^k}\hat{\varphi}\mid_{\xi=0} = \sum_{\ell=0}^{k-1} \binom{k}{\ell} \hat{\varphi}^{(\ell)}(0) m_0^{(k-\ell)}(0)$$

and, from $\hat{\varphi}(0) = 1$, we can compute $(\frac{d}{d\xi})^k\hat{\varphi}$ at any order by iteration of formula (11), since the derivatives of m_0 are obvious to compute.

ii) Tabulation : If $\hat{\varphi} \in L^1$ (so that φ is continuous), it is very easy to tabulate φ. Indeed, we have a two-scale equation

$$(12) \qquad \varphi(\frac{x}{2}) = \sum_{k=A}^{B} a_k \varphi(x-k) \quad \text{where} \quad \operatorname{supp}\varphi = [A,B]$$

so that if we know φ on $\mathbb{Z}$, we may compute iteratively φ on $\mathbb{Z}/2^j$. Thus we just have to know $\varphi(k)$, $A+1 \leq k \leq B-1$. But those values are solutions of a linear system :

$$(13) \qquad \begin{cases} \text{For } A+1 \leq k \leq B-1, \quad \varphi(k) = \displaystyle\sum_{\ell=1}^{B} a_\ell \varphi(2k-\ell) \\ \displaystyle\sum_{k=A+1}^{B-1} \varphi(k) = 1. \end{cases}$$

Write $\varphi(k) = \epsilon_k$, with $\epsilon_k = 0$ if $k \leq A$ or $k \geq N$; then the system (13) has an unique solution $(\epsilon_k)_{k\in\mathbb{Z}}$. Indeed if (ϵ_k) is any solution of (13) (with $\epsilon_k = 0$ for $k \notin \{A+1,\cdots,B-1\}$), then we write $P_0(\xi) = \sum_{k=A+1}^{B-1} \epsilon_k e^{-ik\xi}$ and $\hat{\theta}_0 = P_0(\xi)\chi_K(\xi)$ where K is a Cohen compact set associated to m_0. Then we define θ_{m+1} as $\widehat{\theta_{m+1}}(\xi) = m_0(\frac{\xi}{2})\hat{\theta}\ (\frac{\xi}{2})$. We have $\widehat{\theta_{m+1}}(\xi) = \prod_{j=1}^{n+1} m_0(\frac{\xi}{2^j})\chi_K(\frac{\xi}{2^{n+1}})P_0(\frac{\xi}{2^{n+1}})$, hence $\widehat{\theta_{m+1}} \to P_0(0)\hat{\varphi}$ in L^1 (by dominated convergence theorem), and since $P_0(0) = 1$, $\theta_{m+1} \to \varphi$ in C^0. But we have $\theta_0(k) = \epsilon_k$ for all $k \in \mathbb{Z}$, hence by induction on n :

$$\theta_{n+1}(k) = \sum_{\ell=A}^{B} a_\ell \theta_n(2k-\ell) = \sum_{\ell=A}^{B} a_\ell \epsilon_{2k-\ell}.$$

If $k \in \{A+1, \cdots, B-1\}$, we find $\theta_{n+1}(k) = \epsilon_k$; if $k \leq A$ then $2k - \ell \leq A$ hence $\epsilon_{2k-\ell} = 0$ and $\theta_{n+1}(k) = 0$; if $k \geq B$ then $2k - \ell \geq B$ hence $\epsilon_{2k-\ell} = 0$ and $\theta_{n+1}(k) = 0$.

Thus we may determine φ by solving an eigenvalue problem for a finite matrix. Another way to determine φ is the *cascade algorithm* of I. Daubechies [DAU1]. One approximates φ in C^0 by g_n, where $\widehat{g_{n+1}}(\xi) = m_0(\frac{\xi}{2})\hat{g}_n(\frac{\xi}{2})$ and g_0 is such that $\sum_{k\in\mathbb{Z}} \mid \hat{g}_0(\xi + 2k\pi) \mid \in L^\infty$, $\hat{g}_0(2k\pi) = 0$, $\hat{g}_0(0) = 1$ and $\lim_{\xi\to 0} \sum_{k\in\mathbb{Z}} \mid \hat{g}(\xi + 2k\pi) \mid = 1$; then $\hat{g}_n \to \hat{\varphi}$ in L^1, thus we have a nice approximation of φ. One usually chooses for g_0 a spline function, for sake of simplicity. Nevertheless, the convergence of g_n to φ need not to be rapid. S. Durand [DUR] has shown that in order to have a good convergence one should require on g_0 that $\hat{g}_0$ and $\hat{\varphi}$ should have on $2\pi\mathbb{Z}$ not only the same values but also the same first derivatives.

iii) Derivatives of φ : In order to tabulate $\frac{d\varphi}{dx}$, one may use the fact that $\frac{d\varphi}{dx} = \tilde{\varphi}(x) - \tilde{\varphi}(x-1)$ where $\tilde{\varphi}$ is another scaling function. One may also use the two-scale equation (12) to get the system

$$\text{(14)} \qquad \begin{cases} \text{For } A+1 \leq k \leq B-1, \quad \varphi'(k) = \sum_{\ell=A}^{B} 2a_\ell \varphi(2k-\ell) \\ \sum_{k=A+1}^{B-1} \varphi'(k) = 0 \ \text{ and } \ \sum_{\ell=A+1}^{B-1} k\varphi'(k) = -i\frac{d}{d\xi} m_0(0). \end{cases}$$

iv) Primitives of φ : We may remember that $\int_x^{x+1} \varphi(t)dt$ is a scaling function so that we may compute easily $\int_k^{k+1} \varphi(t)dt$. For $\int_{\frac{k}{2^j}}^{\frac{k+1}{2^j}} \varphi(t)dt$, expand$\varphi$ on the $\varphi_{j,\ell}$ functions.

v) Bilinear integrals : If we want to compute $\int \varphi_1(x-k)\varphi_2(x-\ell)dx$ for two scaling functions, we just have to notice that $\Phi(x) = \int \varphi_1(y)\varphi_2(y-x)dy$ satisfies itself a two-scale equations (with filter $M(\xi) = m_1(\xi)\bar{m}_2(\xi)$). It always satisfies $\hat{\Phi} \in L^1$ (but may fail to generate a Riesz basis, because M may violate Cohen's criterion).

For trilinear integrals, one should work with $2D$-scaling functions

$$\Phi(x,y) = \int \varphi_1(y)\varphi_2(z-x)\varphi_3(z-y)dz \ ;$$

the computations are similar but more heavy.

vi) Quadrature formulas : Now, we want to compute $s_{j,k} = \int f(x)2^{j/2}\varphi^*(2^j x - k)dx$ given the sampling $(f(\frac{k}{2^j}))_{k\in\mathbb{Z}}$. (From the values $s_{j,k}$ estimated for the larger j compatible with the sampling, one deduces the values $s_{\ell,k}$, $\ell < j$, through Mallat's algorithm). The first solution is to use a naive formula

$$s_{j,k} \approx 2^{-j/2} \sum_{\ell\in\mathbb{Z}} f(\frac{\ell}{2^j})\varphi^*(\ell - k). \tag{15}$$

If φ^* reconstructs polynomials up to degree M, we have $(\frac{d}{d\xi})^k\hat{\varphi}^*(2\ell\pi) = 0$ for $\ell \neq 0$ and $0 \leq k \leq M$, hence $\sum_{\ell\in\mathbb{Z}} \ell^k\varphi^*(\ell) = \int x^k\varphi^* dx$. Hence formula (15) is valid on polynomials up to degree M.

We may need further accordance than to degree M, to have a better accuracy for a regular f, or want to shorten the computations (since the length of $\operatorname{Supp}\varphi^*$ may be much longer than $M+1$). We search then for a formula

$$s_{j,k} \approx 2^{-j/2} \sum_{\ell\in\mathbb{Z}} f(\frac{\ell}{2^j})\alpha_{k-\ell} \tag{16}$$

which is exact on polynomials up to degree Q and where $(\alpha_k)_{k\in\mathbb{Z}}$ takes for all but $Q+1$ indexes k the value 0 : $\alpha_0, \alpha_1, \cdots, \alpha_Q \neq 0$ and $\alpha_k = 0$ elsewhere. Then α_k is given by :

$$\text{for } k \in \{0,\cdots,Q\}, \quad \alpha_k = \int \prod_{\ell\neq k,\ \ell=0}^{Q} \left(\frac{x-\ell}{k-\ell}\right)\varphi^*(x)dx \tag{17}$$

and thus may be easily computed since we know how to compute the moments of φ^*.

More elaborate formulas, including Chebychev collocation, have been developped by W. Sweldens [SWE]. Remember also that the coiflets φ (example n° 8 in chapter 5) were constructed with size of support $3M-1$ so that for all $k \in \{0,\cdots,M-1\}$, $\int x^k\varphi(x-\ell)dx = \ell^k$; hence for a coiflet, we may use the formula : $s_{j,k} \approx f(\frac{k}{2^j})$.

5. The BCR algorithm.

Among the applications of wavelets to numerical analysis, we choose to expose the *BCR algorithm* introduced in 1989 by G. Beylkin, R. Coifman and V. Rokhlin [BEY1]. The idea is to represent a linear operator on $L^2(\mathbb{R})$ by a matrix in a wavelet basis. If we discretize $T \in \mathcal{L}(L^2, L^2)$ by $T_N = P_N T P_N$, where P_N is the orthogonal projection operator on the space V_N of a multi-resolution analysis generated by a compactly supported orthogonal scaling function, we are going to see the way to use the wavelet decomposition $V_N = V_{N-J} \oplus \bigoplus_{k=1}^{J} W_{N-k}$ to get a better representation of T.

The *standard form* of T_N is obtained by the change of basis from the scaling functions $(\varphi_{N,k})_{k\in\mathbb{Z}}$ to the wavelet basis $(\varphi_{N-J,k})_{k\in\mathbb{Z}} \cup (\psi_{N-\ell,k})_{\ell\in\{1,\cdots,J\},k\in\mathbb{Z}}$. We have

$$T_N = \sum_{\ell=1}^{J} A_\ell + \sum_{\ell'=1}^{J} B_{\ell'} + \sum_{\ell=1}^{J}\sum_{\ell'=1}^{J} C_{\ell,\ell'}$$

with $A_\ell = P_{N-J}TQ_{N-\ell}$, $B_{\ell'} = Q_{N-\ell'}TP_{N-J}$, $C_{\ell,\ell'} = Q_{N-\ell'}TQ_{N-\ell}$.

The *non-standard form* of T_N is obtained by a change of basis, not in V_N but in the space $V_N \hat{\otimes} V_N$ of matrices onto V_N : we replace the $2D$ scaling function basis $(\varphi_{N,k} \otimes \varphi_{N,k'})$ by the $2D$ wavelet basis

$$(\varphi_{N-J,k} \otimes \varphi_{N-J,K'})_{(k,k')\in\mathbb{Z}^2} \cup (\psi_{N-\ell,k} \otimes \varphi_{N-\ell,k'})_{1\leq\ell\leq J,(k,k')\in\mathbb{Z}^2} \cup$$

$$(\varphi_{N-\ell,k} \otimes \psi_{N-\ell,k'})_{1\leq\ell\leq J,(k,k')\in\mathbb{Z}^2} \cup (\psi_{N-\ell,k} \otimes \psi_{N-\ell,k'})_{1\leq\ell\leq J,(k,k')\in\mathbb{Z}^2}.$$

I.e. we have written :

$$\begin{aligned} T_N &= T_{N-J} \oplus \sum_{\ell=1}^{J} T_{N-\ell+1} - T_{N-\ell} \\ &= T_{N-J} \oplus \sum_{\ell=1}^{J} Q_{N-\ell}TP_{N-\ell} + P_{N-\ell}TQ_{N-\ell} + Q_{N-\ell}TQ_{N-\ell}. \end{aligned}$$

In the non-standard form, we don't let wavelets at different scales interact but replace this interaction by the interaction of wavelets and scaling functions at the same scale.

The non-standard representation is especially interesting for pseudo-differential operators (as e.g. parametrics of elliptic differential operators) or Calderón-Zygmund operators, because for such operators the distribution kernel $K(x,y)$ has its singularities concentrated on the diagonal $x = y$; hence in the $2D$ wavelet basis, its coefficients are concentrated on wavelets located near the diagonal. It means that when one neglects the small coefficients, we obtain a sparse matrix whose number of non-zero entries grows *linearly* with the number of data instead of *growing as the square* of this number. Another interesting feature is the fact that these operators behave almost homogeneously with respect to dilations, so that the obtained matrix can be well handled with a simple preconditioner.

Thus, the non-standard form is efficient for the compression of pseudo-differential operators. The standard form is especially interesting (for pseudo-differential operators) for iterative algorithms, because it allows a fast multiplication of matrices. (Be careful that the non-standard matrix doesn't obey any more to the usual matrix algebra. In order to compute $T(v_N)$ we have to write

$$v_N = v_{N-1} + w_{N-1} = v_{N-2} + w_{N-2} + w_{N-1} = \cdots = v_{N-J} + w_{N-J} + \cdots + w_{N-1}$$

and to write v_N in a non-standard way as :

$$\hat{v}_N = \begin{pmatrix} v_{N-J} \\ w_{N-J} \\ v_{n-J-1} \\ w_{n-J-1} \\ \vdots \\ v_{N-1} \\ w_{N-1} \end{pmatrix}$$

then multiply $\hat{v}_N$ by the non-standard matrix

$$M = \begin{pmatrix} A_{N-J} & B_{N-J} & 0 & 0 & \cdots & \cdots & \cdots \\ C_{N-J} & D_{N-J} & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & B_{N-J-1} & 0 & \cdots & \cdots \\ O & O & C_{N-J-1} & D_{N-J-1} & 0 & \cdots & \cdots \\ \vdots & \vdots & 0 & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & 0 & B_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & C_{N-1} & D_{N-1} \end{pmatrix}$$

(where $A_k = P_kTP_k$, $B_k = P_kTQ_k$, $C_k = Q_kTP_k$, $D_k = Q_kTQ_k$) which gives

$$\hat{\alpha}_N = \begin{pmatrix} \alpha_{N-J} \\ \beta_{N-J} \\ \alpha_{N-J-1} \\ \beta_{N-J-1} \\ \vdots \\ \alpha_{N-1} \\ \beta_{N-1} \end{pmatrix} \qquad \text{where } \alpha_k \in V_k \quad \text{and} \quad \beta_k \in w_k$$

and finally $T(v_N) = \sum_{\ell=1}^{J} \alpha_{N-\ell}+\beta_{N-\ell} = (1,\cdots,1)\hat{\alpha}_N$. Of course we *don't have* $T^2(v_N) = (1,\cdots,1)M^2\hat{v}_N)$.

6. The wavelet shrinkage.

From the very beginning, wavelet transforms have been viewed as a tool for compression or de-noising in signal processing. The idea is to set to 0 the wavelet coefficients which are under a given threshold ϵ. In compression, they are looked at as unsignificant : those wavelet details are neglectible with regard to the coarse approximation P_Jf (where one approximates f by $P_{N+J}f$, then makes a wavelet transform $P_{N+J}f = P_Jf + \sum_{\ell=0}^{N-1} P_{J+\ell}f$). In de-noising, they are looked at as unreliable because lost in the noise.

