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# ASYMPTOTIC ANALYSIS OF VARIATIONAL PROBLEMS WITH CONSTRAINTS OF OBSTACLE TYPE

HEDY ATTOUCH, COLETTE PICARD

The purpose of this work is to give a complete presentation with some improvements and new developpements of a recent paper of E. De Giorgi [14] : Γ-limit of obstacles.

Chapter I - Approximation of convex lower semi-continuous functionals.

1. Yosida approximation
2. Statement of the theorem : approximation of a convex, lsc, functional by an increasing sequence of polyedral functionals
3. The dual statement ; a Galerkin procedure for convex lsc functional

Chapter II - Integral representation of unilateral constraints.

Chapter III - Γ-limit of obstacles. The quadratic case.

Chapter IV - Γ-limit of obstacles. The non quadratic case.
Explicit formula for periodical obstacles.

Chapter V - Γ-limit of bilateral constraints.
Problems with holes.

## INTRODUCTION

The origin of this paper is the following problem :
Let us consider a variational inequality, with a constraint of obstacle-type, the obstacle depending on a parameter $n \in \mathbb{N}$ ; for example

$$(I_n) \qquad \operatorname*{Min}_{\substack{u \geq g_n \\ u \in H_0^1(\Omega)}} \left\{ \int_\Omega |Du|^2\, dx - \int_\Omega fu\, dx \right\} .$$

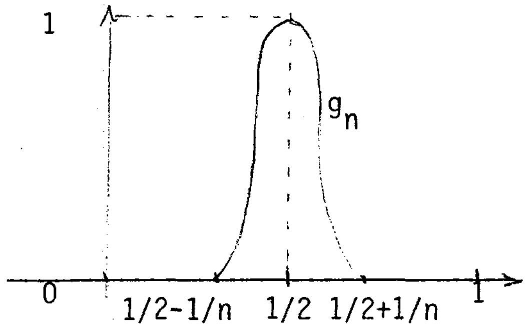
When $g_n$ converges to some function $g$ , what can we say about the solutions $u(g_n)$ of the corresponding problems $(I_n)$ ?
The answer depends obviously on the topology for which the convergence of the sequence $(g_n)_{n \in \mathbb{N}}$ holds.

On a very simple example, one can see that even if the $(g_n)_{n \in \mathbb{N}}$ are regular obstacles and the constraint $u \geq g_n$ is taken almost everywhere one will need tools of potential theory, more precisely of capacity theory in order to interpret the limit problem (I).

Take in one discussion the following $g_n$ ; clearly $g_n \to 0$ almost everywhere but the limit problem B is

$$(I) \qquad \operatorname*{Min}_{\substack{u(\frac{1}{2}) \geq 1 \\ u \in H_0^1(\Omega)}} \left\{ \int_\Omega |Du|^2\, dx - \int_\Omega fu \right\} ,$$

$$\Omega = ]0,1[ \, .$$




i.e. the limit constraint is not taken in the sense almost everywhere, and one has to use the continuous representant of $u$ (more gerenally the quasi-continuous representants of $u$) in order to interpret the limit problem.

Concerning the determination of the limit problem, we can distinguish two types of results : "stability" and "non stability" results.

A. In the first type of results $g_n$ converges to $g$ in a strong enough topology in order the limit problem to be

$$\text{(I)} \qquad \underset{\substack{u \geq g \\ u \in H_0^1(\Omega)}}{\text{Min}} \quad \int_\Omega |Du|^2\, dx - \int_\Omega fu\}$$

i.e. $u(g_n) \to u(g)$ ; that's what we call a <u>stability result</u>.

Concerning this problem one can find an abundant litterature [1], [8], [9], [3] . In [3], using recent results of potential theory, the authors proved the equivalence :

when $g_n$, $g$ are quasi-continuous

$$\forall f \in H^{-1}(\Omega) \qquad u(g_n) \to u(g)$$

$$\Updownarrow$$

$$g_n \to g \text{ in } L^2(C) \text{ i.e. } \int_0^{+\infty} C^2(|g_n - g| > t)\, dt^2 \xrightarrow[n\to+\infty]{} 0 .$$

(All the notions of capacity are relative, in the situation described, to the capacity defined from the norm $\|\cdot\|_{H_0^1(\Omega)}$ ).

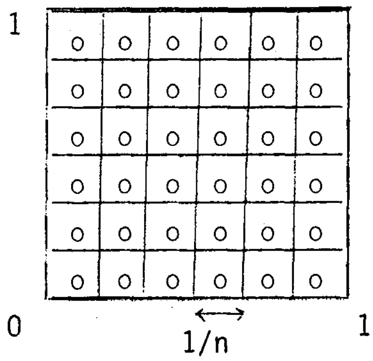
A useful critera which assures the convergence of $g_n$ to $g$ in $L^2(C)$ is the following (cf. [20] , [3]) :

$(g_n \rightharpoonup g$ in $w\text{-}W^{1,p}(\Omega)$ with $p>2) \Longrightarrow (g_n \xrightarrow{L^2(C)} g)$ .

B. The second type of results concern the situations where there is no stability in the sense of A, but for which there exist a limit problem. In [8] , Carbone and Colombini studied in detail the following situation.

In two dimensions

$\Omega = ]0,1[ \times ]0,1[$ ,
let $g_n = 1$ on the balls centered in each small squares, and of radius $a_n = e^{-n^2}$ and $g_n = 0$ elsewhere




In [8] , they proved (cf. also Murat and Cioranescu [11])

$$\min_{\substack{u\geq g_n \\ u\in H_0^1(\Omega)}} \left\{\int_\Omega |Du|^2 - \int_\Omega fu\,dx\right\} \xrightarrow[n\to+\infty]{} \min_{u\geq 0} \left\{\int_\Omega |Du|^2\,dx + 2\pi \int [(u-1)^-]^2\,dx - \int_\Omega fu\,dx\right\}$$

i.e., in the limit problem, we find an extra-term which we can interpret as a penalty-term, with finite values, relatively to the constraint $u\geq 1$

C. A natural problem is to understand the full significance of this phenomena, and to interpret in a unified way the parts A and B : this may be summurized in the following way :

" *What is the closure in variational sense of the constraints of obstacle type* " :

In [14] , De Giorgi gives a first and sharp answer to this problem in its full generality for a quadratic energy functional :

$$(*) \quad \min_{\substack{u\geq g_n \\ u\ H_0^1(\Omega)}} \left\{\int_\Omega |Du|^2 - \int_\Omega fu\,dx\right\} \longrightarrow \min_{u\in H_0^1(\Omega)} \left\{\int_\Omega |Du|^2 dx + \int_\Omega j(x,\tilde{u}(x))\,d\mu(x) - \int_\Omega fu\,dx + \nu(\Omega)\right\}$$

where $\mu,\nu$ are positive Radon measure, $\mu \in H^{-1}$ , $\widehat{u}$ is a quasi-continuous representant of $u$ , $j$ is convex, lsc, decreasing with respect to $u$ .

That's the most general form of the limit problem when starting with problem $(I_n)$ ; one would mention that there is no assumption of convergence on the $g_n$ , and that this result has to be interpreted in such a general setting as compactness result (i.e. $\exists\ (n_k)_{k \in \mathbb{N}}$ such that (*) holds for this subsequence).

In this article, we give a complete presentation of this result ; the basic idea of the proof is the same as in De Giorgi's proof and the technics are relevant of $\Gamma$-convergence theory. We improve the De Giorgi's results in the following directions :

. Introducing new tools in the approximation theory in convex analysis we clarify and simplify the part of functional analysis in the proof and allow the attack, by the same method, of many other problems in variational inequalities.

We extend (but, up to now, not in such a general context) the results to the case where the energy functional is not quadratic (for example, $\int_\Omega \ Du|^p\, dx)$ .

. In the case where the coefficients of the energy functional are rapidly oscillating (for example $\int_\Omega \sum a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\ dx)$ combining the preceeding technics with tools of compactness by compensation (cf. [19]) we can describe the limit problem.

. The general theorems of the $\Gamma$-convergence theory being, by nature, compactness results, we show how to use these results in order to compute precisely the limit problem.

. Finally, we prove that the bilateral constraints problems and particularly problems with equality on "holes" can be deduced very simply from the preceeding unilateral results.

# CH.I APPROXIMATION OF CONVEX LOWER SEMI-CONTINUOUS FUNCTIONALS

Let $V$ be a general real Banach space, $\|.\|_V$ the norm in $V$ ; let us denote by $V'$ its dual and $<.,.>_{(V',V)}$ the pairing between $V$ and $V'$ .

The duality map $H : V \to V'$ is defined by :

(1.1) $$H(v) = \{f \in V' \,/\, \|f\|_{V'} = \|v\|_V \text{ and } <f,v>_{(V',V)} = \|v\|^2\} .$$

From the Hahn-Banach theorem, for every $v \in V$ , $H(v)$ is non void. Moreover, $H = \partial(\frac{1}{2}\|.\|^2)$ , the subdifferential of the functional $v \to \frac{1}{2}\|v\|_V^2$ .

Let $F : V \to ]-\infty,+\infty]$ a proper functional (i.e. $\not\equiv +\infty$) ; for any $\lambda>0$ , we define $F_\lambda$ its Yosida approximation :

(1.2) $$F_\lambda(v) = \operatorname*{Inf}_{z\in V} \{F(z) + \frac{1}{2\lambda}\|v-z\|^2\} .$$

Let us examine the properties of $F_\lambda$ in such a general context.

(1.3) *Lemma*

*Let $V$ a general real Banach space and $F : V \to ]-\infty,+\infty]$ a proper, lower semi-continuous functional on $V$ satisfying :*

$$\exists\, \beta \geq 0 \text{ s.t. } \forall v \in V \quad F(v) + \alpha\|v\|^2 + \beta \geq 0 \text{ ; then,}$$

*a) $\forall v \in V$, $\quad F_\lambda(v) \uparrow F(v)$ as $\lambda$ decreases to zero.*

*b) $\forall \lambda > 0$, $\forall u,v \in V$, $|F_\lambda(u)-F_\lambda(v)| \leq \frac{1}{\lambda} C(\|u\|,\|v\|).\|u-v\|$ .*
*where $C(\|u\|,\|v\|) = c_1\|u\| + c_2\|v\| + c_3$ , with $c_1, c_2, c_3 \in \mathbb{R}^+$*

Proof :

a) By definition, $\forall z \in V$, $F_\lambda(v) \leq F(z) + \frac{1}{2\lambda}\|v-z\|^2$ ; taking $z = v$

we get $F_\lambda(v) \leqslant F(v)$ , and (1.3)bis $\sup\limits_{\lambda>0} F_\lambda(v) \leqslant F(v)$ .

By definition of the inf, for every $\lambda>0$ there exists $z_\lambda \in V$ such that

$$(1.4) \qquad F_\lambda(v) \leqslant F(z_\lambda) + \frac{1}{2\lambda}\|v-z_\lambda\|^2 \leqslant F_\lambda(v) + \lambda .$$

Since $F(z) + \alpha\|z\|^2 + \beta \geqslant 0$ , (1.4) implies

$$\|z_\lambda - v\|^2 \leqslant 2\lambda \left[\sup_\lambda F_\lambda(v)+\lambda+2\alpha\|z_\lambda-v\|^2+2\alpha\|v\|^2+\beta\right] .$$

If $\sup\limits_{\lambda>0} F_\lambda(v) < +\infty$ , this implies :

$$(1.5) \qquad z_\lambda \xrightarrow{s-V} v \text{ as } \lambda \to 0 .$$

From (1.4) $F(z_\lambda) \leqslant \sup\limits_\lambda F_\lambda(v) + \lambda$ ; making $\lambda \to 0$ , from the strong lower semi continuity of $F$ , and (1.5), it follows

$$(1.6) \qquad F(v) \leqslant \sup_{\lambda>0} F_\lambda(v) .$$

From (1.3)bis and (1.6) $F(v) = \sup\limits_{\lambda>0} F_\lambda(v)$ .

If $\sup\limits_{\lambda>0} F_\lambda(v) = +\infty$, from (1.3)bis $F(v) = +\infty$ and there is still equality.

In any case $F_\lambda(v) \uparrow F(v)$ as $\lambda \downarrow 0$ .

b) Let $z_0 \in V$ , $z_0 \in D(F)$ i.e. $F(z_0) < +\infty$ ; let $u,v \in V$

$F_\lambda(v) = \operatorname*{Inf}\limits_{z\in V} \{F(z) + \frac{1}{2\lambda}\|v-z\|^2\}$ ; given $\varepsilon_k > 0$ , $\varepsilon_k \to 0$

for every $k \in \mathbb{N}$ , there exist $z_k \in V$ such that :

$$(1.7) \qquad F_\lambda(v) \leqslant F(z_k) + \frac{1}{2\lambda}\|v-z_k\|^2 \leqslant F_\lambda(v) + \varepsilon_k \leqslant F(z_0) + \frac{1}{2\lambda}\|v-z_0\|^2 + \varepsilon_k .$$

By the same argument as in part a), we get :

$$\|v-z_k\|^2 \leqslant 2\lambda\left[F(z_0)+\varepsilon_k+\alpha\|z_k\|^2+\beta\right] + \|v-z_0\|^2$$

and this implies that

(1.8) $\|z_k\| \leq C(\|v\|)$ independently of $\lambda$ (for $\lambda \in ]0,\Lambda_0[$ , $\Lambda_0<+\infty$) and $k \in \mathbb{N}$ , with $C(\|v\|) = c'_1 \|v\| + c'_2$ $(c'_1, c'_2 \in R^+)$

By definition of $F_\lambda(u)$

$$F_\lambda(u) \leq F(z_k) + \frac{1}{2\lambda}\|u-z_k\|^2$$

$$\leq \{F(z_k)+ \frac{1}{2\lambda}\|v-z_k\|^2\} + \frac{1}{2\lambda}\{\|u-z_k\|^2-\|v-z_k\|^2\} \quad ;$$

so from (1.7)

$$F_\lambda(u) - F_\lambda(v) \leq \varepsilon_k + \frac{1}{2\lambda}\{\|u-z_k\|^2-\|v-z_k\|^2\} \; ;$$

by definition of $H$

$$\frac{1}{2}\|v-z_k\|^2 \geq \frac{1}{2}\|u-z_k\|^2 + \langle H(u-z_k),v-u\rangle_{(V',V)} \; ;$$

so

$$F_\lambda(u) - F_\lambda(v) \leq \varepsilon_k + \frac{1}{\lambda}\|H(u-z_k)\|.\|v-u\|$$

$$\leq \varepsilon_k + \frac{1}{\lambda}\|u-z_k\|.\|v-u\| \; ; \text{ from (1.8)}$$

$$\leq \varepsilon_k + \frac{1}{\lambda}[\|u\|+C(\|v\|)].\|v-u\| .$$

Making $k \longrightarrow +\infty$ , $\varepsilon_k \to 0$ and echanging $v$ and $u$ , we finally get

$$|F_\lambda(u)-F_\lambda(v)| \leq \frac{1}{\lambda} C(\|u\|,\|v\|).\|v-u\|$$

where $C(\|u\|,\|v\|) = c_1 \|u\| + c_2 \|v\| + c_3$.

(1.9) ___Lemma___ : ___Let___ $V$ ___a general Banach space, and,___ $F, G : V \longrightarrow ]-\infty,+\infty]$ ___two convex lower semi-continuous proper functionals satisfying___ :

*(i)* $\quad G \leqslant F$

*(ii)* $\quad \exists (g_i)_{i \in \mathbb{N}}$ <u>*a dense subset of*</u> $V$ <u>*such that*</u> :

$\forall \ i \in \mathbb{N}, \quad G_1(g_i) \geqslant F_1(g_i)$ <u>*then,*</u> $\quad F = G$ .

<u>Proof of Lemma 1.9</u> :

From (i), $F \geqslant G$ ; let us prove the converse inequality :

From (ii) $\forall \ i \in \mathbb{N}, \quad G_1(g_i) \geqslant F_1(g_i)$ .

From the lemma (1.3)b and the density of the sequence $(g_i)_{i \in \mathbb{N}}$ in $V$ it follows :

(1.10) $$\forall \ v \in V, \quad G_1(v) \geqslant F_1(v) .$$

From (i) $G_1 \leqslant F_1$ so, finally, $G_1 = F_1$ .

Now we remark that for any functional $G$

$$(G_\lambda)^* = (G \nabla \frac{1}{2\lambda} \|\cdot\|^2)^* = G^* + 2\lambda(\|\cdot\|^2)^* .$$

So the equality $G_1 = F_1$ implies that

$$G^* + 2(\|\cdot\|^2)^* = F^* + 2(\|\cdot\|^2)^* \quad \text{i.e.}$$

(1.11) $\quad F^* = G^*$ and $F^{**} = G^{**}$ ; since $F$ and $G$ have been assumed convex, l.s.c., $F = F^{**}$ , $G = G^{**}$ and finally

$$F = G .$$

(1.12) <u>Remark</u>

(a) The conclusion of the Lemma 1.9 still holds if, instead of taking in (ii) the Yosida index equal to one, we take it equal to some $\lambda_0 > 0$ .

(b) If $F$ and $G$ are only assumed lower semi-continuous, the conclusion of (1.9) is still valid under the assumptions :

(i) $G \leqslant F$

(ii)$_{bis}$ $\exists\, (g_i)_{i \in \mathbb{N}}$ a dense subset of $V$, $\exists\, (\lambda_j)_{j \in \mathbb{N}}$ a sequence $\lambda_j \downarrow 0$ such that

$$\forall\, (i,j) \in \mathbb{N} \times \mathbb{N}, \quad G_{\lambda_j}(g_i) \geqslant F_{\lambda_j}(g_i) \quad .$$

Now, we can state the main result of this chapter :

(1.13) Definition

Let $V$ a reflexive Banach space with a strictly convex norm, and $F : V \longrightarrow ]-\infty,+\infty]$ a convex, lower semi-continuous proper function. Let $\forall\, \lambda>0$, $\forall\, v \in V$, $F_\lambda(v) = \underset{z\in V}{\operatorname{Inf}} \{F(z) + \frac{1}{2\lambda} \|v-z\|^2\}$, this minimum is achieved at a unique point that we shall denote $J_\lambda^F(v)$ :

$$(1.14) \qquad \forall\, \lambda>0 \;,\; \forall\, v \in V, \;\; F_\lambda(v) = F(J_\lambda^F v) + \frac{1}{2\lambda} \|v-J^F v\|^2 .$$

From the classical theorem of additivity of the subdifferentials, $J_\lambda^F(v)$ satisfies the extremality relation :

$$(1.15) \qquad \partial F(J_\lambda^F v) + \frac{1}{\lambda} H(J_\lambda^F v-v) \ni 0 \quad \text{i.e.}$$

$$(1.15)\text{bis} \qquad \frac{1}{\lambda} H(v-J_\lambda^F v) \in \partial F(J_\lambda^F v) \; .$$

*(1.16) Theorem*

*Let $V$ a reflexive separable Banach space and $F : V \longrightarrow ]-\infty,+\infty]$ a convex, lower semi-continuous proper functional. Let $(g_i)_{i \in \mathbb{N}}$ a dense, denumbrable subset of $V$ : for any $i \in \mathbb{N}$ we define :*

$$(1.17) \qquad u_i = J_1^F(g_i) \quad \text{and}$$

$$(1.18) \qquad \forall\, i \in \mathbb{N}, \; \forall\, v \in \mathbb{N}, \quad F^i(v) = F(u_i) + \langle H(g_i-u_i), v-u_i\rangle_{(V',V)} \; .$$

*Then,* $F = \sup_{i \in \mathbb{N}} F^i$ *; consequently, defining*

$$F^r = \sup_{1 \leq i \leq r} F^i \,, \quad (F^r)_{r \in \mathbb{N}}$$ *is an increasing sequence of polyedral, convex, continuous functionals converging to* $F$ .

Proof of Theorem 1.16 :

From (1.15)bis $H(g_i-u_i) = H(g_i-J_1^F(g_i)) \in \partial F(J_1^F(g_i)) = \partial F(u_i)$ ; so, by definition of $\partial F$ :

$$\forall\, v \in V \,, \qquad F(v) \geq F(u_i) + \langle H(g_i-u_i), v-u_i \rangle = F^i(v) \;;$$

so $\forall\, i \in \mathbb{N}$, $\quad F \geq F^i$ and, (1.19) $\quad F \geq \sup_{i \in \mathbb{N}} F^i = \sup_{r \in \mathbb{N}} F^r$ .

Now let us prove that :

$$(1.20) \qquad (F^i)_1(g_i) = F_1(g_i) \;;$$

by definition, $\quad (F^i)_1(g_i) = \operatorname{Min}_{z \in V} \{F^i(z) + \frac{1}{2\lambda}\|g_i-z\|^2\}$

$$= \operatorname{Min}_{z \in V} \{\langle H(g_i-u_i), z-u_i \rangle + \frac{1}{2\lambda}\|z-g_i\|^2\} + F(u_i) \,.$$

This minimum is achieved at a point $z_i$ such that :

$$H(g_i-u_i) = H(g_i-z_i) \,.$$

Since $H$ is strictly monotone, $z_i = u_i$ and,

$$(F^i)_1(g_i) = \frac{1}{2\lambda}\|u_i-g_i\|^2 + F(u_i) = F_1(g_i) \quad \text{i.e. (1.20).}$$

(1.21) So, $\quad (\sup_j F^j)_1(g_i) \geq (F^i)_1(g_i) = F_1(g_i)$ .

From (1.19) and (1.21) $\quad \begin{cases} \sup_{j \in \mathbb{N}} F^j \leq F \text{ and} \\ \forall\, i \in \mathbb{N},\ \left(\sup_{j \in \mathbb{N}} F^j\right)_1 (g_i) \geq (F)_1(g_i) \end{cases}$

so, from the Lemma (1.9) it follows :

$$F = \sup_{j \in \mathbb{N}} F^j = \lim_{r \to +\infty} \uparrow F^r \quad .$$

(1.22) <u>Remarks</u>

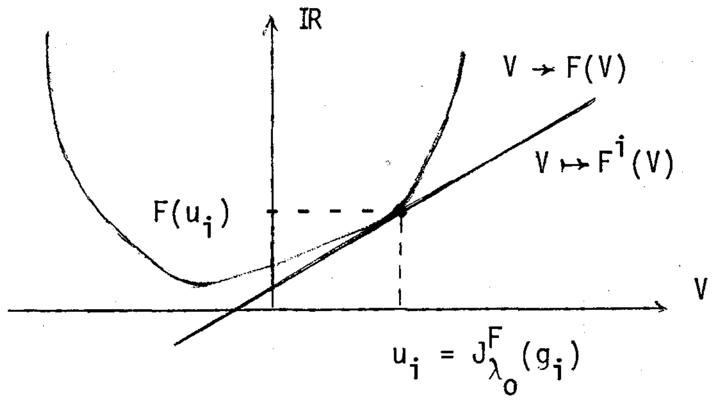
Let us discuss the signification of the Theorme (1.16).

a) We know that every convex lower semi-continuous functional is equal to a supremum of continuous affine functionals :

(1.23) $$F = F^{**} \iff \forall\, v \in V \qquad F(v) = \sup_{f \in V'} \{ \langle v,f \rangle_{(V,V')} - F^*(f) \} \; .$$

The Theorem (1.16) tells us that if the space $V$ is <u>reflexive and separable</u> any convex, l.s.c., proper functional $F$ <u>is equal to the supremum of a denumbrable family of such affine functionals</u> $(F^i)_{i \in \mathbb{N}}$ .
Moreover, we can take for the $(F^i)_{i \in \mathbb{N}}$ affine functionals, whose graph is a supporting hyperplane to the graph of $F$ .
The Theorem tells us how to construct such $(F^i)_{i \in \mathbb{N}}$ :

Take $(g_i)_{i \in \mathbb{N}}$ any denumbrable dense subset of $V$ , and $\lambda_0 > 0$ .
Let $u_i = J^F_{\lambda_0}(g_i)$ and $F^i(v) = F(u_i) + \frac{1}{\lambda_0} \langle H(g_i - u_i), v - u_i \rangle$ .
We remark that $H(g_i - u_i) \in \partial F(u_i)$ i.e. $F^i$ is a supporting hyperplane :



We emphazize on the fact that one has to construct these supporting hyperplanes in a precise way, through the resolution of auxiliary variational problems involving $F$ :

The simpler argument which would consist taking any dense family $(f_i)_{i \in \mathbb{N}}$ in $V'$ and writing (1.23)

$$\forall v \in V, \quad F(v) = \sup \{<v,f_i> - F^*(f_i)\} \text{ is not correct,}$$

since two convex lower semi-continuous functionals may be equal on a dense subset of $V$ and be different ; for example, take

$$V = L^2(\Omega) , \quad F(v) = \|v\|_{H^1(\Omega)} , \; D(F) = H^1(\Omega)$$

$$G(v) = \|v\|_{H_0^1(\Omega)} , \; D(G) = H_0^1(\Omega) ;$$

then $G = F + \Pi_{\{u \in H^1 / u_{|\partial\Omega} = 0\}}$ , $G = F$ on $H_0^1(\Omega)$ which is dense in $L^2$ but $G \neq F$!

b) The interest of such approximation result is that we have succeded writing any convex,l.s.c., proper functional as <u>an increasing limit of a sequence of convex, continuous, polyedral functionals</u> (i.e. regular functionals with a very simple geometry ; by polyedral, we mean a supremum of a finite number of affine functionals).

Moreover, we know to construct in a precise way these polyedral approximations. We shall see in the next chapter how to use this tool in order to obtain a representation theorem for a class of functionals.

Let us examine the geometric interpretation of the Theorem 1.16 when $F$ is equal to the indicator function of a closed, convex, non void set $K$ in $V$ .

