N° 23

Complexe cotangent ; application à la théorie des déformations

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1971

PUBLICATIONS MATHEMATIQUES D'ORSAY

ORSAY
Série A, n°
N° d'ordre :

749

THÈSES

PRESENTEES,

AU CENTRE D'ORSAY
UNIVERSITÉ PARIS-SUD

POUR OBTENIR

LE grade DE DOCTEUR ès-sciences

PAR

ILLUSIE Luc

" Complexe cotangent; application à la théorie des déformations".

1re THESE. —

2e THESE. — Propositions données par la Faculté.

Soutenues le 21 Mai 1971 **devant la Commission d'examen**

MM. CARTAN Président.

GROTHENDIECK
DEMAZURE
SERRE
Examinateurs.

PREFACE

Notre thèse principale fait l'objet d'un ouvrage publié en deux volumes (659 pages) dans la collection des Springer Lecture Notes ([12], [13]).

Le texte qui suit reproduit, avec l'aimable autorisation de Springer-Verlag, un exposé extrait des comptes rendus d'un colloque de Géométrie Algébrique qui s'est tenu à Halifax (Canada) en janvier 1971 (*Toposes*, *Algebraic Geometry and Logic*, Lecture Notes in Mathematics *274*, 1972). Cet exposé constitue un résumé assez complet des résultats qui forment le coeur de l'ouvrage cité supra.

Paris, octobre 1972

COTANGENT COMPLEX AND DEFORMATIONS OF TORSORS AND GROUP SCHEMES

by Luc Illusie (*)

In this exposé we outline some applications of the cotangent complex to obstruction problems concerning the first order deformations of flat group schemes locally of finite presentation and of torsors under such groups. An enlarged version with detailed proofs will appear in [13]. The results presented here were conjectured by Grothendieck in 1968 and 1969. Those dealing with deformations of flat commutative group schemes play an essential role in recent work of his and W. Messing on Barsotti-Tate groups ([11], [16]).

In § 1 we recall some basic facts about cotangent complex theory. For details the reader is referred to [12]. The main result concerns the obstruction to the first order deformation of a flat ringed topos. Thanks to a general method of deformation of diagrams, which owes much to Deligne's cohomological descent theory (SGA 4 VI), we can apply this result to the problems mentioned above. The diagram we use to handle the deformations of flat commutative group schemes with ring of operators was suggested to us by L. Breen's recent work on the structure of $\mathrm{Ext}^i(G_a, G_a)$ ([5]).

1. Review of cotangent complex theory.

1.1. To each map of schemes (more generally, of ringed topoi)

$$f : X \longrightarrow Y$$

is associated a chain complex of flat $\underline{O}_X$-Modules, denoted by

$$L_{X/Y}$$

and called the cotangent complex of f (or X over Y), which generalizes in a natural way the complex associated by André [1] and Quillen [18] to

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(*) part of this research was done while the author was supported by M. I. T.

a map of rings. It is augmented towards $\Omega^1_{X/Y}$, the sheaf of Kähler differentials of f, the augmentation **establishing an isomorphism**

$$H_o(L_{X/Y}) \xrightarrow{\sim} \Omega^1_{X/Y}$$

1.2. Suppose f is a morphism of schemes. Then the homology sheaves $H_i(L_{X/Y})$ are quasi-coherent $\underline{O}_X$-Modules. If f is smooth, $\Omega^1_{X/Y}$ is locally free of finite rank and the augmentation $L_{X/Y} \longrightarrow \Omega^1_{X/Y}$ is a quasi-isomorphism (the converse being true when f is locally of finite presentation). If f is a locally complete intersection map in the sense of Berthelot (SGA 6 VIII 1), then, in the terminology of (SGA 6 I 4.8), $L_{X/Y}$ is of perfect amplitude $\subset [-1,0]$, which means that $L_{X/Y}$ is locally isomorphic, in the derived category D(X), to a complex of locally free sheaves of finite rank concentrated in degrees -1 and 0 .(The converse (due to Quillen) is true provided that Y is locally noetherian and f locally of finite type.) Finally, suppose f admits a factorization

$$X \xrightarrow{i} X' \xrightarrow{f'} Y \quad ,$$

where f' is formally smooth and i is a closed immersion defined by an Ideal I . Then, in D(X) there is a canonical isomorphism

$$t_{[-1}(L_{X/Y}) \simeq (0 \to I/I^2 \xrightarrow{d} i^*(\Omega^1_{X'/Y}) \to 0)$$

where $i^*(\Omega^1_{X'/Y})$ is placed in degree 0, d is induced by the universal derivative $d_{X'/Y} : \underline{O}_{X'} \longrightarrow \Omega^1_{X'/Y}$, and $t_{[n}(L)$, for a complex L, denotes the complex deduced from L by killing $H^i(L)$ for $i < n$, namely $(0 \to L^n/B^n \to L^{n+1} \to L^{n+2} \to \dots)$.

1.3. $L_{X/Y}$ depends functorially on f in the same way as $\Omega^1_{X/Y}$. This means that each (essentially) commutative square of ringed topoi

(1.3.1)
$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ f \downarrow & & \downarrow \\ Y & \longleftarrow & Y' \end{array}$$

gives rise to a map of complexes

(1.3.2)
$$g^* L_{X/Y} \longrightarrow L_{X'/Y'}$$

these maps satisfying a certain coherence condition relative to the composition of squares. Note that $g^*L_{X/Y} = Lg^*L_{X/Y}$ in $D(X)$ since the components of $L_{X/Y}$ are flat.

Proposition 1.3.3 (base change). *Suppose* (1.3.1) *is defined by a cartesian square of schemes such that* $\underline{Tor}_i^Y(\underline{O}_X,\underline{O}_{Y'}) = 0$ for $i > 0$ (*which is the case for example if* X *or* Y' *is flat over* Y). *Then* (1.3.2) *is a quasi-isomorphism*.

1.4. Let

(1.4.1) $$X \xrightarrow{f} Y \longrightarrow Z$$

be maps of ringed topoi. Then there is a canonical, exact triangle in $D(X)$

(1.4.2) $$\begin{array}{ccc} & L_{X/Y} & \\ \swarrow & & \nwarrow \\ f^*L_{Y/Z} & \longrightarrow & L_{X/Z} \end{array}$$

where the maps of degree 0 are those defined by functoriality of the cotangent complex. It is called the *transitivity triangle*. It depends functorially on (1.4.1). The map of degree 1 in (1.4.2) is sometimes denoted by $K(X/Y/Z)$, and called the *Kodaira-Spencer map* (or *class*). When (1.4.1) is defined by smooth morphisms of schemes, $K(X/Y/Z)$ coincides with the usual class in $H^1(X, T_{X/Y} \otimes f^*\Omega^1_{Y/Z})$, where $T_{X/Y}$ is the tangent sheaf of f (dual of $\Omega^1_{X/Y}$).

1.5. Let $f : X \to Y$ be a map of ringed topoi, and let M be an $\underline{O}_X$-Module. By a Y-*extension of* X *by* M we mean a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ f\downarrow & \swarrow & \\ Y & & \end{array}$$

where i is an equivalence on the underlying topoi and on the rings induces a surjective map $\underline{O}_{X'} \to \underline{O}_X$ whose kernel is of square zero and isomorphic to M as an $\underline{O}_X$-Module. Maps of Y-extensions are defined in the obvious way. Note that if f is defined by a map of schemes and M is quasi-coherent, then the above factorization comes from a factorization in the category of schemes.

The interest of the cotangent complex in the theory of deformations comes from :

Theorem 1.5.1. *There exists a canonical, functorial isomorphism between the set of isomorphism classes of* Y-*extensions of* X *by* M, *equipped with the group structure defined by the usual addition law on extensions, and the group* $\mathrm{Ext}^1(L_{X/Y},M)$. *Moreover, the group of automorphisms of any fixed* Y-*extension* X' *of* X *by* M *is canonically isomorphic to* $\mathrm{Ext}^0(L_{X/Y},M)$.

(The last part is, of course, a trivial consequence of the isomorphism $H_o(L_{X/Y}) \xrightarrow{\sim} \Omega^1_{X/Y}$ (1.1)).

1.6. Consider a commutative diagram of ringed topoi

(1.6.1)
$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{j} & Y' \\ \downarrow & \swarrow & \\ S & & \end{array}$$

where j is an S-extension of Y by some $\underline{O}_Y$-Module J and i is a Y'-extension of X by some $\underline{O}_X$-Module I . Such a diagram defines a map of $\underline{O}_X$-Modules

$$(*) \qquad f^*J \longrightarrow I$$

We shall call f' a *deformation of* f *over* Y' if (*) is an isomorphism. If f is flat, then any deformation f' is automatically flat, as a result of the well-known flatness criterion. The key result in deformation theory is the following, which is a formal consequence of (1.4) and (1.5.1) :

Theorem 1.7. *Let*

$$X \xrightarrow{f} Y \longrightarrow S$$

be maps of ringed topoi, and let $j : Y \longrightarrow Y'$ *be an* S-*extension of* Y *by an* $\underline{O}_Y$-*Module* J . *Then* :

(i) There exists an obstruction

$$\omega(f,j) \in \mathrm{Ext}^2(L_{X/Y}, f^*J)$$

whose vanishing is necessary and sufficient for the existence of a deformation f' of f over Y'

(ii) When $\omega(f,j) = 0$, the set of isomorphism classes of deformations f' is an affine space under $\mathrm{Ext}^1(L_{X/Y}, f^*J)$ and the group of automorphisms of a fixed deformation is canonically isomorphic to $\mathrm{Ext}^0(L_{X/Y}, f^*J)$.