Many authors have investigated de-noising through wavelet coefficients thresholding. Among them, we quote the wavelet shrinkage algorithm of Donoho and Johnstone [DON2].

These authors are interested in recovering a function $f(t)$ on $[0,1]$ from the noisy data $y_k = f\left(\frac{k}{2^{N+J}}\right) + \sigma z_k$ where $0 \leq k \leq 2^{N+J}-1$ and the z_k are a white noise. They use the wavelet transform on the interval of Cohen, Daubechies and Vial to get wavelet coefficients $(w_{j,k})_{J\leq j\leq N+J-1, 0\leq k\leq 2^j-1}$ and coarse scale coefficients $(s_{J,k})_{0\leq k\leq 2^J-1}$. Then they make a soft thresholding : $\widehat{w_{j,k}} = \mathrm{sgn}(w_{j,k})(\mid w_{j,k} \mid -t)_+$ with treshold $t = \sqrt{2(N+J)}\sigma$, and compute from $(\hat{w}_{j,k})$ and $(\widehat{s_{j,k}})$ the estimate $\hat{f}(t)$, $t \in [0,1]$.

D. Donoho proved that this estimate is very near to be optimal among all measurable procedures, provided we have an a priori regularity estimate on f. This near-minimaxity result relies heavily on the fact that wavelets provide unconditional bases for many smoothness spaces (as Besov spaces) so that wavelet shrinkage is a smoothing operator with respect to those smoothness classes.

Chapter 9

FURTHER EXTENSIONS OF WAVELET THEORY

In this chapter, we will say a few words on some generalizations of wavelet bases. An immediate generalization is the notion of *multi-resolution analysis with multiplicity* r : the space V_0 is no more generated by the translates of a single scaling function φ, but by the translates of r functions $\varphi_1, \cdots, \varphi_r$. Another generalization is the notion of *wavelet packets,* which associates to an orthonormal scaling filter m_0 a whole library of orthonormal bases of $L^2(\mathbb{R})$ together with a selection algorithm for finding the best basis for a given signal. Such a best basis algorithm has been also developped for the *local sine transform* of Malvar, which allows an adaptative time-frequency analysis, where one constructs an adaptative segmentation of the time axis (whereas in the setting of wavelet packets one constructs an adaptative segmentation of the frequency axis). Finally, those best basis algorithms have been recently criticized by S. Mallat, who introduced a new adaptative time-frequency algorithm, the so-called *matching pursuit.*

1. Multiple scaling functions.

An extension of the notion of multiresolution analysis was proposed in 1992 by Goodmann, Lee and Tang [GOO] :

DEFINITION 1. - *A multiple multiresolution analysis of* $L^2(\mathbb{R})$ *is a family* $(V_j)_{j\in\mathbb{Z}}$ *of closed linear subspaces of* $L^2(\mathbb{R})$ *such that :*

(i) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j \neq \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* L^2
(ii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
(iii) *there exists a finite number of functions* $\varphi_1, \cdots, \varphi_N$ *such that the family* $(\varphi_\epsilon(x - \ell))_{1\leq\epsilon\leq N, k\in\mathbb{Z}}$ *is a Riesz basis of* V_0.

A natural example of such a generalized multiresolution analysis is provided by the spline functions with multiple knots.

Multiple multiresolution analyses share many properties with the "classical" multiresolution analyses. (See for instance Chapter 4 where we deal with such analyses). The scaling filter m_0 is replaced by a $N \times N$ matrix $M_0(\xi)$; its properties have been studied by L. Hervé in [HER].

However, besides spline functions, there is not so much examples of multiple multiresolution analyses in the litterature. A striking example has been given by Hardin, Kessler and Massopust [HAR] : they obtain continuous orthonormal scaling functions with a very short support ($N = 2$, $\operatorname{Supp}\varphi_1 = [0,1]$, $\operatorname{Supp}\varphi_2 \subset [0,2]$).

2. Wavelet packets.

Wavelet packets have been introduced in 1990 by R. Coifman, Y. Meyer, S. Quake and V. Wickerhauser [COIM4]. They have been developped as a tool for coding and compression ; their use in analysis encounters some obstacles because of the lack of control for the frequency localization of the wavelet packets [COIM2].

The idea of wavelet packet comes from the following remark : whereas scaling filters in wavelet theory can be viewed as quadrature mirror filters, the algorithms for which the two families of filters were introduced are very different in the splitting of the phase space they are related to. In case of Mallat's algorithm, V_0 is splitted into $V_{-1} \oplus W_{-1}$: V_0 is roughly speaking a space of "band-limited" signals with bandwidth 2π, and each V_{-1} and W_{-1} have bandwidth π ; then one splits V_{-1} into $V_{-2} \oplus W_{-2}$ while W_{-1} remains unchanged, and so on. In case of the subband coding scheme, one splits the space of band-limited signals with bandwidth 2π into two spaces of bandwidth π, and then one splits again *each* space into two subspaces, and so on. Wavelet packets were introduced as a description of the intermediate algorithms between those two cases.

More precisely, let φ be a compactly supported orthonormal scaling function, with associated scaling filter $m_0(\xi)$. Let m_1 be the associated filter : $m_1(\xi) = e^{-i\xi}\bar{m}_0(\xi+\pi)$, so that one may define the associated orthonormal wavelet ψ by $\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)$.

DEFINITION 2. - *The basic wavelet packets $(w_n(x))_{n\in\mathbb{N}}$ associated to the orthonormal scaling function φ are defined by :*

$$(1) \qquad \hat{w}_n(\xi) = \prod_{j=1}^{N} m_{\epsilon_j}\left(\frac{\xi}{2^j}\right)\cdot\hat{\varphi}\left(\frac{\xi}{2^N}\right), \quad n = \sum_{j=1}^{N}\epsilon_j 2^{j-1}, \epsilon_j \in \{0,1\}.$$

THEOREM 1. - *Let V_0 be the closed linear span of $\varphi(x-k)$, $k \in \mathbb{Z}$. Then :*

i) For all $q \in \mathbb{N}$, the family $(2^{-q/2}w_n(2^{-q}x-k))_{0\leq n<2^q, k\in\mathbb{Z}}$ is an orthonormal basis of V_0.
ii) More generally if $[0,1)$ is decomposed as a disjoint union of finitely many dyadic intervals $[0,1) = \bigcup_{(n,q)\in Q}[\frac{n}{2^q}, \frac{n+1}{2^q})$, the family $(2^{-q/2}w_n(2^{-q}x-k))_{(n,q)\in Q, k\in\mathbb{Z}}$ is an orthonormal basis of V_0.

This theorem can even be generalized to infinite decompositions : in [COIM2], it is shown that if $m_0(\xi)$ is the scaling filter of the Lemarié-Meyer wavelet then for each infinite disjoint decomposition $[0,1) = \bigcup_{(n,q)}[\frac{n}{2^q}, \frac{n+1}{2^q}) \cup D$ (where D is denumerable), then the family $(2^{-q/2}w_n(2^{-q}x-k))_{(n,q)\in Q, k\in\mathbb{Z}}$ is still an orthonormal basis of V_0.

Proof of theorem 1. This theorem is obvious. m_0 and m_1 are two conjugate quadrature mirror filters, this if $\omega \in L^2$ is such that $(\omega(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis of a space Ω, then ω_0 and ω_1 defined by $\hat{\omega}_0(2\xi) = m_0(\xi)\hat{\omega}(\xi)$ and $\hat{\omega}_1(\xi) = m_1(\xi)\hat{\omega}(\xi)$ generate orthonormal bases $\left(\frac{1}{\sqrt{2}}\omega_0(\frac{x}{2}-k)\right)_{k\in\mathbb{Z}}$ of a subspace Ω_0 and $\left(\frac{1}{\sqrt{2}}\omega_1(\frac{x}{2}-k)\right)_{k\in\mathbb{Z}}$ of a subspace Ω_1 such that $\Omega = \Omega_0 \overset{\perp}{\oplus} \Omega_1$.

Thus we may split V_0 into $V_{0,0}$ and $V_{0,1}$, then split $V_{0,0}$ into $V_{0,0,0}$ and $V_{0,0,1}$ and $V_{0,1}$ into $V_{0,1,0}$ and $V_{0,1,1}$, and so on up to step q. Then we have

$$V_0 = \overset{\perp}{\bigoplus_{(\epsilon_1,\cdots,\epsilon_q)\in\{0,1\}^q}} V_{0,\epsilon_1,\cdots,\epsilon_q}$$

and a basis of $V_{0,\epsilon_1,\cdots,\epsilon_q}$ is given by $(2^{-q/2}w_{\sum_1^q \epsilon_j 2^{j-1}}(2^{-q}x-k))_{k\in\mathbb{Z}}$. Thus point i) is proved.

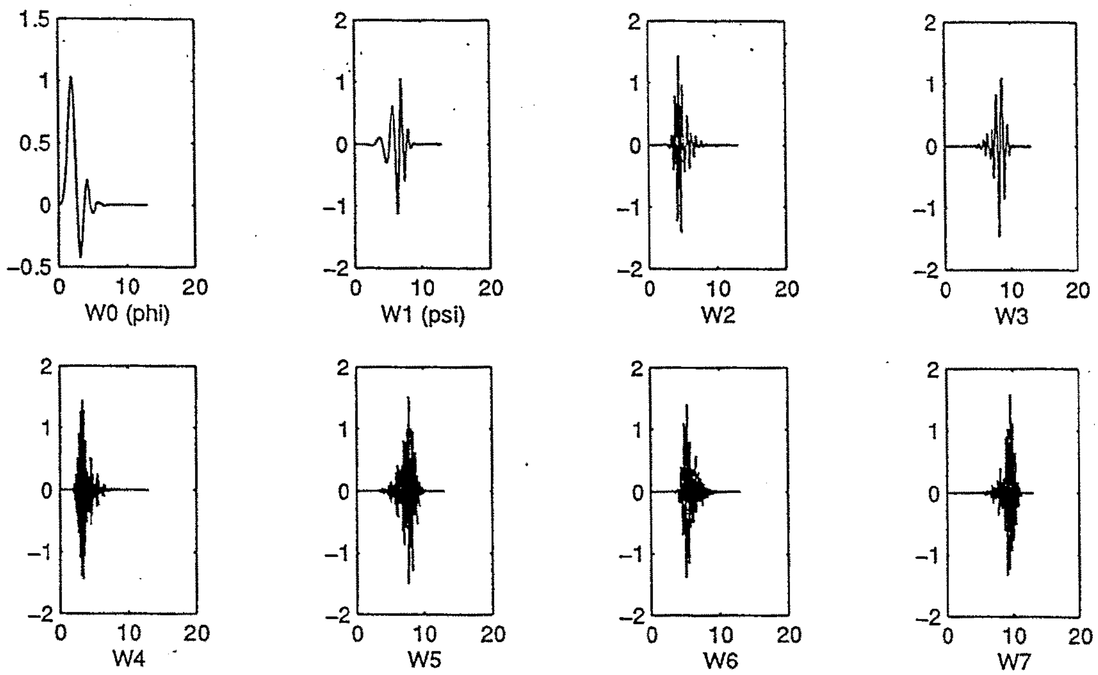
Point ii) is very easy. We may choose for each $V_{0,\epsilon_1,\cdots,\epsilon_q}$ to split it into two parts or to keep it as a single space. Each finite partition of $[0,1)$ corresponds to a finite sequence of choices.

For instance, Mallat's algorithm corresponds to the splitting

$$V_0 = V_{0,0} \oplus V_{0,1} = V_{0,0,0} \oplus V_{0,0,1} \oplus V_{0,1} = \cdots$$

i.e. $V_0 = V_{-N} \oplus W_{-N} \oplus W_{-N+1} \oplus \cdots \oplus W_{-1}$ with $V_{-N} = V_{0,\epsilon_1=0,\cdots,\epsilon_N=0}$ and $W_{-K} = V_{0,\epsilon_1=0,\epsilon_2=0,\cdots,\epsilon_{K-1}=0,\epsilon_K=1}$; thus $[0,1)$ is split into $[0,\frac{1}{2^N}) \cup \bigcup_1^N [\frac{1}{2^k},\frac{2}{2^k})$. In constrast, the subband coding scheme corresponds to the splitting $V_0 = \bigoplus_{(\epsilon_1,\cdots,\epsilon_N)\in\{0,1\}^N} V_{0,\epsilon_1,\cdots,\epsilon_N}$ and $[0,1) = \bigcup_{k=0}^{2^N-1}[\frac{k}{2^N},\frac{k+1}{2^N})$.

Figure 5: The first basic wavelet packets associated to the Daubechies filter of length 14 (N=7)



What is the purpose of introducing such bases ? The idea is that wavelets are well fitted to the analysis of localized singularities (as a Dirac mass for instance) but have a rather poor frequency resolution ; in contrast, the subband coding scheme may focus very well on a given frequency but at the cost of a loose space resolution. Now, having a whole family of bases should allow one to choose for a given signal a "best basis".

What is a "best basis" ? We consider a signal $f \in V_0$ with finite length (i.e. Supp f is compact, or f is periodical, or f is restricted to an interval,$\cdots$) and we would like to compare the expansion of f onto two bases (ϵ_k) and (η_k). The tool which is frequently used is the *entropy* of the expansion :

$$E(f,(\epsilon_k)) = \sum |< f \mid \epsilon_k >|^2 \log_2 \left(\frac{\| f \|_2^2}{|< f \mid \epsilon_k >|^2} \right).$$

If f is proportional to ϵ_{k_0} then $E(f,(\epsilon_k)) = 0$; otherwise

$$E(f,(\epsilon_k)) \geq \| f \|_2^2 \log_2 \left(\frac{\| f \|_2^2}{\operatorname*{Max}_k |< f \mid \epsilon_k >|^2} \right) > 0.$$

Thus the entropy measures the concentration of the expansion of f onto (ϵ_k). The best basis (inside a family of orthonormal bases) will then be the basis which minimizes the entropy.