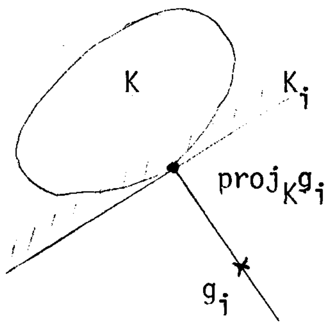
*(1.24)* *<u>Proposition</u>*

*<u>Let</u>* $K$ *<u>a closed convex non void subset of a reflexive Banach space</u>* $V$ .

*Let* $(g_i)_{i\in\mathbb{N}}$ *be a dense denumbrable subset of* $V$ *; then* $K = \bigcap_{i\in\mathbb{N}} K_i$ *where* $K_i$ *is the half-space containing* $K$ *:*

$$K_i = \{v \in V \;/\; \langle H(g_i - \mathrm{proj}_K\, g_i),\ v - \mathrm{proj}_K\, g_i\rangle \leqslant 0\} \quad .$$

Proof of Proposition 1.24 :




From the Theorem 1.16, taking $F = \amalg_K \begin{cases} 0 & \text{on } K \\ +\infty & \text{elsewhere} \end{cases}$, $(g_i)_{i\in\mathbb{N}}$ a dense subset of $V$, $(\lambda_j)_{j\in\mathbb{N}}$ a sequence, $\lambda_j \downarrow 0$, we have :

$$\forall\, v \in V,\quad F(v) = \sup_{i,j} F_{ij}(v) = \sup_{i,j\in\mathbb{N}\times\mathbb{N}} \{F(u_{ij}) + \frac{1}{\lambda_j}\langle H(g_i - u_{ij}), v - u_{ij}\rangle\}$$

where $u_{ij} = J^F_{\lambda_j}(g_i)$ i.e. $u_{ij}$ minimizes : $F(z) + \frac{1}{2\lambda_j}\|z - g_i\|_V^2$ that is to say $u_{ij} = \mathrm{proj}_K\, g_i$ ; since $F(u_{ij}) = 0$ we obtain

$$\forall\, v \in V,\qquad F(v) = \sup_{i,j\in\mathbb{N}\times\mathbb{N}} \{\frac{1}{\lambda_j}\langle H(g_i - \mathrm{proj}_K g_i),\ v - \mathrm{proj}_K g_i\rangle\}$$

$$\amalg_K(v) = F(v) = \sup_{i\in\mathbb{N}} \amalg_{K_i}(v) = \amalg_{\bigcap K_i}(v) \qquad \text{i.e.} \qquad K = \bigcap_i K_i$$

where $K_i = \{v \in V \;/\; \langle H(g_i - \mathrm{proj}_K g_i),\ v - \mathrm{proj}_K g_i\rangle \leqslant 0\}$ .

(1.25) Remarks

a) The Proposition (1.24) tells us that in a reflexive separable Banach space, given a closed, convex, non void set $K$, one can find a dense denumbrable subset of $K$ (in its boundary) such that $K$ is equal to the intersection of supporting hyperplanes to $K$ at these points.

Moreover, one can take such points $(u_i)_{i\in\mathbb{N}}$ in the following way : take $(g_i)_{i\in\mathbb{N}}$ a dense subset of $V$, $u_i = \mathrm{proj}_K g_i$ and for a normal vector of the supporting hyperplane at $u_i$ , $H(g_i-u_i)$ .

b) One can consider the Theorem (1.16) as a corollary of the Proposition (1.24) ; applying the Proposition (1.24) in $V \times \mathbb{R}$ and to the closed convex set equal to the epigraph of $F$ , one can refind (1.16).

Finally, let us look to <u>the dual fromulation of the Theorem</u> (1.16).

*(1.26)* <u>*Proposition*</u>

<u>*Let $V$ a reflexive, separable Banach space .*</u>
<u>*Let $F : V \longrightarrow ]-\infty,+\infty]$ a convex, lower semi-continuous, proper functional and $(g_i)_{i\in\mathbb{N}}$ a dense subset of $V$ .*</u>

<u>*We define for any*</u> $r \in \mathbb{N}$ , $E_r = \overline{Conv}\ [u_i = J_1^F(g_i) \ / \ 1 \leq i \leq r]$ <u>*and*</u>

$$F^r = \begin{cases} F & \text{on } E_r \\ +\infty & \text{elsewhere.} \end{cases}$$

<u>*Then, the sequence*</u> $(F^r)_{r\in\mathbb{N}}$ <u>*is a decreasing sequence and*</u> $F$ <u>*is equal to the lower semi-continuous regularization of*</u> $\operatorname*{Inf}_{r\in\mathbb{N}} F^r$. <u>*In other words,*</u> $F^r$ <u>*converges to*</u> $F$ <u>*in Mosco sense.*</u>

<u>Proof of Proposition 1.26</u> :

Clearly, the sequence $F^r$ decreases to the functional $G$ equal to

$$G = \begin{cases} F & \text{on } \bigcup_{r\in\mathbb{N}}' E_r \\ +\infty & \text{elsewhere .} \end{cases}$$

So, $(F^r)_{r\in\mathbb{N}}$ converges in Mosco sense to the lower semi-continuous regularization of $G$ (cf. Mosco [22]), sc $(G)$ .

Let us prove that $sc(G) = F$ :

Clearly, for every $r \in \mathbb{N}$ , $F \leq F^r$ so, $F \leq \operatorname*{Inf}_{r\in\mathbb{N}} F^r = G$ and since $F$

is l.s.c. , $F \leq sc(G)$ .

Let us assume a moment that we can prove the following property :

(1.27) $\forall\, v \in D(F),\ \exists v_r \in E_r$ such that $v_r \xrightarrow[r\to+\infty]{s-V} v$ and

$$F(v) = \lim_{r\to+\infty} F(v_r) \ .$$

From (1.27), it will follow that

$$\forall\, v \in V, \quad F(v) \geq \limsup F(v_r) \geq \liminf G(v_r) \quad (\text{since } G = F \text{ on } \bigcup E_r)$$
$$\geq sc(G)(v) \qquad (\text{by definition of } sc(G)) \ .$$

Let us prove (1.27) ; let us assume first that $v \in D(\partial F)$ ; then $\exists f \in \partial F(v)$ and $v + \partial F(v) \ni f + v$ i.e. $v = J_1^F(v+f)$ (we assume for simplicity that $V$ is a Hilbert space which is identified with its dual : $H = Id^t$) .

From the density of the sequence $(g_i)_{i \in \mathbb{N}}$ in $V$ , we can find a sequence $(h_r)_{r \in \mathbb{N}}$ , $h_r \in \{g_1,\ldots,g_r\}$ such that $h_r \xrightarrow[r\to+\infty]{} v + f$ .

From the continuity of $J_1^F$

$$J_1^F h_r = v_r \xrightarrow[r\to+\infty]{} J_1^F(v+f) = v \ \text{ and } \ v_r \ \text{ belongs to } \ E_r \ .$$

Moreover, from Lemma (1.3) $F_1(h_r) \xrightarrow[r\to+\infty]{} F_1(v+f)$ ; since from 1.14

$$F_1(h_r) = F(J_1^F h_r) + \frac{1}{2}\|h_r - J_1^F h_r\|^2$$

$$F_1(v+f) = F(J_1^F(v+f)) + \frac{1}{2}\|(v+f) - J_1^F(v+f)\|^2$$

and $\|h_r - J_1^F h_r\|^2 \to \|(v+f) - J_1^F(v+f)\|^2$ it follows that

$$F(J_1^F h_r) \xrightarrow[r\to+\infty]{} F(J_1^F(v+f)) = F(v) \quad \text{i.e.} \quad F(v_r) \to F(v) \ .$$

So, (1.27) is proved when $v \in D(\partial F)$ .

Now we use that if $v \in D(F)$ , there exists a sequence $v_k \in D(\partial F)$ such that $v_k \longrightarrow v$ and $\phi(v_k) \longrightarrow \phi(v)$ :
take for example a sequence $(\lambda_k)_{k \in \mathbb{N}}$ , $\lambda_k \longrightarrow 0$ and

$$v_k = J^F_{\lambda_k} v \xrightarrow[k\to+\infty]{s-V} v \ ;$$

one can verify that

$$F(J^F_{\lambda_k} v) \xrightarrow[k\to+\infty]{} F(v) \ .$$

Now, we can conclude by a classical diagonalisation argument :

$$\begin{array}{ccc} \{v^r_k, F(v^r_k)\} & \xrightarrow[r\to+\infty]{} & \{v_k, F(v_k)\} \\ & & \big\downarrow \ {\scriptstyle k\to+\infty} \\ & & \{v, F(v)\} \end{array} \ .$$

From [1] lemma 1 , there exists a sequence $r \longrightarrow k(r)$ such that

$$(v^r_{k(r)}, F(v^r_{k(r)})) \xrightarrow[r\to+\infty]{} (v, F(v))$$

and we take $v_r = v^r_{k(r)} \in E_r$ : $v_r \xrightarrow[r\to+\infty]{s-V} v$, $F(v_r) \longrightarrow F(v)$ .

(1.28) <u>Remark</u>

Let us explain why the Proposition (1.26) is a dual formulation of the Theorem (1.16) ; by the same way we shall get an other proof of the Proposition (1.26).

$$(1.29) \qquad F = \lim \uparrow F^r \implies F^* = sc[\lim \downarrow (F^r)^*] \ .$$

Let us compute $(F^r)^*$ , that is to say the conjugate functional of a polyedral functional :

$$\forall\, f \in V^* \ , (F^r)^*(f) = \sup_{v\in V} \{<f,v> - \sup_{1\le i\le r} \{F(u_i) + <\partial F(u_i), v-u_i>\}\}$$

where $u_i = J_1^F(g_i)$, $\partial F(u_i) = H(g_i - u_i)$ .

Since $F(u_i) + F^*(\partial F(u_i)) = \langle u_i, \partial F(u_i)\rangle_{(V,V')}$ we get

(1.30) $\forall f \in V^*, \forall r \in \mathbb{N}$, $(F^r)^*(f) = \sup_{v\in V} \inf_{1\leq i\leq r} \{\langle f-\partial F(u_i), v\rangle + F^*(\partial F(u_i))\}$ .

Let us prove that

(1.31) $D((F^r)^*) = \overline{\text{Conv}}\ \{\partial F(u_i) \;/\; i=1,\ldots,r\}$ ;

clearly $\forall i \in \{1,\ldots,r\}$ , from (1.30) $(F^r)^*(\partial F(u_i)) \leqslant F^*(\partial F(u_i))$ ;

since $(F^r)^* \geqslant F^*$ , it follows that :

(1.32) $\forall i \in \{1,\ldots,r\}$, $(F^r)^*(\partial F(u_i)) = F^*(\partial F(u_i))$ ;

since $F^{r*}$ is convex its domain is convex and

(1.31)bis $D(F^{r*}) \supset \overline{\text{Conv}}\ \{\partial F(u_i) \;/\; i=1,\ldots,r\}$ ;

let us prove the opposite inclusion : let $f \notin \overline{\text{Conv}}\ \{\partial F(u_i) \;/\; i,\ldots,r\}$ ;
from the Hahn-Banach theorem there exists $v_0 \in V$ such that :

$$\langle f, v_0\rangle \geqslant \sup_{1\leqslant i\leqslant r} \langle \partial F(u_i), v_0\rangle \quad .$$

So $\inf_{1\leqslant i\leqslant r} \{\langle f-\partial F(u_i), v\rangle > 0$ , which implies that

$$\sup_{v\in V} \inf_{1\leqslant i\leqslant r} \{\langle f-\partial F(u_i), v\rangle + F^*(\partial F(u_i))\} = +\infty$$

and from (1.30) this means that $(F^r)^*(f) = +\infty$; so (1.31) is proved.

Since the complete description of $(F^r)^*$ on its domain is rather complicate, we define the functional $G^r$ :

(1.33) $$G^r(f) = \begin{cases} F^*(f) & \text{if } f \in \overline{\text{Conv}}\ \{\partial F(u_i) \;/\; 1\leqslant i\leqslant r\} \\ +\infty & \text{elsewhere} \end{cases}$$

then,

(1.34) $$F^* \leqslant G^r \leqslant (F^r)^* .$$

The first inequality is clear from (1.33) ; the second one follows from

$$D(G^r) = D((F^r)^*) \quad \text{and} \quad G^r(f) = F^*(f) \leqslant (F^r)^*(f) \quad \text{on} \quad D(G^r) .$$

From (1.34) it follows that :

(1.35) $$F^* = sc \left[\lim \downarrow G^r\right] .$$

Now we remark that

(1.36) $$u_i = J_1^F(g_i) \implies \partial F(u_i) = J_1^{F^*}(g_i)$$

(in order to prove (1.36) one use the definition of $J_1^F$ and the property $(\partial F)^{-1} = \partial F^*$) ; finally

(1.37) $$G^r(f) = \begin{cases} F^*(f) & \text{if } f \in \overline{\text{Conv}}\ \{J_1^{F^*}(g_i) \ / \ 1 \leqslant i \leqslant r\} \\ +\infty & \text{elsewhere.} \end{cases}$$

From (135) and (1.37) taking $G = F^*$ , we refind the conclusions of the Proposition (1.26).

*(1.38) Corollary : A Galerkin procedure for lower semi-continuous functionals.*

*Let $V$ a separable reflexive Banach space, $F : V \to ]-\infty,+\infty]$ a convex, lower semi-continuous, proper functional which is coercive.*
*Let $(g_i)_{i \in \mathbb{N}}$ a dense subset of $V$ ; we define*

$$\forall i \in \mathbb{N}, \quad u_i = J_1^F(g_i) \quad \text{and}$$

$$\forall r \in \mathbb{N}, \quad E_r = \overline{Conv}\ \{J_1^F(g_i) \ / \ 1 \leqslant i \leqslant r\} .$$

*Then, for every $f \in V'$*

$$\underset{v \in E_r}{Min} \{F(v) - <f,v>\} \xrightarrow[r \to +\infty]{} \underset{v \in V}{Min} \{F(v) - <f,v>\} .$$

Proof of Corollary (1.38)

It is a direct application of the Proposition (1.26) :

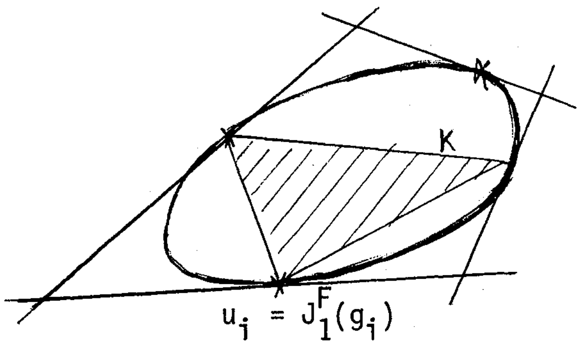
$$F^r = F + \mathbb{I}_{E_r} \text{ converge in Mosco sense to } F .$$

Since $F^r \geqslant F$ and $F$ is coercive it follows (cf. [22]) that

$$\forall f \in V', \qquad \underset{v \in V}{Min} \{F^r(v) - <f,v>\} \longrightarrow \underset{v \in V}{Min} \{F(v) - <f,v>\}$$

(1.39) Remark :

We saw that, geometrically, the Theorem 1.16 corresponds to an external approximation of a closed convex set ; its dual formulation corresponds to an internal approximation :




Let us illustrate the flexibility of the preceding approximation methods : given $K$ a closed, convex, non void set in a separable, reflexive Banach space, we shall denote :

$$\begin{cases} K^h \text{ its external approximation : } K = \bigcap_{h \in \mathbb{N}} K^h \text{ given by Proposition 1.24 .} \\ K_h \text{ its internal approximation : } K = \overline{\bigcup_{h \in \mathbb{N}} K_h} \text{ given by Proposition 1.26 .} \end{cases}$$

*(1.40) Proposition*

*Let $\Phi : V \to ]-\infty,+\infty]$ a convex, lower semi-continuous, proper* *functional.*

a) *Then,* $\phi + \mathbb{I}_{K^h} \uparrow \phi + \mathbb{I}_K$ ; *therefore* $(\phi+\mathbb{I}_{K^h})_{h \in \mathbb{N}}$ *converges to* $\phi + \mathbb{I}_K$ *in Mosco sense and if* $\phi$ *is coercive*

$$\forall f \in V', \qquad \operatorname*{Min}_{v \in K^h} \{\phi(v) - \langle f,v\rangle\} \xrightarrow[h\to+\infty]{} \operatorname*{Min}_{v\in K} \{\phi(v)-\langle f,v\rangle\} .$$

b) *Let us assume moreover that* $\phi$ *is continuous at a point* $v_0 \in K$ ; *then,* $(\phi+\mathbb{I}_{K_h})_{h \in \mathbb{N}}$ *is a decreasing sequence which converges in Mosco sense to* $\phi+\mathbb{I}_K$ ; *therefore if* $\phi$ *is coercive*

$$\forall f \in V', \qquad \operatorname*{Min}_{v \in K_h} \{\phi(v) - \langle f,v\rangle\} \xrightarrow[h\to+\infty]{} \operatorname*{Min}_{v\in K} \langle\phi(v)-\langle f,v\rangle\} .$$

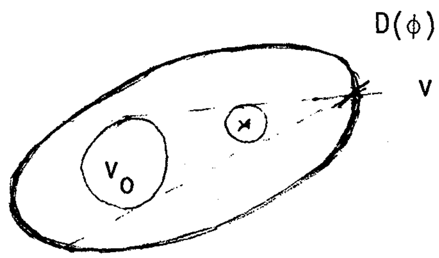
Proof of Proposition (1.40)

a) Clearly $\phi + \mathbb{I}_{K^h} \uparrow \phi + \mathbb{I}_K$ ; it follows by a classical argument (cf. [ ] ) that $(\phi+\mathbb{I}_{K^h})_{h\in\mathbb{N}}$ converges to $\phi + \mathbb{I}_K$ in Mosco sense ;

b) Let us prove that $(\phi+\mathbb{I}_{K_h})_{h\in\mathbb{N}}$ converges to $\phi + \mathbb{I}_K$ in Mosco sense :

$$\forall v_h \xrightarrow{\omega-V} v \ , \quad \liminf\, (\phi+\mathbb{I}_{K_h})(v_h) \geqslant \liminf \phi(v_h) + \liminf \mathbb{I}_{K_h}(v_h)$$
$$\geqslant \phi(v) + \mathbb{I}_K(v) .$$

Let us take $v_0 \in K$ such that $\phi$ is continuous at $v_0$ ; then $v_0 \in \operatorname{Int} D(\phi)$ , $D(\phi) \supset B(v_0,\rho_0)$ ;



Let us take $v \in D(\phi) \cap K$ and let us find a sequence $(v_h)_{h \in \mathbb{N}}$ such that :

$$(\phi+\mathbb{I}_K)(v) = \lim_{h\to+\infty} (\phi+\mathbb{I}_{K_h})(v_h) \quad .$$

Let $v_t = tv + (1-t)v_0$ ; then $v_t \in \mathrm{Int}(D(\phi))$ and $\phi$ is continuous at $v_t$ ; let $(v_h^t)_{h \in \mathbb{N}}$ a sequence, $v_h^t \in K_h$ such that

$$v_h^t \xrightarrow{s-V} v^t \quad \text{(we remark that } v_t \in K \text{ , so } v_h^t \text{ exists!)}$$

then, $\phi(v_h^t) \xrightarrow[h\to+\infty]{} \phi(v^t)$ .

On the other hand, by convexity of $\phi$

$$\phi(v_t) \leqslant t\,\phi(v) + (1-t)\,\phi(v_0) \ .$$

When $t \longrightarrow 1$ , since $\phi(v_0) < +\infty$

$\limsup_{t\to1} \phi\,(v_t) \leqslant \phi(v)$ ; since $v_t \xrightarrow[t\to1]{} v$ by l-semi-continuity of $\phi$

$\phi(v) \leqslant \liminf_{t\to1} \phi(v_t)$ ; so $\phi(v_t) \xrightarrow[t\to1]{} \phi(v)$ ; we have the following diagram :

$$\begin{array}{ccc}
(v_h^t,\phi(v_h^t)) & \xrightarrow[h\to+\infty]{(s-V)\times\mathbb{R}} & (v_t,\phi(v_t)) \\
& & \Big\downarrow {\scriptstyle t\to1} \\
& & (v,\phi(v)) \ .
\end{array}$$

By a classical diagonalisation argument, there exists a sequence $(t_h)_{h \in \mathbb{N}}$ such that

$$(v_h^{t_h},\phi(v_h^{t_h})) \xrightarrow[h\to+\infty]{} (v,\phi(v)) \quad \text{and} \quad v_h^{t_h} \in K_h \ .$$

So $\phi + \mathbb{I}_{K_h} \longrightarrow \phi + \mathbb{I}_K$ in Mosco sense.

(1.41) Remark

a) The interest of the preceding Proposition is that, since one can construct the approximating convex polyedral sets $K^h$ or $K_h$ , one can directly apply this approximation procedure to the minimisation of any convex (continuous at one point when working with $K_h$ ) l.s.c. functional on $K$ .

b) One could also by the same technics approximate evolution problems governed by subdifferentials of convex functionals.

# CH.II INTEGRAL REPRESENTATION OF UNILATERAL CONSTRAINTS

In this paragraph we shall denote

$\mathcal{O}_n$ the family of the open bounded subsets of $\Omega$ .

$\mathcal{B}_n$ the family of the bounded borelian subsets of $\Omega$ .

$$V = W_0^{1,p}(\Omega) \qquad 1<p<+\infty.$$

As we shall see in the next paragraph, any "variational limit" of unilateral constraints has the following properties $(\mathcal{F})$ .
Our purpose, in this paragraph, is to obtain an integral representation theorem for the functionals of $(\mathcal{F})$ :

(2.1) Definition

$\mathcal{F} = \{F : V \times \mathcal{B}_n \to \overline{\mathbb{R}}^+$ satisfying (i), (ii), (iii), (iv), (v)$\}$ :

$(\mathcal{F})$

(i) $\forall\, v \in V$ , $B \mapsto F(v,B)$ is a positive, outer regular, Borel measure.

(ii) $\forall\, \omega \in \mathcal{O}_n$, $v \mapsto F(v,\omega)$ is a convex, l.s.c., proper functional on $V$ .

(iii) $\forall\, \omega \in \mathcal{O}_n$, $v \mapsto F(v,\omega)$ is decreasing.

(iv) $\forall\, u,v \in V$, $\forall\, \omega \in \mathcal{O}_n$ , $u_{|\omega} = v_{|\omega} \Rightarrow F(u,\omega) = F(v,\omega)$ .

(v) $\forall\, u,v \in V$, $\forall\, \omega \in \mathcal{O}_n$ , $F(u\wedge v,\omega) + F(u\vee v,\omega) \leq F(u,\omega) + F(v,\omega)$ .

Remarks

a) It is an open question to know if (V) is a consequence of

(i)...(iv).
Clearly, (v) is not independent of (i)...(iv) : it will follow from the representation theorem that, if $F \in \mathcal{F}$ then, in (v) the equality holds!

b) From the outer regularity of $F(v,.)$ , it follows easily that the properties (iii) and (v) are valid for any $B \in \mathcal{B}_n$ .

In order to state the representation theorem we shall need the following notions of potential theory (cf. [ 2] for example, for more details).

(2.2) <u>Definitions</u>

For any $v \in V = W_0^{1,p}(\Omega)$ , we shall denote by $\tilde{v}$ the class (for the equality quasi-everywhere) of its quasi-continuous representatives. (The notions of capacity are associated with the capacity defined from the norm of $V$).

The positive cone of $V'$, $(W^{-1,p'})^+$, is called the cone of positive, finite energy measures ; the elements of $(V')^+$can be identified with positive Radon measures and if, $\omega_\mu \in (V')^+$ , $\mu$ being the associated Radon measure

$$\forall v \in V , \qquad \langle \omega_\mu , v \rangle_{(V',V)} = \int_\Omega \tilde{v}(x)\, d\mu(x) \quad \text{i.e.} \quad V \underset{\text{continuously}}{\hookrightarrow} L^1(d\mu) .$$

*(2.3)* <u>*Theorem*</u>

<u>*Let*</u> $F \in \mathcal{F}$ *;* <u>*then*</u> $F$ <u>*can be represented as an integral functional*</u> *:*

$$(2.3)bis \quad \forall v \in V, \ \forall \omega \in \mathcal{O}_n , \quad F(v,\omega) = \int_\omega f(x,\tilde{v}(x))\, d\mu(x) + \nu(\omega)$$

<u>*where*</u> *:* *a/* $\mu$ <u>*and*</u> $\nu$ <u>*are two positive Radon measures and*</u> $\mu$ <u>*is a finite energy measure.*</u>

*b/* $f : \Omega_x \times \mathbb{R}_t \to ]-\infty,+\infty]$ <u>*is Borel measurable with respect*</u> *to* $x$ *,* <u>*convex,*</u> *l.s.c.,* <u>*decreasing with respect to*</u> $t$ .

<u>*Moreover, we can take*</u> $\nu(B) = F(u_0,B)$ <u>*with*</u> $u_0 \in V$ , <u>*such that*</u> *:*

$$\forall B \in \mathcal{B}_n , \quad F(u_0,B) < +\infty \text{ and } \forall B \in \mathcal{B}_n , \quad \int_B f(x,t)\, d\mu(x) + \nu(B) \geq 0 .$$

The relation (2.3)bis can be extended to $V \times R(F)$ where $R(F)$ is a rich family of borelian sets (cf. Definition 2.4).

In order to prove the Theorem (2.3) we shall use two types of tools :

a) the approximation result of Ch.I which is relevant of the convex analysis.

b) The measure theory.