(iii) The obstruction $\omega(f,j)$ can be written as a Yoneda cup-product

$$\omega(f,j) = (f^*e(j))K(X/Y/S)$$

where $K(X/Y/S) \in \mathrm{Ext}^1(L_{X/Y}, f^*L_{Y/S})$ is the Kodaira-Spencer class (1.4) and $e(j) \in \mathrm{Ext}^1(L_{Y/S}, J)$ is the class defined by j (1.5.1).

2. Equivariant deformations.

2.1. Fix a scheme S and a group scheme G over S. Let X, Y be G-schemes, and let $f : X \longrightarrow Y$ be a G-equivariant map. We shall assume G or f to be flat. To avoid technical complications, we shall also assume f to be a complete intersection map in the sense of Berthelot (SGA 6 VIII 1) (recall this implies (1.2) that $L_{X/Y}$ is a perfect complex). Taking into account the action of G, and using the base change property (1.3.3), we can define a complex of G-$\underline{O}_X$-Modules, or more precisely an object

(2.1.1) $$L^G_{X/Y} \in \mathrm{ob}\ D^b(BG_{/X})$$

unique up to unique isomorphism, in such a way that the underlying complex of $\underline{O}_X$-Modules is canonically isomorphic to $L_{X/Y}$ in $D(X)$, and $H_0(L^G_{X/Y}) \xrightarrow{\sim} \Omega^1_{X/Y}$ as a G-$\underline{O}_X$-Module [1]. Here BG means the classifying topos (SGA 4 IV 2.4) of G considered as a sheaf of groups for the fpqc topology on the category of

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[1] These conditions do not characterize $L^G_{X/Y}$!

all schemes over S, and $BG_{/X}$ is the topos of objects of BG over X ; BG (hence $BG_{/X}$) is equipped with the canonical ring defined by the structural rings of schemes over S. When X admits an equivariant closed embedding $i : X \longrightarrow X'$ into a smooth G-scheme X' over Y, $L^G_{X/Y}$ can be taken to be the complex of G-$\underline{O}_X$-Modules (cf. (1.2))

$$0 \longrightarrow I/I^2 \xrightarrow{\quad d \quad} i^*(\Omega^1_{X'/Y}) \longrightarrow 0$$

where I is the Ideal of i . In the general case, the definition of (2.1.1) is a little involved ; some indications are given below. Note that if X, Y are trivial G-schemes, then we have $L^G_{X/Y} \simeq L_{X/Y}$ where $L_{X/Y}$ is viewed as a complex of trivial G-$\underline{O}_X$-Modules.

The equivariant cotangent complex $L^G_{X/Y}$ enjoys functorial properties analogous to those satisfied by $L_{X/Y}$. In particular, a composition of G-maps gives rise to an "equivariant transitivity triangle", hence to an "equivariant Kodaira-Spencer class". Details will be omitted.

2.2. Consider a commutative diagram of G-schemes over S of the form (1.6.1) where all maps are G-maps and i (resp. j) is a closed immersion defined by an Ideal I (resp. J) of square zero. So I (resp. J) is a G-$\underline{O}_X$- (resp. G-$\underline{O}_Y$-) Module and the canonical map $f^*J \rightarrow I$ is a G-map. We shall call f' an equivariant deformation of f over Y' if the underlying map of schemes is a deformation of f in the sense of (1.6), or equivalently if the canonical map $f^*J \rightarrow I$ is a G-isomorphism.

The following result is a consequence of (1.7). A sketch of proof will be given below.

Proposition 2.3. *Suppose G and f : X $\longrightarrow$ Y satisfy the hypotheses of (2.1). Fix a closed equivariant embedding j : Y $\rightarrow$ Y' of G-schemes over S, the Ideal J of j being of square zero. Then there is an obstruction*

$$\omega(G,f,j) \in \mathrm{Ext}^2_G(L^G_{X/Y}, f^*J)$$

whose vanishing is necessary and sufficient for the existence of an equivariant deformation f' of f over Y'. When $\omega(G,f,j) = 0$, the set of isomorphism classes of equivariant deformations f' is an affine space under $\mathrm{Ext}^1_G(L^G_{X/Y}, f^*J)$, and the group of automorphisms of a fixed f' is canonically isomorphic to $\mathrm{Ext}^0_G(L^G_{X/Y}, f^*J)$. (Here the notation $\mathrm{Ext}^i_G(L,M)$ for $L, M \in \mathrm{ob}\, D(BG_{/X})$ is short for $\mathrm{Ext}^i(BG_{/X}; L,M)$.)

It is also possible to write $\omega(G,f,j)$ as a cup-product with an equivariant Kodaira-Spencer class. A precise statement will be given in the particular case of torsors, which we are now going to examine.

2.4. From now on we shall assume G to be flat and locally of finite presentation. By the well-known theorem giving the local structure of algebraic groups this implies that $G \to S$ is a (locally) complete intersection map (in the sense of Berthelot [1]). Let $f : X \to Y$ be a G-map of G-schemes over S, the action of G on Y being trivial. Denote by G_Y the group scheme over Y induced by G (i.e. $G_Y = G \times_S Y$). Recall that X is said to be a principal homogeneous space (or torsor [2]) under G_Y if the following conditions are satisfied :

(i) the map $G_Y \times_Y X \longrightarrow X \times_Y X$, $(g,x) \mapsto (gx,x)$ is an isomorphism

(ii) $f : X \to Y$ is faithfully flat and quasi-compact.

These also amount to saying that after some fpqc base change $Y' \to Y$ (for example, f) X becomes isomorphic to G_Y, acting on itself by left multiplication. They imply f is a complete intersection (because the latter property is local for the fpqc topology (SGA 6 VIII 1.6)). So, by (2.1) the equivariant cotangent complex $L^G_{X/Y}$ is defined. Denote by $f^G : BG_{/X} \to Y$ the canonical map (f^G_* is "taking the global sections invariant under G"). It can be shown by descent that $Rf^G_*(L^G_{X/Y})$ is a perfect complex, of perfect amplitude $\subset [-1,0]$ (SGA 6 I 4.8), and that the adjunction map

(1) which here is also the sense of (EGA IV 19.3.6) because of (SGA 6 VIII 1.4).

(2) from the French "torseur".

$$Lf^{G*}Rf_*^G(L_{X/Y}^G) \longrightarrow L_{X/Y}^G$$

(hence, too, the adjunction map $Lf^* Rf_*^G(L_{X/Y}^G) \longrightarrow L_{X/Y}$) is an isomorphism.

Definition 2.5. Let Y be a scheme over S and let $f : X \to Y$ be a torsor under G_Y. The complex $Rf_*^G(L_{X/Y}^G) \in \text{ob } D^b(Y)$ will be denoted by $\ell_{X/Y}$ and called the co-Lie complex of X over Y.

2.5.1. The formation of the co-Lie complex commutes with any base change $Y' \to Y$. If X is trivial over Y, i.e. f admits a section $s : Y \to X$, then we have

$$\ell_{X/Y} \simeq Ls^*(L_{X/Y}) \quad .$$

In particular, take $Y = S$ and X to be G acting on itself by left multiplication. Then we have

$$\ell_{G/S} \simeq Le^*(L_{G/S}).$$

where $e : S \to G$ is the unit section. The complex $\ell_{G/S}$, often denoted simply ℓ_G, is called the co-Lie complex of G, and will be discussed later. It was first introduced by Mazur-Roberts [15] for a finite G. When G is smooth, it coincides with the sheaf of invariant differential forms ω_G, dual to the Lie algebra of G.

Theorem 2.6. In the situation of (2.5), let $j : Y \hookrightarrow Y'$ be an S-extension of Y by a quasi-coherent O_Y-Module J. Then there exists an obstruction

$$\omega(G,f,j) \in H^2(Y, \ell_{X/Y}^{\vee} \overset{L}{\otimes} J) \qquad (1)$$

whose vanishing is necessary and sufficient for the existence of a torsor $f' : X' \to Y'$ under $G_{Y'}$, such that $X' \times_{Y'} Y$ be isomorphic to X (as a torsor under G_Y). When $\omega(G,f,j) = 0$, the set of isomorphism classes of such torsors X' is an affine space under $H^1(Y, \ell_{X/Y}^{\vee} \overset{L}{\otimes} J)$, and the group of automorphisms of a solution is canonically isomorphic to $H^0(Y, \ell_{X/Y}^{\vee} \overset{L}{\otimes} J)$.

(1) For $L \in \text{ob } D(Y)$ we set $L^{\vee} = R\underline{Hom}(L, O_Y)$.