The best basis algorithm for wavelet packets is described in the book by V. Wickerhauser [WIC], and in many papers (see the talk of R. Coifman at Kyoto for instance [COI]). The algorithm is based on *Shannon's formula for entropy* : if a basis (ϵ_k) is decomposed in $(\epsilon_k)_{k\in K_1}$ and $(\epsilon_k)_{k\in K_2}$ and the function f is decomposed in $f_1 = \sum_{K_1} < f_1 \mid \epsilon_k > \epsilon_k$ and $f_2 = \sum_{K_2} < f_2 \mid \epsilon_k > \epsilon_k$, we have obviously :

$$E(f,(\epsilon_k)) = \| f_1 \|_2^2 \log_2 \frac{\| f \|_2^2}{\| f_1 \|_2^2} + \| f_2 \|_2^2 \log_2 \frac{\| f \|_2^2}{\| f_1 \|_2^2} + E(f_1,(\epsilon_k)_{K_1}) + E(f_2,(\epsilon_k)_{K_2}).$$

Now if we replace $(\epsilon_k)_{K_1}$ by another basis $(\eta_k)_{K_1}$ for Span$(\epsilon_k)_{K_1}$ and $(\epsilon_k)_{K_2}$ by $(\eta_k)_{K_2}$, we find that the minimum of

$$E(f,(\epsilon_K)_{K_1},(\epsilon_k)_{K_2}), E(f,(\epsilon_k)_{K_1},(\eta_k)_{K_2}), E(f,(\eta_k)_{K_1},(\epsilon_k)_{K_2}), E(f,(\eta_k)_{K_1},(\eta_k)_{K_2})$$

is determined by the minimum of $E(f_1,(\epsilon_k)_{K_1})$ and $E(f_1,(\eta_k)_{K_1})$ and by the minimum of $E(f_2,(\epsilon_k)_{K_2})$ and $E(f_2,(\eta_k)_{K_2})$. Thus the strategy is clear : we have, at each level K, spaces $V_{0,\epsilon_1,\cdots,\epsilon_K}$ and projections of f on $V_0,\epsilon_1,\cdots,\epsilon_K$ (which we note $f_{\epsilon_1,\cdots,\epsilon_N}$ $\sum_k 2^{-K/2} < f \mid w_n(2^{-K}x-k) > w_n(2^{-K}x-k)$, $n = \sum_1^K \epsilon_j 2^{j-1}$ and bases $b_{\epsilon_1,\cdots,\epsilon_K} = (2^{-K/2} w_n(2^{-K}x-k))$. We begin at level L, with basis $\beta_{\epsilon_1,\cdots,\epsilon_L} = b_{\epsilon_1,\cdots,\epsilon_L}$ for $V_{0,\epsilon_1,\cdots,\epsilon_L}$, then inductively define bases at level K from bases at level $K+1$ by comparing $E = E(f_{\epsilon_1,\cdots,\epsilon_K}, b_{\epsilon_1,\cdots,\epsilon_K})$ to

$$A = \| f_{\epsilon_1,\cdots,\epsilon_K,0} \|_2^2 \log_2 \frac{\| f_{\epsilon_1,\cdots,\epsilon_K} \|_2^2}{\| f_{\epsilon_1,\cdots,\epsilon_K,0} \|_2^2} + \| f_{\epsilon_1,\cdots,\epsilon_K,1} \|_2^2 \log_2 \frac{\| f_{\epsilon_1,\cdots,\epsilon_K} \|_2^2}{\| f_{\epsilon_1,\cdots,\epsilon_K,1} \|_2^2}$$

$$+E(f_{\epsilon_1,\cdots,\epsilon_K,0},\beta_{\epsilon_1,\cdots,\epsilon_K,0})+E(f_{\epsilon_1,\cdots,\epsilon_K,1},\beta_{\epsilon_1,\cdots,\epsilon_K,1}).$$

If $E\le A$, we keep $\beta_{\epsilon_1,\cdots,\epsilon_K}=b_{\epsilon_1,\cdots,\epsilon_K}$; if $E>A$ then we put $\beta_{\epsilon_1,\cdots,\epsilon_K}=\beta_{\epsilon_1,\cdots,\epsilon_K,0}\cup\beta_{\epsilon_1,\cdots,\epsilon_K,1}$ (and we have $E(f_{\epsilon_1,\cdots,\epsilon_K},\beta_{\epsilon_1,\cdots,\epsilon_K})=A$ in that case).

This gives a very fast algorithm for the best basis selection, (due to the embedding properties of the different spaces we deal with).

3. Local sine bases.

Local sine bases have been introduced in 1987 by H. Malvar [MALV] as a way to define an orthonormal basis adapted to the windowed Fourier transform. (Remember that one cannot find an orthonormal basis of $L^2(\mathbb{R})$ of the form $g(x-k)e^{-2i\pi nt}$ such that $xg\in L^2$ and $g'\in L^2$). The Malvar basis can be defined in the following way : choose a window function w such that $w\in C_c^\infty(\mathbb{R})$, $\operatorname{Supp} w\subset[-\frac12,\frac32]$, $0\le w\le 1$, $w(1-t)=w(t)$ and for all $t\in[-\frac12,\frac12]$, $w^2(t)+w^2(-t)=1$; we call such a function a *Malvar window.*

THEOREM 2. - *Let w be a Malvar window. Then the family*

$$(w(t-2\ell))_{\ell\in\mathbb{Z}}\cup(\sqrt{2}w(t-2\ell)\cos(k\pi t))_{\ell\in\mathbb{Z},k\ge1}\cup(\sqrt{2}w(t-2\ell-1)\sin(k\pi t))_{\ell\in\mathbb{Z},k\ge1}$$

is an Hilbertian basis of $L^2(\mathbb{R})$.

Proof. We define W_ℓ by : $f\in W_{2\ell}$ if and only if $f(x+2\ell)\in W_0$, $f\in W_{2\ell+1}$ if and only if $f(x+2\ell)\in W_1$, and :

$$f\in W_0\Leftrightarrow \operatorname{Supp} f\subset[-\frac12,\frac32],\ f(t)=f(-t)\text{ on }[-\frac12,\frac12],\ f(t)=f(2-t)\text{ on }[\frac12,\frac32]$$

$$f\in W_1\Leftrightarrow \operatorname{Supp} f[\frac32,\frac52],\ f(t)=-f(2-t)\text{ on }[\frac12,\frac32],\ f(t)=-f(4-t)\text{ on }[\frac32,\frac52].$$

Then we have $L^2=\overset{\perp}{\bigoplus_{\ell\in\mathbb{Z}}} W_\ell$. The orthogonality between W_ℓ and $W_{\ell'}$ is obvious if $|\ell-\ell'|>1$; if $\ell=\ell'+1$, we write for $f\in W_\ell$ and $g\in W_{\ell'}$

$$\int f\bar{g}dx=\int_{\ell-1/2}^{\ell+1/2}f(x)\bar{g}(x)dx=\int_{\ell-1/2}^{\ell}(f(x)\bar{g}(x)+f(2\ell-x)\bar{g}(2\ell-x))dx=0$$

because one of the functions is symmetric around ℓ and the other one is antisymmetric. The completeness is easily proved : write

$$f=\sum_{\ell\in\mathbb{Z}}f\chi_{[\ell-\frac12,\ell+\frac12]}$$

$$=\sum_{\ell\in\mathbb{Z}}\frac{f(x)\chi_{[\ell-\frac12,\ell+\frac12]}(x)+f(2\ell-x)\chi_{[\ell-\frac12,\ell+\frac12]}(x)}{2}+$$

$$\frac{f(x)\chi_{[\ell-\frac{1}{2},\ell+\frac{1}{2}]}(x) - f(2\ell - x)\chi_{[\ell-\frac{1}{2},\ell+\frac{1}{2}]}(x)}{2}$$

$$= \sum_{\ell\in\mathbb{Z}} g_\ell(x)$$

with

$$g_{2\ell}(x) = \chi_{[2\ell-\frac{1}{2},2\ell+\frac{1}{2}]}(x)\left(\frac{f(x)+f(4\ell-x)}{2}\right) + \chi_{[2\ell-\frac{1}{2},2\ell+\frac{3}{2}]}(x)\frac{f(x)+f(4\ell+2-x)}{2}$$

and

$$g_{2\ell+1}(x) = \chi_{[2\ell-\frac{1}{2},2\ell+\frac{3}{2}]}(x)\frac{f(x)-f(4\ell+2-x)}{2} + \chi_{[2\ell+\frac{3}{2},2\ell+\frac{5}{2}]}(x)\frac{f(x)-f(4\ell+4-x)}{2}.$$

We have $g_\ell \in W_\ell$, and thus $L^2 = \overset{\perp}{\oplus} W_\ell$.

Now we claim that $w(t-2\ell)) \cup (\sqrt{2}w(t-2\ell)\cos(k\pi t))_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_{2\ell}$. We may assume $\ell = 0$; we thus have to show that : $T : \mu(t) \to \mu(t)w(t)$ is an isometry between the space of even periodical functions in $L^2(\mathbb{R}/\mathbb{Z})$ and W_0. But $T\mu \in W_0$ is obvious : on $[-\frac{1}{2}, \frac{1}{2}]$, $T\mu(t) = \mu(t)w(t) = \mu(-t)w(-t)$ and

$$\begin{aligned} T\mu(1-t) &= \mu(1-t)w(1-t) = \mu(1-t)w(t) = \mu(-1+t)w(-t) \\ &= \mu(1+t)w(1+t). \end{aligned}$$

Moreover, we have :

$$\begin{aligned} \int_{-1/2}^{3/2} | T\mu(t) |^2 &= \int_{-1/2}^{1/2} | w(t) |^2 \, (| \mu(t) |^2 + | \mu(1-t) |^2)dt \\ &= \int_0^{1/2} | \mu(t) |^2 + | \mu(1-t) |^2 \, dt \quad (\text{since } \mu(1-t) = \mu(1+t)) \\ &= \int_0^1 | \mu(t) |^2 \, dt = \frac{1}{2}\int_0^2 | \mu(t) |^2 \, dt. \end{aligned}$$

In order to conclude, we have just to check that T is onto. Define for $\varphi \in W_0$,

$$\mu(\varphi) = \sum_{\ell\in\mathbb{Z}} \varphi(t-2\ell)w(t-2\ell) + \sum_{\ell\in\mathbb{Z}} \varphi(2\ell-t)w(2\ell-t)$$

we have

$$T(\mu(\varphi)) = w(t)\left(\sum_{\ell\in\mathbb{Z}} \varphi(t-2\ell)w(t-2\ell) + \sum_{\ell\in\mathbb{Z}} \varphi(2\ell-t)w(2\ell-t)\right) \; ;$$

if $t \in [-\frac{1}{2}, \frac{1}{2}]$ this gives

$$T(\mu(\varphi)(t)) = w(t)\varphi(t)w(t) + \varphi(-t)w(t)w(-t) = 2\varphi(t) ;$$

if $t \in [-\frac{1}{2}, \frac{3}{2}]$, we have :

$$T(\mu(f)(t)) = w(t)\varphi(t)w(t) + \varphi(2-t)w(2-t)w(2-t) = 2\varphi(t) ;$$

thus $T(\mu(\varphi)) = 2\varphi$.

Similarly, $(\sqrt{2}w(t-2\ell-1)\sin(k\pi t))_{k\in\mathbb{Z}}$ is an orthonormal basis of $W_{2\ell+1}$. Indeed, $S : \mu(t) \to \mu(t)w(t-1)$ is an isometry between the space of odd 2-periodical functions in $L^2(\mathbb{R}/\mathbb{Z})$ and W_1. Thus, theorem 2 is proved. ■

Malvar's basis provides then an elegant solution to the problem of finding an orthonormal basis for the windowed Fourier transform. We can have even a better basis : we can adapt the basis to an arbitrary locally finite decomposition of $\mathcal{R}$ into intervals, as it was shown by R. Coifman and Y. Meyer [COIM3] :

THEOREM 3. - *Let $(x_j)_{j\in\mathbb{Z}}$ be a sequence of real numbers such that $x_j < x_{j+1}$, $\lim_{j\to+\infty} x_j = +\infty$ and $\lim_{j\to-\infty} x_j = -\infty$. Let $(\alpha_j)_{j\in\mathbb{Z}}$ be a sequence of positive numbers such that $\alpha_j + \alpha_{j+1} < x_{j+1} - x_j$. And let $(w_j)_{j\in\mathbb{Z}}$ be a sequence of functions such that :*

(i) $\operatorname{Supp} w_j \subset [x_j - \alpha_j, x_{j+1} + \alpha_{j+1}]$
(ii) *on* $[x_{j+1} - \alpha_{j+1}, x_{j+1} + \alpha_{j+1}]$, $\quad w_j(x) = w_{j+1}(2x_{j+1} - x)$
(iii) $0 \le w_j \le 1$ and $\sum_{j\in\mathbb{Z}} w_j^2(x) = 1$.

Then the family $\varphi_{j,k}(x) = \sqrt{\frac{2}{x_{j+1}-x_j}}w_j(x)\cos\left\{\frac{(2k+1)\pi(x-x_j)}{2(x_{j+1}-x_j)}\right\}$, $j \in \mathbb{Z}$, $k \in \mathbb{N}$, *is an orthonormal basis of* $L^2(\mathbb{R})$.

Proof. Write $f = \sum f w_j^2 = \sum f_j w_j$, where on $[x_j + \alpha_j, x_{j+1} - \beta_j]$, $f_j = f$, on $[x_j - \alpha_j, x_j + \alpha_j]$, $f_j(x) = \frac{f(x)w_j(x) + f(2x_j - x)w_j(2x_j - x)}{2}$ and on $[x_{j+1} - \alpha_{j+1}, x_{j+1} + \alpha_{j+1}]$, $f_j(x) = \frac{f(x)w_j(x) - f(2x_{j+1} - x)w_j(2x_{j+1} - x)}{2}$ (notice that on $[x_{j+1} - \alpha_{j+1}, x_{j+1} + \alpha_{j+1}]$, $w_j(x)w_j(2x_{j+1} - x) = w_{j+1}(x)w_{j+1}(2x_{j+1} - x) = w_j(x)w_{j+1}(x)$). We have

$$f_j w_j = \frac{w_j(x)}{2}\left(\sum_{\ell\in\mathbb{Z}} f(x - \ell L_j)(-1)^\ell w_j(x - \ell L_j)\right.$$

$$\left. + \sum_{\ell\in\mathbb{Z}} (-1)^\ell w(2x_j - x - \ell L_j) f(2x_j - x - \ell L_j)\right)$$

where $L_j = 2(x_{j+1} - x_j)$, and thus each f in L^2 can be decomposed as $\sum_{j\in\mathbb{Z}} g_j w_j$ where g_j is L_j-anti-periodical and satisfies $g_j(x) = g_j(2x_j - x)$. Now, g_j can itself be written as

$$g_j = \sum_{k\in\mathbb{Z}} g_{j,k}\cos\frac{(2k+1)\pi(x - x_j)}{L_j},$$

and thus the family $\varphi_{j,k}$ is complete. The orthonormality is obvious : $\varphi_{j,k}\varphi_{j',k'} = 0$ if $|j - j'| \geq 2$,

$$\varphi_{j,k}(x)\varphi_{j+1,k'}(x) = -\varphi_{j,k}(2x_{j+1} - x)\varphi_{j+1,k'}(2x_{j+1} - x),$$

so that $< \varphi_{j,k} \mid \varphi_{j',k'} >= 0$ if $j \neq j'$; now if $j = j'$, we have

$$\begin{aligned}\int \varphi_{j,k}\varphi_{j,k'}dx = & \int_{x_j}^{x_j+\alpha_j} \frac{2}{x_{j+1} - x_j}\cos\frac{(2k+1)\pi(x - x_j)}{L_j}\cos\frac{(2k'+1)\pi(x - x_j)}{L_j} \\ & \cdot (w_j(x)^2 + w_j(2x_j - x)^2)dx \\ & + \int_{x_j+\alpha_j}^{x_{j+1}-\alpha_{j+1}} \frac{2}{x_{j+1} - x_j}\cos\frac{(2k+1)\pi(x - x_j)}{L_j}\cos\frac{(2k'+1)\pi(x - x_j)}{L_j}dx \\ & + \int_{x_{j+1}-\alpha_{j+1}}^{x_{j+1}} \frac{2}{x_{j+1} - x_j}\cos\frac{(2k+1)\pi(x - x_j)}{L_j}\cos\frac{(2k'+1)\pi(x - x_j)}{L_j} \\ & \cdot (w_j(x)^2 + wj\ \ 2x_{j+1} - x)^2)dx \\ = & \int_0^{L/2} \frac{4}{L}\cos\frac{(2k+1)\pi t}{L}\cos\frac{(2k'+1)\pi t}{L}dt = \delta_{k,k'}.\end{aligned}$$

Thus theorem 3 is proved. ■

The main interest of the above construction is the *merging property :* we may easily delete the point x_j in our subdivision of $\mathbb{R}$ by the following procedure : we replace the windows w_{j-1} (defined on $[x_{j-1}-\alpha_{j-1}, x_j+\alpha_j]$) and w_j (defined on $[x_j - \alpha_j, x_{j+1}+\alpha_{j+1}]$) by the window w defined on $[x_{j-1} - \alpha_{j-1}, x_{j+1} + \alpha_{j+1}]$ by $w(x) = \sqrt{w_{j-1}(x)^2 + w_j(x)^2}$. Then we have the following useful relationship :

$$\text{Span}\left(w_j(x)\cos\tfrac{(2k+1)\pi(x-x_j)}{2(x_{j+1}-x_j)}, k \in \mathbb{N}\right) \overset{\perp}{\oplus} \text{Span}\left(w_{j-1}(x)\cos\tfrac{(2k+1)\pi(x-x_{j-1})}{2(x_j-x_{j-1})}, k \in \mathbb{N}\right)$$

$$= \text{Span}\left(w(x)\cos\frac{(2k+1)\pi(x - x_{j-1})}{2(x_{j+1} - x_{j-1})}, k \in \mathbb{N}\right)$$

which allows, as for the wavelet packets, a best basis algorithm (based on Shannon's formula of entropy) for choosing the more suitable segmentation of $\mathbb{R}$ for a given signal.