In some arguments, as we shall see, these two types of tools will be intimately combined ; let us define now the notion of measure theory we shall use :

(2.4) Definitions

a) A subset D of $\mathcal{B}_n$ is dense in $\mathcal{B}_n$ if :

$\forall A,B \in \mathcal{B}_n$ such that $\bar{A} \subset \overset{\circ}{B}$, $\exists D \in \mathcal{D}$ such that $\bar{A} \subset \overset{\circ}{D} \subset \bar{D} \subset \overset{\circ}{B}$.

b) A subset R of $\mathcal{B}_n$ is rich in $\mathcal{B}_n$ if :

For every family $(B_t)_{t \in [0,1]}$ of elements of $\mathcal{B}_n$ such that : $\forall s<t$, $\bar{B}_s \subset \overset{\circ}{B}_t$, the set $\{t \in [0,1] \;/\; B_t \notin R\}$ is denumbrable.

*(2.5) Proposition*

*a) There exists a denumbrable dense subset of $\mathcal{B}_n$ ;*

*b) A denumbrable intersection of rich subsets of $\mathcal{B}_n$ is still rich ;*

*c) Any rich subset is dense ;*

*d) If $R$ is rich, $R \cap \mathcal{C}_n$ is dense.*

The following property justifies the introduction of the notion of rich subset of $\mathcal{B}_n$ :

*(2.6) Proposition*

*Let $\alpha : \mathcal{B}_n \longrightarrow \bar{\mathbb{R}}^+$ an increasing function ; then, the subset of $\mathcal{B}_n$ formed by the sets $B$ satisfying*

$$\sup_{\overset{o}{\bar{A}} \subset B} \alpha(A) = \alpha(B) = \inf_{\overset{o}{\bar{B}} \subset A} \alpha(A)$$

*is rich in* $\mathcal{B}_n$ .

Proof of Theorem (2.3)

Let $F \in \mathcal{F}$

Step 1 : Let $(f_i)_{i \in \mathbb{N}}$ be a dense denumbrable subset of $V$ .
Let $(\omega_k)_{k \in \mathbb{N}}$ be a dense denumbrable family of open sets in $\mathcal{B}_n$ .

For every $k \in \mathbb{N}$ , we define the functional

(2.13) $$F^k(v) = F(v,\omega_k) \ .$$

Since $\omega_k$ is open, bounded, the functional $F^k : V \longrightarrow \overline{\mathbb{R}}^+$ is convex, lower semi-continuous proper ; we can apply the approximation theorem (1.16) of Ch. I :
denoting $A^k = \partial F^k$ , we define

(2.14) $$\forall (i,k) \in \mathbb{N} \times \mathbb{N}, \quad u_{ik} = J_1^{A^k}(f_i)$$

(2.15) $$\forall (i,k) \in \mathbb{N} \times \mathbb{N}, \quad \mu_{ik} = H(u_{ik}-f_i) \in - \partial F^k(u_{ik})$$

($H$ is the duality mapping from $V$ onto $V^*$).

and by Theorem (1.16)

(2.16) $$\forall k \in \mathbb{N}, \ \forall v \in V, \ F(v,\omega_k) = \sup_{i \in \mathbb{N}} \{F(u_{ik},\omega_k) + \langle \mu_{ik},u_{ik}-v\rangle_{(V,V')}\} \ .$$

Let us look in detail to $\mu_{ik}$ ; in a general way, if $\mu \in - \partial F(u,\omega)$ , by definition of $\partial F(.,\omega)$ :

$$\forall v \in V, \qquad F(u+v,\omega) \geqslant F(u,\omega) - \langle \mu,v\rangle \ .$$

If $v \in V^+$, since $z \longmapsto F(z,\omega)$ is decreasing $(\mathcal{F}_{iii})$ it follows that :

$$F(u,\omega) \geqslant F(u+v,\omega) \geqslant F(u,\omega) - \langle\mu,v\rangle \text{ , i.e.}$$

$\forall\, v \in V^+$ , $\langle\mu,v\rangle \geqslant 0$ ; so $\mu$ is a positive energy measure of $V'^+$ ;
Moreover, if $v|_{\overline{\omega}} = 0$ , from $(\mathcal{F}_{iv})$ it follows that :

$$F(u+v,\omega) = F(u,\omega) \geqslant F(u,\omega) - \langle\mu,v\rangle$$

and $\mu$ is supported by $\overline{\omega}$ ;
denoting for any $v \in V$ , by $\tilde{v}$ its quasi-continuous representative, we can write (2.16) in the following way :

$$(2.17)\quad \forall\, k \in \mathbb{N} \ ,\ \forall\, v \in V \ ,\ F(v,\omega_k) = \sup_{i\in\mathbb{N}} \{F(u_{ik},\omega_k) + \int_{\overline{\omega}_k} (\tilde{u}_{ik}-\tilde{v})d\mu_{ik}\}$$

Step 2 : Let us define

$$(2.18)\quad \forall (i,k) \in \mathbb{N} \times \mathbb{N} \ ,\ \forall\, v \in V \ ,\ \forall\, B \in \mathcal{B}_n \ ,$$

$$F_{ik}(v,B) = F(u_{ik},B \cap \omega_k) + \int_B (\tilde{u}_{ik}-\tilde{v})\, d\mu_{ik}$$

and study the properties of the functionals $F_{ik}$ :

*(2.19) Proposition*

$$\forall (i,k) \in \mathbb{N} \ ,\ \forall\, \omega \in \mathcal{O}_n \ ,\ \forall\, v \in V \ ,\quad F_{ik}(v,\omega) \leqslant F(v,\omega)$$

Proof of Proposition (2.19)

We have to prove that

$$(2.20)\quad \forall (i,k) \in \mathbb{N} \ ,\ \forall\, v \in V \ ,\ F(v,\omega) \geqslant F(u_{ik},\omega \cap \omega_k) + \int_\omega (\tilde{u}_{ik}-\tilde{v})d\mu_{ik}$$

In order to prove (2.20), let us prove first :

$$(2.21)\quad \forall\, \phi \geqslant 0 \ ,\quad F(u_{ik}+\phi,\omega \cap \omega_k) \geqslant F(u_{ik},\omega \cap \omega_k) - \int_\omega \tilde{\phi}\, d\mu_{ik}$$

$$(2.22) \qquad \forall \phi \geqslant 0 \ , \quad F(u_{ik}-\phi, \omega \cap \omega_k) \geqslant F(u_{ik}, \omega \cap \omega_k) + \int_\omega \tilde{\phi}\, d\mu_{ik} \ .$$

Let us assume (2.21) and (2.22) proved and, look how (2.20) follows :
In (2.21) we take $\phi = (v-u_{ik})^+$ , in (2.22) $\phi = (u_{ik}-v)^+$ and add (2.21) and (2.22) :

$$F(u_{ik} \vee v, \omega \cap \omega_k) + F(u_{ik} \wedge v, \omega \cap \omega_k) \geqslant 2F(u_{ik}, \omega \cap \omega_k) + \int_\omega (\tilde{u}_{ik} - \tilde{v})\, d\mu_{ik}$$

(since $(u_{ik}-v)^+ - (v-u_{ik})^+ = u_{ik}-v$) ;

then, applying $(\mathcal{F}_v)$ ,

$$F(u_{ik}, \omega \cap \omega_k) + F(v, \omega \cap \omega_k) \geqslant 2F(u_{ik}, \omega \cap \omega_k) + \int_\omega (\tilde{u}_{ik} - \tilde{v})\, d\mu_{ik}$$

and (2.20) follows.

Let us prove (2.21) and (2.22) :

*(2.23)* *__Lemma__*

$$\forall i,k \in \mathbb{N} \ , \ \forall \omega \in \mathcal{O}_n \ , \ \forall \phi \in V \ , \quad \phi \geqslant 0$$

$$(2.21) \qquad F(u_{ik}+\phi, \omega \cap \omega_k) \geqslant F(u_{ik}, \omega \cap \omega_k) - \int_\omega \tilde{\phi}\, d\mu_{ik} \ .$$

$$(2.22) \qquad F(u_{ik}-\phi, \omega \cap \omega_k) \geqslant F(u_{ik}, \omega \cap \omega_k) + \int_\omega \tilde{\phi}\, d\mu_{ik} \ .$$

__Proof__ __of__ __Lemma__ (2.23)

Let $(\mathcal{O}_n)_{n \in \mathbb{N}}$ an increasing sequence of open sets in $\omega$ such that

$$\mathcal{O}_n \subset \bar{\mathcal{O}}_n \subset \mathcal{O}_{n+1} \subset \ldots \subset \omega \quad \text{and} \quad \bigcup_n \mathcal{O}_n = \omega \ .$$

From the Urysohn lemma, there exist $\theta_n \in D(\Omega)$ satisfying :

$$\theta_n = 1 \ \text{ on } \ \bar{\mathcal{O}}_n \ , \quad \theta_n = 0 \ \text{ on } \ \complement \omega \ , \quad 0 \leqslant \theta_n \leqslant 1 \ .$$

Let $\phi_n = \phi \cdot \theta_n$ ; $\phi_n$ satisfies :

$$\phi_n = \phi \text{ on } \bar{\mathcal{O}}_n \; , \phi_n = 0 \text{ on } \complement\omega \; , \quad 0 \leq \phi_n \leq \phi \; .$$

By definition of $\mu_{ik}$ :

$$(2.24) \qquad F(u_{ik}+\phi_n,\omega_k) \geq F(u_{ik},\omega_k) - \int \tilde{\phi}_n \, d\mu_{ik}$$

$$(2.25) \qquad F(u_{ik}-\phi_n,\omega_k) \geq F(u_{ik},\omega_k) + \int \tilde{\phi}_n \, d\mu_{ik} \; .$$

. Let us look first to (2.24) :

$$F(u_{ik}+\phi_n,\omega_k) = F(u_{ik}+\phi_n,\omega_k \cap \mathcal{O}_n) + F(u_{ik}+\phi_n,\omega_k \setminus \mathcal{O}_n) \; .$$

From $(\mathcal{F})_{iii}$ and $(\mathcal{F})_{iv}$ ,

$$(2.26) \qquad F(u_{ik}+\phi_n,\omega_k) \leq F(u_{ik}+\phi,\omega_k \cap \mathcal{O}_n) + F(u_{ik},\omega_k \setminus \mathcal{O}_n) \; .$$

Moreover

$$(2.27) \qquad F(u_{ik},\omega_k \setminus \mathcal{O}_n) \leq F(u_{ik},\omega_k) < +\infty \quad (\text{since } v \mapsto F(v,\omega_k) \text{ is proper,}$$

and from the definition of $u_{ik}$) . Combining (2.24), (2.26) we get

$$(2.28) \qquad F(u_{ik}+\phi,\omega_k \cap \mathcal{O}_n) + F(u_{ik},\omega_k \setminus \mathcal{O}_n)$$

$$\geq F(u_{ik},\omega_k \cap \mathcal{O}_n) + F(u_{ik},\omega_k \setminus \mathcal{O}_n) - \int \tilde{\phi}_n \, d\mu_{ik} \; ,$$

and using (2.27), we can substract $F(u_{ik},\omega_k \setminus \mathcal{O}_n)$ to this inequality :

$$F(u_{ik}+\phi,\omega_k \cap \mathcal{O}_n) \geq F(u_{ik},\omega_k \cap \mathcal{O}_n) - \int \tilde{\phi}_n \, d\mu_{ik}$$

$$\geq F(u_{ik},\omega_k \cap \mathcal{O}_n) - \int_\omega \tilde{\phi} \, d\mu_{ik} \; .$$

When $n \to +\infty$ , since $\bigcup_n (\omega_k \cap \mathcal{O}_n) = \omega_k \cap (\cap \mathcal{O}_n) = \omega_k \cap \omega$

$$F(u_{ik}+\phi,\omega_k \cap \omega) \geq F(u_{ik},\omega_k \cap \omega) - \int_\omega \tilde{\phi} \, d\mu_{ik} \quad \text{i.e. (2.21)} \; .$$

. Let us look now to (2.25) :

$$F(u_{ik}-\phi_n,\omega_k) = F(u_{ik}-\phi_n,\omega_k \cap \bar{\omega}) + F(u_{ik}-\phi_n,\omega_k \setminus \bar{\omega}) \ .$$

From $(\mathcal{F})_{iii}$ and $(\mathcal{F})_{iv}$ since $-\phi_n \geqslant -\phi$ and $\phi_n = 0$ on $\complement\, \bar{\omega}$ ,

$$(2.29) \qquad F(u_{ik}-\phi_n,\omega_k) \leqslant F(u_{ik}-\phi,\omega_k \cap \bar{\omega}) + F(u_{ik},\omega_k \setminus \bar{\omega}) \ .$$

Combining (2.25) and (2.29)

$$(2.30) \qquad F(u_{ik}-\phi,\omega_k \cap \bar{\omega}) \geqslant F(u_{ik},\omega_k \cap \bar{\omega}) + \int \tilde{\phi}_n \, d\mu_{ik}$$

$$\geqslant F(u_{ik},\omega_k \cap \omega) + \int_{\mathcal{O}_n} \tilde{\phi} \, d\mu_{ik} \ .$$

Since $\bigcup_n \mathcal{O}_n = \omega$ , making $n \to +\infty$ , we get

$$(2.31) \qquad F(u_{ik}-\phi,\omega_k \cap \bar{\omega}) \geqslant F(u_{ik},\omega_k \cap \omega) + \int_{\omega} \tilde{\phi} \, d\mu_{ik} \ .$$

Applying the Proposition (2.6), (2.5) to the increasing set function

$$\omega \longmapsto F(u_{ik}-\phi,\omega_k \cap \omega) \ ,$$

the family $R$ of the open sets $\omega$ satisfying

$$F(u_{ik}-\phi,\ \omega_k \cap \omega) = \operatorname*{Inf}_{\overset{\circ}{A} \supset \bar{\omega}} F(u_{ik}-\phi,\omega_k \cap A) \quad \text{is dense.}$$

For such an $\omega$ , $F(u_{ik}-\phi,\omega_k \cap \bar{\omega}) = F(u_{ik}-\phi,\omega_k \cap \omega)$ ;

so (2.31) turns into

$$(2.32) \quad \forall\, \omega \in R. \qquad F(u_{ik}-\phi,\omega_k \cap \omega) \geqslant F(u_{ik},\omega_k \cap \omega) + \int_{\omega} \tilde{\phi} \, d\mu_{ik}$$

Let us remark that $R$ depends on $\phi$ , but that's enough in order to conclude since any open set is regular with respect to a borelian measure and (2.32) can be extended to any $\omega \in \mathcal{O}_n$ .

*(2.33) Proposition*

$$\forall \omega \in \mathcal{O}_n\ , \quad \forall v \in V,$$

$$\begin{aligned} F(v,\omega) &= \sup_{(i,k) \in \mathbb{N} \times \mathbb{N}} F_{ik}(v,\omega) \\ &= \sup_{(i,k) \in \mathbb{N} \times \mathbb{N}} \{F(u_{ik}, \omega \cap \omega_k) + \int_\omega (\widetilde{u_{ik}} - \widetilde{v})\, d\mu_{ik}\} \ . \end{aligned}$$

Proof of Proposition (2.33)

From the Proposition (2.19), $\forall (i,k) \in \mathbb{N} \times \mathbb{N}\ ,\ \forall v \in V\ ,\ \forall \omega \in \mathcal{O}_n$

$$F(v,\omega) \geqslant F_{ik}(v,\omega) \ ;$$

so

$$F(v,\omega) \geqslant \sup_{(i,k) \in \mathbb{N} \times \mathbb{N}} F_{ik}(v,\omega) \ .$$

Let us prove the opposite inequality :

$$\forall k \in \mathbb{N}\ , \quad F(v,\omega_k) = \sup_{i \in \mathbb{N}} \{F(u_{ik},\omega_k) + \int_{\overline{\omega}_k} (\widetilde{u_{ik}} - \widehat{v})\, d\mu_{ik}\} \ .$$

If $B \supset \overline{\omega}_k$ ,

$$= \sup_{i \in \mathbb{N}} \{F(u_{ik},\omega_k \cap B) + \int_B (\widetilde{u_{ik}} - \widetilde{v})\, d\mu_{ik}\}$$

$$= \sup_{i \in \mathbb{N}} F_{ik}(v,B) \ .$$

So $\forall B \supset \overline{\omega}_k \quad F(v,\omega_k) \leqslant \sup_{i,j \in \mathbb{N} \times \mathbb{N}} F_{ij}(v,B)$ ;

therefore

$$\sup_{\overline{\omega}_k \subset B} F(v,\omega_k) \leqslant \sup_{i,j \in \mathbb{N} \times \mathbb{N}} F_{ij}(v,B) \ .$$

Taking $B \in \mathcal{O}_n$ , since $F(v,.)$ is a borelian measure, any open set is inner regular and

$\forall \omega \in \mathcal{O}_n\ , \quad F(v,\omega) \leqslant \sup_{i,j \in \mathbb{N} \times \mathbb{N}} F_{ij}(v,\omega)$ ; so, the equality holds.

Step 3 : Using the Proposition (2.33) we are going to prove the representation theorem.

(2.34) Definition

$\forall r \in \mathbb{N}$ , let us define $F_r : V \times \mathcal{B}_n \to \overline{\mathbb{R}}^+$ :

$$\forall r \in \mathbb{N} , \forall v \in V , \quad F_r(v,B) = \sup \left\{ \sum_{i,k=1}^{r} F_{ik}(v,B_{ik}) \ / \ \sum_{i,k=1}^{r} B_{ik} \subset B \right\}$$

i.e. $B \mapsto F_r(v,B)$ is the smallest positive measure which is greater than all the measures $B \mapsto F_{ik}(v,B)$ , $1 \leq i,k \leq r$ .

(By $\sum$ we mean that the $B_{i,k}$ are taken disjoint).

*(2.35) Proposition*

*a)* $\forall r \in \mathbb{N}$ , $F_r \in \mathcal{F}$

*b)* $\forall r \in \mathbb{N}$ , $F_r \leq F_{r+1}$

*c)* $\forall v \in V$ , $\forall \omega \in \mathcal{O}_n$ , $\quad F(v,\omega) = \lim_{(r\uparrow+\infty)} \uparrow F_r(v,\omega)$

*d)* $\forall u,v \in V$ , $\forall B \in \mathcal{B}_n$ ; $\quad |F_r(u,B)-F_r(v,B)| \leq \int_B |\tilde{u}-\tilde{v}| \, d\mu_r$

*with* $\mu_r = \sum_{i,k=1}^{r} \mu_{ik}$ *($\mu_r$ is a positive finite energy measure).*

a) For every $B \in \mathcal{B}_n$ , the functional $v \mapsto F_r(v,B)$ , as a supremum of convex, lower semi-continuous functional is still convex lower semi-continuous ;
moreover

$$\forall r \in \mathbb{N} , \ \forall \omega \in \mathcal{O}_n , \ \forall v \in V , \quad F_r(v,\omega) \leq F_{r+1}(v,\omega) \leq F(v,\omega) .$$

The last inequality follows from (2.33), and the definition of $F_r$ :

From (2.33) $\forall v \in V$ , $\forall \omega \in \mathcal{O}_n$ , $F(v,\omega) \geq F_{ik}(v,\omega)$ .

This inequality can be extended to $\mathcal{B}_n$ since

$$\forall B \in \mathcal{B}_n \ , \quad F(v,B) = \inf_{\omega \supset B} F(v,\omega)$$

and since $B \longrightarrow F_{ik}(v,B)$ is a borelian measure which is finite on any compact subset of $\omega$, it is a regular measure and

$$\forall B \in \mathcal{B}_n \ , \quad F_{ik}(v,B) = \inf_{\omega \supset B} F_{ik}(v, \omega);$$

so the inequality

$$F(v,B) \geqslant F_{ik}(v,B)$$

holds for any $(i,k) \in \mathbb{N} \times \mathbb{N}$ , $v \in V$ , $B \in \mathcal{B}_n$ ;
since $B \mapsto F(v,B)$ is a positive borelian measure by definition of $F_r$ it follows

$$\forall r \in \mathbb{N} \ , \ \forall v \in V \ , \ \forall B \in \mathcal{B}_n \ , \quad F(v,B) \geqslant F_r(v,B) \ .$$

So $$F(v,B) \geqslant \sup_{r \in \mathbb{N}} F_r(v,B) \ ;$$

on the other hand

$$\forall i,k \in \mathbb{N} \ , \ \forall \ 1 \leqslant i,k \leqslant r \ , \quad F_r(v,B) \geqslant F_{ik}(v,B) \ ;$$

so, $$\sup_{r \in \mathbb{N}} F_r(v,B) \geqslant F_{ik}(v,B) \ ;$$

since this is true for every $(i,k) \in \mathbb{N}$ , $\sup\limits_r F_r(v,B) \geqslant \sup\limits_{i,k} F_{i,k}(v,B)$

and from (2.33)

$$\sup_r F_r(v,B) \geqslant \sup_{i,k} F_{ik}(v,B) = F(v,B) \quad \text{if} \quad B \in \mathcal{O}_n \ .$$

Finally, $\forall v \in V \ , \ \forall \omega \in \mathcal{O}_n \ , \quad F(v,\omega) = \sup\limits_{r \in \mathbb{N}} F_r(v,\omega) \ .$

This implies that every functional $F_r(.,\omega)$ is proper since $F(.,\omega)$ is proper.

From Definition (2.34) one easily verify that properties (iii) and (iv) of $\mathcal{F}$ are satisfied.

The only property $\mathcal{F}$ which is not obvious to verify is the property (v) :

Take $(B_{ik})$ and $(C_{ik})$ two subdivisions of $B$ ; then

$$I = \sum_{i,k} \left[F(u_{ik},B_{ik}\cap\omega_k) + \int_{B_{ik}} (\widetilde{u_{ik}}-\widetilde{u\wedge v})d\mu_{ik}\right]$$

$$+ \sum_{i,k} \left[F(u_{ik},C_{ik}\cap\omega_k) + \int_{C_{ik}} (\widetilde{u_{ik}}-\widetilde{u\vee v})d\mu_{ik}\right]$$

$$= \sum_{i,k} \left[F(u_{ik},B_{ik}\cap\omega_k) + F(u_{ik},C_{ik}\cap\omega_k)\right]$$

$$+ \sum_{i,k} \int_{B_{ik}\cap\{u<v\}} (\widetilde{u_{ik}}-\widetilde{u})d\mu_{ik} + \int_{C_{ik}\cap\{u\geqslant v\}} (\widetilde{u_{ik}}-\widetilde{u})d\mu_{ik}$$

$$+ \sum_{i,k} \int_{B_{ik}\cap\{u\geqslant v\}} (\widetilde{u_{ik}}-\widetilde{v})d\mu_{ik} + \int_{C_{ik}\cap\{v>u\}} (\widetilde{u_{ik}}-\widetilde{v})d\mu_{ik} \; .$$

Taking $D_{ik} = [B_{ik}\cap\{u<v\}] \cup [C_{ik}\cap\{u\geqslant v\}]$ , $\sum D_{ik} \subset B$

$$E_{ik} = [B_{ik}\cap\{u<v\}] \cup [C_{ik}\cap\{v>u\}] \; , \quad \sum E_{ik} \subset B$$

and remarking that these sets are two by two disjoint, we can write

$$I = \left[\sum_{i,k} F(u_{ik},D_{ik}\cap\omega_k) + \int_{D_{ik}} (\widetilde{u_{ik}}-\widetilde{u})d\mu_{ik}\right]$$

$$+ \left[\sum_{i,k} F(u_{ik},E_{ik}\cap\omega_k) + \int_{E_{ik}} (\widetilde{u_{ik}}-\widetilde{v})d\mu_{ik}\right]$$

So, $$\sum_{i,k=1}^{r} F_{ik}(u\wedge v,B_{ik}) + \sum F_{ik}(u\vee v,C_{ik}) \leqslant \sum F_{ik}(u,D_{ik}) + \sum F_{ik}(v,E_{ik})$$

$$\leqslant F_r(u,B) + F_r(v,B) \; .$$

Since this is true for any $(B_{ik})$ and $(C_{ik})$ , it follows

$$F_r(u\wedge v,B) + F_r(u\vee v,B) \leqslant F_r(u,B) + F_r(v,B) \; .$$

Finally, starting from $F \in \mathcal{F}$, we have been able to write $F$ as

$$F = \lim \uparrow F_r$$

with $F_r$ belonging still to $\mathcal{F}$ ; the interest of this approximation is that the $F_r$ enjoy strong continuity properties (d) :

Let $u, v \in V$ and $B \in \mathcal{B}_n$ ; for every $(B_{ik})_{1 \leq i,k \leq r}$ with $\sum_{i,k=1}^{r} B_{ik} \subset B$

$$F_{ik}(v, B_{ik}) - F_{ik}(u, B_{ik}) = \int_{B_{ik}} (\tilde{u}-\tilde{v}) d\mu_{ik} \leq \int_{B_{ik}} |\tilde{u}-\tilde{v}| d\mu_r$$

with $\mu_r = \sum_{i,k=1}^{r} \mu_{ik}$ .