In effect, it is easily seen that a torsor X' under $G_{Y'}$ inducing X on Y is the same as an equivariant deformation of f over Y' (2.2). Therefore the theorem follows from (2.3), since by descent we have

$$\mathrm{Ext}^i_G(L^G_{X/Y}, f^* J) \simeq \mathrm{Ext}^i(\chi_{X/Y}, J)$$
$$\simeq H^i(Y, \chi^\vee_{X/Y} \overset{L}{\otimes} J) \qquad (\chi_{X/Y} \text{ being perfect}).$$

Remark 2.6.1. Suppose G is smooth. Then $\chi^\vee_{X/Y}$ is nothing but the Lie Algebra t_{G_Y} of G_Y twisted by the torsor X via the adjoint action of G_Y :

$$\chi^\vee_{X/Y} \simeq t_{X/Y} \overset{dfn}{=} X \overset{G_Y}{\times} t_{G_Y} ,$$

and (2.6) is easy to prove directly. Observe that equivariant deformations of X locally exist and that any two deformations are always locally isomorphic, the sheaf of automorphisms of a given deformation being identified with $t_{X/Y}$. Hence the obstruction in H^2 is obtained by a classical cocycle calculation best expressed in Giraud's language of "gerbes" [9].

2.7. It is possible by descent to construct from the Kodaira-Spencer class (1.4) a canonical class, called the Atiyah class of X,

(2.7.1) $\quad at(X/Y/S) \in \mathrm{Ext}^1(\chi_{X/Y}, tL_{Y/S})$

where $tL_{Y/S}$ stands for the pro-object of truncated complexes "$\varprojlim_n$" $t_{[n} L_{Y/S}$ (1). As in (1.7), the obstruction $\omega(G,f,j)$ of (2.6) can be written as a cup-product

(2.7.2) $\quad \omega(G,f,j) = e(j)at(X/Y/S)$

where $e(j) \in \mathrm{Ext}^1(L_{Y/S}, J) = \mathrm{Ext}^1(tL_{Y/S}, J)$ is the class of the Y-extension j. When both G and Y are smooth, $at(X/Y/S)$ coincides with the class in $H^1(Y, t_{X/Y} \otimes \Omega^1_{Y/S})$ constructed by Atiyah in [3].

2.8. Sketch of proof of (2.3). a) Recall that to any G-scheme X over S is associated in a functorial way a simplicial scheme over S, called its nerve :

$$\mathrm{Ner}(G,X) = (\ldots G^n \times X \overset{\to}{\underset{\to}{\to}} \ \ldots G \times X \rightrightarrows X)$$

(1) for the notation $t_{[n}$, see (1.2).

(products are taken over S, faces and degeneracies are given by the standard formulas :

$$d_o(g_1,\dots,g_n,x) = (g_2,\dots,g_n,g_1x)$$
$$d_1(g_1,\dots,g_n,x) = (g_1g_2,g_3,\dots,g_n,x) \quad ,$$

etc.). So, in the situation of (2.3), we have a commutative diagram of simplicial schemes over S :

$$\begin{array}{ccc} \mathrm{Ner}(G,X) & & \\ \downarrow & & \\ \mathrm{Ner}(G,Y) & \longrightarrow & \mathrm{Ner}(G,Y') \\ \downarrow & \swarrow & \\ \mathrm{Ner}(G) & & \end{array} \qquad (2.8.1)$$

where Ner(G) is short for Ner(G,S). Thanks to the hypotheses and the flatness criterion, it is easily checked that an equivariant deformation of X over Y' is the same as a deformation of Ner(G,X) over Ner(G,Y') as a simplicial scheme, i.e. a commutative square of simplicial schemes

$$\begin{array}{ccc} \mathrm{Ner}(G,X) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \mathrm{Ner}(G,Y) & \longrightarrow & \mathrm{Ner}(G,Y') \end{array}$$

such that, for each $n \in \mathbb{N}$, $Z_n \longrightarrow G^n \times Y'$ is a deformation of $G^n \times X \longrightarrow G^n \times Y$ (as a map of schemes).

b) Recall that any diagram D ($i \mapsto D_i$) of ringed spaces defines (SGA 4 VI) a ringed topos still denoted by D and called the <u>total topos</u> of D . A Module on D consists of a family of Modules E_i on the D_i together with transition maps $f^*E_j \longrightarrow E_i$ satisfying certain compatibility conditions. So (2.8.1) defines a diagram of ringed topoi, in which the triangle is a Ner(G)-extension of Ner(G,Y) by an Idéal of square zero, which will still be denoted by J for simplicity. Moreover, a deformation Z as above is the same as a deformation of Ner(G,X) $\to$ Ner(G,Y) over Ner(G,Y') as a map of ringed topoi. Therefore we can apply (1.7) and the problem boils down to identifying the groups $\mathrm{Ext}^i(L_{\mathrm{Ner}(G,X)/\mathrm{Ner}(G,Y)}, f^*J)$ with the groups $\mathrm{Ext}^i_G(L^G_{X/Y}, f^*J)$ of (2.3).

c) First of all we have to define $L^G_{X/Y}$. Let us indicate very briefly how to do this. By an analogue of the nerve construction, we get a map of ringed topoi $\mathrm{Ner}(G,X) \longrightarrow BG_{/X}$, which, by descent, induces a fully faithful functor

$$(*) \qquad D^b(BG_{/X})_{qcoh} \longrightarrow D^b(\mathrm{Ner}(G,X))$$

where $D^b(BG_{/X})_{qcoh}$ is the full sub-category of $D(BG_{/X})$ defined by those complexes whose cohomology is bounded and quasi-coherent. Now, thanks to the hypotheses and the base changed property (1.3.3), it can be shown that $L_{\mathrm{Ner}(G,X)/\mathrm{Ner}(G,Y)}$ is isomorphic to the image under (*) of an object $L^G_{X/Y}$ of $D^b(BG_{/X})_{qcoh}$ (unique up to unique isomorphism). The identification desired at the end of b) follows from the fact that (*) is fully faithful. **This completes the** proof of (2.3). The Atiyah class (2.7.1) and the formula (2.7.2) are easily deduced from the Kodaira-Spencer class of the vertical composition in (2.8.1) and from (1.7 (iii)).

Remark 2.9. There is an alternate approach to the results of this section, which is based on Deligne's theory of Picard stacks (SGA 4 XVIII). It also yields interesting refinements, for example the following, which will be appreciated by the specialist : in the situation of (2.6), assume $\omega(G,f,j) = 0$: then the Picard stack of equivariant deformations of X over Y' is represented by the complex $j_*(\mathcal{X}^{\vee}_{X/Y} \overset{L}{\otimes} J)[1]$.

3. Deformations of non-commutative, flat group schemes.

In this section, we fix a scheme S and a flat, locally of finite presentation group scheme G over S.

3.1. Using the action of G on itself by left multiplication, we have defined in (2.5.1) the co-Lie complex of G, $\mathcal{X}_G \in \mathrm{ob}\, D(S)$, from which we can reconstruct $L_{G/S}$ by means of the canonical isomorphism (2.4.1)

$$Lf^*(\mathcal{X}_G) \xrightarrow{\sim} L_{G/S}$$

where $f : G \to S$ is the projection. We could as well have used the action of G on itself by right multiplication to define analogously an object χ'_G of D(S), but χ_G and χ'_G would have been canonically isomorphic since both have to be canonically isomorphic to $Le^*(L_{G/S})$ where $e : S \to G$ is the unit section. Yet, if we denote by G^o the opposite group and let $G \times G^o$ [1] act on G by $(g,h)x = gxh$, we can define a finer object than χ_G, namely

(3.1.1) $$\underline{\underline{\chi}}_G \overset{dfn}{=} Rf_*^{G^o}(L_{G/S}^{GxG^o}) \in \text{ob } D(BG)$$

where $L_{G/S}^{GxG^o} \in \text{ob } D(B(GxG^o)_{/G})$ is the equivariant cotangent complex of the GxG^o-scheme G (2.1.1) and $f^{G^o} : B(GxG^o)_{/G} \longrightarrow BG$ is the canonical map defined by f ($f_*^{G^o}$ is "taking the sheaf of global sections invariant under G^o "). It follows easily from the definition that the object of D(S) deduced from $\underline{\underline{\chi}}_G$ by forgetting the action of G is canonically isomorphic to χ_G. When G is smooth over S, $\underline{\underline{\chi}}_G$ is nothing but ω_G, the sheaf of right invariant differential forms of degree 1, equipped with the adjoint action of G.

From $\underline{\underline{\chi}}_G$ we can reconstruct the co-Lie complex of any G-torsor. In effect, let Y be a scheme over S and X be a torsor on Y under G_Y. By the classifying property of BG, X defines a map $p : Y \to BG$ such that $p^*(PG) \simeq X$, where PG is the universal torsor on BG. Then we have

(3.1.2) $$\chi_{X/Y} \simeq p^*\underline{\underline{\chi}}_G ,$$

which generalizes the formula of (2.6.1).

3.2. Suppose now S is a T-scheme and we are given a T-extension S' of S by a quasi-coherent $\underline{O}_S$-Module I, $i : S \to S'$. We want to pinpoint the obstruction to the existence of a deformation of G over S', by which we mean a flat group scheme G' over S' together with an isomorphism $G' \times_{S'} S \xrightarrow{\sim} G$ (note that if G is commutative, G' need not be commutative). Before stating the main result, we need a notation.