The references for this local sine transform and the associated best basis algorithm are the same than for the wavelet packets [WIC], [COI] ; demo software for such algorithms is available by anonymous *ftp* at Yale University (ceres.math.yale.edu).

4. The matching pursuit algorithm.

The matching pursuit algorithm was introduced in 1992 by S. Mallat and Z. Zhang [MALL2] for representing highly non-stationary signals, for which the best basis algorithms described in the previous sections cannot be optimal.

The idea is to expand a given signal in a small number of time-frequency atoms, and in order to make this number of atoms as small as possible to select atoms that match the different structures included in the signal. In the setting of best basis algorithms, one seeks a global optimization of the entropy of the representation of the signal on a basis selected in a given library, but this global optimization may fail to be efficient if the signal contains very different structures : each structure may be better represented in its own basis. Thus, after having determined a first component of the signal through the use of a first "best" basis, one should analyze the remaining signal through a second "best" basis, and so on. Then, one doesn't deal with (global) bases any more, but only with superposition of atoms of various shapes and properties.

The matching pursuit algorithm of S. Mallat is then the following one. Given a separable Hilbert space H and a "dictionary" D, i.e. a family $(e_d)_{d\in D}$ of unitary vectors in H such that the linear span of the e_d's is dense in H, one tries to represent a given $f \in H$ by a superposition of the e_d's by selecting the best e_d that approximates f, and then the best one that approximates the remainder, and so on ; since H is (a priori) infinite-dimensional, this best e_d may not exist, so we fix $\alpha \in (0,1)$ and seek e_{d_0} such that:

$$|< f \mid e_{d_0} >| \geq \alpha \sup_{d\in D} |< f \mid e_d >|$$

and write $f =< f \mid e_{d_0} > e_{d_0} + g_1 = f_1 + g_1$; then we define inductively g_n (and $f_n = f - g_n$) by $g_n =< g_n \mid e_{d_n} > e_{d_n} + g_{n+1}$ where $|< g_n \mid e_{d_n} >| \geq \alpha \sup_{d\in D} |< g_n \mid e_d >|$. Of course we have $\| g_n \|_H^2 = |< g_n \mid e_{d_n} >|^2 + \| g_{n+1} \|_H^2$; and thus (writing $f = g$)

$$\sum_{n=0}^{+\infty} |< g_n \mid e_{d_n} >|^2 \leq \| f \|_H^2 \, .$$

As a matter of fact, adapting a proof of Jones [JON] for the statistical algorithm named the *projection pursuit regression*, Mallat and Zhang proved that $\| g_n \|_H \to 0$ as $n \to +\infty$, so that :

$$f = \sum_{n=0}^{+\infty} < g_n \mid e_{d_n} > e_{d_n} \quad \text{and} \quad \| f \|_H^2 = \sum_{n=0}^{+\infty} |< g_n \mid e_{d_n} >|^2 \, .$$

Moreover, though the summands are not pairwise orthogonal, we have the following control on the partial sums

$$f_n = f - g_n = \sum_{k=0}^{n-1} < g_k \mid e_{d_k} > e_{d_k}$$

$$\| f - f_n \|_H^2 = \| g_n \|_H^2 = \sum_{k=n}^{+\infty} |< g_k \mid e_{d_k} >|^2 \, .$$

The dictionary used by Mallat and Zhang, called "the time-frequency atoms", is the collection of the functions

$$g_{s,\xi,u}(t) = \frac{1}{\sqrt{s}} g\left(\frac{t-u}{s}\right) e^{i\xi t}$$

where g is a fixed single window function in $L^2(\mathbb{R})$ (usually a Gaussian). Thus, we have a family of analyzing functions which is invariant by translations, dilations (as for the wavelet transform) and modulations (as for the windowed Fourier transform).

Of course this family has to be discretized in order to get an implementable algorithm. (Mallat chooses the dyadic sampling for the scales : $s = 2^j$ ($j \in \mathbb{Z}$), the related uniform sampling for the frequency : $\xi = k2^{-j}$ ($k \in \mathbb{Z}$) for $s = 2^j$, and as uniform sampling $u = k\delta_0$ for the position, where the mesh δ_0 corresponds to the initial sampling mesh for the analyzed signal).

As a conclusion, let's have a few words on complexity. For a given signal of size N, we can compute its wavelet transform with an $0(N)$ complexity ; if we compute its representation in the best basis (in the wavelet packets library or the local sines one), we need an algorithm with complexity $0(N \log N)$ to find the best basis. Of course, the matching pursuit algorithm requires much more computations : $\cdots$

It means that we have to pay a high cost when we want to develop a time-frequency analysis of a signal with no a priori informations (i.e. a signal whose structural components are not assumed to belong to a given family of signals). Reversely, in signal analysis, one often analyzes a well determined class of signals (having a physical meaning) and one reduces drastically the computational cost by choosing a devoted family of analyzing functions instead of an universal one. It means that the dream of an universal time-frequency algorithm is probably just utopistic, and that the collection of wavelet bases (or algorithms) is much wider than the one we reviewed in those few pages.

Chapter 10

SOME EXAMPLES OF APPLICATIONS OF WAVELETS TO ANALYSIS

In this concluding chapter, we give some examples of how wavelets can be used in analysis. Of course, this is not an exhaustive treatment of the applications which have already been developped.

We focus on two points. The first one is the analysis of the so-called Calderón-Zygmund operators. We prove the $T(1)$ *theorem of G. David and J. L. Journé* and expose some properties of the paraproduct operators (including the *div-curl theorem)*.

The second point (which will be only sketched) is the *local analysis of the Riemann function* $\sum_1^\infty \frac{1}{n^2} e^{i\pi n^2 x}$, which has been an important test function for wavelet tools in microlocal analysis.

1. Wavelets and para-products.

Para-products have been introduced by J. M. Bony [BON] as a tool for the study of propagation of singularities in non-linear PDE's. They played a key role in the study of Calderón-Zygmund operators [DAV]. In the setting of wavelet theory, the para-product $\pi(b, f)$ for $b \in \mathrm{BMO}$ and $f \in L^2$was defined by Y. Meyer as :

$$(1) \qquad \pi(b,f) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{\alpha=1}^{2^d-1} 2^{2jd} < b \mid \psi_\alpha(2^j x - k) >< f \mid \varphi(2^j x - k) > \psi_\alpha(2^j x - k)$$

for a basis of orthonormal wavelets $(2^{jd/2}\psi_\alpha(2^j x - k))_{j\in\mathbb{Z}, k\in\mathbb{Z}^d, 1\le\alpha\le 2^d-1}$ of $L^2(\mathbb{R}^d)$ with an associated orthonormal scaling function φ such that ψ_α and φ have rapid decay in $L^2(\mathbb{R}^d)$ and belong to the space C^ϵ for some $\epsilon > 0$.

We are going to use a generalization of such operators :

PROPOSITION 1. - *Let $\mathcal{B}$ be a set of functions in $L^2(\mathbb{R}^d)$ such that :*

(i) *for all $f \in \mathcal{B}$, $\mathrm{Supp}\, f \subset K$ (where K is a fixed compact set) ;*

(ii) *there is an $\epsilon > 0$ such that for all $f \in \mathcal{B}$, $f \in H^\epsilon$ and $\| f \|_{H^\epsilon} \le 1$ and let $\mathcal{B}_0 = \{f \in \mathcal{B} / \int f\, dx = 0\}$. Then if $(\alpha_{j,k})$, $(\beta_{j,k})$, $(\gamma_{j,k})$ ($j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$) are families of functions in $\mathcal{B}$ (in $\mathcal{B}_0$ for $\beta_{j,k}$, $\gamma_{j,k}$) then for all $f \in L^2$ and $b \in \mathrm{BMO}(\mathcal{R}^d)$ we have*

$$(2) \qquad \left\| \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} 2^{2jd} < b \mid \beta_{j,k}(2^j x - k) >< f \mid \alpha_{j,k}(2^j x - k) > \gamma_{j,k}(2^j x - k) \right\|_2 \le$$

$$C_{\mathcal{B}} \| b \|_{\mathrm{BMO}} \| f \|_2$$

where $C_{\mathcal{B}}$ depends only on $\mathcal{B}$ (i.e. on K and ϵ).

The proof lies on the following lemma :

LEMMA 1. - Let $Q_{j,k} = \{x/2^j x - k \in [0,1]^d\}$. Then we have for all open set Ω :

$$\sum_{Q_{j,k}} \sum_{\subset \Omega} 2^{jd} \mid < b \mid \beta_{j,k}(2^j x - k) > \mid^2 \leq C_{\mathcal{B}} \parallel b \parallel^2_{\mathrm{BMO}} \mid \Omega \mid \tag{3}$$

where $C_{\mathcal{B}}$ depends only on $\mathcal{B}$.

Proof of lemma 1. If $\tilde{\Omega}$ in an open set such that $Q_{j,k} \subset \tilde{\Omega} \Rightarrow \operatorname{Supp} \beta_{j,k}(2^j x - k) \subset \tilde{\Omega}$, we have, writing $\tilde{b} = (b - \frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} b\, dx)\chi_{\tilde{\Omega}}$:

$$I = \sum_{\Omega_{j,k} \subset \Omega} \sum 2^{jd} \mid < b \mid \beta_{j,k}(2^j x - k) > \mid^2 = \sum_{Q_{j,k} \subset \Omega} \sum 2^{jd} \mid < \tilde{b} \mid \beta_{j,k}(2^j x - k) > \mid^2 .$$

Now, we apply the vaguelettes lemma (theorem 3 of chapter 2) to see that

$$I \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} 2^{jd} \mid < \tilde{b} \mid \beta_{j,k}(2^j x - k) > \mid^2 \leq C_{\mathcal{B}} \parallel \tilde{b} \parallel_2^2 \leq C_{\mathcal{B}} \mid \tilde{\Omega} \mid \parallel b \parallel^2_{\mathrm{BMO}} .$$

Thus it is enough to show that we may choose $\tilde{\Omega}$ with $\mid \tilde{\Omega} \mid \leq C_{\mathcal{B}} \mid \Omega \mid$. Choose M such that $K \subset [-M, M-1]^d$ and decompose Ω into maximal dyadic cubes $\Omega = \bigcup_{(j_0,k_0)\in\Lambda} Q_{j_0,k_0}$. If $Q_{j,k} \subset Q_{j_0,k_0}$, then $\operatorname{Supp} \beta_{j,k} \subset \tilde{\Omega}_{j_0,k_0}$ where $\tilde{\Omega}_{j_0,k_0} = \{x/2^{j_0} x - k_0 \in [-M, M]^d\}$; thus we define $\tilde{\Omega} = \bigcup_{(j_0,k_0)\in\Lambda} \tilde{\Omega}_{j_0,k_0}$ and we see that we have

$$\mid \tilde{\Omega} \mid \leq \sum_{(j_0,k_0)\in\Lambda} (2M)^d \mid Q_{j_0,k_0} \mid = (2M)^d \mid \Omega \mid .$$

Proof of proposition 1. We apply again the vaguelettes lemma to see that :

$$J = \left\| \sum_j \sum_k 2^{2jd} < b \mid \beta_{j,k}(2^j x - k) >< f \mid \alpha_{j,k}(2^j x - k) > \gamma_{j,k}(2^j x - k) \right\|_2$$
$$\leq C_{\mathcal{B}} \sum_j \sum_k 2^{3jd} \mid < b \mid \beta_{j,k}(2^j x - k) > \mid^2 \mid < f \mid \alpha_{j,k}(2^j x - k) > \mid^2 .$$

Now, we define

$$\alpha^*(x) = \sup_{x \in Q_{j,k}} 2^{jd} \mid < f \mid \alpha_{j,k}(2^j x - k) > \mid .$$

Since $\alpha_{j,k} \in H^\epsilon$, we have $\alpha_{j,k} \in L^r$ with $0 < \frac{1}{2} - \frac{1}{r} \leq \frac{2\epsilon}{d}$, hence writing $\frac{1}{r} + \frac{1}{s} = 1$ (with $s < 2$) :

$$2^{jd} \mid < f \mid \alpha_{j,k}(2^j x - k) > \mid \leq (2^{jd} \int_{\tilde{\Omega}_{j,k}} \mid f \mid^s dx)^{1/s} C_\epsilon,$$

hence

$$\alpha^* \leq C \sup_{x\in Q}\left(\frac{1}{\mid Q\mid}\int_Q \mid f\mid^s dx\right)^{1/s} = Cf_s^*(x),$$

and

$$\| \alpha^* \|_2 \leq C \| f_s^* \|_2 \leq C_s \| f \|_2 .$$

Now we define $\Omega_N = \{x/\alpha^*(x) > 2^N\}$ and $W_N = \{x/2^N < \alpha^*(x) \leq 2^{N+1}\}$. We have :

$$\sum_{N\in\mathbb{Z}} 2^{2N} \mid W_N \mid \leq \int \alpha^{*2} dx \leq 4 \sum_{N\in\mathbb{Z}} 2^{2N} \mid W_N \mid$$

and

$$\sum_{N\in\mathbb{Z}} 2^{2N} \mid \Omega_N \mid = \sum_{N\in\mathbb{Z}} \sum_{K\geq N} 2^{2N} \mid W_K \mid = \frac{4}{3} \sum_{K\in\mathbb{Z}} 2^{2K} \mid W_K \mid .$$

We define moreover

$$\Lambda_N = \{(j,k)/\ 2^{jd} \mid < f \mid \alpha_{j,k}(2^j x - k) > \mid \in (2^N, 2^{N+1}]\}.$$

Then

$$(j,k) \in \Lambda_N \Rightarrow Q_{j,k} \subset \Omega_N,$$

hence :

$$\sum_j \sum_k 2^{3jd} \mid < b \mid \beta(2^j x - k) > \mid^2 \mid < f \mid \alpha_{j,k}(2^j x - k) > \mid^2$$

$$\leq \sum_N \sum_{(j,k)\in\Lambda^N} 2^{jd} \mid < b \mid \beta(2^j x - k) > \mid^2 4^{N+1}$$

$$\leq \sum_N \mid \Omega_N \mid 4^{N+1} C \| b \|^2_{\mathrm{BMO}} \quad \text{(by lemma 1)}$$

$$\leq C' \| b \|^2_{\mathrm{BMO}} \| \alpha^* \|_2^2$$ and Proposition 1 is proved. ■

COROLLARY. - *Under the hypotheses of Proposition 1, we have that*

$$(f,g) \to \sum_j \sum_k 2^{2jd} < f \mid \alpha_{j,k}(2^j x - k) >< g \mid \gamma_{j,k}(2^j x - k) > \beta_{j,k}(2^j x - k)$$

is a bounded bi-linear mapping from $L^2 \times L^2$ into the Hardy space $H^1(\mathbb{R}^d)$.