(One may take more precisely $\mu_r = \sup(\mu_{ik} \ / \ i,k = 1,\ldots,r)$ ;

after sommation,

$$\begin{aligned}
\sum_{i,k} F_{ik}(v, B_{ik}) &\leq \sum_{i,k} F_{ik}(u, B_{ik}) + \sum_{i,k} \int_{B_{ik}} |\tilde{u}-\tilde{v}| d\mu_r \\
&\leq \sum_{i,k} F_{ik}(u, B_{ik}) + \int_{B} |\tilde{u}-\tilde{v}| d\mu_r \\
&\leq F_r(u,B) + \int_{B} |\tilde{u}-\tilde{v}| d\mu_r \ ;
\end{aligned}$$

since this is true for every $(B_{ik})_{i,k=1}^{r}$ it follows :

$$F_r(v,B) \leq F_r(u,B) + \int_B |\tilde{u}-\tilde{v}| d\mu_r$$

and

$$|F_r(v,B) - F_r(u,B)| \leq \int_B |\tilde{u}-\tilde{v}| d\mu_r \ .$$

The integral representation of $F_r$ and hence of $F$ will follow from the following proposition :

(2.36) *Proposition*

*Let* $F \in \mathcal{F}$ *and satisfying :* $F : V \times \mathcal{B}_n \longrightarrow \mathbb{R}^+$ *and* $\exists \mu$ *, a positive Radon measure of finite energy, such that :*

$$\forall u,v \in V \ , \quad \forall B \in \mathcal{B}_n \ , \qquad |F(u,B)-F(v,B)| \leq \int_B |\tilde{u}-\tilde{v}|\, d\mu \ .$$

*Then, taking* $u_o \in V$ *, let us denote*

$$\nu(B) = F(u_o,B)$$

$$\theta(B) = \lim_{t \to -\infty} \frac{1}{|t|} F(u_o+t,B) \ .$$

*Then,* $\nu$ *and* $\theta$ *are two positive Radon measures and* $\theta \leq \mu$ *(i.e.* $\theta$ *is of finite energy) ; for every* $t \in \mathbb{R}$ *the measure*

$$B \mapsto F(u_o+t,B) - F(u_o,B) \ \text{ is absolutely continuous with respect to } \ \theta \ .$$

*By the Radon-Nikodym theorem, there exists a function* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying :*

a) $\forall t \in \mathbb{R}$ , $\quad x \longrightarrow f(x,t)$ *is a borelian function and*

$$F(u_o+t,B) - F(u_o,B) = \int_B f(x,t)\, d\theta(x) \ , \quad \forall B \in \mathcal{B}_n \ .$$

b) *For* $\theta$ *a.e.* $x \in \Omega$ , $\ t \mapsto f(x,t)$ *is convex, decreasing*

c) $\forall x \in \Omega$ , $\ f(x,u_o(x)) = 0$ .

*Moreover,*

$$\forall u \in V \ , \quad \forall B \in \mathcal{B}_n \ , \quad F(u,B) = \int_B f(x,\tilde{u}(x))\, d\theta(x) + \nu(B) \ .$$

Proof of Proposition (2.36)

Let us first remark that there is no ambiguity in the notion $F(u_o+t,B)$ since the value of $F(v,B)$ does depend only of the value of $v$ on $B$ ($u_o+t \notin V$!) .

Then, we remark that the function $t \to F(u_0+t,B)$ is convex, decreasing, so

$t<0$. $\dfrac{F(u_0+t,B - F(u_0)}{|t|}$ is increasing and positive.

We denote by $\theta(B) = \lim\limits_{t\to-\infty} \uparrow \dfrac{F(u_0+t,B) - F(u_0)}{|t|} = \lim\limits_{(t\to-\infty)} \uparrow \dfrac{F(u_0+t,B)}{|t|}$ ;

$B \mapsto \theta(B)$ as an increasing limit of positive Radon measure is still a positive Radon measure and from

$$\frac{F(u_0+t,B) - F(u_0)}{|t|} \leqslant \int_B d\mu, \text{ it follows}$$

$0 \leqslant \theta(B) \leqslant \mu(B)$ i.e. $\theta$ is a positive, finite energy measure (since $\mu$ is positive, finite energy measure!).

A posteriori, from the final formula (2.32) it will appear that $\theta$ is the smallest positive measure $\mu$ such that

$$\forall u,v \in V \ , \ \forall B \in \mathcal{B}_n, \ |F(u,B)-F(v,B)| \leqslant \int_B |\tilde{u}-\tilde{v}| d\mu \ .$$

From $0 \leqslant F(u_0+t,B) - F(u_0) \leqslant |t| \ \theta(B)$ , it follows that the measure $B \mapsto F(u_0+t,B) - F(u_0)$ is absolutely continuous with respect to $\mu$ and by R.N. theorem there exits a function $f_t(x)$ borelian-measurable, integrable relatively to $\theta$ such that :

$$\forall B \in \mathcal{B}_n \ , \ F(u_0+t,B) - F(u_0,B) = \int_B f_t(x) \ d\theta(x) \ ;$$

denoting $f(x,t) = f_t(x)$ we have

$$\forall B \in \mathcal{B}_n \ , \quad F(u_0+t,B) - F(u_0,B) = \int_B f(x,t) \ d\theta(x) \ .$$

. From $F(u_0+s,B) - F(u_0,B) = \int_B f(x,s) \ d\theta(x)$

we get, taking the difference

$$\int_B [f(x,t)-f(x,s)]d\theta(x) \leq |F(u_0+t,B)-F(u_0+s,B)| \leq |t-s| \int_B d\mu(x) .$$

Since $\theta \leq \mu$ , $\theta$ is absolutely continuous with respect to $\mu$ and there exists $h$ a borelian functional, $0 \leq h \leq 1$ , such that : $\theta = h\mu$ .

From,

$$\frac{1}{\mu(B)} \int_B [f(x,t)-f(x,s)] h(x) d\mu(x) \leq |t-s| .$$

Taking for $B$ a decreasing sequence of neighbourhood of $x_0$ , we get :

$$\forall s,t \in \mathbb{R} \times \mathbb{R} , \qquad h(x)|f(x,t)-f(x,s)| \leq |t-s| \quad \mu\text{-ppx} .$$

Since $t \mapsto F(u_0+t,B) - F(u_0)$ is convex, decreasing it follows easily that $t \mapsto f(t,x)$ is convex, decreasing : for example,
let $t > s$ ; then

$$\forall B \in \mathcal{B}_n , \quad \int_B f(x,t)\, d\theta(x) \leq \int_B f(x,s)\, d\theta(x)$$

it follows that $\qquad f(x,t) \leq f(x,s) \quad \theta\text{-a.e. } x.$

Let us prove, to end the proof of the Proposition (2.36), the integral representation :

From $\qquad |F(u_0+u,B)-F(u_0,B)| \leq \int_B |\tilde{u}(x)|\, d\mu(x)$

it follows that the measure $B \mapsto F(u_0+u,B) - F(u_0,B)$ is absolutely continuous with respect to the measure $|\tilde{u}|\, d\mu$ and by the Radon-Nikodym theorem, for every $u \in V$ there exists a function $g_u \in L^1(|\tilde{u}|d\mu)$ such that

$$\forall B \in \mathcal{B}_n , \qquad F(u_0+u,B) - F(u_0,B) = \int_B g_u(x)\, |\tilde{u}(x)|d\mu(x) .$$

Taking $u \equiv t$ ,

$$F(u_0+t,B) - F(u_0,B) = \int_B g_t(x)\, |t|d\mu(x) = \int_B f(x,t)d\theta(x) = \int_B f(x,t)\, h(x)d\mu(x) .$$

Since this is true for every $B \in \mathcal{B}_n$ , $\forall\ t \in \mathbb{R}$ , $g_t(x)|t| = f(x,t)\ hx)$ $\mu$-ae.x

Let us now explicit $g_u$ : take $x_0 \in \Omega$ and B a neighbourood of $x_0$ .

$$|F(u_0+u,B)-F(u_0+\tilde{u}(x_0),B)| \leq \int_B |\tilde{u}(x)-\tilde{u}(x_0)|\ d\mu(x)$$

$$\int_B [g_u(x)\,|\tilde{u}(x)|-g_{\tilde{u}(x_0)}(x)\,|\tilde{u}(x_0)|]\ d\mu(x) \leq \int_B |\tilde{u}(x)-\tilde{u}(x_0)|\ d\mu(x)\ .$$

Dividing by $\mu(B)$ and making $\mathcal{B} \downarrow \{x_0\}$ , we get :

$$\mu\ ppx_0 \qquad g_u(x_0)\,|\tilde{u}(x_0)| = g_{\tilde{u}(x_0)}(x_0)\,|\tilde{u}(x_0)|\ .$$

Finally, we get

$$F(u_0+u,B) - F(u_0,B) = \int_B g_u(x)\,|\tilde{u}(x)|\,d\mu(x) = \int_B g_{\tilde{u}(x)}(x)\,|\tilde{u}(x)|\ d\mu(x)$$

and since $\qquad g_{\tilde{u}(x)}(x)\ |\tilde{u}(x)| = f(x,\tilde{u}(x))\ h(x)$

$$F(u_0+u,B) - F(u_0,B) = \int_B g_u(x)\,|\tilde{u}(x)|\,d\mu(x) = \int_B f(x,\tilde{u}(x))\ h(x)\ d\mu(x)$$

and, by definition of h , $\theta = hd\mu$

$$F(u_0+u,B) - F(u_0,B) = \int_B f(x,\tilde{u}(x))\ d\theta(x).$$

So, $\qquad F(u_0+u,B) - F(u_0,B) = \int_B f(x,\tilde{u}(x))\ d\theta(x)$

and $\qquad F(u,B) = \int_B f(x,\tilde{u}(x)-\tilde{u}_0(x))\ d\theta(x) + F(u_0,B)\ .$

<u>End of the proof of Theorem</u> 2.3

For Proposition (2.33) , $\qquad F = \sup F_r$ ; more precisely

$$\forall\ \omega \in \mathcal{O}_n\ ,\ \forall\ v \in V\ , \qquad F(v,\omega) = \lim_{r\to+\infty} \uparrow F_r(v,\omega)\ .$$

Since for every $r \in \mathbb{N}$ , $F_r$ satisfies the hypothesis of the Proposition

(2.36), (2.35(d)), it can be represented as an integral functional :

$$\forall v \in V\ ,\ \forall B \in \mathcal{B}_n\ ,\quad F_r(v,B) = \int_B f_r(x,\tilde{v}(x))\,d\theta_r(x) + \nu_r(B)$$

with $\theta_r$ a positive measure of finite energy and $\nu_r(B) = F_r(u_0,B)$ .
Let us choose $u_0 \in V$ such that :

$B \mapsto F(u_0,B)$ is a positive Radon measure (i.e. $\forall K$ compact, $K \subset \Omega$, $F(u_0,K) < +\infty$) and denote $\nu(B) = F(u_0,B)$ .
Since $\nu_r \leq \nu$ and $\nu$ is a Radon measure, by the Radon-Nikodym theorem, we can write $\nu_r = k_r\, d\nu$ ; so

$$F_r(v,B) = \int_B f_r(x,\tilde{v}(x))\; d\theta_r(x) + \int_B k_r(x)\; d\nu(x)\ .$$

Let us now consider the other term of $F_r(v,B)$ and let us rewrite it also as an integral functional with respect to a fixed measure (independent of $r \in \mathbb{N}$) :

$$\int_B f_r(x,\tilde{v}(x))\; d\theta_r(x) = \int_B 2^r\|d\theta_r\|_{V'}\,.\; f_r(x,\tilde{v}(x))\; \frac{d\theta_r}{2^r\|d\theta_r\|_{V'}}\ .$$

Let us define $dp_r = \sum_{k=1}^{r} \frac{d\theta_k}{2^k\|d\theta_r\|_{V'}}$ and $dp = \lim_{r\to+\infty} dp_r = \sum_{k=1}^{+\infty} \frac{d\theta_k}{2^k\|d\theta_k\|_{V'}}$ .

By construction, $dp$ is a positive Radon measure of finite energy and

$$\forall r \in \mathbb{N}\ ,\qquad \frac{d\theta_r}{2^r\|d\theta_r\|_{V'}} \leq dp\ .$$

Since $dp$ is a positive Radon measure, it follows by application of the Radon-Nikodym theorem that

$$\forall r \in \mathbb{N}\ ,\quad \exists h_r \in L^1(dp)\quad \text{s.t}\qquad \frac{d\theta_r}{2^r\|d\theta_r\|} = h_r.dp\ ;\quad \text{so}$$

$$F_r(v,B) = \int_B 2^r.\|d\theta_r\|.fr(x,\tilde{v}(x))\ h_r(x)\ dp + \int_B k_r(x)\ d\nu\ .$$

Taking $X = p+\nu$ , since $X$ is a positive Radon measure and $p \leq X$ , $\nu \leq X$ applying once more the R.N. theorem

$$p = hX\ ,\quad \nu = kX \quad \text{and finally}$$

$$F_r(v,B) = \int_B \left[2^r \|d\theta_r\|.f_r(x,\tilde{v}(x))\ h_r(x)\ h(x) + k_r(x)\ k(x)\right]\ dX(x)\ .$$

Denoting $g_r(x,t) = 2^r\|d\theta_r\|.f_r(x,t)\ h_r(x)\ h(x) + k_r(x)\ k(x)$ .

We have that for $X$-almost every $x$ , $(g_r(x,\tilde{v}(x)))_{r \in \mathbb{N}}$ is an increasing sequence (This follows from the growth of the sequence $(F_r)_{r \in \mathbb{N}}$) .

By the Beppo-Levy theorem denoting

$g(x,t) = \lim \uparrow g_r(x,t)$, we obtain for every $\omega \in \mathcal{O}_n$

$$F(v,\omega) = \int_\omega g(x,\tilde{v}(x))\ dX(x)\ ;$$

more precisely, since $\nu_r = k_r\ d\nu \uparrow d\nu$ , $k_r \uparrow 1$ , and,

$$2^r\|d\theta_r\|.f_r(x,t)\ h_r\ h \xrightarrow[r \to +\infty]{} g(x,t) - k(x) = f(x,t)\ h(x)\ ;$$

finally (with the convention $f(x,t) = 0$ if $h(x) = 0$ )

$$F(v,\omega) = \int_\omega f(x,\tilde{v}(x))\ h(x)\ dX(x) + \int_\omega k(x)\ dX(x)\ ,$$

$$F(v,\omega) = \int_\omega f(x,\tilde{v}(x))\ dp(x) + \int_\omega \nu(x) = \int_\omega f(x,\tilde{v}(x))\ dp(x) + F(u_0,\omega)$$

which is the conclusion of Theorem 2.3.

*(2.37)* *Corollary*

*Let* $\psi$ *be a function from* $\Omega$ *into* $\overline{\mathbb{R}}$ *and let us assume that*

*(i)* $\exists u_0 \in V$ *s.t* $u_0(x) \geq \psi(x)$ , *V-quasi everywhere.*

*Let us define :*

$$(v,B) \in V \times \mathcal{B}_n , \quad F(v,B) = \begin{cases} 0 & \text{if } v(x) \geqslant \psi(x) \text{ q.e. on } B \\ +\infty & \text{elsewhere.} \end{cases}$$

Then $F_{|V \times \mathcal{O}_n}$ belongs to $\mathcal{F}$ ; consequently, there exists $\mu$ a positive measure which does not charge the polar sets, and an integrand $f$ borelian with respect to $x$ , convex decreasing with respect to $t$ , such that :

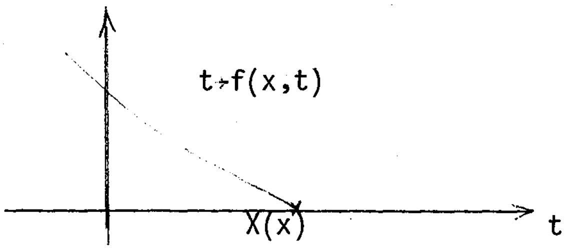
$$\forall v \in V , \forall \omega \in \mathcal{O}_n , \quad \tilde{v}(x) \geqslant \psi(x) \text{ q.e. on } \omega \Longleftrightarrow \int_\omega f(x,\tilde{v}(x))\, d\mu(x) = 0 .$$

This last equality turns into $f(x,\tilde{v}(x)) = 0$ , $\mu$ a.e. $x$

(Let us remark that in the integral representation $\nu(\omega) = F(u_0,\omega) = 0$) .

Since $t \mapsto f(x,t)$ is convex, decreasing, and positive

$$f(x,t) = 0 \Longleftrightarrow t \geqslant X(x) .$$




Finally, there exists a function $x \to X(x)$ borelian such that

(2.38) $\forall v \in V , \forall \omega \in \mathcal{O}_n , \quad \tilde{v}(x) \geqslant \psi(x)$ q.e. on $\omega \Longleftrightarrow \tilde{v}(x) \geqslant X(x)$ $\mu$ a.e. on $\omega$.

Let us remark that $\tilde{v}(x) \geqslant \psi(x)$ q.e. on $\omega \Longrightarrow \tilde{v}(x) \geqslant \psi(x)$ $\mu$ a.e. on $\omega$ (since $\mu$ does not charge sets of zero capacity); consequently

$$\forall v \in V , \forall \omega \in \mathcal{O}_n , \quad \tilde{v}(x) \geqslant X(x)\ \mu \text{ a.e. on } \omega \Longrightarrow \tilde{v}(x) \geqslant \psi(x)\ \mu\text{.a.e. on } \omega,$$

$$\text{and} \quad \tilde{X}(x) \geqslant \psi(x) \quad \mu \text{ a.e. on } \omega ,$$

where $\tilde{X}$ (resp. $\tilde{\psi}$) is the quasi-s.c.s. regularization of $X$ (resp. $\psi$) . So

$$(2.39) \qquad \tilde{X} \geqslant \tilde{\psi} \ \mu \text{ a.e. on } \omega .$$

On the other hand, there exists a sequence $v_n \in V \cap \mathscr{C}_c(\Omega)$ such that $v_n(x) \downarrow \tilde{\psi}(x)$ quasi everywhere.

Since $\tilde{v}_n \geqslant \psi$ q.e. $\Longrightarrow \tilde{v}_n \geqslant X(x)$ $\mu$ a.e. on $\omega$, going to the limit as $n \to +\infty$, we obtain

$$(2.40) \qquad \tilde{\psi} \geqslant X \ \mu \text{ a.e. on } \omega .$$

This implies that $\tilde{\psi} \geqslant \tilde{X}$ $\mu$ a.e. on $\omega$ and finally

$$\tilde{\psi} = \tilde{X} \ \mu \text{ a.e. on } \omega .$$

Since $\tilde{v} \geqslant X$ $\mu$ a.e. on $\omega \Longleftrightarrow \tilde{v} \geqslant \tilde{X}$ $\mu$ a.e. on $\omega$ we finally get

$$(2.41) \ \forall v \in V, \forall \omega \in \mathcal{O}_n , \ \tilde{v}(x) \geqslant \psi(x) \text{ q.e. on } \omega \Longleftrightarrow \tilde{v}(x) \geqslant \tilde{\psi}(x) \ \mu \text{ a.e. on } \omega.$$

If we start with $\psi$ which is quasi s.c.s. the formulation is simpler :

$$(2.42) \ \forall v \in V, \forall \omega \in \mathcal{O}_n, \ \tilde{v}(x) \geqslant \psi(x) \text{ q.e. on } \omega \Longleftrightarrow \tilde{v}(x) \geqslant \psi(x) \ \mu \text{ a.e. on } \omega.$$

So for any obstacle $\psi$ there exists a measure $\mu_\psi$ (which depends on $\psi$!) such that, $\mu_\psi$ is a positive Radon measure of finite energy and, it is equivalent to take the constraint $\tilde{u} \geqslant \psi$ in capacity sense or $\mu$-measure sense.

If $\psi$ is regular, i.e. $\psi \in V$ we refind a classical result of potential theory ; taking $\mu = dx$ the Lebesgue measure and $\omega$ an open set

$$\tilde{v} \geqslant 0 \text{ q.e. on } \omega \Longleftrightarrow \tilde{v} \geqslant 0 \ \mu \text{ a.e. on } \omega .$$

Let us observe finally that (2.42) can be extended to $V \times R(F)$ where $R(F)$ is a rich family of borelian subsets of $\Omega$ .

# CH.III $\Gamma$-LIMITS OF OBSTACLES

In this paragraph, we still denote

$$V = W_0^{1,p}(\Omega) \ , \quad 1<p<+\infty \ ,$$

and introduce two classes of functionals : the energy functionals $\mathcal{E}_p$ and the constraint functionals $\mathcal{F}_\gamma$.

(3.1) $\mathcal{E}_p$ is the family of functionals $\phi : V \to \mathbb{R}^+$ of the following type : $\phi(v) = \int_\Omega f(x,Dv(x))dx$ with

$$(3.2) \qquad \begin{cases} \lambda|z|^p \leqslant f(x,z) \leqslant M(1+|z|^p) \\ x \mapsto f(x,z) \text{ is borelian measurable.} \\ z \mapsto f(x,z) \text{ is convex continuous.} \end{cases}$$

For any $\omega \in \mathcal{O}_n$ and $v \in V$ we denote $\phi(v,\omega) = \int_\omega f(x,Dv(x))dx$ .

We recall the following compactness result concerning the family $\mathcal{E}_p$ :

(3.3) Given a sequence $(\phi_h)_{h \in \mathbb{N}}$ of functionals of $\mathcal{E}_p$ , one can extract a subsequence $(\phi_{h_k})_{k \in \mathbb{N}}$ such that :

$\forall v \in V$ , $\forall \omega \in \mathcal{O}_n$ , $\Gamma^-(s\text{-}L^p(\Omega)) \lim\limits_{k\to+\infty} \phi_{h_k}(v,\omega) = \phi(v,\omega)$ exist,

and $\phi$ still belongs to $\mathcal{E}_p$ .

(3.4) $\mathcal{F}_\gamma$ is the family of the functionals of $\mathcal{F}$ satisfying :

$$\forall\, \omega \in \mathcal{O}_n \ , \qquad \operatorname*{Inf}_{v\in V} \left[F(v,\omega)+\|v\|_V^p\right] \leqslant \gamma(\omega) < +\infty \ .$$

Now we can give the statement of the main result of this chapter :

(3.5) *Theorem* (*)

*Let* $(\phi_h)_{h\in\mathbb{N}}$ *a sequence of functionals of the class* $\mathcal{E}_p$ .
*Let* $(F_h)_{h\in\mathbb{N}}$ *a sequence of functionals of the class* $\mathcal{F}_\gamma$ .
*Then, there exists a subsequence* $(h_k)_{k\in\mathbb{N}}$ *, a rich family* $R$ *of borelian subsets of* $\Omega$ *, and two functionals* $\phi$ *and* $F$ *belonging respectively to* $\mathcal{E}_p$ *and* $\mathcal{F}_\gamma$ *such that*

*(i)* $\forall\, v \in V$ , $\qquad \phi(v) = \Gamma^-(s\text{-}L^p(\Omega)) \lim \phi_{h_k}(v)$

*(ii)* $\forall\, v\in V$ , $\forall\,\omega \in \mathcal{O}_n \cap R$ ,

$$\phi(v) + F(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega)) \lim_{k\to+\infty} \left[\phi_{h_k}(v)+F_{h_k}(v,\omega)\right] \ .$$

*The functional* $F$ *can be represented :*

$$F(v,\omega) = \int_\omega h(x,\tilde{v}(x))\,d\mu(x) + \nu(\omega)$$

*where* $\mu$ *and* $\nu$ *are two positive Radon measures,* $\mu$ *of finite energy ;* $h(x,t)$ *is borelian with respect to* $x$ *, convex decreasing and lower semi-continuous with respect to* $t$ .

*(3.5)bis Corollary*

*Let* $(\psi_h)_{h\in\mathbb{N}}$ *a sequence of functions* $\psi_h : \Omega \to \overline{\mathbb{R}}$ *such that :*

$$\exists\, (v_h)_{h\in\mathbb{N}},\ \forall\, h\in\mathbb{N}\ ,\quad v_h \in W_0^{1,p}\ ,\ v_h \geqslant \psi_h \ \text{ and } \sup_{h\in\mathbb{N}} \|v_h\|_{W_0^{1,p}} < +\infty \ ,$$

$$(p \in\, ]1,\infty[) \ .$$

---

(*) In this paragraph we shall prove the Theorem (3.5) only in the case of quadratic energy functionals.

*Then, there exists a subsequence* $(h_k)_{k\in\mathbb{N}}$ *, a rich family* $R$ *of borelian subsets of* $\Omega$ *, such that :*

$\forall f \in W^{-1,p'}(\Omega)$ , $\forall \omega \subset R$

$$\underset{\substack{v\in W_o^{1,p}(\Omega)\\ v\geq\psi_{h_k} \text{ on } \omega}}{\mathrm{Min}} \left\{\int_\Omega |Dv|^p dx - \langle f,v\rangle\right\} \xrightarrow[k\to+\infty]{} \underset{v\in W_o^{1,p}(\Omega)}{\mathrm{Min}} \left\{\int_\Omega |Dv|^p dx + \int_\omega h(x,v(x))\,d\mu(x) - \langle f,v\rangle\right\}.$$

*where* $\mu$ *is a positive finite energy measure,* $h$ *is borelian with respect to* $x$ *, convex, l.s.c., decreasing with respect to* $t$ .

Proof of Theorem (3.5)

Step 1 : From (3.3) we can extract a subsequence $(\phi_{h_k})_{k\in\mathbb{N}}$ such that

$$\forall v\in V ,\ \forall \omega\in \mathcal{O}_n ,\ \Gamma^-(s\text{-}L^p(\Omega))\lim \phi_{h_k}(v,\omega) = \phi(v,\omega) \quad \text{with} \quad \phi\in\mathcal{E}_p .$$

From now on, we shall work on this subsequence and therefore may assume that the $\Gamma^-$lim of the sequence $(\phi_h)_{h\in\mathbb{N}}$ exists.