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[1] unless otherwise stated, products are taken over S.

3.3. Let $u : X \longrightarrow Y$ be a map of ringed topoi, which for simplicity we shall assume to be flat. Denote by Ab the category of abelian groups. In ([12] III 4.10) we define an exact functor

$$R\Gamma(Y/X,-) : D^{+}(Y) \longrightarrow D^{+}(Ab)$$

with the property that, for $E \in \mathrm{ob}\, D^{+}(Y)$, there exists a canonical, functorial, exact triangle

$$(3.3.1)\qquad \begin{array}{ccc} & R\Gamma(X,u^{*}E) & \\ \swarrow_{+1} & & \nwarrow^{*} \\ R\Gamma(Y/X,E) & \longrightarrow & R\Gamma(Y,E) \end{array}$$

where $*$ is the restriction map. The cohomology groups

$$H^{i}(Y/X,E) \overset{\mathrm{dfn}}{=} H^{i}R\Gamma(Y/X,E)$$

are called the relative cohomology groups of E mod. Y . They are related to the absolute cohomology groups by the exact sequence of cohomology of (3.3.1) :

$$\cdots \rightarrow H^{i}(Y/X,E) \longrightarrow H^{i}(Y,E) \longrightarrow H^{i}(X,u^{*}E) \longrightarrow H^{i+1}(Y/X,E) \rightarrow \cdots \quad .$$

We can now state :

Theorem 3.4. *In the situation of* (3.2), *there exists an obstruction*

$$\omega(G,i) \in H^{3}(BG/S,\underline{\ell}_{G}^{\vee} \overset{L}{\otimes} I) \qquad (^{1})$$

whose vanishing is necessary and sufficient for the existence of a deformation of G *into a flat group scheme* G' *over* S'. *When* $\omega(G,i) = 0$, *the set of isomorphism classes of deformations* G' *is an affine space under* $H^{2}(BG/S,\underline{\ell}_{G}^{\vee} \overset{L}{\otimes} I)$, *and the group of automorphisms of a given deformation is canonically isomorphic to* $H^{1}(BG/S,\underline{\ell}_{G}^{\vee} \overset{L}{\otimes} I)$. *Moreover, there exists a canonical class, depending only on* $G \rightarrow S \rightarrow T$,

$$c(G/S/T) \in H^{2}(BG/S,\underline{\ell}_{G}^{\vee} \overset{L}{\otimes} tL_{S/T}) \quad , (^{2})$$

whose cup-product with the class $e(i) \in \mathrm{Ext}^{1}(L_{S/T},I)$ *defined by the*

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(1) For $L \in \mathrm{ob}\, D(BG)$, $L^{\vee} \overset{\mathrm{dfn}}{=} \underline{R\mathrm{Hom}}(L,\underline{O})$. The relative cohomology groups are taken with respect to the unit section map $S \rightarrow BG$.

(2) As in (2.7), $tL_{S/T}$ stands for the pro-object "$\varprojlim$" $t_{[n}L_{S/T}$.

T-<u>extension</u> i <u>yields the obstruction</u> $\omega(G,i)$:

$$\omega(G,i) = e(i)c(G/S/T) \qquad .$$

3.5. The above result improves those of Demazure in (SGA 3 III) where, **aside from** the fact that only the smooth case was discussed, the problem of deforming G as a group scheme was not considered as a whole, but rather broken down into successive steps : (i) deform G as a scheme (ii) deform the multiplication $G \times G \to G$ (iii) render it associative. As Grothendieck pointed out, the partial obstructions encountered in (loc. cit.) turn out to be the images of $\omega(G,i)$ into the successive quotients of the filtration of $H^3(BG/S, \check{\ell}_{\underline{G}} \overset{L}{\otimes} I)$ arising from the "Moore spectral sequence" :

$$E_1^{pq} = \begin{cases} H^q(G^p, \check{\ell}_G \overset{L}{\otimes} I) & \text{if } p \geqslant 1 \\ 0 & \text{if } p \leqslant 0 \end{cases} \Longrightarrow H^*(BG/S, \check{\ell}_{\underline{G}} \overset{L}{\otimes} I)$$

A complete discussion will be found in [13].

3.6. <u>Sketch of proof of</u> (3.4). a) Let Y be a scheme. The functor $G \mapsto \mathrm{Ner}(G)$ (2.8 a)) from the category of group schemes over Y to the category of simplicial schemes over Y is fully faithful, and it is easy to see that its essential image consists of exactly those simplicial schemes X over Y which satisfy the following exactness conditions :

(i) $X_o \to Y$ is an isomorphism ;

(ii) for $n \geqslant 2$, the canonical map

$$X_n \to X_1 \times_{X_o} \cdots \times_{X_o} X_1 \qquad \text{(n factors)} \qquad ,$$

obtained by writing the interval $[0,n]$ as an amalgamated sum, is an isomorphism ;

(iii) the squares

$$\begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ {\scriptstyle d_2}\downarrow & & \downarrow{\scriptstyle d_1} \\ X_1 & \xrightarrow{d_1} & X_o \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ \downarrow{\scriptstyle d_o} & & \downarrow{\scriptstyle d_o} \\ X_1 & \xrightarrow{d_o} & X_o \end{array}$$

are cartesian. ((ii) expresses the fact that X comes from a category object,

(i) + (ii) that X comes from a monoïd, (ii) + (iii) that X comes from a groupoïd.)

b) Using a) and the flatness criterion, we see that a deformation of G into a flat group scheme over S' is essentially the same as a deformation of Ner(G) into a flat simplicial scheme over S' or, equivalently, a deformation over S' of the corresponding ringed topos (2.8 b)). Therefore we can again apply (1.7), and what remains is to identify the groups $\mathrm{Ext}^i(L_{\mathrm{Ner}(G)/S}, f^*I)$ for $0 \leq i \leq 2$. Now, by looking at the exact triangle of the composition $\mathrm{Ner}(G,G) \to \mathrm{Ner}(G) \to S$, it is not too hard to prove that there is a canonical isomorphism

$$\mathrm{Ext}^*(L_{\mathrm{Ner}(G)/S}, f^*I) \xrightarrow{\sim} H^{*+1}(BG/S, \underline{\ell}_G^\vee \overset{L}{\otimes} I)$$

(actually I could be replaced by any complex with bounded, quasi-coherent cohomology), and that concludes the proof.

4. Interlude on Lie and co-Lie complexes.

The group schemes considered in this section are assumed to be flat and locally of finite presentation over a fixed scheme S .

4.1. Notations. Let G be a group scheme over S . Recall that the co-Lie complex ℓ_G is of perfect amplitude $\subset [-1,0]$, hence has only two interesting cohomology sheaves, namely

$$\omega_G = H^0(\ell_G) \quad , \quad n_G = H^{-1}(\ell_G)$$

The dual of ℓ_G, i.e. $\ell_G^\vee = \underline{\mathrm{RHom}}(\ell_G, \underline{O}_S)$, is called the Lie complex of G . It is perfect, of perfect amplitude $\subset [0,1]$. Its two possibly non zero cohomology sheaves will be denoted by

$$t_G = H^0(\ell_G^\vee) \quad , \quad \nu_G = H^1(\ell_G^\vee) \quad .$$

Note we have

$$t_G = (\omega_G)^\vee \quad , \quad n_G = (\nu_G)^\vee$$

(where $-^\vee = \underline{\mathrm{Hom}}(-, \underline{O}_S)$). Therefore the basic invariants are ω_G , ν_G (and the class in $\mathrm{Ext}^2(\omega_G, n_G)$ (resp. $\mathrm{Ext}^2(\nu_G, t_G)$) defined by ℓ_G (resp. $\ell_G^\vee$)).

We shall first give two general methods for computing ℓ_G .

4.2. Suppose we are given a closed embedding of G into a smooth group scheme over S, $i : G \longrightarrow G'$, and denote by I the Ideal of i . Then, by (1.2) we have

$$L_{G/S} \simeq (0 \to I/I^2 \xrightarrow{d} i^* \Omega^1_{G'/S} \to 0)$$

hence, by (2.5.1) :

$$\chi_G \simeq Le^*(L_{G/S}) \simeq (0 \to e^*(I/I^2) \longrightarrow \omega_{G'} \to 0)$$

where $e : S \to G$ is the unit section. Moreover, as i is a homomorphism of groups, G acts naturally on the right hand side, and the complex of G-$\underline{O}_S$-Modules thus defined represents $\underline{\chi}_G$ in D(BG).