Proof. Since $H^1(\mathbb{R}^d)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{\,\mathrm{BMO}(\mathbb{R}^d)}$, this is deduced by duality from Proposition 1. ■

We then obtain the following nice result of S. Dobyinsky [DOB] :

THEOREM 1. - *Let* (V_j), (V_j^*) *be two bi-orthogonal multi-resolution analyses of* $L^2(\mathbb{R}^d)$ *(associated to dyadic dilations), such that they have dual compactly supported scaling functions* φ, φ^* *and* φ, φ^* *belong to the Hölder space* C^ϵ *for some* $\epsilon > 0$. *Let* (ψ_α), $(\psi_\alpha^*)_{1 \le \alpha \le 2^d - 1}$ *be associated compactly supported dual wavelets. Then for any* $f, g \in L^2(\mathbb{R}^d)$

$$q(f,g) = fg - \sum_{\alpha=1}^{2^d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} 2^{2jd} < f \mid \psi_\alpha^*(2^j x - k) >< g \mid \bar{\psi}_\alpha(2^j x - k) >$$

$$\psi_\alpha(2^j x - k)\bar{\psi}_\alpha^*(2^j x - k)$$

belongs to the Hardy space $H^1(\mathbb{R}^d)$, *and* $q(\cdot,\cdot)$ *defines a bounded bi-linear mapping from* $L^2 \times L^2$ *to* H^1.

Proof. We write $q(f,g) = A_1(f,g) + A_2(f,g) + A_3(f,g)$, where :

$$A_1(f,g) = \sum_{\alpha=1}^{2^d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{\beta=1}^{2^d-1} \sum_{\ell=-\infty}^{j-1} \sum_{r \in \mathbb{Z}^d} 2^{jd} 2^{\ell d} < f \mid \psi_\beta^*(2^\ell x - r) >< g \mid \bar{\psi}_\alpha(2^j x - k) >$$

$$\psi_\beta(2^\ell x - r)\bar{\psi}_\alpha^*(2^j x - k)$$

$$= \sum_{\alpha=1}^{2^d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{p \in \mathbb{Z}^d} 2^{2jd} < f \mid \varphi^*(2^j x - p) >< g \mid \bar{\psi}_\alpha(2^j x - k) >$$

$$\varphi(2^j x - p)\bar{\psi}_\alpha^*(2^j x - k)$$

$$A_2(f,g) = \sum_{\alpha=1}^{2^d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{p \in \mathbb{Z}^d} 2^{2jd} < f \mid \psi_\alpha^*(2^j x - k) >< g \mid \bar{\varphi}(2^j x - p) >$$

$$\psi_\alpha(2^j x - k)\bar{\varphi}^*(2^j x - p)$$

$$A_3(f,g) = \sum_{\alpha=1}^{2^d-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{\beta=1}^{2^d-1} \sum_{p \in \mathbb{Z}^d, (\beta,p) \ne (\alpha,k)} 2^{2jd} < f \mid \psi_\alpha^*(2^j x - k >$$

$$< g \mid \bar{\psi}_\beta(2^j x - \beta), \psi_\alpha(2^j x - k)\bar{\psi}_\beta^*(2^j x - p).$$

Now we may apply the corollary of Proposition 1 to each term A_1, A_2, A_3 to conclude.

If we assume only $\varphi, \varphi^* \in H^\epsilon$ (instead of C^ϵ), theorem 1 is still valid provided that moreover for all $k \in \mathbb{Z}^d$, $\varphi(x)\bar{\varphi}^*(x-k)$ belongs to H^ϵ. For instance, theorem 1 is valid for the Haar basis ($\varphi = \varphi^* = \chi_{[0,1]^d}$). ■

2. The div-curl theorem.

The Hardy space $H^1(\mathbb{R}^d)$ enjoys much better properties than $L^1(\mathbb{R}^d)$. For instance, it is a dual space and thus enjoys weak compactness properties which are useful in analysis. A striking example for the study of Navier-Stokes on Euler equation on $\mathbb{R}^2$ is discussed by Evans in [EVS]. A key tool is then the *div-curl theorem* of P. L. Lions and Y. Meyer [COIL] :

THEOREM 2. - *If $f,g \in L^1_{\rm loc}(\mathbb{R}^2)$ are such that $\vec\nabla f$, $\vec\nabla g$ are square-integrable, then $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ belongs to the Hardy space $H^1(\mathbb{R}^2)$ and*

$$\| \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \|_{H^1} \leq C \| \vec\nabla f \|_2 \| \vec\nabla g \|_2 .$$

Proof. We adapt here the proof by S. Dobyinsky [DOB]. We select a bi-orthogonal multiresolution analysis of $L^2(\mathbb{R})$ with compactly supported dual scaling functions φ, φ^*, and compactly supported wavelets ψ, ψ^*. Moreover we assume that $\varphi \in C^{1+\epsilon}$, $\varphi^* \in C^\epsilon$, and write $\tilde\varphi = \sum_{k\geq 0} \varphi'(x-k)$, $\tilde\varphi^* = \int_x^{x+1} \varphi^*(t)dt$, $\tilde\psi(x) = \psi'(x)$ and $\tilde\psi^* = \int_x^{+\infty} \psi^*(t)dt$. We assume all these functions to be real-valued.

We note $\varphi^{[0],[0]} = \varphi$, $\varphi^{[0],[1]} = \tilde\varphi$, $\varphi^{[1],[0]} = \psi$, $\varphi^{[1],[1]} = \tilde\psi$, $\varphi^{*[0],[0]} = \varphi^*$, $\varphi^{*[0],[1]} = \tilde\varphi^*$, $\varphi^{*[1],[0]} = \psi^*$, $\varphi^{*[1],[1]} = \tilde\psi^*$ and

$$\varphi_{j,k}^{(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)} = 2^j \varphi^{[\epsilon_1][\eta_1]}(2^j x - k_1)\varphi^{[\epsilon_2][\eta_2]}(2^j y - k_2)$$

and the same for $\varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)}$. Then for each $(\eta_1,\eta_2) \in \{0,1\}^2$ we have bi-orthogonal bases of $L^2(\mathbb{R}^d)$ given by :

$$(\varphi_{j,k}^{(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)})_{j\in\mathbb{Z},k\in\mathbb{Z}^2,(\epsilon_1,\epsilon_2)\neq(0,0)},$$

$$(\varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)})_{j\in\mathbb{Z},k\in\mathbb{Z}^2,(\epsilon_1,\epsilon_2)\neq(0,0)}.$$

Thus we know that if $\alpha, \beta \in L^2$, then $q_{(\eta_1,\eta_2)}(\alpha,\beta) \in H^1(\mathbb{R}^2)$ where

$$q_{(\eta_1,\eta_2)}(\alpha,\beta) = \alpha\beta - \sum_{(\epsilon_1,\epsilon_2)\neq(0,0)} \sum_j \sum_\ell < \alpha \mid \varphi_{j,k}^{(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)} >$$
$$< \beta \mid \varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)} > \varphi_{j,k}^{(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)} \varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(\eta_1,\eta_2)}$$

(by theorem 1). We then write :

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = q_{(0,1)}(\frac{\partial f}{\partial x}, \frac{\partial g}{\partial y}) - q_{(1,0)}(\frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}) + R.$$

We thus have to show that the remainder R belongs to $H^1(\mathbb{R}^2)$. We rewrite R following the rules :

$$< \frac{\partial f}{\partial x} \mid \varphi_{j,k_1,k_2}^{(0,\epsilon_2)(0,\eta_2)} >= 2^j \left(< f \mid \varphi_{j,k_1+1,k_2}^{(0,\epsilon_2)(1,\eta_2)} > - < f \mid \varphi_{j,k_1,k_2}^{(0,\epsilon_2)(1,\eta_2)} >\right)$$
$$< \frac{\partial f}{\partial x} \mid \varphi_{j,k_1,k_2}^{(1,\epsilon_2)(0,\eta_2)} >= -2^j < f \mid \varphi_{j,k_1,k_2}^{(1,\epsilon_2)(1,\eta_2)} >$$
$$< \frac{\partial g}{\partial x} \mid \varphi_{j,k_1,k_2}^{*(0,\epsilon_2)(1,\eta_2)} >= 2^j \left(< g \mid \varphi_{j,k_1,k_2}^{*(0,\epsilon_2)(0,\eta_2)} > - < g \mid \varphi_{j,k_1-1,k_2}^{*(0,\epsilon_2)(0,\eta_2)} >\right)$$
$$< \frac{\partial g}{\partial x} \mid \varphi_{j,k_1,k_2}^{*(1,\epsilon_1)(1,\eta_2)} >= 2^j < g \mid \varphi_{j,k_1,k_2}^{*(1,\epsilon_2)(0,\eta_2)} >$$

and similar rules for $\frac{\partial f}{\partial y}$ and $\eta_2 = 0$, or $\frac{\partial g}{\partial y}$ and $\eta_2 = 1$. Thus, we obtain :
$R = \sum_{(\epsilon_1,\epsilon_2)\neq(0,0)} \sum_j \sum_k 4^j < f \mid \varphi_{j,k}^{(\epsilon_1,\epsilon_2)(1,1)} >< g \mid \varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(0,0)} > 2^{2j}\omega_{(\epsilon_1,\epsilon_2)}(2^j x - k)$
with

$$\omega_{(1,0)} = \begin{array}{l} -\psi(x)\psi^*(x)\tilde{\varphi}(y)\tilde{\varphi}^*(y) + \psi(x)\psi^*(x)\tilde{\varphi}(y+1)\tilde{\varphi}^*(y+1) \\ +\tilde{\psi}(x)\psi^*(x)\varphi(y)\varphi^*(y) - \tilde{\psi}(x)\psi^*(x)\varphi(y-1)\varphi^*(y-1) \end{array}$$

$$\omega_{(0,1)} = \begin{array}{l} -\psi(x)\varphi^*(x)\tilde{\psi}(y)\tilde{\psi}^*(y) + \varphi(x-1)\varphi^*(x-1)\tilde{\psi}(y)\tilde{\psi}^*(y) \\ +\tilde{\varphi}(x)\tilde{\varphi}^*(x)\psi(y)\psi^*(y) - \tilde{\varphi}(x+1)\tilde{\varphi}^*(x+1)\psi(y)\psi^*(y) \end{array}$$

and

$$\omega_{(1,1)} = -\psi(x)\psi^*(x)\tilde{\psi}(y)\tilde{\psi}^*(y) + \tilde{\psi}(x)\tilde{\psi}^*(x)\psi(y)\psi^*(y).$$

Now, the $\omega_{(\epsilon_1,\epsilon_2)}$ are compactly supported, C^ϵ and have mean zero :

$$\int\int \omega_{(\epsilon_1,\epsilon_2)}(x,y)\,dx\,dy = 0.$$

Thus they belong to $B_1^{0,1}$ and we have :

$$\begin{aligned} \| R \|_{B_1^{0,1}} &\leq C \sum_{(\epsilon_1,\epsilon_2)\neq(0,0)} \sum_j \sum_k 4^j \,|< f \mid \varphi_{j,k}^{(\epsilon_1,\epsilon_2)(1,1)} >< g \mid \varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(0,0)} >| \\ &\leq C \left(\sum_{(\epsilon_1,\epsilon_2)\neq(0,0)} \sum_j \sum_k 4^j \,|< f \mid \varphi_{j,k}^{(\epsilon_1,\epsilon_2)(1,1)} >|^2 \right)^{1/2} \\ &\quad \left(\sum_{(\epsilon_1,\epsilon_2)\neq(0,0)} \sum_j \sum_k 4^j \,|< g \mid \varphi_{j,k}^{*(\epsilon_1,\epsilon_2)(0,0)} >|^2 \right)^{1/2} \\ &\leq C' \,\| \vec{\nabla} f \|_2 \| \vec{\nabla} g \|_2 \,. \end{aligned}$$

(We have used the following easy corollary of the vaguelettes lemma : if $\omega \in C^\epsilon$ is compactly supported and satisfies $\int \omega(x,y)dy = \int x\omega(x,y)dy = 0$, then

$$\sum_j \sum_k 4^j \,|< f \mid 2^j\omega(2^j x - k) >|^2 \leq C \,\| \frac{\partial f}{\partial x} \|_2^2).$$

Since $B_1^{0,1} \subset H^1(\mathbb{R}^2)$, theorem 2 is proved. ∎

3. Calderón-Zygmund operators.

Paraproduct operators $\pi(b,\cdot)$, with $b \in \mathrm{BMO}$, are examples of a wider class of operators which we have already encountered : the Calderón-Zygmund operators.

We recall some basic definitions on this topic.

DEFINITION 1. - *A singular integral operator T is a (bounded) linear operator from $C_c^\infty(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ such that there exists a continuous function $K(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ such that :*

(i) $\forall f \in C_c^\infty$, $\forall g \in C_c^\infty$ *with* $\operatorname{Supp} f \cap \operatorname{Supp} g = \emptyset$, $< Tf \mid g >= \int\int K(x,y) f(y) \bar{g}(x) dx\, dy$

(ii) *For a constant C and for all* $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $x \neq y$, $\mid K(x,y) \mid \leq C \frac{1}{|x-y|^d}$

(iii) *For a constant C, a constant $\delta \in (0,1]$ and for all $(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ such that $x \neq y$ and $\mid z \mid < \frac{1}{2} \mid x - y \mid$:*

$$\mid K(x, y+z) - K(x,y) \mid + \mid K(x,y) - K(x+z, y) \mid \leq C \frac{\mid z \mid^\delta}{\mid x - y \mid^{d+\delta}}.$$

The kernel K describes the behaviour of T for functions f, g with disjoint supports. If we want information near the diagonal $x = y$, we have to introduce another estimate :

DEFINITION 2. - *A singular integral operator T is said to satisfy the weak boundedness property (which is written "$T \in WBP$") if there exists a compact set K_0 with non-empty interior, a constant C_0 and a number N such that for all $f, g \in C_c^\infty$ with support included in K_0, all $x_0 \in \mathbb{R}^d$ and all $\lambda > 0$:*

$$(4) \quad |< T(\frac{1}{\lambda^{d/2}} f(\frac{x-x_0}{\lambda})) \mid \frac{1}{\lambda^{d/2}} g(\frac{x-x_0}{\lambda}) >| \leq C_0 \sum_{|\alpha| \leq N} \| \frac{\partial^\alpha f}{\partial x^\alpha} \|_\infty \sum_{|\alpha| \leq N} \| \frac{\partial^\alpha g}{\partial x^\alpha} \|_\infty .$$

The last definition we shall need is the definition of the distribution $T(1)$ associated to a singular integral operator.