Let $D$ be a dense denumbrable family of open sets ; from the classical compactness theorem ( cf. [15] ), and a diagonalisation argument, we can extract a subsequence (that we still denote $h_k$) such that :

$$(3.6)\quad \forall v\in V ,\ \forall \omega\in D ,\quad \Gamma^-(s\text{-}L^p(\Omega))\lim_{k\to+\infty} [\phi_{h_k}(v)+F_{h_k}(v,\omega)] \text{ exists .}$$

Let us define two functionals $F^+$ and $F^-$ : $\forall v\in V$ , $\forall \omega\in\mathcal{O}_n$

$$(3.7)\quad \begin{cases} \phi(v) + F^+(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega))\lim\sup_{k\to+\infty}[\phi_{h_k}(v)+F_{hk}(v,\omega)] \\ \phi(v) + F^-(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega))\lim\inf_{k\to+\infty}[\phi_{h_k}(v)+F_{h_k}(v,\omega)] . \end{cases}$$

By definition of the $\Gamma^-$lim , and from (3.6), (3.7), we have :

(3.8) $\forall\, v \in V\ ,\quad \forall\, \omega \in D\ ,\qquad F^{+}(v,\omega) = F^{-}(v,\omega)\ .$

From the classical properties of the $\Gamma^{-}$limit and the properties of the functionals $(F_h)_{h \in \mathbb{N}}$ and $(\phi_h)_{h \in \mathbb{N}}$ it follows easily that the functionals $F^{+}$ and $F^{-}$ are lower semi continuous with respect to $v$ and positive increasing with respect to $\omega$.

Let us now show how these regularity properties are enough, in order to extend the equality (3.8) to $V \times R(F)$, where $R(F)$ is a rich family of borelian sets :

(3.9) Definition

Let $\mathcal{G}$ the class of the functionals $G : V \times \mathcal{B}_n \rightarrow \mathbf{R}^{+}$ satisfying :

(i) $\forall\, v \in V\ ,\ B \longmapsto G(v,B)$ is positive increasing

(ii) $\forall\, \mathcal{O} \in \mathcal{O}_n\ ,\ v \longmapsto G(v,\mathcal{O})$ is lower semi-continuous on $V$.

In order to extend the property (3.5) we introduce :

We shall denote $B_e(G)$ the class of all borelian sets $B$ satisfying

$$\forall\, v \in V\ ,\ \forall\, \lambda > 0\ ,\qquad G_\lambda(v,B) = \inf_{\bar{B} \supset \mathring{A}} G_\lambda(v,A)$$

and by $B_i(G)$ the class of all borelian sets $B$ satisfying :

$$\forall\, v \in V\ ,\ \forall\, \lambda > 0\ ,\qquad G_\lambda(v,B) = \sup_{\mathring{B} \supset \bar{A}} G_\lambda(v,A)$$

and $B(G) = B_e(G) \cap B_i(G)$ .

*(3.10) Proposition*

*For any functional $G$ of $\mathcal{G}$, the sets $B_e(G)$, $B_i(G)$ and $B(G)$ are rich in $\mathcal{B}_n$.*

Proof of Proposition (3.10)

Let us take a sequence $(\lambda_i)_{i \in \mathbb{N}}$ , $\lambda_i > 0$ , $\lambda_i \xrightarrow[i\to+\infty]{} 0$ and, since $V$ is separable a sequence $(v_j)_{j \in \mathbb{N}}$ dense in $V$ .
For each $(i,j) \in \mathbb{N} \times \mathbb{N}$ , the set $R_{i,j}$ of all borelian sets $B$ satisfying

$$(3.11) \qquad \sup_{\bar{A} \subset \overset{\circ}{B}} G_{\lambda_i}(v_j,A) = G_{\lambda_i}(v_j,B) = \inf_{\bar{B} \subset \overset{\circ}{A}} G_{\lambda_i}(v_j,A)$$

is rich in $\mathcal{B}_n$ , by Proposition (2.6).
Let us take $R = \bigcap_{(i,j) \in \mathbb{N} \times \mathbb{N}} R_{i,j}$ ; by Proposition (2.5) $R$ is still rich in $\mathcal{B}n$ and (3.11) is satisfied for any $B \in R$ and any $(i,j) \in \mathbb{N} \times \mathbb{N}$ ; using the lipschitz continuity properties of $G_\lambda(v,B)$ with respect to $v$ and $\lambda$ , we can go to the limit on (3.11) and obtain

$$\forall v \in V , \forall \lambda > 0 , \forall B \in R , \quad \sup_{\bar{A} \subset \overset{\circ}{B}} G_\lambda(v,A) = G_\lambda(v,B) = \inf_{\overset{\circ}{A} \supset \bar{B}} G_\lambda(v,A) ;$$

so, $B(G) \supset R$ and $B(G)$ is rich.

In the following proposition, we shall see that the sets $B(G)$ enjoy some continuity prolongation properties relatively to the functional $G$ :

*(3.12) Proposition*

*Let $G^1$ and $G^2$ two functionals of the class $\mathcal{G}$ and let us suppose that :*
*$\forall v \in V$ , $\exists$ $D(v)$ a dense subset of $\mathcal{B}_n$ (which may depend on $v$) such that :*

$$\forall D \in D(v) , \quad G^1(v,D) = G^2(v,D) .$$

*Then, $B(G^1) = B(G^2)$, and $\forall B \in B(G)$ , $\forall \lambda > 0$ , $\forall v \in V$ , $G^1_\lambda(v,B) = G^2_\lambda(v,B)$ ; consequently :*
*$\forall v \in V$ , $\forall \mathcal{O} \in B(G) \cap \mathcal{O}_n$ , $\quad G^1(v,\mathcal{O}) = G^2(v,\mathcal{O})$ .*

Proof of Proposition (3.12)

Let us prove that $\forall \lambda>0$ , $\forall v \in V$ , $\forall B \in \mathcal{B}_n$

$$(3.13) \qquad \inf_{\overset{\circ}{A}\supset\bar{B}} G^1_\lambda(v,A) \geqslant \inf_{\overset{\circ}{A}\supset\bar{B}} G^2_\lambda(v,A) \geqslant \sup_{\overset{\circ}{B}\supset\bar{A}} G^2_\lambda(v,A) \geqslant \sup_{\overset{\circ}{B}\supset\bar{A}} G^1_\lambda(v,A) .$$

If $B \in \mathrm{B}(G^1)$ then (3.13) implies clearly that $B \in B(G^2)$ i.e. $B(G^1) \subset B(G^2)$ ; by symmetry, we can prove that $B(G^2) \subset B(G^1)$, and the equality $B(G^1) = B(G^2)$ will follow. Moreover, we shall get

$$(3.14) \qquad \forall \lambda>0 , \forall B \in B(G) , \forall v \in V , \qquad G^1_\lambda(v,B) = G^2_\lambda(v,B) .$$

Going to the limit on (3.14) when $\lambda$ goes to zero, we obtain :

$$(3.15) \qquad \forall B \in B(G) \cap \mathcal{O}_n , \forall v \in V , \qquad G^1(v,B) = G^2(v,B) .$$

So, let us prove (3.13) and let us begin by proving the left inequality :

$$(3.16) \qquad \inf_{\overset{\circ}{A}\supset\bar{B}} G^1_\lambda(v,A) \geqslant \inf_{\overset{\circ}{A}\supset\bar{B}} G^2_\lambda(v,A) .$$

By definition of the infimum, given $\varepsilon_i > 0$ , $\varepsilon_i \to 0$ , for every $i \in \mathbb{N}$ we can find $A_i \in \mathcal{B}_n$ , $\overset{\circ}{A}_i \supset \bar{B}$ such that

$$\inf_{\overset{\circ}{A}\supset\bar{B}} G^1_\lambda(v,A) \geqslant G^1_\lambda(v,A_i) - \varepsilon_i ;$$

by definition of $G^1_\lambda(v,A_i)$

$$\forall i \in \mathbb{N} , \exists\, v_i \in V \text{ s.t. : } \qquad G^1_\lambda(v,A_i) \geqslant G^1(v_i,A_i) + \frac{1}{2\lambda} \|v-v_i\|^2_V - \varepsilon_i .$$

Combining the two last inequalities we get

$$\inf_{\overset{\circ}{A}\supset\bar{B}} G^1_\lambda(v,A) \geqslant G^1(v_i,A_i) + \frac{1}{2\lambda} \|v-v_i\|^2_V - 2\varepsilon_i .$$

By assumption, there exists a dense subset of $\mathcal{B}_n$ ,$D(v_i)$ such that

$$\forall D \in D(v_i) , \qquad G^1(v_i,D) = G^2(v_i,D) .$$

Since $\overset{\circ}{A}_i \supset \overline{B}$ and $D(v_i)$ is dense :

$$\forall i \in \mathbb{N} , \ \exists D_i \in D(v_i) \ \text{ s.t. : } \overline{B} \subset \overset{\circ}{D}_i \subset \overline{D}_i \subset \overset{\circ}{A}_i \subset A_i .$$

Since $G^1(v_i,.)$ is increasing and $A_i \supset D_i$

$$\begin{aligned}
\inf_{\overset{\circ}{A}\supset\overline{B}} G^1_\lambda(v,A) &\geqslant G^1(v_i,D_i) + \frac{1}{2\lambda} \|v-v_i\|^2_V - 2\varepsilon_i \\
&\geqslant G^2(v_i,D_i) + \frac{1}{2\lambda} \|v-v_i\|^2_V - 2\varepsilon_i \\
&\geqslant \operatorname*{Inf}_{z\in V} \{G^2(z,D_i) + \frac{1}{2\lambda}\|v-v_i\|^2_V\} - 2\varepsilon_i = G^2_\lambda(v,D_i) - 2\varepsilon_i \\
&\geqslant \operatorname*{Inf}_{\overset{\circ}{A}\supset\overline{B}} G^2_\lambda(v,A) - 2\varepsilon_i \quad \text{and since this is true } \forall i \in \mathbb{N} ,
\end{aligned}$$

$$\operatorname*{Inf}_{\overset{\circ}{A}\supset B} G^1_\lambda(v,A) \geqslant \operatorname*{Inf}_{\overset{\circ}{A}\supset\overline{B}} G^2_\lambda(v,A) \quad \text{(and by symmetry there is equality!).}$$

Let us now prove the right inequality of (3.13)

$$\sup_{\overset{\circ}{B}\supset\overline{A}} G^2_\lambda(v,A) \geqslant \sup_{\overset{\circ}{B}\supset\overline{A}} G^1_\lambda(v,A) . \tag{3.17}$$

Let $A$ be fixed, $\overline{A} \subset \overset{\circ}{B}$ and let us prove that

$$G^1_\lambda(v,A) \leqslant \sup_{\overline{C}\subset\overset{\circ}{B}} G^2_\lambda(v,C) . \tag{3.18}$$

Since $\overline{A} \subset \overset{\circ}{B}$ there exist $C \in \mathcal{B}_n$ such that $\overline{A} \subset \overset{\circ}{C} \subset \overline{C} \subset \overset{\circ}{B}$ ;

By definition of $G^1_\lambda(v,A)$ ,

$$\forall z \in V , \qquad G^1_\lambda(v,A) \leqslant G^1(z,A) + \frac{1}{2\lambda} \|v-z\|^2 .$$

For every $z \in V$ , there exists by definition a dense subset $D(z)$ of $\mathcal{B}_n$

such that $\forall B \in D(z)$ , $G^1(z,B) = G^2(z,B)$ .

Since $D(z)$ is dense and $\bar{A} \subset \mathring{C}$, there exist $D_z \in D(z)$ such that

$$\bar{A} \subset \mathring{D}_z \subset \bar{D}_z \subset \mathring{C} \subset \bar{C} \subset \mathring{B} \text{ , therefore}$$

$$\forall z \in V \text{ , } \quad G^1_\lambda(v,A) \leqslant G^1(z,D_z) + \frac{1}{2\lambda} \|v-z\|^2$$
$$\leqslant G^2(z,D_z) + \frac{1}{2\lambda} \|v-z\|^2$$
$$\leqslant G^2(z,C) + \frac{1}{2\lambda} \|v-z\|^2 \text{ .}$$

Since this is true for every $z \in V$

$$G^1_\lambda(v,A) \leqslant \operatorname*{Inf}_{z \in V} \left\{G^2(z,C) + \frac{1}{2\lambda} \|v-z\|^2\right\} = G^2_\lambda(v,C)$$

Since $\bar{C} \subset \mathring{B}$ , we finally get

$$G^1_\lambda(v,A) \leqslant \sup_{\bar{C} \subset \mathring{B}} G^2_\lambda(v,C) \quad \text{i.e. (3.18).}$$

End of Step 1 : We first remark that, by assumption, since the functionals $F_h$ belong to $\mathcal{F}$ $v \to F_h(v,\omega)$ is lower semi-continuous for any $\omega \in \mathcal{O}_n$!

From (3.8) and (3.12) it follows that

$$(3.19) \quad \mathcal{B}(F^+) = \mathcal{B}(F^-) \text{ and } \forall\, v \in V \text{ , } \forall \omega \in \mathcal{B}(F^+) \cap \mathcal{O}_n = \mathcal{B}(F^-) \cap \mathcal{O}_n \text{ ,}$$

$$F^+(v,\omega) = F^-(v,\omega) \text{ ,}$$

so the $\Gamma$-limit exists for any $v \in V$ and $\omega \in \mathcal{B}(F) \cap \mathcal{O}_n$ with $B(F)$ a rich family.

In order to obtain a precise representation of the $\Gamma$-limit we need more information about the dependance of $F^+$ and $F^-$ with respect to $B \in \mathcal{B}_n$ and $v \in V$ .

## Step 2 : Properties of $F^+$ and $F^-$

*(3.24) Lemma*

$$\forall v \in V\ ,\ \forall A,B \in \mathcal{B}_n\ ,\quad F^+(v,A\cup B) \leqslant F^+(v,A) + F^+(v,B)\ .$$

Proof of Lemma (3.24)

By definition of $\Gamma^-(s\text{-}L^p(\Omega))$ lim sup , there exist two sequences $(v_h^A)_{h\in\mathbb{N}}$ , $(v_h^B)_{h\in\mathbb{N}}$ :

$$(3.25)\quad \begin{cases} \phi(v) + F^+(v,A) = \limsup\limits_{h\to+\infty} \left[\phi^h(v_h^A)+F^h(v_h^A,A)\right] , v_h^A \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} v \\ \phi(v) + F^+(v,B) = \limsup\limits_{h\to+\infty} \left[\phi^h(v_h^B)+F^h(v_h^B,B)\right] , v_h^B \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} v\ . \end{cases}$$

Since $v_h^A \vee v_h^B \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} v$ , it follows from the definition of the $\Gamma^-(s\text{-}L^p(\Omega))$ lim sup :

$$(3.26)\quad \phi(v) + F^+(v,A\cup B) \leqslant \limsup_{h\to+\infty} \left[\Phi^h(v_h^A\vee v_h^B)+F^h(v_h^A\vee v_h^B,A\cup B)\right]\ .$$

From the additivity of $A \longmapsto F^h(v,A)$

$$(3.27)\qquad F^h(v_h^A\vee v_h^B,A\cup B) \leqslant F^h(v_h^A\vee v_h^B,A) + F^h(v_h^A\vee v_h^B,B)\ .$$

From the decreasing property of $v \longmapsto F^h(v,A)$

$$(3.28)\qquad \begin{cases} F^h(v_h^A\vee v_h^B,A) \leqslant F^h(v_h^A,A) \\ F^h(v_h^B\vee v_h^B,B) \leqslant F^h(v_h^B,B)\ . \end{cases}$$

From

$$(3.29)\qquad \phi^h(v_h^A\vee v_h^B) + \phi^h(v_h^A\wedge v_h^B) = \phi^h(v_h^A) + \phi^h(v_h^B)$$

it follows combining (3.26), (3.27), (3.28), (3.29) :

$$(3.30)\quad \phi(v) + F^+(v,A\cup B) \leqslant \limsup_{h\to+\infty} [\phi^h(v_h^A)+\phi^h(v_h^B)-\phi^h(v_h^A\wedge v_h^B)+F^h(v_h^A,A)+F^h(v_h^B,B)]$$

$$\leqslant \limsup_{h\to+\infty} [\phi^h(v_h^A)+F^h(v_h^A,A)]+\limsup[\phi^h(v_h^B)+F^h(v_h^B,B)]$$

$$+ \limsup_{h\to+\infty} [-\phi^h(v_h^A\wedge v_h^B)] \ .$$

Since, $v_h^A \wedge v_h^B \xrightarrow[h\to+\infty]{s-L^p(\Omega)} v$ and $\phi = \Gamma^-(s\text{-}L^p(\Omega)) \lim \phi^h$

$$\phi(v) \leqslant \liminf \phi^h(v_h^A \wedge v_h^B) \ ; \text{ so}$$

$$(3.31)\qquad \limsup [-\phi^h(v_h^A\wedge v_h^B)] \leqslant -\phi(v) \ .$$

Finally combining (3.25), (3.30) and (3.31)

$$\phi(v) + F^+(v,A\cup B) \leqslant \phi(v) + F^+(v,A) + \phi(v) + F^+(v,B) - \phi(v) \quad \text{i.e.}$$

$$F^+(v,A\cup B) \leqslant F^+(v,A) + F^+(v,B) \ .$$

*(3.32) Lemma*

*$\forall v \in V$ , $\forall A,B$ open sets satisfying : $\bar{A} \cap \bar{B} = \emptyset$ and $A \cup B \in \mathcal{B}(F^-)$ (which is the rich family of regular borelian sets with respect to $F^-$)*

$$F^-(v,A\cup B) \geqslant F^-(v,A) + F^-(v,B) \ .$$

Proof of Lemma (3.32)

Let $A_\varepsilon \downarrow A$ , $B_\varepsilon \downarrow B$ as $\varepsilon \downarrow 0$ such that $\mathring{A}_\varepsilon \supset \bar{A}$ , $\mathring{B}_\varepsilon \supset \bar{B}$ , $A_\varepsilon \cap B_\varepsilon = \emptyset$ and

$$(3.32)\text{bis}\qquad F^-(v,A\cup B) = \lim_{(\varepsilon\downarrow 0)} \downarrow F^-(v, A_\varepsilon \cup B_\varepsilon) \ ;$$

Let us fix $\varepsilon>0$ ; by definition of $F^-(v,A_\varepsilon \cup B_\varepsilon)$ :

(3.33) $\exists\ v_h^\varepsilon \xrightarrow[h\to+\infty]{} v$ in $s\text{-}L^p(\Omega)$ such that

$$\phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon) = \liminf_{h\to+\infty} [\phi^h(v_h^\varepsilon)+F_h(v_h^\varepsilon,A_\varepsilon \cup B_\varepsilon)] \quad .$$

From the additivity of $F_h(v_h^\varepsilon,.)$ ,

(3.34) $$\phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon) = \liminf_{h\to+\infty} [\phi^h(v_h^\varepsilon)+F^h(v_h^\varepsilon,A_\varepsilon)+F^h(v_h^\varepsilon,B_\varepsilon)] \quad .$$

Let $X_A^\varepsilon \in W^{1,\infty}(\Omega)$ such that $X_A^\varepsilon = 1$ on $A$, $X_A^\varepsilon = 0$ on $\Omega\setminus A_\varepsilon$, $0 \leq X_A^\varepsilon \leq 1$

$X_B^\varepsilon \in W^{1,\infty}(\Omega)$ such that $X_B^\varepsilon = 1$ on $B$, $X_B^\varepsilon = 0$ on $\Omega\setminus B_\varepsilon$, $0 \leq X_B^\varepsilon \leq 1$ .

On the other hand, let

(3.35) $$\begin{cases} z_h^A \to v \text{ in } s\text{-}L^p(\Omega) \text{ with } \phi^h(z_h^A,\Omega\setminus\bar{A}) \to \phi(v,\Omega\setminus\bar{A}) \ . \\ z_h^B \to v \text{ in } s\text{-}L^p(\Omega) \text{ with } \phi^h(z_h^B,\Omega\setminus\bar{B}) \to \phi(v,\Omega\setminus\bar{B}) \ . \end{cases}$$

Let us define

(3.36) $$\begin{cases} v_h^{A,\varepsilon} = X_A^\varepsilon\, v_h^\varepsilon + (1-X_A^\varepsilon) z_h^A \\ v_h^{B,\varepsilon} = X_B^\varepsilon\, v_h^\varepsilon + (1-X_B^\varepsilon) z_h^B \end{cases}$$

and compute $\phi^h(v_h^{A,\varepsilon})$ and $\phi^h(v_h^{B,\varepsilon})$ :

$$Dv_h^{A,\varepsilon} = X_A^\varepsilon.Dv_h^\varepsilon + (1-X_A^\varepsilon)Dz_h^A + (v_h^\varepsilon-z_h^A)DX_A^\varepsilon \quad .$$

For any $t \in ]0,1[$

$$t\, Dv_h^{A,\varepsilon} = tX_A^\varepsilon.Dv_h^\varepsilon + t(1-X_A^\varepsilon)Dz_h^A + (1-t)\,\frac{t}{1-t}\,(v_h^\varepsilon-z_h^A)DX_A^\varepsilon \quad .$$

From the continuity of $f_h$ and the majoration $f_h(x,z) \leq M(1+|z|^p)$ ,

$$f_h(x,tDv_h^{A,\varepsilon}) \leq tX_A^\varepsilon\, f_h(x,Dv_h^\varepsilon) + t(1-X_A^\varepsilon)\, f_h(x,Dz_h^A) + (1-t)\, M[\int_\Omega 1+|v_h^\varepsilon-z_h^A|^p|DX_A^\varepsilon|^p]$$

So

$$(3.37)\quad \begin{cases} \phi^h(tv_h^{A,\varepsilon}) \leqslant \int_{A_\varepsilon} f_h(x,Dv_h^\varepsilon)dx + \int_{\Omega\setminus\bar{A}} f_h(x,Dz_h^A)dx \\ \qquad\qquad + (1-t)M \int_\Omega [1+|v_h^\varepsilon - z_h^A|^p |DX_A^\varepsilon|^p]\, dx \\ \phi^h(tv_h^{B,\varepsilon}) \leqslant \int_{B_\varepsilon} f_h(x,Dv_h^\varepsilon)dx + \int_{\Omega\setminus\bar{B}} f_h(x,Dz_h^B)dx \\ \qquad\qquad + (1-t)M \int_\Omega [1+|v_h^\varepsilon - z_h^B|^p\, DX_B^\varepsilon|^p]\, dx\ . \end{cases}$$

From (3.34) ,

$$(3.38)\quad \phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon)$$
$$\geqslant \liminf\Big[\int_{A_\varepsilon} f_h(x,Dv_h^\varepsilon)dx + \int_{B_\varepsilon} f_h(x,Dv_h^\varepsilon)dx + F_h(v_h^\varepsilon,A_\varepsilon) + F_h(v_h^\varepsilon,B_\varepsilon)\Big]$$
$$+ \liminf\Big[\int_{\Omega\setminus(A_\varepsilon\cup B_\varepsilon)} f_h(x,Dv_h^\varepsilon)dx\Big] \quad ;$$

moreover

$$(3.39)\qquad \liminf_{h\to+\infty} \int_{\Omega\setminus(A_\varepsilon\cup B_\varepsilon)} f_h(x,Dv_h^\varepsilon)dx \geqslant \int_{\Omega\setminus(A_\varepsilon\cup B_\varepsilon)} f(x,Dv)dx\ .$$

Combining (3.37), (3.38),

$$(3.40)\quad \phi(v) + F^-(v,A_\varepsilon\cup B_\varepsilon) \geqslant \liminf_{h\to+\infty}[\phi^h(tv_h^{A,\varepsilon})+F_h(v_h^\varepsilon,A_\varepsilon)]$$
$$+ \liminf_{h\to+\infty}[\phi^h(tv_h^{B,t})+F_h(v_h^\varepsilon,B_\varepsilon)]$$
$$+ \liminf_{h\to+\infty}\Big[-(1-t)M\ \{\int_\Omega 2+|v_h^\varepsilon - z_h^A|^p|DX_A^\varepsilon|^p + |v_h^\varepsilon - z_h^B|^p|DX_B^\varepsilon|^p\}\Big]$$
$$+ \liminf_{h\to+\infty}\Big[-\int_{\Omega\setminus\bar{A}} f_h(x,Dz_h^A)dx - \int_{\Omega\setminus\bar{B}} f_h(x,Dz_h^B)dx\Big]$$

$$+ \liminf_{h\to+\infty} \Big[\int_{\Omega\setminus(A_\varepsilon \cup B_\varepsilon)} f_h(x,Dv_h^\varepsilon)dx\Big].$$

From (3.33), (3.35), (3.39), (3.40) turns into

$$\begin{aligned}\phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon) \geqslant\ & \liminf_{h\to+\infty}\big[\phi^h(tv_h^{A,\varepsilon}) + F_h(v_h^\varepsilon,A_\varepsilon)\big] \\ & + \liminf_{h\to+\infty}\big[\phi^h(tv_h^{B,\varepsilon}) + F_h(v_h^\varepsilon,B_\varepsilon)\big] \\ & - \int_{\Omega\setminus A} f(x,Dv)dx - \int_{\Omega\setminus B} f(x,Dv)dx + \int_{\Omega\setminus(A_\varepsilon \cup B_\varepsilon)} f(x,Dv)dx .\end{aligned}$$

We now use the minorations : $F_h(v_h^\varepsilon,A_\varepsilon) \geqslant F_h(v_h^\varepsilon,A) = F_h(v_h^{\varepsilon,A},A)$

since $v_h^\varepsilon = v_h^{\varepsilon,A}$ on $A$

$$F_h(v_h^\varepsilon,B_\varepsilon) \geqslant F_h(v_h^\varepsilon,B) = F_h(v_h^{\varepsilon,B},B) .$$

On the other hand, we use $\phi^h(tz) = |t|^p\ \phi^h(z)$(*) and get

$$\begin{aligned}\phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon) \geqslant\ & t^p \liminf_{h\to+\infty}\big[\phi^h(v_h^{A,\varepsilon}) + F_h(v_h^{\varepsilon,A},A)\big] \\ & + t^p \liminf\big[\phi^h(v_h^{B,\varepsilon}) + F_h(v_h^{\varepsilon,B},B)\big] \\ & - \int_{\Omega\setminus A} f(x,Dv)dx - \int_{\Omega\setminus B} f(x,Dv)dx + \int_{\Omega\setminus(A_\varepsilon \cup B_\varepsilon)} f(x,Dv)dx .\end{aligned}$$

From the definition of $F^-(v,A)$ and $F^-(v,B)$ , since $v_h^{A,\varepsilon} \xrightarrow{s-L^p(\Omega)} v$ and $v_h^{B,\varepsilon} \xrightarrow{s-L^p(\Omega)} v$

$$\begin{aligned}\phi(v) + F^-(v,A_\varepsilon \cup B_\varepsilon) \geqslant\ & t^p\big[\phi(v) + F^-(v,A)\big] + t^p\big[\phi(v) + F^-(v,B)\big] \\ & - \int_{\Omega\setminus A} f(x,Dv)dx - \int_{\Omega\setminus B} f(x,Dv)dx + \int_{\Omega\setminus(A_\varepsilon \cup B_\varepsilon)} f(x,Dv)dx .\end{aligned}$$

---

(*) When $\phi^h$ does not verify this condition, it is easy to adapt the prove since

$$\phi^h(v_h^{A,\varepsilon}) - \phi^h(tv_h^{A,\varepsilon}) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 1^- \quad \text{uniformly in } h.$$

Making $t \to 1$ and $\varepsilon \to 0$, from (3.32)bis

$$\phi(v) + F^-(v,A\cup B) \geqslant 2\phi(v) + F^-(v,A) + F^-(v,B)$$

$$- \int_{\Omega\setminus A} f(x,Dv)dx - \int_{\Omega\setminus B} f(x,Dv)dx + \int_{\Omega\setminus(A\cup B)} f(x,Dv)dx$$

i.e. $\qquad F^-(v,A\cup B) \geqslant F^-(v,A) \cup F^-(v,B)$ .