4.3. Suppose now we have an exact sequence of group schemes over S

(4.3.1) $$1 \to G \longrightarrow G' \longrightarrow G'' \to 1$$

("exact" being taken with respect to the fpqc topology). Then, as Mazur-Roberts observed in [15], there is defined a canonical exact triangle in D(S) :

(4.3.2) $$\begin{array}{ccc} & \chi_G & \\ \swarrow & & \nwarrow \\ \chi_{G''} & \longrightarrow & \chi_{G'} \end{array}$$

where the horizontal map is induced by $G' \longrightarrow G''$ by functoriality of the co-Lie complex. It is indeed an immediate consequence of (1.3.3) and (1.4.2). Particularly interesting is (4.3.2) when G', G" are smooth. We then obtain :

(4.3.3) $$\chi_G \simeq (0 \to \omega_{G''} \longrightarrow \omega_{G'} \to 0)$$

<u>Examples</u> 4.3.4. a) Take $G = (\mu_n)_S$. We have the exact sequence :

$$0 \longrightarrow (\mu_n)_S \longrightarrow (G_m)_S \xrightarrow{n} (G_m)_S \longrightarrow 0$$

Hence (4.3.3) yields :

$$\chi_{\mu_{nS}} \simeq (0 \to \underline{O}_S \xrightarrow{n} \underline{O}_S \to 0)$$

In particular, we have $\chi_{\mu_{nS}} = 0$, i.e. μ_{nS} is étale, if and only if n is invertible on S, which is of course well known.

b) Suppose $p1_{\underline{S}} = 0$, where p is a prime number. The group scheme α_p on S is defined by the exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow (G_a)_S \xrightarrow{F} (G_a)_S \longrightarrow 0$$

where F is the Frobenius map $x \mapsto x^p$. Hence (4.3.3) gives

$$\ell_{\alpha_p} \simeq (0 \to \underline{O}_S \xrightarrow{0} \underline{O}_S \to 0) \quad .$$

c) Suppose G is a finite, locally free, commutative group scheme over S. Let A be the bi-algebra defining G, and $A^\vee = \underline{\mathrm{Hom}}(A, \underline{O}_S)$ denote the dual of A, which is also the bi-algebra of the Cartier dual G^* . It is well known (see for instance ([16] II 3.2.4)) that there is a canonical, functorial, closed embedding of G into $W(A^\vee)^+ = G'$ ([1]) , hence we have a canonical, functorial exact sequence $0 \to G \to G' \to G'' \to 0$ with G' and G" smooth, which by (4.3.3) provides a canonical, functorial way of calculating ℓ_G .

We shall now discuss some general properties of the Lie and co-Lie complexes in the commutative case. First of all, we have the following result, which generalizes the fact that the adjoint action of a commutative group on its Lie algebra is trivial :

Proposition 4.4. Let G be a commutative group scheme over S and $p : BG \to S$ denote the canonical projection. There is a canonical, functorial isomorphism in D(BG) :

$$\underline{\ell}_G \simeq Lp^*(\ell_G) \quad .$$

Proof. See [13]. The basic observation is that the multiplication $m : G \times G \longrightarrow G$, being a group homomorphism, induces a map $B(G \times G) \longrightarrow BG$, for which the inverse image of G as a G-object by left multiplication is G as a (G x G)-object by left and right multiplication.

4.5. Let Y be a scheme over S, and $j : Y \to Y'$ be an S-extension of Y by a quasi-coherent Module J . Let G be a commutative group scheme over S. Using the result mentioned at the end of (2.9), it can be shown that there is a canonical, functorial isomorphism of $D(\mathbb{Z}_{Y'})$:

(4.5.1) $$j_*(\ell^\vee_{G_Y} \overset{L}{\otimes} J) \simeq (0 \to G_{Y'} \xrightarrow{d} j_* G_Y \to 0)$$

([1]) i.e. the group of invertible elements in $A^\vee$.

where $G_{Y'}$ is placed in degree 0 and d is the adjunction map ($D(\mathbb{Z}_{Y'})$ means the derived category of $\mathbb{Z}$-Modules on the large fpqc site of Y'). This formula was conjectured by Grothendieck after his reading [15], and proven by Deligne. It yields in particular a canonical, functorial description of the Lie complex $\ell_G^\vee$ as an object of $D(\mathbb{Z}_S)$, since we can take for j the inclusion of S into the scheme of dual numbers on $\underline{O}_S$. Observing that $R^1 j_*(G_Y) = 0$, we derive from (4.5.1) an exact sequence

$$\ldots \to H^i(Y',G_{Y'}) \longrightarrow H^i(Y,G_Y) \longrightarrow H^{i+1}(Y,\ell_{G_Y} \overset{L}{\otimes} J) \to \ldots$$

for $i \leq 1$, which shows again (cf. (2.6)(1)) that the obstruction to deforming a G_Y-torsor over Y' lies in $H^2(Y,\ell_{G_Y}^\vee \overset{L}{\otimes} J)$ (moreover, the above sequence can be interpreted as an exact sequence of relative cohomology (3.3), hence the other parts of (2.6)).

4.6. To conclude these generalities, let us mention a very striking formula for the Lie complex in the finite case, which is due to Grothendieck (see [11]). Let G be a finite, locally free, commutative group scheme over S, and G^* denote its Cartier dual. Let I be a quasi-coherent $\underline{O}_S$-Module. Then, there is a canonical, functorial isomorphism in D(S) :

$$(4.6.1) \qquad \ell_G^\vee \overset{L}{\otimes} I \simeq t_{1]}\, R\underline{Hom}_{\mathbb{Z}}(G^*,I) \quad (\simeq t_{1]}\, R\underline{Hom}_{\underline{O}_S}(G^* \overset{L}{\otimes}_{\mathbb{Z}} \underline{O}_S, I))$$

where $t_{n]}L$, for a complex L, means the truncated complex obtained by killing $H^i(L)$ for $i > n$, i.e. $t_{n]}L = (\ldots \to L^{n-2} \to L^{n-1} \to Z^n \to 0)$. In particular, with the notations of (4.1), we deduce from (4.6.1) canonical, functorial isomorphisms :

$$(4.6.2) \qquad t_G \simeq \underline{Hom}(G^*,I) \qquad , \qquad \nu_G \simeq \underline{Ext}^1(G^*,I) \qquad .$$

The above formulas are helpful in the study of the co-Lie and Lie complexes of truncated Barsotti-Tate groups. For details the reader is referred to [11], where he will also find interesting developments concerning the

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(1) Note that (3.1.2) and (4.4) imply $\ell_{X/Y} \simeq \ell_{G_Y}$.

relationship between Lie complexes and Dieudonné modules.

5. Deformations of commutative, flat group schemes.

As in § 4, all our group schemes will be assumed to be flat and locally of finite presentation, unless otherwise stated. We fix a scheme S.

5.1. Let A be a ring scheme over S, not necessarily commutative, but associative and unitary. We don't assume the underlying scheme to be flat or locally of finite presentation. By an A-module scheme over S we mean a commutative group scheme G over S, endowed with an A-module structure, i.e. a bi-linear map $A \times_S G \longrightarrow G$, $(a,g) \mapsto ag$, such that $a(bg) = (ab)g$, $1g = g$, for any T-valued points a, b of A, g of G. We have especially in mind the case where A is the constant ring scheme $\mathbb{Z}_S$ (resp. $(\mathbb{Z}/n\mathbb{Z})_S$), in which case an A-module scheme is simply a commutative group scheme (resp. a commutative group scheme killed by n). But other cases may be of interest, e. g. $A = \mathbb{Z}_S[\Gamma]$ (Γ being a discrete group or monoïd), $A = \underline{O}_S$, $A = \underline{\underline{W}}_S$, the universal Witt scheme over S ([17] p. 179).

Let G be an A-module scheme over S. Thanks to the action of A, it is possible to define a finer object than ℓ_G, namely an object

(5.1.1) $$\ell_G^A \in \text{ob } D(A \otimes_{\mathbb{Z}} \underline{O}_S) \qquad (*)$$

whose image under the forgetful functor $D(A \otimes_{\mathbb{Z}} \underline{O}_S) \longrightarrow D(S)$ is canonically isomorphic to ℓ_G. If G is smooth, we can take for ℓ_G^A the sheaf of invariant differential forms ω_G endowed with its natural A-linear structure. The definition of (5.1.1) in full generality is a little sophisticated (as were the definitions of $L_{X/Y}^G$ (2.1.1) or $\underline{\underline{\ell}}_G$ (3.1.1)). One method consists in interpreting $\check{\ell}_G[1]$ as the stack of equivariant deformations of G over the dual numbers on S (4.5) and observing that the latter stack has a natural $A \otimes_{\mathbb{Z}} \underline{O}_S$-linear structure.

(*) (ajouté en octobre 1972) Cette construction n'est valable que si $\mathrm{Tor}_1^{\mathbb{Z}}(A,\underline{O}_S) = 0$; dans le cas général, on doit remplacer $A \otimes_{\mathbb{Z}} \underline{O}_S$ par l'anneau dérivé $A \overset{L}{\otimes}_{\mathbb{Z}} \underline{O}_S$, voir ([13] VII 4.1), mais cette modification n'a pas d'incidence sur l'énoncé des théorèmes 5.3, 5.4, et 5.5 ci-dessous.

hence defines, by the dictionary of (SGA 4 XVIII), a complex of $A \otimes_{\mathbb{Z}} \underline{O}_S$-Modules of length 1, which is $R\underline{Hom}_{\underline{O}_S}(\chi_G^A, \underline{O}_S)[1]$. Another method, in the style of (2.8 c)), uses a large diagram describing the structure of A-module of G (see [13] and (5.8) below).

We shall sometimes write χ_G instead of χ_G^A. The notation $\chi_G^{\vee}$ will mean $R\underline{Hom}_{\underline{O}_S}(\chi_G^A, \underline{O}_S)$ ($\in$ ob $D(A \otimes_{\mathbb{Z}} \underline{O}_S)$).