DEFINITION 3. - *If T is a singular integral operator with kernel $K(x,y)$, then $T(1)$ is the linear functional on*

$$\mathcal{D}_0 = \{ g \in C_c^\infty(\mathbb{R}^d) / \int g\, dx = 0 \}$$

defined by :

$$(5) \quad < T(1) \mid g >=< T(\chi_g) \mid g > + \int\int (K(x,y) - K(x_g, y) \bar{g}(x)(1 - \chi_g(y)) dx\, dy$$

where χ_g is any function in C_c^∞ such that $\chi_g \equiv 1$ in the neighborhood of $\operatorname{Supp} g$ *and* x_g *is any point in* $\operatorname{Supp} g$.

(Of course $< T(1) \mid g >$ doesn't depend on the choice of χ_g or x_g).

A *Calderón-Zygmund operator* is a singular integral operator which can be extended as a bounded operator from L^2 to L^2. Such operators are characterized by the celebrated *$T(1)$ theorem of David and Journé* [DAV] :

THEOREM 3. - *Let T be a singular integral operator on $\mathbb{R}^d$. Then T has a bounded extension on L^2 if and only if $T \in WBP, T(1) \in BMO, T^*(1) \in BMO$.*

Proof. The necessity is a classical result : if T is a Calderón-Zygmund operator, then T maps L^∞ to BMO, Hence $T(1) \in$ BMO(and $T^*(1) \in$ BMO). For the converse implication, one may assume $T(1) = T^*(1) = 0$: indeed replace T by $T_0 - \pi(T(1), \cdot) - \pi(T^*(1), \cdot)^*$, where $\pi(b, \cdot)$ is the paraproduct operator with $b \in$ BMO. We have already seen that $\pi(b, \cdot) \in \mathcal{L}(L^2, L^2)$ and it is obvious that $\pi(b, \cdot)(1) = b$ in BMO.

Now, we use a wavelet basis of $L^2(\mathbb{R}^d)$ $(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k))$ $(1 \leq \epsilon \leq 2^d - 1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d)$, such that $\psi_\epsilon \in C^N$ and is compactly supported (where N is given by condition (4) of weak boundedness property). $T(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k))$ is thus well defined. We are going to show that we may apply the vaguelettes lemma to the family $(T(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k))_{1 \leq \epsilon \leq 2^d-1, j \in \mathbb{Z}, k \in \mathbb{Z}^d}$ (and thus obtain the L^2-boundedness of T).

Because of the invariance of (4) or of the estimates on the size and the regularity of K through dilation and translation (the operator T and $A_{\lambda,x_0}^{-1} T A_{\lambda,x_0}$ satisfy the same estimates, where $A_{\lambda,x_0}(f) = \lambda^{-\frac{d}{2}} f(\frac{x-x_0}{\lambda}))$, it is enough to prove that $\int T\psi_\epsilon dx = 0$, $T(\psi_\epsilon) \in L^2((1+ \mid x \mid)^{\frac{n}{2}+\alpha}dx) \cap H^\alpha$ for some $\alpha > 0$ and that :

$$\left(\int (\mid x \mid +1)^{d+2\alpha} \mid T(\psi_\epsilon) \mid^2 dx \right)^{1/2} + \| T(\psi_\epsilon) \|_{H^\alpha} \leq C_1,$$

where C_1 depends only on

$$\|\mid x - y \mid^d K(x,y) \|_\infty, \quad \sup_{z < \frac{1}{2}(x-y)} \left| \frac{\mid x - y \mid^{d+\delta}}{\mid z \mid^\delta} \mid K(x,y) - K(x, y+z) \mid \right|,$$

$$\sup_{z < \frac{1}{2}(x-y)} \left| \frac{\mid x - y \mid^{d+\delta}}{\mid z \mid^\delta} \mid K(x+z, y) - K(x,y) \mid \right|, \quad K_0, \quad N$$

and

$$\sup_\lambda \sup_{x_0} \sup_{f,g \subset C_0^\infty(K_0)} \left\{ \frac{\frac{1}{\lambda^d} \mid < T(f(\frac{x-x_0}{\lambda})) \mid g(\frac{x-x_0}{\lambda}) > \mid}{\sum_{\mid\alpha\mid \leq N} \| \frac{\partial^\alpha f}{\partial x^\alpha} \|_\infty \sum_{\mid\alpha\mid \leq N} \| \frac{\partial^\alpha g}{\partial x^\alpha} \|_\infty} \right\}.$$

Now, this can be easily shown :

- $\int T(\psi_\epsilon) dx = 0$ because $T^*(1) = 0$
- If $\mathrm{Supp}\, \psi_\epsilon \subset [-M, M]^d$ and $\mid x \mid > 2M$, we have :

$$T(\psi_\epsilon(x)) = \int (K(x,y) - K(x,0)) \psi_\epsilon(y) dy,$$

and thus

$$\mid T(\psi_\epsilon)(x) \mid \leq \| \psi_\epsilon \|_2 \left(\int_{[-M,M]^d} C \frac{\mid y \mid^{2\delta}}{\mid x - y \mid^{2d+2\delta}} dy \right)^{1/2} \leq C' \| \psi_\epsilon \|_2 \frac{1}{\mid x \mid^{d+\delta}}$$

and thus

$$\int_{\mid x \mid > 2M} \mid x \mid^{d+\delta} \mid T(\psi_\epsilon)(x) \mid^2 dx < +\infty.$$

• The next step is to show that for f, g, h compactly supported functions of class C^N we have :

$$(6) \qquad < T(fg) \mid h > - < T(f) \mid \bar{g}h > = \int\int K(x,y)(g(y)-g(x))f(y)\bar{h}(x)\,dx\,dy$$

as a consequence of the weak boundedness property. Indeed, if we use a resolution of identity $\sum_{k\in\mathbb{Z}^d}\varphi(x-k) = 1$ with $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} &< T(fg) \mid h > \\ &= \sum_{\operatorname{Supp}\varphi(Ax-k)} \ \sum_{\cap \operatorname{Supp}\varphi(Ax-\ell)=\emptyset} \int\int K(x,y)f(y)g(y)\varphi(Ay-k)\bar{h}(x)\varphi(Ax-\ell)\,dx\,dy \\ &+ \sum_{\operatorname{Supp}\varphi(Ax-k)} \ \sum_{\cap \operatorname{Supp}\varphi(Ax-\ell)\neq\emptyset} < T(fg\varphi(Ax-k)) \mid h\cdot\varphi(dx-\ell) > ; \end{aligned}$$

the second sum is rewritten as

$$\begin{aligned} &\sum_{\operatorname{Supp}\varphi(Ax-k)} \ \sum_{\cap \operatorname{Supp}\varphi(Ax-\ell)\neq\emptyset} g(\frac{k}{A}) < Tf\varphi(Ax-k) \mid h\varphi(Ax-\ell) > + \\ &\sum_{\operatorname{Supp}\varphi(Ax-k)} \ \sum_{\cap \operatorname{Supp}\varphi(Ax-\ell)\neq\emptyset} < T(f(g-g(\frac{k}{A}))\varphi(Ax-k) \mid h\varphi(Ax-\ell) > . \end{aligned}$$

Due to the weak boundedness property, we may estimate the last sum by $C(\mid \operatorname{supp} f \mid A^{-d})\cdot A^{-1}\cdot A^d$ where A^{-d} corresponds to the number of indexes in the sum, A^{-1} to the norm of $\left\|g(\frac{x+k}{A}) - g(\frac{k}{A})\right\|_{C^N}$ $(A \geq 1)$ and A^d to the weak boundedness property. Letting A go to $+\infty$, we obtain (6).

- One sees easily that if $\omega \equiv 1$ on a neighborhood of 0, then $\left\|T(\omega(\frac{x-x_0}{\lambda}))\right\|_\infty \leq C$ where C doesn't depend on x_0 or λ : just use $T(1) = 0$ to see that on a neighborhood of x_0

  $$T(\omega(\frac{x-x_0}{\lambda})) = C_{x_0,\lambda} - \int (K(x,y) - K(x_0,y))(1-\omega(\frac{y-x_0}{\lambda}))dy$$

  and estimate the constant $C_{x_0,\lambda}$ through the weak boundedness property.

- Thus we see that $T(\psi_\epsilon)$ is bounded : if $\omega \equiv 1$ on $\operatorname{Supp}\psi_\epsilon$ then

  $$T(\psi_\epsilon) = \psi_\epsilon T(\omega) + \int K(x,y)(\psi_\epsilon(y) - \psi_\epsilon(x))\omega(y)dy.$$

- Now, we just have to see that $T(\psi_\epsilon) \in H^\alpha$ for $\alpha \in (0,\delta)$. Indeed, if x and h are given in $\mathbb{R}^d$ and $\omega \equiv 1$ on a neighborhood of both x and $x+h$, we have :

  $$\begin{aligned} T\psi_\epsilon(x+h) - T\psi_\epsilon(x) &= \int (K(x+h,y) - K(x,y))(\psi_\epsilon(y) - \psi_\epsilon(x))(1-\omega(y))dy \\ &- \int K(x+h,y)(\psi_\epsilon(y) - \psi_\epsilon(x+h))\omega(y)dy \\ &+ \int K(x,y)(\psi_\epsilon(y) - \psi_\epsilon(x))\omega(y)dy \\ &+ (\psi_\epsilon(x+h) - \psi(x))T\omega(x). \end{aligned}$$

Hence we have (choosing $\beta \in (\alpha, \delta)$ and $\gamma \in (0, \alpha)$) :

$$\begin{aligned}
| T\psi_\epsilon(x) - T\psi_\epsilon(x+h) | \leq C \int_{|x-y|>2|h|} & \frac{| h |^\delta}{| x-y |^{d+\delta}} | \psi_\epsilon(y) - \psi_\epsilon(x) | \, dy \\
&+ C \int_{|x-y|>5|h|} \frac{1}{| x-y |^d} | \psi_\epsilon(y) - \psi_\epsilon(x) | \, dy \\
&+ C \int_{|x-y|<5|h|} \frac{1}{| x+h-y |^d} | \psi_\epsilon(y) - \psi_\epsilon(x+h) | \, dy \\
&+ C | \psi_\epsilon(x) - \psi_\epsilon(x+h) |
\end{aligned}$$

$$\begin{aligned}
\leq C' \Bigg\{ \int_{|x-y|>2|h|} & \frac{| h |^{2\beta}}{| x-y |^{d+2\beta}} | \psi_\epsilon(y) - \psi_\epsilon(x) |^2 \, dy \\
&+ \int_{|x-y|<5|h|} \frac{| h |^{2\gamma}}{| x-y |^{d+2\gamma}} | \psi_\epsilon(y) - \psi_\epsilon(x) |^2 \, dy \\
&+ \int_{|x-y|<5|h|} \frac{| h |^{2\gamma}}{| x+h-y |^{d+2\gamma}} | \psi_\epsilon(y) - \psi_\epsilon(x+h) |^2 \, dy \\
&+ | \psi_\epsilon(x) - \psi_\epsilon(x+h) |^2 \Bigg\}^{1/2}
\end{aligned}$$

and obtain

$$\int \int | T(\psi_\epsilon)(x) - T(\psi_\epsilon)(x+h) |^2 \, dx \frac{dh}{| h |^{d+2\alpha}} \leq$$

$$C' \int \int | \psi_\epsilon(x) - \psi_\epsilon(y) |^2 \frac{dx \, dy}{| x-y |^{d+2\alpha}}.$$

Thus theorem 3 is proved.

Another way to prove theorem 3 when $T(1) = T^*(1) = 0$ is to estimate the size of the coefficients of the matrix T in the orthonormal wavelet basis $(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k))$, i.e. to majorate

$$|< T(2^{j\frac{d}{2}}\psi_\epsilon(2^j x - k) \mid 2^{j'\frac{d}{2}}\psi_{\epsilon'}(2^{j'} x - k') > .$$

As a matter of fact, one can apply Schur's lemma on the coefficients of the matrix of T, even when one uses the Haar basis instead of a C^N basis [COIJ]. Moreover the Haar basis can be easily modified to prove the $T(b)$ theorem (where the test-function 1 in theorem 3 is replaced by a para-accretive functions b).

4. The Riemann function.

Riemann's function $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{i\pi n^2 t}$ was proposed to Weierstrass as an example of continuous and nowhere differentiable function. But neither Riemann nor Weierstrass could prove its non-differentiability. In 1916, Hardy proved the non-differentiability on irrationals and on a certain class of rationals. Then in 1970, J. Gerver proved very surprisingly that the function was indeed differentiable on rationals $\frac{p}{q}$ with p and q odd numbers and non-differentiable elsewhere.

M. Holschneider tried to give a wavelet based proof of the result of Gerver. He used the continuous wavelet transform of J. Morlet, with the analytical wavelet $\psi(t) = \frac{1}{(t+i)^2}$ [HOT]. Then the wavelet transform of $W(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{i\pi n^2 t}$ is the function $F(b+ia) = C_0 a^{3/2}(\theta(b+ia) - 1)$, where θ is the Jacobi function $\theta(z) = 1 + 2\sum_1^{\infty} e^{i\pi n^2 z}$. Since $\mid \theta(b+ia) \mid \leq Ca^{-1/2}$ for $a \leq 1$, we find that W is 1/2-Hölderian.

Now, if x is irrational, Hardy and Littlewood proved that for some sequence of positive numbers $a_n \to 0$ and some positive constant C we have $\mid \theta(x+ia_n) \mid a_n^{1/4} \geq C$, hence $\mid F(x+ia_n) \mid \geq C\, a_n^{\frac{3}{2}-\frac{1}{4}} = C\sqrt{a_n} a_n^{\frac{3}{4}}$ and W cannot be better than $C^{\frac{3}{4}}$ at x.

For the rational points, one uses the theta group G_θ generated by $K^2 : z \hookrightarrow z+1$ and $U : z \to -\frac{1}{z}$. The rational points are split in two orbits : the orbit of 1 consists of exactly the rational $\frac{p}{q}$ with p and q odd numbers and the orbit of 0 is the rest. The rôle played by G_θ comes from the well-known formulas :

$$\theta(K^2 z) = \theta(z) \quad \text{and} \quad \theta(Uz) = \sqrt{-iz}\,\theta(z).$$

At any point x of the orbit of 0, we have

$$a^{\frac{3}{2}}\theta(x+b+ia) = C_x a^{\frac{3}{2}} \sqrt{\frac{i}{b+ia}} + 0\left(\sqrt{a}(b+ia)^{\frac{3}{2}-\epsilon}\right)$$

where the constant C_x is given by $C_0 = 1$, $C_{K^2 x} = C_x$ and $C_{Ux} = \sqrt{\frac{-i}{x}} C_x$. The idea is to compare this formula to the wavelet transform of $C_x^- \mid t-x \mid_-^{\alpha} + C_x^+ \mid t-x \mid_+^{\alpha}$, with $C_x^+ = e^{i\alpha\pi} C_x^-$.