*(3.41)* <u>*Lemma*</u>

$$\forall\, B \in \mathcal{B}_n\,,\quad v \longmapsto F^+(v,B) \ \ \underline{\textit{is}}\ \underline{\textit{decreasing}}.$$

<u>Proof of Lemma</u> (3.41)

Let $B \in \mathcal{B}_n$ and $v^1, v^2 \in V$ with $v^1 \leqslant v^2$ ; by definition of $F^+(v^1,B)$

(3.42) $\quad \phi(v^1) + F^+(v^1,B) = \limsup\limits_{h\to+\infty}[\phi^h(v^1_h)+F^h(v^1_h,B)]\ ,\quad v^1_h \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} v^1$ .

Let us consider the sequence $v^1_h \vee v^2_h \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} v^1 \vee v^2 = v^2$ ; (*)

(3.42) $\quad \phi(v^2) + F^+(v^2,B) \leqslant \limsup\limits_{h\to+\infty}[\phi^h(v^1_h \vee v^2_h)+F^h(v^1_h \vee v^2_h,B)]$ .

From

(3.43) $\quad \phi^h(v^1_h \vee v^2_h) + \phi^h(v^1_h \wedge v^2_h) = \phi^h(v^1_h) + \phi^h(v^2_h)$ and

(3.44) $\qquad F^h(v^1_h \vee v^2_h,B) \leqslant F^h(v^1_h,B)$ , it follows :

$$\phi(v^2) + F^+(v^2,B) \leqslant \limsup[\phi^h(v^1_h)+F^h(v^1_h,B)] + \limsup[\phi^h(v^2_h)]$$

$$+ \limsup[-\phi^h(v^1_h \wedge v^2_h)]$$

$$\leqslant \phi(v^1) + \phi(v^2) + F^+(v^1,B) - \phi(v^1)\,.$$

---

(*) The sequence $(v^2_h)_{h\in\mathbb{N}}$ is taken such that $\phi^h(v^2_h) \to \phi(v^2)$ .

We have use the inequality $\limsup\,[-\phi^h(v_h^1 \wedge v_h^2)] \leqslant -\phi(v^1)$ , or, equivalently, $\phi(v^1) \leqslant \liminf \phi^h(v_h^1 \wedge v_h^2)$ ; this last inequality follows from : $v_h^1 \wedge v_h^2 \longrightarrow v^1 \wedge v^2 = v^1$ in $s\text{-}L^p(\Omega)$ and $\phi = \Gamma^-(s\text{-}L^p(\Omega))\lim \phi^h$.

Finally,

$$F^+(v^2,B) \leqslant F^+(v^1,B) \ .$$

*(3.45)* *Lemma*

$$\forall\, \omega \in \mathcal{O}_n \ , \quad v \longmapsto F^+(v,\omega) \quad \underline{\textit{is convex}} \quad (p=2)$$

Proof of Lemma (3.45)

We assume that $\phi^h(v,\omega) = \displaystyle\int_\omega \sum_{i,j=1}^N a_{ij}^h(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx$

$$\phi(v,\omega) = \int_\omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx$$

with $a_{ij}^h = a_{ji}^h$ , $\forall\, h \in \mathbb{N}$ , $\forall\, i,j=1,\ldots,N$ ; we denote

$$A^h v = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}^h \frac{v}{\partial x_j}\right) \ , \quad Av = -\sum \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial v}{\partial x_j}\right)$$

$\phi^h(v,\Omega) = \phi^h(v) = (A^h v,v) = a_h(v,v)$

$\phi(v,\Omega) = \phi(v) = (Av,v) = a(v,v)$ .

For any $v \in V$ , let us define $v_h^*$ by :

(3.46) $\qquad A^h v_h^* = Av \quad$ i.e. $\quad v_h^* = (A^h)^{-1} Av = L_h(v)$ .

Then,

(3.47) $\quad v_h^* \xrightarrow[(h\to+\infty)]{\omega\text{-}H_0^1} v \quad$ and $\quad \forall\, i=1,\ldots,N \quad \sum_j a_{ij}^h \dfrac{\partial v_h^*}{\partial x_j} \xrightarrow[h\to+\infty]{\omega\text{-}L^2} \sum_j a_{ij} \dfrac{\partial v}{\partial x_j}$ .

Let us prove that :

(3.48) $\forall v \in V, \ \forall \omega \in \mathcal{O}_n,$

$$F^+(v,\omega) = \underset{\{z_h \xrightarrow{s-L^2} 0\}}{\mathrm{Min}} \ \limsup_{h\to+\infty} \left[\phi^h(z_h)+F_h(z_h+L_h(v),\omega)\right] .$$

By definition of $F^+(v,\omega)$ ,

$$\phi(v) + F^+(v,\omega) = \underset{\{v_h \xrightarrow{s-L^2} v\}}{\mathrm{Min}} \ \limsup_{h\to+\infty} \left[\phi^h(v_h)+F_h(v_h,\omega)\right] .$$

Let $v_h^*$ defined by (3.47) and let us write $v_h = v_h^* + z_h$ ; then

$$\phi(v) + F^+(v,\omega) = \underset{\{z_h \to 0\}}{\mathrm{Min}} \ \limsup_{h\to+\infty} \left[\phi^h(v_h^*+z_h)+F_h(v_h^*+z_h,\omega)\right] .$$

Let us compute

$$\phi^h(v_h^*+z_h) = a_h(v_h^*+z_h, v_h^*+z_h)$$

$$= a_h(v_h^*,v_h^*) + 2a_h(v_h^*,z_h) + a_h(z_h,z_h)$$

$$= \phi_h(v_h^*) + \phi_h(z_h) + 2a_h(v_h^*,z_h) .$$

By (3.47), $\phi_h(v_h^*) \xrightarrow[h\to+\infty]{} \phi(v)$ ; let us prove that

(3.49) $$a_h(v_h^*,z_h) \xrightarrow[h\to+\infty]{} 0 :$$

$$a_h(v_h^*,z_h) = \int_\Omega \sum_i \Big( \sum_j a_{ij}^h \frac{\partial v_h^*}{\partial x_j}\Big) \cdot \frac{\partial z_h}{\partial x_i} \, dx .$$

By assumption $\dfrac{\partial z_h}{\partial x_i} \xrightarrow{\omega-L^2} 0$

By construction $\displaystyle\sum_j a_{ij}^h \frac{\partial v_h^*}{\partial x_j} \xrightarrow{\omega-L^2} \sum_j a_{ij} \frac{\partial v}{\partial x_j}$ .

Moreover, $\quad \mathrm{rot}\,\left(\frac{\partial z_h}{\partial x_i}\right)_{i=1,\dots,N} = 0$ ,

$$- \mathrm{div}\left(\sum_j a_{ij}^h \frac{\partial v_h^*}{\partial x_j}\right)_{i=1,\dots,N} = Av \text{ is compact in } H^{-1} .$$

So we are in the situation where we can apply the theorem of compactness by compensation of Murat-Tartar [19] and

$$a_h(v_h^*, z_h) \xrightarrow[h\to+\infty]{} 0 .$$

([19] tells us that $\sum_i \left(\sum_j a_{ij}^h \frac{\partial v_h^*}{\partial x_j}\right) \frac{\partial z_h}{\partial x_i} \to 0$ in $D'(\Omega)$ ;

since it is bounded in $L^2(\Omega)$ , it converges weakly to zero in $L^2(\Omega)$ and $\int_\Omega \sum_{i,j} a_{ij}^h \frac{\partial v_h^*}{\partial x_j} \frac{\partial z_h}{\partial x_i} \to 0$ ).

Finally $\quad \phi(v) + F^+(v,\omega) = \underset{\{z_h \xrightarrow{s-L^2} 0\}}{\mathrm{Min}} \ \limsup_{h\to+\infty} \left[\phi(v)+\phi_h(z_h)+F_h(v_h^*+z_h,\omega)\right]$

and

$$F^+(v,\omega) = \underset{\{z_h \xrightarrow{s-L^2} 0\}}{\mathrm{Min}} \ \limsup_{h\to+\infty} \left[\phi_h(z_h)+F_h(z_h+L_h(v),\omega)\right] .$$

Now, we remark that $v \to L_h(v)$ is a linear operator ; the convexity of $F^+(.,\omega)$ follows easily : let

$$F^+(v_1,\omega) = \limsup_{h\to+\infty} \{\phi_h(z_h^1) + F_h[z_h^1+L_h(v_1),\omega]\} , \qquad z_h^1 \xrightarrow{s-L^2} 0$$

$$F^+(v^2,\omega) = \limsup_{h\to+\infty} \{\phi_h(z_h^2)+F_h[z_h^2+L_h(v_2),\omega]\} , \qquad z_h^2 \xrightarrow{s-L^2} 0$$

So, $\quad \lambda z_h^1 + (1-\lambda) z_h^2 \xrightarrow{s-L^2} 0$ and

$$F^+(\lambda v_1+(1-\lambda)v_2,\omega)$$

$$\leqslant \limsup_{h\to+\infty} \{\phi_h(\lambda z_h^1+(1-\lambda)z_h^2) + F_h[\lambda z_h^1+(1-\lambda)z_h^2+L_h(\lambda v_1+(1-\lambda)v_2),\omega]\}$$

$$\leqslant \lambda \limsup_{h\to+\infty} \{\phi_h(z_h^1) + F_h[z_h^1+L_h(v_1),\omega]\}$$

$$+ (1-\lambda) \limsup_{h\to+\infty} \{\phi_h(z_h^2) + F_h[z_h^2+L_h(v_2),\omega]\}$$

$$\leqslant \lambda\, F^+(v,\omega) + (1-\lambda)F^+(v_2,\omega) .$$

Step 3 ; Properties of $F^*$

(3.50) Definition

For any $v \in V$ and $\omega \in \mathcal{O}_n$ we define :

$$F^*(v,\omega) = \sup_{B\subset\omega} F^+(v,B) = \sup_{B\subset\omega} F^-(v,B) .$$

The last equality results clearly from the equality of $F^+(v,.)$ and $F^-(v,.)$ on a dense subset of $\mathcal{D}_n$ (cf. 3.19)).

*(3.51) Proposition*

$$F^* : V\times \mathcal{O}_n \to \overline{\mathbb{R}}^+ \text{ belongs to } \mathcal{F} .$$

Proof of Proposition (3.51)

a) By definition, $\phi(v) + F^+(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega)) \limsup[\phi^h(v)+F_h(v,\omega)]$ ;

it follows from the general properties of the $\Gamma$-limit that

$v \mapsto \phi(v) + F^+(v,\omega)$ is lower semi-continuous for the topology $s\text{-}L^p(\Omega)$ (or equivalently, $w\text{-}W_0^{1,p}(\Omega)$).

Since $v \mapsto \phi(v)$ is continuous for the strong topology of $V = W_0^{1,p}(\Omega)$

it follows that $v \mapsto F^+(v,\omega)$ is lower semi-continuous for the strong topology of $V$. By Lemma (3.41) and (3.45) it is convex $(p=2)$ and decreasing.

Clearly $F^*(.,\omega)$ as a supremum of convex, l.s.c., decreasing functional is still convex, l.s.c. and decreasing.

Let us now prove that $F^*(.,\omega)$ is proper :

By assumption, $\inf\limits_{v\in V} [\phi_h(v.)+F_h(v,\omega)] \leqslant \gamma(\omega) < +\infty$, so there exist

$v_h^\omega = v_h$ in $V$ such that $\phi_h(v_h) + F_h(v_h,\omega) \leqslant \gamma(\omega) + 1$.

Since $F_h$ is positive and the $\phi_h$ equicoercive, the $v_h$ are bounded in $V$ ;

let $v_{h_k} \xrightarrow{w-V} v^\omega$ ; taking $\overline{v_h} = \begin{cases} v_{h_k} & \text{if } h \in \{h_k\}_{k \in \mathbb{N}} \\ v & \text{otherwise} \end{cases}$, $\overline{v_h} \xrightarrow{w-V} v$

and

$$F^-(v^\omega,\omega) + \phi(v^\omega) \leqslant \liminf_{h\to+\infty} \{F_h(\overline{v}_h,\omega) + \phi_h(\overline{v}_h)\}$$

$$\leqslant \limsup_{k\to+\infty} \{F_{h_k}(v_{h_k},\omega) + \phi_{h_k}(v_{h_k})\}$$

$$\leqslant \gamma(\omega) + 1 .$$

So, $F^-(v^\omega,\omega) < +\infty$ and $F^*(v^\omega,\omega) \leqslant F^-(v^\omega,\omega) < +\infty$, i.e. $F^*(.,\omega)$ is proper.

b) Let us prove that $F^*(v,.)$ is a measure :

let $\omega = \omega_1 \cup \omega_2$ with $\omega_1,\omega_2 \in \mathcal{O}_n$ and $\omega_1 \cap \omega_2 = \emptyset$ ; then

(3.52) $\qquad F^*(v,\omega) \leqslant F^*(v,\omega_1) + F^*(v,\omega_2)$

(3.53) $\qquad F^*(v,\omega) \geqslant F^*(v,\omega_1) + F^*(v,\omega_2)$ .

Let us prove first (3.52) : let $\overline{B} \subset \omega_1 \cup \omega_2 = \omega$ ; then $B = (B\cap\omega_1) \cup (B\cap\omega_2)$ and $\overline{B\cap\omega_1} \subset \omega_1$, $\overline{B\cap\omega_2} \subset \omega_2$ ; so from (3.24)

$$F^+(v,B) \leqslant F^+(v,B\cap\omega_1) + F^+(v,B\cap\omega_2)$$

$$\leqslant F^*(v,\omega_1) + F^*(v,\omega_2)$$

and $F^*(v,\omega) = \sup\limits_{\overline{B}\subset\omega} F^+(v,B) \leqslant F^+(v,\omega_1) + F^+(v,\omega_2)$ .

Let us prove now (3.53) :

Let $\overline{B_1}\subset\omega_1$ , and $\overline{B}_2\subset\omega_2$ ; since $\overline{B}_1\cup\overline{B}_2\subset\omega_1\cup\omega_2$ , by definition of a rich family there exist $\Pi\in\mathcal{B}(F^-)$ such that

$$\overline{B}_1\cup\overline{B}_2\subset\Pi\subset\overline{\Pi}\subset\omega_1\cup\omega_2 .$$

Since $\Pi = (\Pi\cap\omega_1)\cup(\Pi\cap\omega_2)$ and $(\overline{\Pi\cap\omega_1})\cap(\overline{\Pi\cap\omega_2}) = \emptyset$ from Lemma (3.32)

$$F^-(v,\Pi) \geqslant F^-(v,\Pi\cap\omega_1) + F^-(v,\Pi\cap\omega_2) .$$

Since $\Pi\cap\omega_1\supset B_1$ and $\Pi\cap\omega_2\supset B_2$ and $F^-(v,.)$ is increasing

$$F^-(v,\Pi) \geqslant F^-(v_1,B_1) + F^-(v,B_2) ;$$

Since $\overline{\Pi}\subset\omega_1\cup\omega_2$ , $F^*(v,\omega_1\cup\omega_2) \geqslant F^-(v,B_1) + F^-(v,B_2)$ ;

Since this inequality is true for any $\overline{B}_1\subset\omega_1$ and $\overline{B}_2\subset\omega_2$ ,

$$F^*(v,\omega_1\cup\omega_2) \geqslant F^*(v,\omega_1) + F^*(v,\omega_2) .$$

Finally, $\forall\,\omega_1,\omega_2\in\mathcal{O}_n$ / $\omega_1\cap\omega_2 = \emptyset$ ,

$$F^*(v,\omega_1\cup\omega_2) = F^*(v,\omega_1) + F^*(v,\omega_2) .$$

The $\sigma$-additivity will follow from the continuity property of $F^*(v,.)$ on increasing sequences :

(3.54) $\quad \forall \omega_n \uparrow \omega \ , \ \forall v \in V \ , \quad F^*(v,\omega_n) \uparrow F^*(v,\omega) \ :$

let $\bar{B} \subset \cup \omega_n = \omega$ ; from the Borel-Lebesgue theorem, there exist $n_o \in \mathbb{N}$ such that $\bar{B} \subset \omega_{n_o}$ ; so

$$\forall \bar{B} \subset \omega \ , \qquad F^+(v,B) \leqslant F^*(v,\omega_{n_o}) \leqslant \lim_{n\to+\infty} \uparrow \ F^*(v,\omega_n)$$

and
$$F^*(v,\omega) \leqslant \lim_{n\to+\infty} \uparrow F^*(v,\omega_n)$$

since $\forall n \in \mathbb{N}$ , $\quad F^*(v,\omega_n) \leqslant F^*(v,\omega)$ the converse inequality is true and (3.54) follows.

The extension of $F^*(v,.)$ to all the borelian sets as an outer measure makes of $F^*(v,.)$ a borelian measure :

(3.55) $$\forall v \in V \ , \forall \mathcal{B} \in \mathcal{B}_n \ , \quad F^*(v,B) = \inf_{\begin{cases}\omega \supset B \\ \omega \in \mathcal{O}_n\end{cases}} F^*(v,\omega) \ .$$

c) Now let us prove that $F^*(.,\omega)$ is local :

*(3.56)* *<u>Lemma</u>*

$$\forall \omega \in \mathcal{O}_n \ , \forall u,v \in V \ , \quad (u_{|\omega} = v_{|\omega}) \implies (F^*(u,\omega) = F^*(v,\omega)) \ .$$

<u>Proof of Lemma</u> (3.56)

Let $A,B \in \mathcal{O}_n$ , such that $\bar{A} \subset B \subset \bar{B} \subset \omega$ ; by definition of $F^+(u,A)$ , there exist $v_h \xrightarrow[h\to+\infty]{L^p(\Omega)} v$ such that

(3.58) $$\phi(v) + F^+(v,B) = \limsup_{h\to+\infty} \{\phi^h(v_h)+F_h(v_h,B)\} \ .$$

Let us define $X \in W^{1,\infty}(\Omega)$ such that $X = \begin{cases} 1 \ \text{ on } \ A \\ 0 \leqslant X \leqslant 1 \\ 0 \ \text{ on } \Omega \setminus B \end{cases}$

and

(3.59) $\quad z_h \longrightarrow u$ in $s\text{-}L^p(\Omega)$ such that $\phi^h(z_h,\Omega\setminus\bar{A}) \longrightarrow \phi(u,\Omega\setminus\bar{A})$ .

We define $\bar{u}_h = Xv_h + (1-X)z_h$ and remark that

$$\bar{u}_h \xrightarrow[h\to+\infty]{} \begin{cases} Xv + (1-X)u = u \text{ on } \omega \text{ (since } v=u \text{ on } \omega), \text{ i.e. } \bar{u}_h \xrightarrow[h\to+\infty]{s\text{-}L^p(\Omega)} u \text{ ;} \\ u \text{ on } \Omega\setminus\omega \text{ .} \end{cases}$$

By definition of $F^+(u,A)$ ,

$$\phi(u) + F^+(u,A) \leqslant \limsup_{h\to+\infty} \{\phi^h(\bar{u}_h)+F_h(\bar{u}_h,A)\} \text{ .}$$

Since $\bar{u}_h = v_h$ on $A$ and $A \subset B$

$$\phi(u) + F^+(u,A) \leqslant \limsup_{h\to+\infty} \{\phi^h(\bar{u}_h)+F_h(v_h,B)\} \text{ .}$$

Now let us compute $\phi^h(\bar{u}_h)$ : as in the proof of the Lemma (3.32) we get :

$$\phi^h(t\bar{u}_h) \leqslant \int_B f_h(x,Dv_h)dx + \int_{\Omega\setminus\bar{A}} f_h(x,Dz_h)dx + (1-t)M\int_\Omega [1+|v_h-z_h|^p|DX|^p]dx \text{ .}$$

So,

$$\begin{aligned} t^p[\phi(u)+F^+(u,A)] \leqslant & \limsup_{h\to+\infty} \Big[\int_\Omega f_h(x,Dv_h)dx+F_h(v_h,B)\Big] \\ & + \limsup_{h\to+\infty} \Big[\int_{\Omega\setminus\bar{A}} f_h(x,Dz_h)dx\Big] + \limsup_{h\to+\infty} \Big[-\int_{\Omega\setminus B} f_h(x,Dv_h)dx\Big] \\ & + \limsup_{h\to+\infty} \Big[(1-t)M\int_\Omega [1+|v_h-z_h|^p|DX|^p|dx\Big] \text{ .} \end{aligned}$$

Now, we remark that :

(3.60) $$\limsup \Big[-\int_{\Omega\setminus B} f_h(x,Dv_h)dx\Big] \leqslant -\int_{\Omega\setminus B} f(x,Dv)dx \text{ , and}$$

(3.61) $$\int_\Omega |v_h-z_h|^p|DX|^p dx = \int_\omega |v_h-z_h|^p|DX|^p dx \xrightarrow[h\to+\infty]{} 0 \text{ since}$$

$v_h \xrightarrow{s\text{-}L^p} v$ , $z_h \xrightarrow{s\text{-}L^p} u$ and $v = u$ on $\omega$ ; using (3.58), (3.59), (3.60) and (3.61) :

$$t^p\left[\phi(u)+F^+(u,A)\right] \leqslant \phi(v) + F^+(v,B) - \int_{\Omega\setminus B} f(x,Dv)dx + \int_{\Omega\setminus A} f(x,Du)dx$$
$$+ (1-t)M\int_\Omega dx \ .$$

Making $t$ converging to one, we obtain

$$(3.62) \qquad F^+(u,A) + \int_A f(x,Du)dx \leqslant F^+(v,B) + \int_B f(x,Dv)dx$$

and (3.62) is true for any $A,B$ with $\overline{A} \subset B \subset \overline{B} \subset \omega$ .

Since $\overline{B} \subset \omega$ , (3.62) implies

$$F^+(u,A) + \int_A f(x,Du)dx \leqslant F^*(v,\omega) + \int_\omega f(x,Dv)dx \ .$$

Taking the supremum with respect to $A$, $\overline{A} \subset \omega$:

$$F^*(u,\omega) + \int_\omega f(x,Du)dx \leqslant F^*(v,\omega) + \int_\omega f(x,Dv)dx \ .$$

Since $u = v$ on $\omega$ , $F^*(u,\omega) \leqslant F^*(v,\omega)$ and interverting the role of $u$ and $v$ , we obtain

$$F^*(u,\omega) = F^*(v,\omega) \ .$$

d) Let us end the proof of the Proposition (3.51) by proving the following lemma :

*(3.63)* <u>*Lemma*</u>

$$\forall\, u,v \in V \ , \forall\, \omega \in \mathcal{O}_n, \quad F^*(u\vee v,\omega) + F^*(u\wedge v,\omega) \leqslant F^*(u,\omega) + F^*(v,\omega) \ .$$

<u>Proof of Lemma</u> (3.63)

Let $B \in \mathcal{O}_n$ and

$$\phi(u) + F^+(u,B) = \limsup_{h\to+\infty} [\phi^h(u_h)+F_h(u_h,B)] \quad , \quad u_h \xrightarrow{s-L^p(\Omega)} u$$

$$\phi(v) + F^+(v,B) = \limsup_{h\to+\infty} [\phi^h(v_h)+F_h(v_h,B)] \quad , \quad v_h \xrightarrow{s-L^p(\Omega)} v \ .$$

Since $u_h \wedge v_h \xrightarrow{s-L^p(\Omega)} u \wedge v$ and $u_h \vee v_h \xrightarrow{s-L^p(\Omega)} u \vee v$ , by definition of $F^-$ ,

$$\phi(u\wedge v) + F^-(u\wedge v,B) \leqslant \liminf_{h\to+\infty} [\phi^h(u_h \wedge v_h)+F_h(u_h \wedge v_h,B)]$$

$$\phi(u\vee v) + F^-(u \vee v,B) \leqslant \liminf [\phi^h(u_h \vee v_h)+F_h(u_h \vee v_h,B)] \ .$$

By addition of these two last inequalities

$$\phi(u\wedge v) + \phi(u\vee v) + F^-(u\wedge v,B) + F^-(u\vee v,B)$$

$$\leqslant \liminf_{h\to+\infty} [\phi^h(u_h\wedge v_h)+\phi^h(u_h\vee v_h)+F_h(u_h\wedge v_h,B)+F_h(u_h\vee v_h,B)]$$

$$\leqslant \liminf \ [\phi^h(u_h)+F_h(u_h,B)+\phi^h(v_h)+F_h(v_h,B)]$$

$$\leqslant \limsup \ [\phi^h(u_h)+F_h(u_h,B)] + \limsup \ [\phi^h(v_h)+F_h(v_h,B)]$$

$$\leqslant \phi(u) + \phi(v) + F^+(u,B) + F^+(v,B)$$

since, $\phi(u \wedge v) + \phi(u \vee v) = \phi(u) + \phi(v)$ it follows that

$$F^-(u \wedge v,B) + F^-(u \vee v,B) \leqslant F^+(u,B) + F^+(v,B) \ . \tag{3.64}$$

Let us take now $\omega \in \mathcal{O}$ and $A,B \in \mathcal{O}_n$ such that $\overline{A} \subset \omega$ , $\overline{B} \subset \omega$ .