Some of the results of § 4 admit natural refinements. For instance, an exact sequence (4.3.1) of A-module schemes gives rise to an exact triangle (4.3.2) in $D(A \otimes_{\mathbb{Z}} \underline{O}_S)$, and the isomorphism (4.5.1), for an A-module scheme G, comes from an isomorphism in D(A).

5.2. We suppose now S is over some fixed scheme T, and we are given a T-extension $i : S \to S'$ by a quasi-coherent $\underline{O}_S$-Module I. We suppose, moreover, that A is induced by a flat ring scheme A' over S', $A = A' \times_{S'} S$. Concerning the deformations of A-module schemes, our main result is the following :

Theorem 5.3. *In the situation of* (5.2), *let* G *be an* A-*module scheme over* S. *There is an obstruction*

$$\omega(G,i) \in \mathrm{Ext}^2_A(G, \chi_G^{\vee} \overset{L}{\otimes} I) \qquad (^1)$$

whose vanishing is necessary and sufficient for the existence of an A'-*module scheme* G' *over* S' *deforming* G, *i.e. equipped with an isomorphism of* A-*module schemes* $G' \times_{S'} S \xrightarrow{\sim} G$. *When* $\omega(G,i) = 0$, *the set of isomorphism classes of deformations* G' *is an affine space under* $\mathrm{Ext}^1_A(G, \chi_G^{\vee} \overset{L}{\otimes} I)$, *and the group of automorphisms of a given deformation is canonically isomorphic to* $\mathrm{Ext}^0_A(G, \chi_G^{\vee} \overset{L}{\otimes} I)$. *Moreover, if* A, A' *are induced by a fixed, flat ring scheme* B *over* T, *there is defined a canonical class, depending only on* B *and the composition* $G \to S \to T$,

$$c(B, G/S/T) \in \mathrm{Ext}^1_A(G, \chi_G^{\vee} \overset{L}{\otimes} tL_{S/T}) \quad ,$$

(1) Ext are taken with respect to the fpqc topology on the category of schemes over S. For the notation $tL_{S/T}$ below, see (2.7).

<u>such that</u> $\omega(G,i)$ <u>is given by the cup-product</u>

$$\omega(G,i) = e(i)c(B,G/S/T)$$

<u>where</u> $e(i) \in \mathrm{Ext}^1(L_{S/T},I)$ <u>is the class of</u> i .

We also have two results concerning the deformations of morphisms of A-module schemes :

<u>Theorem</u> 5.4. <u>Let</u> F', G' <u>be</u> A'-<u>module schemes over</u> S', <u>and let</u> $f : F \longrightarrow G$ <u>be a morphism of</u> A-<u>module schemes</u>, <u>where</u> $F = F' \times_{S'} S$, $G = G' \times_{S'} S$. <u>There is a canonical obstruction</u>, <u>depending functorially on</u> F', G', f :

$$\omega(F',G',f,i) \in \mathrm{Ext}^1_A(F,\chi^\vee_G \overset{L}{\otimes} I)$$

<u>whose vanishing is necessary and sufficient for the existence of a morphism of</u> A'-<u>module schemes</u> $f' : F' \to G'$ <u>such that</u> $f' \times_{S'} S = f$. <u>When</u> $\omega(F',G',f,i) = 0$, <u>the set of solutions</u> f' <u>is an affine space under</u> $\mathrm{Ext}^0_A(F,\chi^\vee_G \overset{L}{\otimes} I)$.

<u>Theorem</u> 5.5. <u>Let</u> F' <u>be an</u> A'-<u>module scheme over</u> S', G <u>an</u> A-<u>module scheme over</u> S, <u>and</u> $f : F \longrightarrow G$ <u>a morphism of</u> A-<u>module schemes</u>, <u>where</u> $F = F' \times_{S'} S$. <u>Denote by</u> $C(f) \in \mathrm{ob}\, D(A)$ <u>the mapping-cylinder of</u> f . <u>There exists a canonical obstruction</u>, <u>depending functorially on</u> F', f :

$$\omega(F',f,i) \in \mathrm{Ext}^2_A(C(f),\chi^\vee_G \overset{L}{\otimes} I)$$

<u>whose vanishing is necessary and sufficient for the simultaneous existence of an</u> A'-<u>module scheme</u> G' <u>deforming</u> G <u>in the sense of</u> (5.3) <u>and a morphism of</u> A'-<u>module schemes</u> $f' : F' \longrightarrow G'$ <u>such that</u> $f' \times_{S'} S = f$. <u>When</u> $\omega(F',f,i) = 0$, <u>the set of isomorphism classes of solutions</u> (G',f') <u>is an affine space under</u> $\mathrm{Ext}^1_A(C(f),\chi^\vee_G \overset{L}{\otimes} I)$ <u>and the group of automorphisms of a given solution is canonically isomorphic to</u> $\mathrm{Ext}^0_A(C(f),\chi^\vee_G \overset{L}{\otimes} I)$.

5.6. As we said in the introduction, (5.3) and (5.4) were conjectured by Grothendieck (letter to the author, 12/2/69) and form a basic tool in the study of Barsotti-Tate groups on a general base ([11] and [16]). As for (5.5), it came from an attempt to prove by means of our theory an unpublished (1) result of Oort, that was kindly brought to our attention by Mazur, and says the following : in the situation of (5.5), suppose S' is affine, $A' = \mathbb{Z}_{S'}$, F' is finite over S', G is an abelian scheme over S, f is a closed immersion, then the obstruction $\omega(F',f,i)$ vanishes, in other words there exist an abelian scheme G' on S' lifting G and an embedding $f' : F' \to G'$ lifting f . Note that in this situation $C(f) = G/F$ is an abelian scheme. It should be the case, at least when 2 is invertible on S, that the Ext^2 of an abelian scheme with a locally free sheaf of finite rank is zero. If this is true, then Oort's result follows from (5.5).

Among (5.3), (5.4), (5.5) there are some compatibilities that we should like to discuss briefly.

a) In the situation of (5.4), suppose F = G and f is the identity. Then $\omega(F',G',f,i) \in \mathrm{Ext}^1_A(G, \chi_G^{\vee} \overset{L}{\otimes} I)$ is the difference, in the sense of (5.3), between the classes of the deformations F', G' of G .

b) In the situation of (5.5), suppose there exists an A'-module scheme G' deforming G . Then it follows from (5.3) that (F',f,i) lies in

$$\mathrm{Coker}(\mathrm{Ext}^1_A(G,M) \xrightarrow{f^*} \mathrm{Ext}^1_A(F,M)) \hookrightarrow \mathrm{Ext}^2_A(C(f),M)$$

where $M = \chi_G^{\vee} \overset{L}{\otimes} I$ and f^* is the map induced by f . It may be observed that the above cokernel corresponds to the obstruction to lifting f into a map $f' : F' \longrightarrow G'$ (5.4) modulo the indeterminacy (5.3) in the choice of G' .

5.7. Before we turn to the proof of the above results, we shall mention a reassuring compatibility between the obstruction $\omega(G,i)$ of (5.3) for A, A' = the constant ring scheme $\mathbb{Z}$ (obstruction to deforming G as a commutative group scheme) and the obstruction $\omega(G,i)$ of (3.4) (obstruction to deforming G as a (possibly non-commutative) group scheme. Recall that, for $M \in \mathrm{ob}\, D^+(\mathbb{Z}_S)$, there is a canonical, functorial map

$$\mathrm{RHom}_{\mathbb{Z}}(G,M) \longrightarrow R\Gamma(BG/S,p^*M)[1]$$

(where $p : BG \to S$ is the projection), which is defined by identifying $R\Gamma(BG/S,p^*M)$ with $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}(\mathrm{Ner}(G))/\mathbb{Z}, M)$ (where $\mathbb{Z}(-)$ denotes the free

(1) to our knowledge

abelian group functor) and using the canonical epimorphism $\mathbb{Z}(G) \to G$, which defines a map $\mathbb{Z}(\mathrm{Ner}(G))/\mathbb{Z} \longrightarrow G[1]$. In particular, thanks to (4.4) we have a map

$$\mathrm{RHom}_{\mathbb{Z}}(G, \check{\chi}_G \overset{L}{\otimes} I) \longrightarrow R\Gamma(BG/S, \check{\chi}_{\underline{G}} \overset{L}{\otimes} I)[1] \quad ,$$

hence a map

$$(*) \qquad \mathrm{Ext}^{*}_{\mathbb{Z}}(G, \check{\chi}_G \overset{L}{\otimes} I) \longrightarrow H^{*+1}(BG/S, \check{\chi}_{\underline{G}} \overset{L}{\otimes} I) \quad .$$

The compatibility says that the image of the obstruction $\omega(G,i)$ of (5.3) under (*) is the obstruction $\omega(G,i)$ of (3.4).

5.8. <u>Sketch of proof</u>. As in (2.8) and (3.6), the idea is to reduce the problem to a problem of deformation of a suitable ringed topos or map of ringed topoi. Thus (5.3) will be reduced to (1.7), while (5.4) and (5.5) will be reduced to variants of (1.7), namely ([12] III 2.2.4) and ([12] III 2.3.2) respectively. Since the technique is the same in the three cases, we shall restrict ourselves to (5.3). The proof is long and rather involved. We shall only outline the main steps. For details, the reader is again referred to [13] .