One find that such a transform is given by :

$$T_\alpha(x+b+ia) = C_\alpha C_x^+ a^{\frac{3}{2}} \left(\frac{i}{b+ia}\right)^{1-\alpha}$$

so that we may write

$$W(t) = C_x^- \mid t-x \mid_-^{1/2} + C_x^+ \mid t-x \mid_+^{1/2} + \rho(t).$$

The remainder is indeed differentiable (since it has wavelet transform $0(\sqrt{a}(b+ia)^{\frac{3}{2}-\epsilon})$.

The orbit of 1 is even easier to handle. Indeed, we have

$$a^{\frac{3}{2}}\theta(1+b+ia) = a^{\frac{3}{2}}(2\theta(4b+4ia) - \theta(b+ia));$$

and developping around 0 we find that the principal part is 0, so that

$$a^{\frac{3}{2}}\theta(1+b+ia) = 0(\sqrt{a}(b+ia)^{\frac{3}{2}-\epsilon}).$$

Thus M. Holschneider, in a joint work with Ph. Tchamitchian, could give a wavelet-based proof that :

i) $W(t)$ is $C^{1/2}$

ii) $W(t)$ is not better than $C^{3/4}$ at any irrational

iii) $W(t+x) = C_x^+ \mid t-x \mid_+^{1/2} + C_x^- \mid t-x \mid_-^{1/2} + W(x) + a_x t + o(t)$ for x on the orbit of 0 under G_θ

iv) W is differentiable at x for x on the orbit of 1 under G_θ.

This result was recently improved in two divergent directions by Y. Meyer and S. Jaffard.

Y. Meyer made a more precise analysis of the behaviour of W on the orbit of 1 [JAM : if x is on the orbit of 1, then

$$W(x+t) = v(t) + \sum_{n\geq 0} \mid t \mid_+^{\frac{3}{2}+n} v_+^n(\frac{1}{t}) + \sum_{n\geq 0} \mid t_- \mid^{\frac{3}{2}+n} v_-^n(\frac{1}{t})$$

where $v \in C^\infty$, $v_\pm^n$ are 2π-periodical functions, $\int_0^{2\pi} v_\pm^n dt = 0$ and $v_\pm^n \in C^{\frac{1}{2}+n}$. This "chirps" expansion of W is based on the wavelet transform of W performed by Holschneider and Tchamitchian, and on the ridge and skeleton extraction algorithm of Torresani [TCH] which was introduced for the wavelet analysis of asymptotic signals. For such signals, the signal is directly recovered as the restriction of the wavelet transform to a subset of the time-frequency space, called the ridge of the transform.

S. Jaffard on the other hand drew his attention to the irrational points [JAF5]. He proved the very beautiful following result :

If $x \notin Q$ and if $\frac{p_n}{q_n}$ is the sequence of its approximations by continued fractions, define $\tau(x) = \sup(\tau / \mid x - \frac{p_m}{q_m} \mid \leq \frac{1}{q_m^\tau}$ for infinitely many m's such that p_m and q_m are not both odd}.

Then :

i) $\alpha(x) = \frac{1}{2} + \frac{1}{2\tau(x)}$ where $\alpha(x) = \sup\{\beta / \limsup_{\substack{y\to x\\ \neq}} \frac{|w(y)-w(x)|}{|x-y|^\beta} < +\infty\}$

ii) the Hausdorff dimension $d(\alpha)$ of $\{x \in \mathbb{R}/\alpha(x) = \alpha\}$ is given by :

$$\begin{aligned} d(\alpha) &= 4\alpha - 2 \quad \text{for} \quad \frac{1}{2} \leq \alpha \leq \frac{3}{4} \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

(We may write as well $d(\alpha) = \frac{2}{\tau}$ where τ is the common value $\tau(x)$ of those x such that $\alpha(x) = \alpha$).

His result is based on the microlocal analysis by wavelets [JAF3] (transformed into a tool for multifractal analysis in [JAF4]) applied to the wavelet transform of $W(t)$ of Holschneider and Tchamitchian.

Thus the original ideas of Holschneider led eventually to a deep renewal of our knowledge of the Riemann function, due to the new technology developed recently in microlocal analysis.

REFERENCES

Bibliographical references on wavelets can be found in several bibliographies, as : R. Creutzburg, Bibliography of wavelets in digital signal processing and related topics, in *Progress in wavelet analysis and applications,* Y. Meyer and S. Roques ed., Frontières, Gif-sur-Yvette, 1993, 747-785 (849 entries) or S. Pittner, J. Schneid and C. W. Ueberhuber, *Wavelet literature survey,* Technical University, Vienna, 1993 (976 entries). They can also be obtained on-line with the wavelet server of the University of South Carolina (ftp address : ftp.math.scarolina.edu(/pub/wavelet) ; Gopher address : gopher.math.scarolina.edu ; WWW address : http://www.math.scarolina.edu/~wavelet).

[ARG] ARGOUL, F., ARNEODO, A., GRASSEAU, G. GAGNE, Y., HOPFINGER, E. J. and FRISCH, U. : Wavelet analysis of turbulence reveals the multifractal nature of the Richardson cascade. *Nature* 338 (1989), 51-53.

[ARN] ARNEODO, A., ARGOUL, F. et GRASSEAU, G. : Transformation en ondelettes et renormalisation, in *Les ondelettes en 1989,* P. G. Lemarié ed., Lecture Notes in Math. n° 1438. Berlin, Springer-Verlag, 1990.

[ASL] ASLAKSEN E. W. and KLAUDER, J. R. : Unitary representations of the affine group. *J. Math. Phys.* 9 (1968), 206-211.

[AUS1] AUSCHER, P. : *Ondelettes, fractals et applications.* Thèse, Univ. Paris IX, 1989.

[AUS2] AUSCHER, P. : Solution of two problems on wavelets. Preprint, IRMAR, Univ. Rennes I, 1992 Submitted to *Ann. of Math.*

[BAA] BAASTIANS, M. J. : Gabor's signal expansion and degrees of freedom of a signal. Proc. IEEE 68 (1980), 538-539.

[BAC] BACRY, E. ARNEODO, A. and MUZY, J. F. : Singularity spectrum of fractal signals from wavelet analysis : exact results. *J. Stat. Physics* 70 (1993).

[BAL] BALIAN, R. : Un principe d'incertitude fort en théorie du signal ou en mécanique quantique. *C. R. Acad. Sci. Paris* 292 (1981), 1357-1362.

[BAT1] BATTLE G. : A block spin construction of ondelettes. Part I : Lemarié functions. *Comm. Math. Phys.* 110 (1987), 601-615.

[BAT2] BATTLE G. : A block spin construction of ondelettes. Part II : the QFT connection. *Comm. Math. Phys.* 114 (1988), 93-102.

[BAT3] BATTLE G. : Heisenberg proof of the Balian-low theorem. *Lett. Math. Phys.* 15 (1988), 175-177.

[BAT4] BATTLE G. : A block spin construction of ondelettes. Part III : a note on pre-ondelettes. Preprint, TAMU, Texas, 1990.

[BER] BERKOLAIKO, M. and NOVIKOV, I. : On infinitely smooth almost wavelets with compact support. *Dokl. Rus. Acad. Nauk* 326 (1992), 935-938 (in russian).

[BEY1] BEYLKIN, G., COIFMAN, R. R. and ROKHLIN, V. : Fast wavelet transforms and numerical algorithms I. *Comm. Pure & Appl. Math.* 44 (1991), 141-183.

[BEY2] BEYLKIN, G. : On the representation of operators in bases of compactly supported wavelets. *SIAM J. Numer. Anal.* 29 (1992), 1716-1740.

[BON] BONY, J. M. : Calcul symbolique et propagation des singularités pour les EDP non linéaires. *Ann. Sci. Ec. Norm. Sup.* 14 (1981), 209-246.

[BOR] BOURDAUD, G. : Ondelettes et espaces de Besov. To appear in *Revista Mat. Iberoamer.*

[BOU] BOURGAIN, J. : A remark on the uncertainty principle for Hilbertian basis. *J. Funct. Anal.* 79 (1988), 136-143.

[BUR] BURT, P. J. and ADELSON, E. H. : The Laplacian pyramid as a compact image code. *IEEE Trans. on Comm.* 31 (1983), 532-540.

[CAL] CALDERÓN, A. P. : Intermediate spaces and interpolation, the complex method. *Studia Math.* 24 (1964), 113-190.

[CHS] CHUI, C. K. and SHI, X. L. : Inequalities of Littlewood-Paley type for frames and wavelets. *SIAM J. Math. Anal.* 24 (1993), 263-277.

[CHW1] CHUI, C. K. and WANG, J. Z. : A cardinal spline approach to wavelets. *Proc. Amer. Math. Soc.* 113 (1991), 785-793.

[CHW2] CHUI, C. K. and WANG, J. Z. : On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.* 330 (1992), 903-915.

[CHW3] CHUI, C. K. and WANG, J. Z. : A general framework of compactly supported spline and wavelets. *J. Approx. Theory* 71 (1992), 263-304.

[COD1] COHEN, A. and DAUBECHIES, I. : A stability criterion for bi-orthogonal wavelets and their related subband coding scheme. *Duke Math. J.* 68 (1992), 313-335.

[COD2] COHEN, A. and DAUBECHIES, I. : Non-separable bidimensional wavelet bases 1. *Revista Mat. Iberoamericana* 9 (1993), 51-133.

[CODF] COHEN, A., DAUBECHIES, I. and FEAUVEAU, J. C. : Bi-orthogonal bases of compactly supported wavelets. *Comm. Pure & Appl. Math.* 45 (1992), 485-560.

[CODV] COHEN, A., DAUBECHIES, I. and VIAL, P. : Wavelets on the interval and fast wavelet transforms. *Appl. and Comp. Harm. Anal.* 1 (1993), 54-81.

[COH] COHEN, A. : *Ondelettes, analyses multirésolutions et traitement numérique du signal.* Thèse, Univ. Paris IX, 1990.

[COHC] COHEN, A. et CONZE, J. P. : Régularité des bases d'ondelettes et mesures ergodiques. *Revista Mat. Iberoamer.* 8 (1992), 351-366.

[COHD] COHEN, A. and DYN, N. : Non-stationary subdivision schemes and multi-resolution analysis. Submitted to *SIAM J. Math. Anal.*

[COI] COIFMAN, R. R. : Adapted multiresolution analysis, computation, signal processing and opeartor theory, in *Internat. Congress Math.* Kyoto 1990, vol. II. Berlin, Springer Verlag, 1990, 879-887.

[COIJ] COIFMAN, R. R., JONES, P. & SEMMES, S. : Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves. *J. Amer. Math. Soc.* 2 (1989), 553-564.

[COIL] COIFMAN, R. R., LIONS, P. L., MEYER, Y. & SEMMES, S. : Compacité par compensation et espaces de Hardy. *C. R. Acad. Sci. Paris* 309 (1989), 945-949.

[COIM1] COIFMAN, R. R. & MEYER, Y. : *Au delà des opérateurs pseudo-différentiels.* Astérisque n° 57, Soc. Math. France, 1978.

[COIM2] COIFMAN, R. R. & MEYER, Y. & WICKERHAUSER, M. V. : Size properties of wavelet packets, in *Wavelets and their applications,* M. B. Ruskai et al. Eds., Jones and Bartlett Publ., Boston, 1991, 453-470.

[COIM3] COIFMAN, R. R. & MEYER, Y. : Remarques sur l'analyse de Fourier à fenêtre. *C. R. Acad. Sci. Paris* 312 (1992), 259-261.

[COIM4] COIFMAN, R. R. & MEYER, Y., QUAKE, S. & WICKERHAUSER, M. V. : Signal processing and compression with wavelet packets, in *Progress in wavelet analysis and applications,* Y. Meyer and S. Roques eds., Frontières, Gif/Yvette, 1993.

[COIWE] COIFMAN, R. R. & WEISS, G. : Extensions of Hardy spaces and their uses in analysis. *Bull. Amer. Math. Soc.* 83 (1977), 569-645.

[COIWI] COIFMAN, R. R. & WICKERHAUSER, M. V. : Entropy-based algorithms for best basis selection. *IEEE Trans. on Inform. Theory* 38 (1992), 713-718.

[COM] COMBES, J. M., GROSSMANN, A. & TCHAMITCHIAN, P. eds. : *Wavelets, time-frequency methods and phase space,* Lecture Notes on IPTI. Berlin, Springer-Verlag, 1989.

[DAU1] DAUBECHIES, I. : Orthogonal bases of compactly supported wavelets. *Comm. Pure & Appl. Math.* 41 (1988), 909-996.

[DAU2] DAUBECHIES, I. : The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Inform. Theory* 36 (1990), 961-1005.

[DAU3] DAUBECHIES, I. : *Ten lectures on wavelets.* CBMS-NSF Regional Conference Series in Applied Math. 61, SIAM Press, Philadelphia, 1992.

[DAU4] DAUBECHIES, I. : Orthonormal bases of compactly supported wavelets. II. Variation on a theme. *SIAM J. Math. Anal.* 24 (1993), 499-519.

[DAUG] DAUBECHIES, I., GROSSMANN, A. & MEYER, Y. : Painless nonorthogonal expansions. *J. Math. Phys.* 27 (1986), 1271-1283.

[DAUL] DAUBECHIES, I. & LAGARIAS, J. : Two-scale difference equation. I : Existence and global regularity of solutions. *SIAM J. Math. Anal.* 22 (1991), 1388-1410.

[DAV] DAVID, G. & JOURNE, J. L. : A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. Math.* 120 (1984), 371-397.

[DES] DESLAURIERS, G. & DUBUC, S. : Interpolation dyadique, in *Fractals, dimensions non-entières et applications*, G. Cherbit ed. Paris, Masson, 1987.

[DOB] DOBYINSKY, S. : *Ondelettes, renormalization du produit et applications à certains opérateurs bilinéaires.* Thèse, Univ. Paris-IX, 1992.

[DON1] DONOHO, D. L. : Non-linear solution of linear inverse problems via wavelet-vaguelettes decomposition. Tech. Report, Dept. Statistics, Stanford Univ., 1992.

[DON2] DONOHO, D. L. : Wavelet shrinkage and W.V.D. : a 10-minute tour, in *Progress in wavelet analysis and applications*, Y. Meyer and S. Roques eds. Gif/Yvette, Frontières, 1993, 109-128.

[DUF] DUFFIN, R. J. & SCHAEFFER, A. C. : A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.* 72 (1952), 341-366.

[DUR] DURAND, S. : *Etude de la convergence de l'algorithme en cascade intervenant dans la construction des ondelettes.* Thèse, Univ. Paris IX, 1993.