Let us take $\mathcal{O} \in \mathcal{O}_n$ such that $A \cup B \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \omega$ ; from (3.64)

$$F^-(u \wedge v, \mathcal{O}) + F^-(u \vee v, \mathcal{O}) \leq F^+(u,\mathcal{O}) + F^+(v,\mathcal{O}) \leq F^*(u,\omega) + F^*(v,\omega)$$

Since $A \subset \mathcal{O}$ and $B \subset \mathcal{O}$

$$F^-(u \wedge v, A) + F^-(u \vee v, B) \leq F^*(u,\omega) + F^*(v,\omega) .$$

Taking the supremum with respect to $A$ and $B$ ,

$$F^*(u \wedge v,\omega) + F^*(u \vee v,\omega) \leq F^*(u,\omega) + F^*(v,\omega) .$$

<u>End of the proof of Theorem</u> (3.5)

From (3.19) $\forall \omega \in \mathcal{B}(F^+) \cap \mathcal{O}_n = \mathcal{B}(F^-) \cap \mathcal{O}_n$ , $F^+(v,\omega) = F^-(v,\omega)$ .
By definition

$$F^*(v,\omega) = \sup_{\substack{\bar{B} \subset \omega \\ B \subset \mathcal{O}_n}} F^-(v,B) .$$

For $v$ fixed, by Proposition (2.6) the set of $A \in \mathcal{B}_n$ such that

$$F^-(v,A) = \sup_{\bar{B} \subset A} F^-(v,B) \quad \text{is rich in} \quad \mathcal{B}_n \ ;$$

so for any $v \in V$ there exists a dense subset of open sets such that

$$F^*(v,\omega) = F^-(v,\omega) .$$

By Proposition (3.12) it follows that

$$\mathcal{B}(F^*) = \mathcal{B}(F^-) \quad \text{and} \quad \forall\, v \in V \ , \forall \omega \in \mathcal{B}(F^*) \cap \mathcal{O}_n \ , \quad F^*(v,\omega) = F^-(v,\omega) .$$

We can do the same deduction with $F^+$ and

$\mathcal{B}(F^*) = \mathcal{B}(F^-) = \mathcal{B}(F^+)$ and $\forall v \in V, \forall \omega \in \mathcal{B}(F^*) \cap \mathcal{O}_n$, $F^-(v,\omega) = F^+(v,\omega) = F^*(v,\omega)$ .
Since $F^* \in \mathcal{F}$ we can apply the conclusion of Ch.II and

$$\forall v \in V \ , \ \forall\, \omega \in \mathcal{O}_n \ , \quad F^*(v,\omega) = \int_\omega h(x,\tilde{v}(x))\, d\mu(x) + \nu(\omega) .$$

Finally, we have construct a subsequence $(h_k)_{k \in \mathbb{N}}$ and a limit functional $F$ still belonging to $\mathcal{F}_\gamma$ such that

$\forall v \in V$, $\forall\, \omega \in \mathcal{O}_n \cap \mathcal{B}(F^*)$ ,

$$F^+(v,\omega) = F^-(v,\omega) = F(v,\omega) = \int_\omega h(x,\tilde{v}(x))\, d\mu(x) + \nu(\omega) \ .$$

The conclusion of the Theorem (3.5) follows immediately.

Let us now examine in detail the properties of the limit functional $F$ when the $(F_h)_{h \in \mathbb{N}}$ are pure obstacles :

*(3.65)* *__Proposition__*

*__Let__* $F(v) + \|v\|^p_{W_0^{1,p}(\Omega)} = \Gamma^-(s\text{-}L^p(\Omega)) \lim\limits_{h\to+\infty} \{F_h(v) + \|v\|^p_{W_0^{1,p}(\Omega)}\}$

*__where the__* $(F_h)_{h\in\mathbb{N}}$ *__are pure obstacle functionals__* :

$$F_h(v) = \begin{cases} 0 & \underline{\text{if}}\quad v \geqslant \psi_k \quad \underline{\text{on}}\quad \Omega \\ +\infty & \underline{\text{otherwise}}. \end{cases}$$

*__The two following statements are equivalent__* :

(i) $F$ *__is a pure obstacle functional__*

$\Updownarrow$

(ii) $(F_h)_{h\in\mathbb{N}}$ *__converges in Mosco sense in__* $W^{1,p}$ .

*__Then__* $(F_h)_{h\in\mathbb{N}}$ *__converges in Mosco sense to__* $F$ .

__Proof of Proposition__ (3.65)

(i) $\Longrightarrow$ (ii) :

Let us assume that $F$ is a pure obstacle functional, $F = \mathbb{I}_{K_\psi}$ ; then, for every $v \in K_\psi$ , there exists $v_h \in K_{\psi_h}$ such that :

$$\begin{cases} \|v_h\|^p_{W_0^{1,p}(\Omega)} \xrightarrow[h\to+\infty]{} \|v\|_{W_0^{1,p}(\Omega)} \\ \\ v_h \xrightarrow[h\to+\infty]{} v \text{ in } s\text{-}L^p(\Omega) . \end{cases}$$

Therefore $v_h \longrightarrow v$ in $s\text{-}W_0^{1,p}(\Omega)$ .

On the other hand if $v_h \in K_{\psi_h}$ , $v_h \xrightarrow{w\text{-}W_0^{1,p}(\Omega)} v$ ,

$$F(v) + \|v\|^p_V \leqslant \underline{\lim} \|v_h\|^p_V < +\infty$$

and $F(v) < +\infty$ i.e. $v \in K_\psi$ ; so $K_{\psi_h} \longrightarrow K_\psi$ in Mosco sense ;

(ii) $\Longrightarrow$ (i) :

If $(F_h)_{h\in\mathbb{N}}$ converges in Mosco sense in $W^{1,p}$ , its limit functional $F$ takes only the values zero or $+\infty$ ; therefore $F$ is the indicator functional of a closed convex non void set $K$ : $F = 1_K$ ; moreover $K$ clearly will be unilateral i.e.,

$$\begin{cases} K \text{ stable for the inf-operation} \\ K + V^+ \subset K . \end{cases}$$

So, $K = K_\psi = \{v \in V \ / \ \tilde{v}(x) \geqslant \Psi(x) \ \text{ q.e.}\}$ (cf. [2] ), and, $K_{\psi_h} \rightarrow K_\psi$ in Mosco sense.

It follows from [1] that

$$(\|\cdot\|^p_V + \mathbb{I}_{K_{\psi_h}})_{h\in\mathbb{N}} \quad \text{converges to} \quad (\|\cdot\|^p + \mathbb{I}_{K_\psi}) \quad \text{in Mosco sense}$$

and $\|\cdot\|^p + \mathbb{I}_{K_\psi} = \|\cdot\|^p + F$ i.e. $F = \mathbb{I}_{K_\psi}$ .

Moreover, $F_h \xrightarrow[h\to+\infty]{} F$ in Mosco sense in that case.

Let us examine now how $F$ depends on the energy functional :

(3.66) Example

For simplicity, let us take $F^h(v,\omega) = \begin{cases} 0 & \text{if } \tilde{v} \geqslant \psi_h \text{ q.e. on } \omega \\ +\infty & \text{elsewhere} \end{cases}$

where $\psi_h = \begin{cases} 0 & \text{on } \Omega_h \\ -\infty & \text{elsewhere,} \end{cases}$

and let us denote :

$$\|v\|^p_{W_0^{1,p}(\Omega)} + F(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega)) \lim \{\| v\|^p_{W_0^{1,p}(\Omega)} + F^h(v,\omega)\} \quad .$$

Then, $\forall\, \lambda>0$

$$\lambda\|v\|^p_{W_0^{1,p}(\Omega)} + \lambda F(v,\omega) = \Gamma^-(s\text{-}L^p(\Omega)) \lim \{\lambda\|v\|^p_{W_0^{1,p}(\Omega)} + F^h(v,\omega)\} \quad .$$

Proof of (3.66)

We remark that with the choice we made of $\psi_h$ ,

$$\|\lambda v\|^p + F^h(\lambda v,\omega) = \lambda^p \{\|v\|^p + F^h(v,\omega)\}$$

i.e. $$(\|\cdot\|^p + F^h(\cdot,\omega))(\lambda v) = \lambda^p \{\|\cdot\|^p + F^h(\cdot,\omega)\}(v) \ .$$

This property of homogeneity is clearly preserved by $\Gamma$-limit process so

$$\|\lambda v\|^p + F(\lambda v,\omega) = \lambda^p \|v\|^p + \lambda^p F(v,\omega) \quad \text{i.e.}$$

$$F(\lambda v,\omega) = \lambda^p F(v,\omega)$$

and if $F$ is not a pure obstacle, i.e. if we are not in the situation where $(F^h)_{h\in\mathbb{N}}$ $\Gamma$-converges to $F$ , then the limit functional $F$ depends on the energy functional!

Let us take $p=2$ ; then, from the Theorem 3.5

$$\forall\, v\in V \ , \ \forall\, \omega\in \mathcal{O}_n \ , \qquad F(v,\omega) = \int_\omega h(x,\tilde{v}(x))\, d\mu(x) \ .$$

Since $F(\lambda v,\omega) = \lambda^2 F(v,\omega)$ it follows that

$$\forall t \in \mathbf{R} \ , \ \forall \lambda>0 \ , \quad h(x,\lambda t) = \lambda^2 h(x,t) \ .$$

Clearly, in the situation studied in (3.66) $h(x,t) = 0$ if $t>0$ ; so

$$F(v,\omega) = \int_\omega h(x,\tilde{v}^-(x))\, d\mu(x) = \int_\omega \tilde{v}^{2}(x)\, h(x,-1)\, d\mu(x) = \int_\omega \tilde{v}^{-2}(x)\, d\nu(x) \ .$$

(3.67) Remark

Suppose that the $F^h$ are obstacle functionals and $p=2$ :

$$\|v\|^2 + F(v,\omega) = \Gamma^-(s\text{-}L^2(\Omega)) \lim_{h\to+\infty} \{\|v\|^2+F^h(v,\omega)\} \ .$$

From the equality

$$\|v\|^2 + F^h(v,\omega) = [\|v\|+F^h(v,\omega)]^2 \quad \text{since} \quad F^h = \begin{cases} 0 \\ +\infty \end{cases}$$

it follows that

$$\Gamma^-(s\text{-}L^2(\Omega)) \lim_{h\to+\infty} [\|v\|+F^h(v,\omega)] = \{\Gamma^-(s\text{-}L^2(\Omega)) \lim_{h\to+\infty} [\|v\|^2+F^h(v,\omega)]\}^{1/2}$$
$$= [\|v\|^2+F(v,\omega)]^{1/2}$$
$$= \|v\| + G(v,\omega)$$

with $G^2(v,\omega) + 2\|v\|G(v,\omega) - F(v,\omega) = 0$ i.e.

$$G(v,\omega) = \sqrt{\|v\|^2+F(v,\omega)} - \|v\| = [\int_\Omega |Dv|^2 dx+\int_\omega h(x,\tilde{v}(x)d\mu(x)]^{1/2} - (\int_\Omega |Dv|^2 dx)^{1/2}$$

i.e. $G$ does not enjoy special properties and it is not easy to give directly a description of $G$ . It is only for particular energy functionals that it will be possible to give a simple description of the limit term $F$!

(3.68) Corollary of theorem 3.5.

The same statement of theorem 3.5 is still true when the hypothesis (iii) of the definition of $\mathcal{F}_\gamma$ is replaced by the hypothesis:

$\forall \omega \in \mathcal{O}_n$ , $v \longmapsto F(v,\omega)$ is increasing .

# CH.IV STUDY OF THE NON QUADRATIC CASE

(Variational problems associated with a non quadratic energy functional and highly oscillating potentials).

Let us consider the situation described by Carbone and Colombini in [8] .

Let $\psi_\varepsilon \in L^p_{loc}(\mathbb{R}^n)$ be a sequence of p-locally integrable functions on $\mathbb{R}^n$ such that :

$(\mathcal{H})$

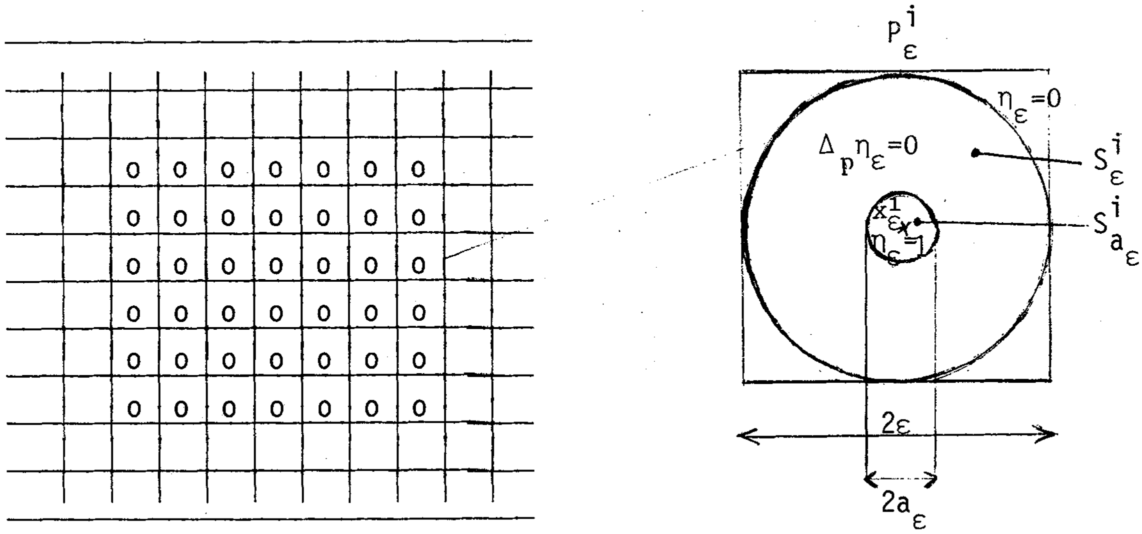
(i) $\exists\, \psi$ <u>regular</u> s.t. $\psi_\varepsilon \longrightarrow \psi$ in $L^p_{loc}(\mathbb{R}^n)$

(ii) $\exists \eta_\varepsilon \geqslant \psi_\varepsilon - \psi$ such that $\eta_\varepsilon \longrightarrow 0$ in $L^p_{loc}(\mathbb{R}^n)$ and

$$\limsup_{\varepsilon\to 0} \int_\omega |D\eta_\varepsilon|^p \, dx \leqslant \nu(\omega) \; , \qquad (1<p<+\infty)$$

holds with $\nu$ a positive Radon measure and for any open bounded set $\omega$ such that $\nu(\partial\omega) = 0$.

<u>A typical situation corresponding to</u> $(\mathcal{H})$ <u>is the following</u> :




Let $\psi_\varepsilon = 1$ on $\bigcup\limits_{i\in\mathbf{N}} S^i_{a_\varepsilon}$ where $S^i_{a_\varepsilon}$ is the open ball of radius $a_\varepsilon$ centered at $x^i_\varepsilon$ in the square $P^i_\varepsilon$ and $\psi_\varepsilon = 0$ elsewhere.

Let us compute $\int_{P^i_\varepsilon} (\Sigma\, |\frac{\partial\eta_\varepsilon}{\partial x_i}|^2)^{\frac{p}{2}}\, dx$ , where

$$\eta_\varepsilon = \begin{cases} 1 \quad \text{on } S^i_{a_\varepsilon} \\ \Delta_p\eta_\varepsilon = 0 \quad \text{on } S^i_\varepsilon \setminus S^i_{a_\varepsilon} \\ 0 \quad \text{on } P^i_\varepsilon \setminus S^i_\varepsilon \end{cases}$$

$S^i_\varepsilon$ is the open ball of center $x^i_\varepsilon$ and radius $\varepsilon$ ; $\eta_\varepsilon$ is radial, so

$$\int_{P^i_\varepsilon} (\Sigma|\frac{\partial\eta_\varepsilon}{\partial x_i}|^2)^{\frac{p}{2}} dx = \int_{P^i_\varepsilon} |\frac{d\eta_\varepsilon}{d\rho}|^p \rho^{n-1}\, d\rho\, d\theta$$

$$= 2\pi \int_{a_\varepsilon}^{\varepsilon} |\frac{d\eta_\varepsilon}{d\rho}|^p \rho^{n-1}\, d\rho \quad .$$

We take $\eta_\varepsilon$ minimizing this integral with the boundary conditions $\eta_\varepsilon(a_\varepsilon) = 1$ , $\eta_\varepsilon(\varepsilon) = 0$ ; the Euler equation is

$$\frac{d}{d\rho}\left(\rho^{n-1}\left|\frac{d\eta}{d\rho}\right|^{p-2}\frac{d\eta}{d\rho}\right) = 0 \quad \text{i.e.}$$

$\rho^{n-1}\left|\frac{d\eta}{d\rho}\right|^{p-2}\frac{d\eta}{d\rho} = C_1$ so $\frac{d\eta}{d\rho}$ has a constant sign and $\frac{d\eta}{d\rho} < 0$

$$\left(-\frac{d\eta}{d\rho}\right)^{p-1} = \frac{C_1}{\rho^{n-1}}, \quad C_1>0 .$$

So $+\frac{d\eta}{d\rho} = -\frac{C_1^{\frac{1}{p-1}}}{\rho^{\frac{n-1}{p-1}}} \Longrightarrow \eta(\rho) = +\frac{C_2}{\rho^{\frac{n-p}{p-1}}} + C_3$

and $\begin{cases} +\frac{C_2}{a_\varepsilon^{\frac{n-p}{p-1}}} + C_3 = 1 \\ +\frac{C_2}{\varepsilon^{\frac{n-p}{p-1}}} + C_3 = 0 \end{cases} \Longrightarrow C_2\left(-\frac{1}{\varepsilon^{\frac{n-p}{p-1}}} + \frac{1}{a_\varepsilon^{\frac{n-p}{p-1}}}\right) = 1$

Since $\frac{d\eta}{d\rho} = -\left(\frac{n-p}{p-1}\right) C_2\rho^{-\frac{n-1}{p-1}}$,

$$\int_{a_\varepsilon}^{\varepsilon}\left|\frac{d\eta}{d\rho}\right|^p \rho^{n-1}\,d\rho = C_2^p\left(\frac{n-p}{p-1}\right)^p\int_{a_\varepsilon}^{\varepsilon}\frac{\rho^{n-1}}{\rho^{\left(\frac{n-1}{p-1}\right)}}\,d\rho$$

Hence,

$$\int_{P_\varepsilon^i}\left(\Sigma\left|\frac{\partial\eta_\varepsilon}{\partial x_i}\right|^2\right)^{\frac{p}{2}}dx = C_2^p\left(\frac{n-p}{p-1}\right)^p\frac{1}{\left(\frac{p-n}{p-1}\right)}\left[\rho^{\frac{p-n}{p-1}}\right]_{a_\varepsilon}^{\varepsilon} = C_2^p\left(\frac{n-p}{p-1}\right)^{p-1}\left[a_\varepsilon^{\frac{n-p}{p-1}} - \varepsilon^{\frac{p-n}{p-1}}\right]$$

$$= \frac{(a_\varepsilon\,\varepsilon)^{\left(\frac{n-p}{p-1}\right)p}}{\left(\varepsilon^{\frac{n-p}{p-1}} - a_\varepsilon^{\frac{n-p}{p-1}}\right)^p}\left(a_\varepsilon^{\frac{p-n}{p-1}} - \varepsilon^{\frac{p-n}{p-1}}\right) \quad 2\pi\left(\frac{n-p}{p-1}\right)^{p-1}$$

$$= \frac{a_\varepsilon^{\,n-p}\,\varepsilon^{\left(\frac{n-p}{p-1}\right)p} - \varepsilon^{\,n-p}\,a_\varepsilon^{\left(\frac{n-p}{p-1}\right)p}}{\left(\varepsilon^{\frac{n-p}{p-1}} - a_\varepsilon^{\frac{n-p}{p-1}}\right)^p} \quad 2\pi\left(\frac{n-p}{p-1}\right)^{p-1} .$$

Let $P \in \mathcal{T}_n$ then there exist $\sim \dfrac{m(P)}{\varepsilon^n}$ square $P_\varepsilon^i$ in $P$.

$$\int_P \Big(\Sigma\Big|\frac{\partial\eta_\varepsilon}{\partial x_i}\Big|^2\Big)^{\frac{p}{2}} dx \sim \frac{2\Pi\, m(P)}{2^n} \times \Big(\frac{n-p}{p-1}\Big)^{p-1} \times \frac{a_\varepsilon^{(n-p)}\varepsilon^{(\frac{n-p}{p-1})p} - \varepsilon^{(n-p)}\, a_\varepsilon^{(\frac{n-p}{p-1})p}}{\varepsilon^n \left(\varepsilon^{\frac{n-p}{p-1}} - a_\varepsilon^{\frac{n-p}{p-1}}\right)^p} .$$

Let us take $a_\varepsilon = \varepsilon^k$, $\underline{k>1}$ then

$$\int_P \Big(\Sigma\Big|\frac{\partial\eta_\varepsilon}{\partial x_i}\Big|^2\Big)^{\frac{p}{2}} dx \sim \frac{2\Pi}{2^n}\, m(P) \Big(\frac{n-p}{p-1}\Big)^{p-1} \times \frac{\varepsilon^{k(n-p)\,+\,(\frac{n-p}{p-1})p} - \varepsilon^{(n-p)\,+\,kp(\frac{n-p}{p-1})}}{\varepsilon^{(\frac{n-p}{p-1})p+n}}$$

but $k(n-p) + (\frac{n-p}{p-1})p < (n-p) + kp(\frac{n-p}{p-1})$ , $\underline{\underline{n>p}}$

$$\Longleftrightarrow k + \frac{p}{p-1} < 1 + \frac{kp}{p-1} \Longleftrightarrow k\Big(\frac{p}{p-1} - 1\Big) > \frac{p}{p-1} - 1$$

$$\Longleftrightarrow k > 1 \qquad \underline{\text{O.K.}}$$

$$\text{So } \int_P \Big(\Sigma\Big|\frac{\partial\eta_\varepsilon}{\partial x_i}\Big|^2\Big)^{\frac{p}{2}} dx \sim \frac{\Pi}{2^{n-1}}\, m(P) \Big(\frac{n-p}{p-1}\Big)^{p-1} \frac{\varepsilon^{k(n-p)\,+\,(\frac{n-p}{p-1})p}}{\varepsilon^{(\frac{n-p}{p-1})p+n}}$$

We are in the confitions of $(\mathcal{H})$ if $k(n-p) \geqslant n$ i.e. $\boxed{k \geqslant \frac{n}{n-p}}$

If $k = \frac{n}{n-p}$ , $\int_P |D\eta_\varepsilon|^p\, dx \leqslant C(n,p) \int_P \big(\Sigma\big|\frac{\partial\eta_\varepsilon}{\partial x_i}\big|^2\big)^{\frac{p}{2}} dx$

$$\xrightarrow[\varepsilon\to 0]{} 2\pi\, m(P) \Big(\frac{n-p}{p-1}\Big)^{p-1} C(n,p) = \nu(P) .$$

Remark

If $p>n$ , and if we ask the $(\eta_\varepsilon)_{\varepsilon>0}$ to be bounded in $W^{1,p}_{loc}(\mathbb{R}^n)$ , by Sobolev inclusions, they will converge uniformly to zero and the conclusion follows : the limit problem is associated with the obstacle zero.