5.8.1. <u>Diagrams</u>. Let T be a category. By a diagram of T of type I we mean a functor $X : I \longrightarrow T$. Let X, Y be diagrams of T of types I, J respectively. By definition a map $f : X \to Y$ is a pair (u,v) where $v : I \to J$ is a functor and $u : X \to Yv$ is a functor morphism. Thus the diagrams of T form a category denoted by $\mathrm{Diagr}_1(T)$. We define $\mathrm{Diagr}_n(T)$, for $n \geqslant 0$, by the formulas :

$$\mathrm{Diagr}_o(T) = T \quad , \quad \mathrm{Diagr}_n(T) = \mathrm{Diagr}(\mathrm{Diagr}_{n-1}(T)) \quad .$$

The category $\mathrm{Diagr}_n(T)$ is called the <u>category of n-diagrams</u> of T . Observe that if T possesses finite products, the same is true for Diagr(T), hence for $\mathrm{Diagr}_n(T)$: if $X : I \to T$, $Y : J \to T$ are diagrams, then X x Y is the diagram of type I x J defined by $(i,j) \mapsto X_i \times Y_j$.

5.8.2. <u>Spectra</u>. Let T be a topos. As we have seen above, the nerve functor gives an embedding of the category of groups of T into the category of simplicial objects of T :

$$\mathrm{Ner} : \mathrm{Group}(T) \hookrightarrow \mathrm{Simpl}(T) \quad ,$$

whose essential image consists exactly of those simplicial objects which satisfy the exactness properties (i), (ii), (iii) of (3.6 a)) (Y replaced by the final object of T). Now, if G is a commutative group of T, the multiplication $G \times G \to G$ is a group homomorphism, hence Ner(G) is actually a simplicial object in the category of commutative groups of T, or equivalently a commutative group of the topos Simpl(T), therefore we can iterate the nerve construction and define Ner(Ner(G)), Ner(Ner(Ner(G))), etc. Denote by $\mathbb{Z}$-Mod(T) the category of commutative groups of T, and by n-Simpl(T), for $n \in \mathbb{N}$, the category of n-simplicial objects of T (i.e. the category of functors from $\Delta^{o} \times \dots \times \Delta^{o}$ (n times) to T where Δ is the category of finite, non void, totally ordered sets). For $G \in \mathrm{ob}\ \mathbb{Z}$-Mod(T), we define inductively $G\langle n \rangle \in \mathrm{ob}$ n-Simpl(T) by

$$(5.8.2.1) \qquad G\langle 0 \rangle = G \quad , \quad G\langle n \rangle = \mathrm{Ner}(G\langle n-1 \rangle) \quad \text{for } n \geqslant 1 \quad .$$

The functor

$$(5.8.2.2) \qquad \mathbb{Z}\text{-}\mathrm{Mod}(T) \longrightarrow n\text{-}\mathrm{Simpl}(T) \quad , \quad G \longmapsto G\langle n \rangle \quad ,$$

is faithful for n = 0, fully faithful for $n \geqslant 1$, and, for $n \geqslant 2$, its essential image consists exactly of those n-simplicial objects Y possessing the following property : the simplicial objects deduced from Y by fixing all variables but one satisfy the conditions (i), (ii), (iii) recalled above (this last assertion is an easy consequence of the well known fact that a group in the category of groups is a commutative group).

Let G be a commutative group of T . Observe that $G\langle n \rangle$ corresponds, by the normalization functor [8] , to the n-complex concentrated in degree (-1,...,-1) with value G. In particular, we can identify $G\langle n \rangle$ with each of the "faces" $G\langle n+1 \rangle(*,\dots,[1],*,\dots,*)$, and we get an augmented, strictly cosimplicial object in Diagr(T) :

$$(5.8.2.3) \qquad \underline{\underline{G}} = (G \to G\langle 1 \rangle \rightrightarrows G\langle 2 \rangle \ \dots \ G\langle n \rangle \underset{n+1 \text{ arrows}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}} G\langle n+1 \rangle \ \dots)$$

which we shall call the spectral diagram (or, simply, spectrum) of G, by analogy with the Eilenberg-Mac-Lane spectra.

Let us return to (5.8.1) for a moment. To any n-diagram X of T is associated an (n-1)-diagram of (Cat) (the category of categories), called the type of X , and denoted by Typ(X). It depends functorially on X and is defined inductively by : Typ(X) = I if X : I $\to$ T is a diagram of T, and, if X : I $\to$ $\mathrm{Diagr}_{n-1}(T)$ is an n-diagram with $n \geqslant 2$, Typ(X) is the diagram $i \mapsto \mathrm{Typ}(X_i)$. If t is a type of n-diagram (i.e. an (n-1)-diagram of (Cat)), we define the category of diagrams of type t, $\mathrm{Diagr}_t(T)$, as the category whose objects are the diagrams of type t and maps are the maps of diagrams inducing the identity on t . Denote by

$$(5.8.2.4) \qquad \mathbb{D} = (\mathrm{pt} \to \Delta^{\circ} \rightrightarrows (\Delta^{\circ})^2 \ \ldots \quad (\Delta^{\circ})^n \ \underset{(x_1,\ldots,x_n) \mapsto (x_1,\ldots,[1],\ldots,x_n)}{\overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}}} \ (\Delta^{\circ})^{n+1} \quad \ldots)$$

the type of any spectral diagram $\underline{\underline{G}}$. We have an embedding

$$(5.8.2.5) \qquad \mathbb{Z}\text{-Mod}(T) \hookrightarrow \mathrm{Diagr}_{\mathbb{D}}(T) \quad , \quad G \longmapsto \underline{\underline{G}} \quad ,$$

whose essential image consists exactly of the diagrams Y of type $\mathbb{D}$ such that, for each $n \in \mathbb{N}$, Y_n is in the essential image of (5.8.2.2) and the maps of n-simplicial objects induced by the n+1 maps $Y_n \longrightarrow Y_{n+1}$ are isomorphisms.

As the normalization functor commutes with external tensor products, we get, for X, Y $\in$ ob $\mathbb{Z}$-Mod(T), p, q $\in \mathbb{N}$, a canonical map

$$(5.8.2.6) \qquad X\langle p\rangle \times Y\langle q\rangle \longrightarrow (X \otimes Y)\langle p+q\rangle \quad ,$$

which is associative in the obvious sense. Let A be a ring object in T (associative and unitary), and let G be a left A-Module. Then, thanks to the maps (5.8.2.6), $\underline{\underline{A}}$ becomes a monoïd object in $\mathrm{Diagr}_2(T)$ (associative and unitary), and $\underline{\underline{G}}$ becomes a (left) $\underline{\underline{A}}$-object. In particular, we can consider the nerves

$$\mathrm{Ner}(\underline{\underline{A}}) = (\dots\ \underline{\underline{A}} \times \underline{\underline{A}} \rightrightarrows \underline{\underline{A}} \rightrightarrows e) \qquad (e = \text{the final object of } T)\ ,$$

$$\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) = (\dots\ \underline{\underline{A}} \times \underline{\underline{A}} \times \underline{\underline{G}} \rightrightarrows \underline{\underline{A}} \times \underline{\underline{G}} \rightrightarrows \underline{\underline{G}})$$

which are certain objects of $\mathrm{Diagr}_3(T)$. There is a natural projection

$$(5.8.2.7) \qquad \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) \longrightarrow \mathrm{Ner}(\underline{\underline{A}}) \quad ,$$

and, as above, it is again possible to characterize A-Mod(T), the category of A-Modules of T, as a certain category of diagrams over $\mathrm{Ner}(\underline{\underline{A}})$ satisfying certain exactness conditions.

5.8.3. Reduction to (1.7). Applying the above to the situation of (5.3), we get a diagram

$$\begin{array}{ccc} \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) & & \\ \Big\downarrow f & & \\ \mathrm{Ner}(\underline{\underline{A}}) & \hookrightarrow & \mathrm{Ner}(\underline{\underline{A}}') \end{array} \quad ,$$

which in turn can be interpreted as a diagram of ringed topoi (like in (2.8)), in which the horizontal map is a T-extension of $\mathrm{Ner}(\underline{\underline{A}})$ by the inverse image of I, still denoted by I . From the generalities of (5.8.2) and the flatness criterion it follows that an A'-module scheme G' deforming G is essentially the same as a deformation of f over $\mathrm{Ner}(\underline{\underline{A}}')$ as a map of ringed topoi. Therefore we can apply (1.7), and it remains to calculate the groups $\mathrm{Ext}^i(L_{\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})/\mathrm{Ner}(\underline{\underline{A}})}, f^*I)$ (or, more generally, the analogous groups with I replaced by a complex of $\underline{O}_S$-Modules with bounded, quasi-coherent cohomology, e. g. a truncation of $L_{S/T}$). This is, however, far from being easy. We shall briefly indicate the main points.