[DYN] DYN, N. & RON, A. : Multiresolution analysis by infinitely differentiable compactly supported functions. Preprint, Univ. Wisconsin 1992.

[EIR] EIROLA, T. : Sobolev characterization of solutions of dilation equations. *SIAM J. Math. Anal.* 23 (1992), 1015-1030.

[EST] ESTEBAN, D. & GALAND, C. : Application of quadrature mirror filters to split-band voice coding schemes. *Proc. IEEE Internat. Conf. ASSP*, Hartford, Connecticut, 1977, 191-195.

[EVG] EVANGELISTA, G. : Wavelet transforms and wave digital filters, in *Wavelets and applications*, Y. Meyer ed. Paris, Masson, 1992, 396-412.

[EVS] EVANS, L. C. & MÜLLER, S. : Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity. *J. Amer. Math. Soc.* 7 (1994), 199-220.

[FAB] FABER, G. : Über die Orthogonalfunktionen des Herrn Haar. *Jahresber.Deutsch Math. Verein.* 19 (1910), 104-112.

[FEA] FEAUVEAU, J. C. : *Analyse multirésolution par ondelettes non-orthogonales et bancs de filtres numériques.* Thèse, Univ. Paris XI, 1990.

[FEF] FEFFERMAN, C. & STEIN, E. M. : H^p spaces of several variables. *Acta Math.* 129 (1972), 137-193.

[FIX] FIX, G. & STRANG, G. : Fourier analysis of the finite element method in Ritz-Galerkin theory. *Stud. Appl. Math.* 48 (1969), 265-273.

[FRA] FRANKLIN, P. : A set of continuous orthogonal functions. *Math. Annalen* 100 (1928), 522-529.

[FRJ] FRAZIER, M. & JAWERTH, B. : The φ-transform and applications to distribution spaces, in *Function spaces and applications*, M. Cwikel et al. eds., Lecture Notes in Math. 1302, 1988, 223-246.

[FRJW] FRAZIER, M., JAWERTH, B. & WEISS, G. : *Littlewood-Paley theory and the study of functional spaces.* CBMS-conference lecture notes 79, Amer. Math. Soc., 1991.

[GAB] GABOR, D. : Theory of communication. *J. Inst. Elect. Engin.* 93 (1946), 429-457.

[GOL] GOLDBERG, D. : A local version of real Hardy spaces. *Duke Math. J.* 46 (1979), 27-42.

[GOO] GOODMAN, T. N. T., LEE, S. L. & TANG, W. S. : Wavelets in wandering subspaces. *Trans. Amer. Math. Soc.* 271 (1992), 451-467.

[GRI1] GRIPENBERG, G. : Unconditional bases of wavelets for Sobolev spaces. *SIAM J. Math. Anal.* 24 (1993), 1030-1042.

[GRI2] GRIPENBERG, G. : Wavelet bases in $L^p(\mathbb{R})$. *Studia Math.* 106 (1993), 175-187.

[GROE1] GRÖCHENIG, K. : Analyse multi-échelles et bases d'ondelettes. *C. R. Acad. Sci. Paris* 305 (1987), 13-15.

[GROE2] GRÖCHENIG, K. : Describing functions : atomic decomposition versus frames. *Mh. Math.* 112 (1991), 1-42.

[GROEM] GRÖCHENIG, K. & MADYCH, W. R. : Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbb{R}^n$. *IEEE Trans. Inf. Theory* 38 (1992), 556-568.

[GROS] GROSSMANN, A. & MORLET, J. : Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.* 15 (1984), 723-736.

[HAAR] HAAR, A. Zur Theorie der orthogonalen Funktionensysteme. *Math. Ann.* 69 (1910), 331-371.

[HAR] HARDIN, D., KESSLER, B. & MASSOPUST, P. R. : Multiresolution analysis based on fractal functions. *J. Approx. Theory* 71 (1992), 104-120.

[HER] HERLEY, C. & VETTERLI, M. : Wavelets and recursive filter banks. *IEEE Trans. on Signal Proc.* 41 (1993), 2536-2556.

[HEM] HERRMANN, O. : On the approximation problem in non recursive digital filter design. *IEEE Trans. on Circuit Theory,* May 1971, 411-413.

[HER] HERVE, L. : *Méthodes d'opérateurs quasi-compacts en analyse multi-résolution, applications à la construction de bases d'ondelettes et à l'interpolation.* Thèse, Univ. Rennes I, 1992.

[HOL] HOLSCHNEIDER, M. : *L'analyse d'objects fractals et leur transfosrmation en ondelettes.* Thèse, Univ. Provence, Marseille, 1988.

[HOT] HOLSCHNEIDER, M. & TCHAMITCHIAN, P. : Pointwise analysis of Riemann's "non-differentiable" function. *Invest. Math.* 105 (1991), 157-175.

[JAF1] JAFFARD, S. : *Construction et propriété des bases d'ondelettes. - Remarques sur la contrôlabilité exacte.* Thèse, Ecole Polytechnique, Palaiseau, 1989.

[JAF2] JAFFARD, S. : Exposants de Hölder en des points donnés et coefficients d'ondelettes. *C. R. Acad. Sci. Paris* 308 (1989), 79-81.

[JAF3] JAFFARD, S. : Pointwise smoothness, two-microlocalization and wavelet coefficients. *Publicacions Matemàtiques* 35 (1991), 155-168.

[JAF4] JAFFARD, S. : Multifractal formalism for functions. Part I : Results valid for functions. Preprint, CERMA, ENPC, La Courtine, 1993.

[JAF5] JAFFARD, S. : The spectrum of singularities of Riemann's function. Preprint, 1994.

[JAFM] JAFFARD, S. & MEYER, Y. : Pointwise behaviour of functions. Preprint, CERMA, ENPC, La Courtine, 1993.

[JIA] JIA, R. Q. & MICCHELLI, C. A. : Using the refinement equation for the construction of pre-wavelets. V : Extensibility of trigonometric polynomials. *Computing* 48 (1992), 61-72.

[JON] JONES, L. K. : On a conjecture of Huber concerning the convergence of projection pursuit regression. *Ann. Statistics* 15 (1987), 880-882.

[JOU] JOUINI, A. & LEMARIE-RIEUSSET, P. G. : Analyses multi-résolutions bi-orthogonales sur l'intervalle et applications. *Ann. I.H.P. (Analyse non-linéaire)* 10 (1993), 453-476.

[LAM] LAM, T. Y. : *Serre's conjecture.* Lecture Notes in Math. N° 635. Berlin, Springer-Verlag, 1978.

[LEM1] LEMARIE, P. G. : Ondelettes à localisation exponentielle. *J. Math. Pures & Appl.* 67 (1988), 227-236.

[LEM2] LEMARIE, P. G. : Bases d'ondelettes sur les groupes de Lie stratifiés. *Bull. Soc. Math. France* 117 (1989), 211-232.

[LEM3] LEMARIE, P. G. : Fonctions à support compact dans les analyses multi-résolutions. *Revista Mat. Iberoamer.* 7 (1991), 157-182.

[LEM4] LEMARIE-RIEUSSET, P. G. : Analyses multi-résolutions non orthogonales, commutation entre projecteurs et dérivation et ondelettes vecteurs à divergence nulle. *Revista Mat. Iberoamer.* 8 (1992), 221-237.

[LEM5] LEMARIE-RIEUSSET, P. G. : Sur l'existence des analyses multi-résolutions en théorie des ondelettes. *Revista Mat. Iberoamer.* 8 (1992), 457-474.

[LEM6] LEMARIE-RIEUSSET, P. G. : Ondelettes généralisées et fonctions d'échelle à support compact. *Revista Mat. Iberoamer.* 9 (1993), 333-371.

[LEM7] LEMARIE-RIEUSSET, P. G. : Une remarque sur les propriétés multi-résolutions des fonctions splines. *C. R. Acad. Sci. Paris* 317 (1993), 1115-1117.

[LEM8] LEMARIE-RIEUSSET, P. G. : Ondelettes et poids de Muckenhoupt. *Studia Math.* 108 (1994), 127-147.

[LEM9] LEMARIE-RIEUSSET, P. G. : Projecteurs invariants, matrices de dilatation, ondelettes et analyses multi-résolutions. *Revista Mat. Iberoamer.* 10 (1994), 283-347.

[LEM10] LEMARIE-RIEUSSET, P. G. : Distributions dont tous les coefficients d'ondelettes sont nuls. *C. R. Acad. Sci. Paris* 318 (1994), 1083-1.

[LEMA1] LEMARIE, P. G. & MALGOUYRES, G. : Ondelettes sans peine. Unpublished manuscript, 1989.

[LEMA2] LEMARIE-RIEUSSET, P. G. & MALGOUYRES, G. : Support des fonctions de base dans une analyse multi-résolution. *C. R. Acad. Sci. Paris* 313 (1991), 377-380.

[LEME] LEMARIE, P. G. & MEYER, Y. : Ondelettes et bases hilbertiennes. *Revista Mat. Iberoamer.* 2 (1986), 1-18.

[LOW] LOW, F. : Complete set of wave packets, in *A passion for physics-Essays in honor of Geoffrey Chew.* Singapore, World Scientific, 1985, 17-22.

[MADA] MADAY, Y., PERRIER, V. & RAVEL, J. C. : Adaptativité dynamique sur les bases d'ondelettes pour l'approximation d'équations aux dérivées partielles. *C. R. Acad. Sci. Paris* 312 (1991), 405-410.

[MADY] MADYCH, W. R. : Polyharmonic splines, multiscale analysis and entire functions, in *Multivariate approximation and interpolation* (Duisburg 1989), Basel, Birkhäuser, 1990, 205-216.

[MAG] MALGOUYRES, G. : Ondelettes à support compact et analyse multi-résolution sur l'intervalle, in *Progress in wavelet analysis and applications*, Y. Meyer and S. Roques eds. Gif/Yvette, Frontières, 1993.

[MALL1] MALLAT, S. : Multiresolution approximation and wavelets. *Trans. Amer. Math. Soc.* 315 (1989), 69-88.

[MALL2] MALLAT, S. & ZHANG, Z. : Matching pursuit within time-frequency dictionaries. *IEEE Trans. on Signal Processing* 41 (1993), 3397-3415.

[MALV] MALVAR, H. S. : Lapped transforms for efficient transforms/subband coding. *IEEE Trans. ASSP* 38 (1990), 969-978.

[MAR] MARR, D. : *Vision, a computational investigation into the human representation and processing of visual information.* New York, W. H. Freeman and Co., 1982.

[MEY1] MEYER, Y. : Principe d'incertitude, des bases hilbertiennes et algèbres d'opérateurs. *Sém. Bourbaki,* n° 662 (1986), 209-223.

[MEY2] MEYER, Y. : *Ondelettes et opérateurs. I : ondelettes.* Paris, Hermann, 1990. (English edition : Cambridge Univ. Press, 1992).

[MEY3] MEYER, Y. : *Ondelettes et opérateurs. II : opérateurs de Calderón-Zygmund.* Paris, Hermann, 1990.

[MEY4] COIFMAN, R. R. & MEYER, Y. : *Ondelettes et opérateurs. III : opérateurs multilinéaires.* Paris, Hermann, 1990.

[MEY5] MEYER, Y. : Ondelettes sur l'intervalle. *Revista Mat. Iberoamer.* (1991), 115-134.

[MEY6] MEYER, Y. : *Wavelets and applications, proceedings of the international conference,* Marseille (France), May 1989. Paris, Masson, 1992.

[MEY7] MEYER, Y. : *Ondelettes : algorithmes et applications.* Paris, Armand Colin, 1992. (English translation : *Wavelets : algorithms and applications*, SIAM Press, Philadelphia, 1993).

[MEY8] MEYER, Y. (reviewer) : An introduction to wavelets, by Charles K. Chui. *Ten lectures on wavelets,* by I. Daubechies. *Bull. Amer. Math. Soc.* 28 (1993), 350-360.

[MUR] MURENZI, R. : *Ondelettes multidimensionnelles et applications à l'analyse d'images.* Thèse, Univ. Catholique de Louvain-la-Neuve, 1990.

[RIE] RIEMENSCHNEIDER, S. D. & SHEN, Z. W. : Box splines, cardinal series and wavelets, in *Approximation theory and functional analysis,* C. K. Chui ed. Boston, Academic Press, 1991, 133-150.

[RIO] RIOUL, O. : *Ondelettes régulières : application à la compression d'images fixes.* Thèse, CNET, 1993.

[RVA] RVACHEV, V. A. : Compactly supported solution of functional-differential equations and their applications. *Russian Math. Surveys* 45 (1990), 87-120.

[SCH] SCHOENBERG, I. J. : *Cardinal spline interpolation,* CBMS, vol. 12, SIAM, Philadelphia, 1973.

[SMI] SMITH, M. J. T. & BARNWELL III, T. P. : Exact reconstruction techniques for tree-structured subband coders. *IEEE Trans. ASSP* 34 (1986), 434-441.

[STE1] STEIN, E. M. : *Singular integrals and differentiability properties of functions.* Princeton, Princeton Univ. Press, 1970.

[STE2] STEIN, E. M. *Harmonic analysis : real-variable methods, orthogonality and oscillatory integrals.* Princeton, Princeton Univ. Press, 1993.

[STEW] STEIN, E. M. & WEISS, G. : *Introduction to Fourier analysis on Euclidean spaces.* Princeton, Princeton Univ. Press, 1971.

[STR] STRÖMBERG, J. O. : A modified Haar system and higher order spline systems, in *Conference in harmonic analysis in honor of A. Zygmund II.* Wadworth Math. Series, 1983, 475-493.

[SWE] SWELDENS, W. & PIESSENS, R. : Calculation of the wavelet decomposition using quadrature formulae. *CWI Quaterly* 1 (1992), 33-52.

[TCH] TCHAMITCHIAN, P. & TORRESANI, B. : Ridge and skeleton extraction from the wavelet transform, in *Wavelets and their applications*, M. B. Ruskai et al. eds. Boston, Jones and Bartlett, 1992, 123-151.

[TEN] TENENBAUM, E., LATTO, A. & RESNIKOFF, H. L. : The evaluation of connection coefficients of compactly supported wavelets, in *Proceedings of the French-USA workshop on Wavelets and turbulence*, Y. Maday ed., Princeton, Princeton Univ. Press, 1992.

[VIL] VILLE, J. : *Théorie et applications de la notion de signal analytique*. C et T, 2ème A. N° 1, 1948.

[VIM1] VILLEMOES, L. F. : Wavelet analysis of two-scale difference equations. *SIAM J. Math. Anal.* (to appear).

[VIM2] VILLEMOES, L. F. : Continuity of nonseparable quincunx wavelets. *Applied and Comp. Harm. Anal.* 1 (1994), 180-187.

[VOL] VOLKNER, H. : On the regularity of wavelets. *IEEE Trans. on Inf. Theory* 38 (1992), 29-42.

[WIC] WICKERHAUSER, M. L. : *Adapted wavelet analysis from theory to software*, A. K. Peters ltd, 1994. (A list of errata is available on the ftp.site WUARCHIVE, with access path /doc/techreports/wustl.edu/math/papers/awafts-errata.tex).

Index

The numbers refer to chapters and sections.