5.8.4. A duality formula. Fix a topos T . To a diagram $X : I \longrightarrow T$ there are associated two topoi : Top(X), $\mathrm{Top}^{\circ}(X)$. The first one is the total topos of (SGA 4 VI) defined by the fibered topos $i \mapsto T_{/X_i}$; its objects are families E consisting of a sheaf E_i on X_i for each object i of I together with a map $X_f^* E_j \longrightarrow E_i$ for each map $f : i \rightarrow j$ of I, these maps satisfying certain transitivity relations. The other one, $Top^o(X)$, is defined in the same way, except that we reverse the sense of the transition arrows, namely give ourselves a map $E_i \rightarrow X_f^* E_j$ for each $f : i \rightarrow j$; in other words, $Top^o(X)$ is the topos of diagrams of type I over X. The construction of Top(X), $Top^o(X)$ easily extends to n-diagrams.

Fix a Ring $\underline{O}$ of T . If X is an n-diagram of T, we equip Top(X), $Top^o(X)$ with the Rings induced by $\underline{O}$. Now, if L is a Module on Top(X) and M a Module on T, we can define a Module $\underline{Hom}^!(L,M)$ on $Top^o(X)$ with the properties that it depends functorially on L, M and induces the ordinary $\underline{Hom}$ on each piece of the diagram. For example, if X is a 1-diagram $I \rightarrow T$, we define $\underline{Hom}^!(L,M)$ as the family $i \mapsto \underline{Hom}(L_i, M_{X_i})$ with the obvious transition maps, and this generalizes trivially to n-diagrams. Moreover, we can derive the construction of $\underline{Hom}^!$, namely define

$$R\underline{Hom}^!(L,M) \in \text{ob } D(Top^o(X))$$

as a bi-functor of $L \in \text{ob } D(Top(X))$, $M \in \text{ob } D^+(T)$. It is not hard to prove the following "duality formula" :

<u>Proposition</u> 5.8.4.1. <u>With the above notations, there exists a canonical, functorial isomorphism</u>

$$RHom(Top(X);L,M_X) \xrightarrow{\sim} R\Gamma(Top^o(X);R\underline{Hom}^!(L,M))$$

<u>for</u> $L \in \text{ob } D^-(Top(X))$, $M \in \text{ob } D^+(T)$.

5.8.5. <u>Passing to the tangent complex.</u> Our goal is to calculate

$$(*) \qquad RHom(L_{Ner(\underline{\underline{A}},\underline{\underline{G}})/Ner(\underline{\underline{A}})}, f^*M)$$

where M is a complex of $\underline{O}_S$-Modules with bounded, quasi-coherent cohomology. By definition (5.8.3), (*) is an RHom of complexes of $\underline{O}$-Modules on the total topos of $Ner(\underline{\underline{A}},\underline{\underline{G}})$ obtained by associating to each scheme of the diagram its small Zariski topos. But, as both arguments inside the RHom have quasi-coherent cohomology, they can, in a natural way, be extended into objects of $D(\mathrm{Top}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})))$ where now $\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})$ is viewed as a 3-diagram in the (large) fpqc topos of S, and by descent the RHom does not change under this extension. Now we can apply (5.8.4.1) and we get an isomorphism

$$(**)\quad \mathrm{RHom}(L_{\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})/\mathrm{Ner}(\underline{\underline{A}})}, f^{*}M) \simeq R\Gamma(\mathrm{Top}^{o}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})), R\underline{\mathrm{Hom}}^{!}(L_{\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})/\mathrm{Ner}(\underline{\underline{A}})}, M))$$

which brings $(*)$ nearer to calculation, because $R\underline{\mathrm{Hom}}^{!}(L_{\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})/\mathrm{Ner}(\underline{\underline{A}})}, M)$ has a very simple interpretation. In effect, denote by $\mathrm{Lie}(G) = \check{\chi}_G$ (5.1) the Lie complex of G, and by $p : \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) \longrightarrow \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{O}})$ the canonical map defined by projecting G to zero. The construction $G \mapsto \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})$ extends trivially to complexes of A-Modules, and it is easily deduced from the Mazur-Roberts exact triangle (4.3.2) that there is a canonical isomorphism of $D^{b}(\mathrm{Top}^{o}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})))$:

$$(***)\quad R\underline{\mathrm{Hom}}^{!}(L_{\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})/\mathrm{Ner}(\underline{\underline{A}})}, M) \simeq p^{*}\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{\underline{\mathrm{Lie}}}}(G)) \overset{L}{\otimes} f^{*}M$$

(the tensor product on the right being of course taken over the ring induced by $\underline{O}_S$). In view of (5.8.3), (5.3) will follow from the combination of $(**)$, $(***)$, and the following canonical isomorphism

$$R\Gamma(\mathrm{Top}^{o}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})), p^{*}\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{\underline{\mathrm{Lie}}}}(G)) \overset{L}{\otimes} f^{*}M) \simeq \mathrm{RHom}_{A}(G, \mathrm{Lie}(G) \overset{L}{\otimes} M) \quad ,$$

which is itself a consequence of the more general

<u>Theorem</u> 5.8.6. <u>Fix a topos</u> T, <u>an associative and unitary ring</u> A <u>of</u> T, <u>a commutative</u> (<u>and unitary</u>) <u>ring</u> R <u>of</u> T, $G \in \mathrm{ob}\ A\text{-Mod}(T)$, $E \in \mathrm{ob}\ D(A \otimes_{\mathbb{Z}} R)$, $M \in \mathrm{ob}\ D^{b}(R)$. <u>Assume</u> E <u>is of finite flat amplitude as a complex of</u> R-<u>Modules</u>. <u>Denote by</u> $p : \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) \to \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{O}})$, $f : \mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}}) \to T$ <u>the canonical projections</u>. <u>Then there exists a canonical</u>, <u>functorial isomorphism</u>

$$R\Gamma(\mathrm{Top}^{o}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})), p^{*}\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{E}}) \overset{L}{\otimes}_{R} f^{*}M) \simeq \mathrm{RHom}_{A}(G, E \overset{L}{\otimes}_{R} M) \quad .$$

<u>Proof</u>. We shall just sketch the idea. For simplicity, we shall assume M = R . Denote by $\mathbb{Z}^{st}(-)$ the functor deduced by stabilization from the functor $X \mapsto \mathbb{Z}(X)$ on the category of abelian groups of T, where $\mathbb{Z}(X)$ means the free abelian group on the underlying sheaf of sets. For $X \in \mathrm{ob}\ \mathbb{Z}\text{-Mod}(T)$, $\mathbb{Z}^{st}(X)$ is a simplicial abelian group of T, defined by

$$\mathbb{Z}^{st}(X) = \varinjlim \mathbb{Z}(X[n])[-n]$$

where the shifts of degrees are performed simplicially thanks to the Dold-Puppe equivalence, and the direct limit is taken with respect to the "suspension maps". Using the pairings (5.8.2.6), it is possible to turn $\mathbb{Z}^{st}(A)$ into an associative and unitary simplicial ring of T, and $\mathbb{Z}^{st}(G)$ into a $\mathbb{Z}^{st}(A)$-Module. Moreover, we have a canonical map of rings $\mathbb{Z}^{st}(A) \longrightarrow A$, and a $\mathbb{Z}^{st}(A)$-linear map $\mathbb{Z}^{st}(G) \longrightarrow G$. Now, the proof of (5.8.6) breaks down into two parts :

a) First, using standard resolutions, we show there is a canonical isomorphism

$$R\Gamma(\mathrm{Top}^{\circ}(\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{G}})),p^{*}\mathrm{Ner}(\underline{\underline{A}},\underline{\underline{E}})) \simeq R\mathrm{Hom}_{A}(\mathbb{Z}^{st}(G) \overset{L}{\otimes}_{\mathbb{Z}^{st}(A)} A\ ,\ E)$$

(This is the hard part of the proof).

b) Second, we show that the canonical map

$$(5.8.6.1) \qquad \mathbb{Z}^{st}(G) \overset{L}{\otimes}_{\mathbb{Z}^{st}(A)} A \longrightarrow G$$

is an isomorphism. This last result is essentially due to Mac-Lane [14], as was explained to us by L. Breen. It is easy to prove by dévissage and reduction to the case G = A .

This concludes the (sketchy) proof of (5.8.6), and therefore demonstrates (5.3).

Remark 5.9. It is easy to deduce from (5.8.6.1) a canonical, functorial resolution of G of the form desired by Grothendieck in (SGA 7 VII 3.5.4).

Remark 5.10. Deligne has indicated a method based on the theory of Picard stacks that should yield another proof of the results of this section.

(ajouté en octobre 1972) Le calcul esquissé à partir de (5.8.5) ayant buté sur des difficultés techniques, nous avons dû substituer au diagramme envisagé en (5.8.3) un diagramme voisin, obtenu en remplaçant $\underline{\underline{X}}$ par l'objet simplicial $C.(\underline{\underline{X}})$ tel que $C_n(\underline{\underline{X}})$ soit la somme disjointe des $X\langle i_o+1\rangle$ suivant les chaînes strictement croissantes $[i_o] \longrightarrow \dots \longrightarrow [i_n]$ d'ensembles finis totalement ordonnés. Le principe du calcul reste néammoins le même, nous renvoyons le lecteur à ([13] VII 4.1.6) pour les détails.

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