Now let us return to the general situation exposed at the beginning of the chapter (assumptions $\mathcal{H}$) ; from Carbone and Colombini [8], Theorem 1 and Theorem 2 :

. Let $A_n^\nu$ be the family of open bounded subsets of $\mathbb{R}^n$ such that

$$m(\partial\Omega) + \nu(\partial\Omega) = 0 .$$

. Then, there exists a subsquence $\varepsilon_k$ such that denoting

$$F_\varepsilon(u,\Omega) = \begin{cases} 0 & \text{if } u(x) \geqslant \psi(x) \text{ a.e. on } \Omega \\ +\infty & \text{elsewhere} \end{cases}$$

we get $\forall\, \Omega \in A_n^\nu$, $\forall\, u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ such that $u(x) \geqslant \psi(x)$ a.e. on $\Omega$

$$(4.1)\quad \Gamma^-(s\text{-}L^p(\Omega)) \lim_{k\to+\infty} \{\int_\Omega |Du|^p dx + F_{\varepsilon_k}(u,\Omega)\} = \int_\Omega j(x,u(x),Du(x))\, d\mu(x)$$

with $\begin{cases} \mu = \nu + dx \\ j : \mathbb{R}^n_x \times \mathbf{R}_u \times \mathbf{R}^n_{Du} \longrightarrow \overline{\mathbb{R}^+} \text{ is a convex normal integrand.} \end{cases}$

If $u(x) < \psi(x)$ for some $x \in \Omega$, then these quantities are equal to $+\infty$. Moreover,

$$(4.2)\qquad \int_\Omega j(x,u(x),Du(x))\, d\mu(x) \leqslant C\Big[\int_\Omega (1+|Du|^p)dx + \int_\Omega d\nu\Big].$$

We shall prove in this chapter that $j$ splits :

$$(4.3)\qquad j(x,u(x),Du(x)) = |Du|^p + j_1(x,u(x)) .$$

*(4.3) Theorem*

*Under the hypotheses ($\mathcal{H}$) of Carbone and Colombini, there exists a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that :*

*$\forall\, \Omega \in A_n^\nu$, $\forall\, u \in W^{1,\infty}(\mathbb{R}^n)$ satisfying $u(x) \geqslant \psi(x)$ a.e. on $\Omega$*

$$\Gamma^-(s\text{-}L^p(\Omega)) \lim_{k\to+\infty} \{\int_\Omega |Du|^p dx + F_{\varepsilon_k}(u,\Omega)\} = \int_\Omega |Du|^p dx + \int_\Omega j(x,u(x))\, d\mu(x)$$

*with* $\begin{cases} \mu = \nu + dx \\ j : \mathbb{R}^n \times \mathbb{R} \longrightarrow \overline{\mathbb{R}}^+ \text{ is a convex, normal, decreasing (with } u\text{), integrand.} \end{cases}$

*If* $u(x) < \psi(x)$ *for some* $x \in \Omega$ *, then these quantities are equal to* $+\infty$ .

Proof of Theorem (4.3)

From (4.1)

(4.4) $$\Gamma^-(s\text{-}L^p(\Omega)) \lim_{k\to+\infty} \{\int_\Omega |Du|^p dx + F_{\varepsilon_k}(u,\Omega)\} = \int_\Omega j(x,u(x),Du(x)\, d\mu(x) = \int_\Omega |Du|^p dx + F(u,\Omega) .$$

So $F(u,\Omega) = \int_\Omega j(x,u(x),Du(x))\, d\mu(x) - \int_\Omega |Du|^p\, dx$ ; from the Radon-Nikodym theorem $dx = h(x)\, d\mu$ , so,

$$F(u,\Omega) = \int_\Omega \{j(x, u,Du)-h(x)|Du|^p\}\, d\mu = \int_\Omega k(x,u(x),Du(x))\, d\mu(x) .$$

Moreover, from (4.2) , $\forall\, u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ such that $u(x) \geq \psi(x)$ a.e.

$$0 \leq \int_\Omega j(x,u(x),Du(x))\, d\mu(x) \leq C_1 [\int_\Omega (1+|Du|^p)\, dx + \int_\Omega d\nu] .$$

Let $x_0 \in \Omega$ a Lebesgue point of $j$ such that $u(x_0) > \psi(x_0)$ .
Taking $\omega_0$ an open neighbourhood of $x_0$ , $\omega_0$ sufficiently small

$$\forall\, x \in \omega_0 , \qquad u(x) \geq \psi(x) + \varepsilon_0 , \qquad \varepsilon_0 > 0 .$$

If $v \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ and $\|v-u\|_{L^\infty(\omega_0)} \leq \varepsilon_0$ then

$\forall x \in \omega_0$ ,  $v(x) \geqslant \psi(x)$ .

Let $\omega \subset \omega_0$ ; for any $k \in \mathbb{R}^+$ , let us define

$$|||v-u|||_{k,\omega} = \frac{1}{\varepsilon_0}\|v-u\|_{L^\infty(\omega)} + \frac{1}{k}\|Dv-Du\|_{L^p(\omega)} .$$

If $|||v-u|||_{k,\omega} \leqslant 1$ , then $\|v-u\|_{L^\infty(\omega)} \leqslant \varepsilon_0$ , so $v(x) \geqslant \psi(x)$ on $\omega$

and $\|Dv\|_{L^p(\omega)} \leqslant \|Du\|_{L^p(\omega)} + \frac{1}{k}$ ; so

$$0 \leqslant \int_\omega j(x,v(x),Dv(x))\, d\mu(x) = G(v,\omega) \leqslant C\left[\int_\omega (1+|Dv|^p)dx + \int_\omega d\nu\right] \leqslant C_1$$

and $v \longmapsto G(v,\omega)$ is a convex, positive functional which is bounded on the ball $|||v-u|||_{k,\omega} \leqslant 1$ ; so, it is lipschitz on any ball of radius strictly less than $1$ and it follows that :

$$\forall v_1,v_2 \in W^{1,\infty}_{loc}(\mathbb{R}^n) \text{ such that } \|v_1-u\|_{L^\infty(\omega)} < \varepsilon_0 , \quad \|Dv_1\|_{L^p(\omega)} \leqslant k$$
$$\|v_2-u\|_{L^\infty(\omega)} < \varepsilon_0 , \quad \|Dv_2\|_{L^p(\omega)} \leqslant k ,$$

there exists a constant $C_k$ such that :

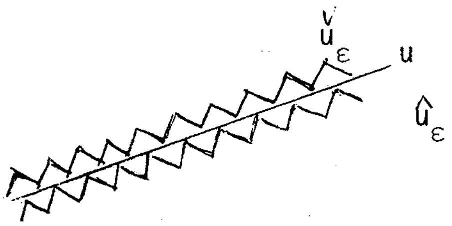
$$(4.5) \quad \left|\int_\omega \{j(x,v_1(x),Dv_1(x)) - j(x,v_2(x),Dv_2(x)\}\, d\mu(x)\right|$$
$$\leqslant C_k\, |||v_1-v_2|||_{k,\omega}$$
$$\leqslant C_k\left[\frac{1}{\varepsilon_0}\|v_1-v_2\|_{L^\infty(\omega)} + \frac{1}{k}\|Dv_1-Dv_2\|_{L^p(\omega)}\right] .$$

. The same argument as for Lemma 3.41 tells us that :

(4.6) $\forall \Omega \in \mathcal{O}_n$ ,  $v \longmapsto F(v,\Omega)$ is a decreasing functional.

Let us prove in a first step that (4.4), (4.5), (4.6) imply that the integrand $k$ is an affine function of $Du$ :

Let us take $u(x) = \xi + <x, \frac{z_1+z_2}{2}>$ , $u(x_0) > \psi(x_0)$ .




Let $u \leqslant \check{u}_\varepsilon \leqslant u+\varepsilon$ , with $D\check{u}_\varepsilon$ oscillating between $z_1$ and $z_2$ . Similarly, we define $\hat{u}_\varepsilon = u_\varepsilon-\varepsilon$ . Since $F(.,\Omega)$ is decreasing

$$(4.7) \qquad \forall \varepsilon>0 , \quad F(\check{u}_\varepsilon,\Omega) \leqslant F(u,\Omega) \leqslant F(\hat{u}_\varepsilon,\Omega) .$$

Taking $\Omega$ sufficiently small, $\Omega \ni x_0$ , we shall get

$$\check{u}_\varepsilon(x) \geqslant \hat{u}_\varepsilon(x) \geqslant \psi(x) \quad , \quad \forall x \in \Omega .$$

From the definition of $F$ , (4.7) can be written

$$(4.8) \ \forall \varepsilon>0 , \int_\Omega k(x,\check{u}_\varepsilon(x),D\check{u}_\varepsilon(x)) \, d\mu(x) \leqslant F(u,\Omega) \leqslant \int_\Omega k(x,\hat{u}_\varepsilon(x),D\hat{u}_\varepsilon(x)) \, d\mu(x) .$$

From (4.5), remarking that $|D\hat{u}_\varepsilon|_{L^p(\Omega)} \leqslant k$ (since $D\hat{u}_\varepsilon \xrightarrow{L^\infty *} \frac{z_1+z_2}{2}$ )

$$(4.9) \left|\int_\Omega \{k(x,\hat{u}_\varepsilon(x),D\hat{u}_\varepsilon(x))-k(x,u(x),D\hat{u}_\varepsilon(x))\}d\mu\right| \leqslant C_k[\frac{1}{\varepsilon_0} \|\hat{u}_\varepsilon-u\|_{L^\infty(\Omega)}] \xrightarrow[\varepsilon\to 0]{} 0$$

From (4.8) and (4.9) ,

$$\lim_{\varepsilon\to 0} \int_\Omega k(x,u(x),D\check{u}_\varepsilon(x) \, d\mu(x) \leqslant F(u,\Omega) \leqslant \lim_{\varepsilon\to 0} \int_\Omega k(x,u(x),D\hat{u}_\varepsilon(x)) \, d\mu(x) .$$

Since $D\hat{u}_\varepsilon$ and $D\check{u}_\varepsilon$ oscillate between $z_1$ and $z_2$ it follows that

$$F(u,\Omega) = \int_\Omega k(x,u(x),\frac{z_1+z_2}{2}) \, d\mu(x) = \int_\Omega \frac{k(x,u(x),z_1)+k(x,u(x),z_2)}{2} \, d\mu(x)$$

i.e. $\forall \Omega \ni x_0$ , $\Omega$ sufficiently small, $\xi > \psi(x_0)$ ,

$$\int_\Omega k(x,\xi+<x,\frac{z_1+z_2}{2}>,\frac{z_1+z_2}{2})\ d\mu(x) = \frac{1}{2}\int_\Omega k(x,\xi+<x,\frac{z_1+z_2}{2}>,z_1) + k(x,\xi+<x,\frac{z_1+z_2}{2}>,z_2))\ d\mu(x).$$

Dividing by $|\mu(\Omega)|$ , and making $\Omega \downarrow \{x_0\}$ , we get :

$$k(x,\xi+<x,\frac{z_1+z_2}{2}>,\frac{z_1+z_2}{2}) = \frac{1}{2}[k(x,\xi+<x,\frac{z_1+z_2}{2}>,z_1) + k(x,\xi+<x,\frac{z_1+z_2}{2}>,z_2)].$$

Since this is true for any $\xi$ such that $\xi + <x_0, \frac{z_1+z_2}{2}> > \psi(x_0)$ we get :

(4.10) $\mu$ pp $x$, $\forall\ \xi>\psi(x)$, $k(x,\xi,\frac{z_1+z_2}{2}) = \frac{1}{2}[k(x,\xi,z_1)+k(x,\xi,z_2)]$ and

$z \longmapsto k(x,\xi,z)$ is affine.

Step two : Let us prove that $k$ is independent of $z$ .
From (4.10), $\forall\ \Omega \in A_n^\nu$

$$\forall\ u>\psi , \qquad F(u,\Omega) = \int_\Omega [\sum_{i=1}^n g_i(x,u(x)) \frac{\partial u}{\partial x_i} + q(x,u(x))]\ d\mu(x) .$$

Let $\xi > \psi(x_0)$ and $z \in \mathbb{R}^n$ ; taking $\Omega \ni x_0$ , $\Omega$ sufficiently small the function $u(x) = \xi + <x-x_0,z>$ will satisfy

$$\forall\ x\in\Omega , \qquad u(x) \geqslant \eta \geqslant \psi(x) \quad \text{with} \quad \eta = \frac{\xi+\psi(x_0)}{2} .$$

Since $v \longmapsto F(v,\Omega)$ is decreasing

$$F(u,\Omega) \leqslant F(\eta,\Omega) \quad \text{i.e.}$$

(4.11) $$\int_\Omega [\sum_{i=1}^n g_i(x,u(x))z_i+q(x,u(x))]\ d\mu(x) \leqslant \int_\Omega q(x,\eta)\ d\mu(x) .$$

$$\leqslant \quad C_1\ \mu(\Omega) \quad \text{(from 4.2)} .$$

Dividing (4.11) by $\mu(\Omega)$ and making $\mu(\Omega)$ go to zero :

$$(4.12) \quad \mu\text{ - pp } x\ , \quad \sum_{i=1}^{n} g_i(x,\xi)z_i + q(x,\xi) \leqslant q(x,\eta) \leqslant C_1\ .$$

Since this is true for any $z \in \mathbb{R}^n$, (4.12) implies that

$$\forall\ \xi > \psi(x)\ , \qquad g_i(x,\xi) = 0 \qquad (\forall\ i=1,\ldots,n)\ .$$

So $F(u,\Omega) = \int_\Omega q(x,u(x))\ d\mu(x)\ ,\ \forall\ u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ with $u>\psi$ .

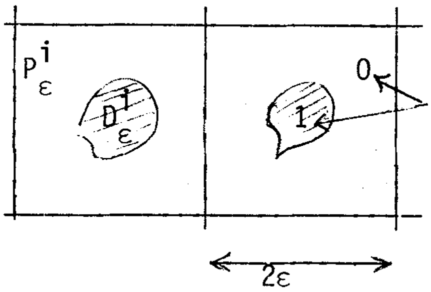
From the definition of the integrand $j$ (cf. [8] ), it follows that

$$F(u,\Omega) = \int_\Omega q(x,u(x))\ d\mu(x)\ ,\ \forall\ u \in W^{1,\infty}_{loc}(\mathbb{R}^n) \text{ with } u \geqslant \psi\ .$$

Let us come back to the situation described at the beginning of this paragraph :

we are going to compute the limit functional by a compactness argument :

we assume that the obstacle is periodic ; in each all $\psi_\varepsilon$ is given by :




$$\psi_\varepsilon = 1 \text{ on } \bigcup_i D^i_\varepsilon$$

$$\psi_\varepsilon = 0 \text{ on } \bigcup_i P^i_\varepsilon \setminus D^i_\varepsilon\ .$$

Let

$$\Gamma^-(s\text{-}L^p(\Omega)) \lim \{\int_\Omega |Du|^p dx + F_\varepsilon(u,\omega)\} = \int_\Omega |Du|^p dx + \int_\Omega j(x,\tilde{u}(x))\ d\mu(x)$$

with $F_\varepsilon(u,\omega) = \begin{cases} 0 & \text{if } u(x) \geqslant \psi_\varepsilon(x) \text{ on } \omega \\ +\infty & \text{elsewhere.} \end{cases}$

Clearly, $j(x,t) = 0$ it $t>1$ ; by an homogeneity argument (the homogeneity is preserved by $\Gamma$-limit)

$$j(x,t) = a(x)\ [(t-1)^-]^p\ .$$

So the limit term can be written $\int_\Omega |Du|^p\, dx + \int_\omega [(u(x)-1)^-]^p\, a(x)\, d\mu(x)$ .

The problem being clearly invariant by translation, the measure $a d\mu$ is the Haar measure on $\mathbb{R}^n$ , so $\exists$ $C$ s.t. $a d\mu = C\, dx$ ; finally

$$\Gamma^-(s\text{-}L^p(\Omega)) \lim \left[\int_\Omega |Du|^p dx + F_\varepsilon(u,\omega)\right] = \int_\Omega |Du|^p dx + C\int_\omega ([u(x)-1]^-)^p\, dx \,.$$

The constant $C$ depends on the shape and the size of the $D_\varepsilon^i$ and on $p$ and $n$ .

By definition of the $\Gamma^-(s\text{-}L^p)$ limit

$$(4.13)\qquad \forall u \in W_0^{1,p}(\Omega)\ ,\quad \int_\Omega |Du|^p\, dx + C\int_\Omega [(u-1)^-]^p\, dx = \underset{\begin{cases} u_\varepsilon \xrightarrow{s\text{-}L^p} u \\ u_\varepsilon \geqslant \psi_\varepsilon \text{ on } \Omega \end{cases}}{\mathrm{Min}} \ \liminf_{\varepsilon\to 0} \int_\Omega |Du_\varepsilon|^p\, dx \,.$$

In fact (4.13) holds for a subsequence $\varepsilon_k$ . The convergence result for the whole sequence will follow from the identification of $C$ , i.e. from the independence of $C$ from the subsequence $\varepsilon_k$ . In order to compute $C$ it is sufficient to compute the right hand side for a particular function and a particular domain.

Let us take $u=0$ and $\Omega = D$ ; remarking that $\mathrm{mes}(D) = 1$

$$(4.14)\qquad C = \underset{\substack{u_\varepsilon \xrightarrow{s\text{-}L^p} 0 \\ u_\varepsilon \geqslant \psi_\varepsilon \text{ on } D}}{\mathrm{Min}} \ \liminf_{\varepsilon\to 0} \int_D |Du_\varepsilon|^p\, dx \,.$$

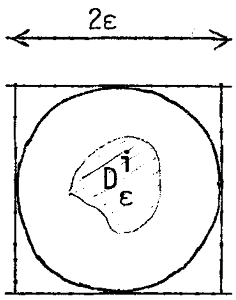
We assume that $C$ is finite i.e. :

$$(4.15)\qquad \exists\ v_\varepsilon \geqslant \psi_\varepsilon\ ,\quad v_\varepsilon \xrightarrow{s\text{-}L^p} 0 \ \text{ with } \|v_\varepsilon\|_{W_0^{1,p}(D)} \leqslant C \,.$$

That's the case in the situation described at the beginning of this chapter when
$$\begin{cases} \psi_\varepsilon = 1 \text{ on } \bigcup S_{a_\varepsilon}^i\ , & \psi_\varepsilon = 0 \text{ elsewhere} \\ a_\varepsilon = \varepsilon^{\frac{n}{n-p}}\ , & 1 \leqslant p < n \end{cases}$$

From (4.14) clearly $C = \underset{\substack{u_\varepsilon \to 0 \\ u_\varepsilon \geq 0 \\ u_\varepsilon = 1 \text{ on } \cup D_\varepsilon^i}}{\operatorname{Min}} \quad \liminf_{\varepsilon \to 0} \int_D |Du_\varepsilon|^p \, dx$ : this follows from the fact that the troncature $r \longrightarrow r \wedge 1$ operates on $W_0^{1,p}(D)$ .

Let $\eta_\varepsilon$ be the function defined at the begining of this chapter.




$P_\varepsilon^i$

$S_\varepsilon^i$

$$\eta_\varepsilon \quad \begin{cases} 1 \text{ on } D_\varepsilon^i \\ 0 \text{ on } P_\varepsilon^i \setminus S_\varepsilon^i \\ -\Delta p \, \eta_\varepsilon = 0 \text{ on } S_\varepsilon^i \setminus D_\varepsilon^i \end{cases}$$

such that $\eta_\varepsilon \longrightarrow 0$ in $s - L^p(D)$ .

(4.15) Then $$C \leq \liminf_{\varepsilon \to 0} \int_D |D\eta_\varepsilon|^p \, dx .$$

In [11], when $p=2$ and when the $D_\varepsilon^i$ are spheres, Murat and Cioranescu proved that the converse inequality is true

(4.16) $$C = \lim_{\varepsilon \to 0} \int_D |D\eta_\varepsilon|^2 \, dx .$$

It seems reasonable to conjecture, for any $p$ and any $D_\varepsilon^i$ , that $C$ is given by :

$$C = \lim_{\varepsilon \to 0} \int_D |D\eta_\varepsilon|^p \, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{(2\varepsilon)^n} \overset{p}{\operatorname{Cap}} D_\varepsilon^i$$

with $\overset{p}{\operatorname{Cap}} D_\varepsilon^i = \underset{\substack{u=1 \text{ on } D_\varepsilon^i \\ u=0 \text{ on } \partial P_\varepsilon^i}}{\operatorname{Min}} \int |Du|^p \, dx$ .

In the situation described at the beginning of the chapter :

$$C = \frac{\Pi}{2^{n-1}} \left(\frac{n-p}{p-1}\right)^{p-1} \quad \text{for } D_\varepsilon^i \text{ of radius } a_\varepsilon = \varepsilon^k, \; k = \frac{n}{n-p}, \; 1 \leq p < n .$$

Let us interpret very simply this constant :

by a change of scale, when $D_\varepsilon^i$ is the homotetic of coefficient $a_\varepsilon$ of $D \subset \mathbb{R}^n$, we get :

$$C = \lim_{\varepsilon \to 0} \frac{1}{(2\varepsilon)^n} \operatorname*{Min}_{\begin{cases} u=1 \text{ on } D_\varepsilon \\ u \in W_0^{1,p}(S_\varepsilon) \end{cases}} \int_{S_\varepsilon} |Du|^p \, dx .$$

Taking $u(x) = v(\frac{x}{a_\varepsilon})$,

$$C = \lim_{\varepsilon \to 0} \frac{1}{(2\varepsilon)^n} \operatorname*{Min}_{\begin{cases} v=1 \text{ on } D \\ v \in W_0^{1,p}\left(S_{\frac{\varepsilon}{a_\varepsilon}}\right) \end{cases}} \int_{S_{\frac{\varepsilon}{a_\varepsilon}}} |Dv(\tfrac{x}{a_\varepsilon})|^p \frac{1}{(a_\varepsilon)^p} \, d(\tfrac{x}{a_\varepsilon}) \times a_\varepsilon^n$$

$$C = \lim_{\varepsilon \to 0} \frac{1}{2^n} \frac{a_\varepsilon^{n-p}}{\varepsilon^n} \operatorname*{Min}_{\begin{cases} v=1 \text{ on } D \\ v \in W_0^{1,p}\left(S_{\frac{\varepsilon}{a_\varepsilon}}\right) \end{cases}} \int_{S_{\frac{\varepsilon}{a_\varepsilon}}} |Dv(x)|^p \, dx .$$

We refind that $C = 0$ if $a_\varepsilon \ll \varepsilon^{\frac{n}{n-p}}$

$C = +\infty$ if $a_\varepsilon \gg \varepsilon^{\frac{n}{n-p}}$

and when $a_\varepsilon = \varepsilon^{\frac{n}{n-p}}$,

$$C = \frac{1}{2^n} \lim_{\varepsilon \to 0} \operatorname*{Min}_{\begin{cases} v=1 \text{ on } D \\ v \in W_0^{1,p}\left(S_{\frac{\varepsilon}{a_\varepsilon}}\right) \end{cases}} \int_{\mathbb{R}^n} |Dv|^p \, dx .$$

Clearly $K_\varepsilon = \left\{ v \in W^{1,p}(\mathbb{R}^n) \ / \ v = 1 \text{ on } D \, , \ v = 0 \text{ outside of } S_{\frac{\varepsilon}{a_\varepsilon}} \right\}$ converge in Mosco sense to

$$K = \{ v \in W^{1,p}(\mathbb{R}^n) \ / \ v = 1 \ \text{ on } \ D \}$$

(we remark that $\frac{\varepsilon}{a_\varepsilon} \to +\infty$ when $\varepsilon \to 0$) ; so

$$C = \frac{1}{2^n} \underset{\substack{v=1 \text{ on } D \\ v \in W^{1,p}(\mathbb{R}^n)}}{\operatorname{Min}} \int_{\mathbb{R}^n} |Dv|^p \, dx = \frac{1}{2^n} \operatorname{Cap}^p_{\mathbb{R}^n} D \, .$$

<u>So</u> C <u>can be interpreted as the capacity in</u> $\mathbb{R}^n$ <u>of the set</u> D .

# CH.V BILATERAL CONSTRAINTS

*Theorem 1*

*Let* $V = W_0^{1,p}(\Omega)$ , $\psi \in V$ *and two sequences of functionals* $(F_h^1)_{h\in\mathbb{N}}$ *and* $(F_h^2)_{h\in\mathbb{N}}$ *satisfying (i), (ii), (iv), (v),* $F_h^1$ *decreasing,* $F_h^2$ *increasing,* $F_h^1(\psi,\omega) = F_h^2(\psi,\omega) = 0$ *for every* $\omega \in \mathcal{O}_n$ .
*Then there exist* $F^1$ *and* $F^2$ *satisfying (i), (ii), (iv), (v),* $F^1$ *decreasing,* $F^2$ *increasing and there exists a subsequence* $(h_k)$ *such that :*

$$\forall u \in V, \quad \forall \omega \in \mathcal{O}_n ,$$

$$\|u\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega) = \Gamma^-(s\text{-}L^p \lim_{v\to u} [\|v\|^p + F^1_{n_k}(v,\omega) + F^2_{n_k}(v,\omega)] \quad .$$

*In fact*

$$\|u\|^p + F^1(u,\omega) = \Gamma^-(s\text{-}L^p) \lim_{v\to u} [\|v\|^p + F^1_{n_k}(v,\omega)]$$

$$\|u\|^p + F^2(u,\omega) = \Gamma^- \ s\text{-}L^p) \lim_{v\to u} [\|v\|^p + F^2_{n_k}(v,\omega)] \quad .$$

Proof

By Theorem 3.5 and corollary 3.68 , there exist $F^1$ , $F^2$ and $h_k$ (denoted by $h$ for simplicity) such that

$$\|\cdot\|^p + F^1(.,\omega) = \Gamma^-(s\text{-}L^p) \lim [\|.\|^p + F_h^1(.,\omega)]$$

$$\|.\|^p + F^2(.,\omega) = \Gamma^-(s\text{-}L^p) \lim [\|.\|^p + F_h^2(.,\omega)] \quad .$$

We shall use the following very simple lemma :

*Lemma*

*Let* $\psi \in V$ *and* $F : V \longrightarrow \bar{\mathbb{R}}^+$ *such that* $F(\psi) = 0$ *and*

$$\forall u,v \in V \,, \quad F(u \vee v) + F(u \wedge v) \leq F(u) + F(v) \,.$$

*Then if* $F$ *is devreasing (resp. increasing) for every* $u \in V$

$$F(u) = F(u \wedge \psi) \qquad (\textit{resp.} \quad F(u) = F(u \vee \psi)) \,.$$

Proof of the Lemma

We have $F(u \wedge \psi) + F(u \vee \psi) \leq F(u)$ .
If $F$ is decreasing $F(u) \leq F(u \wedge \psi)$ and if $F$ is increasing $F(u) \leq F(u \vee \psi)$ . The result follows.

Let us prove now the Theorem. Let $u \in V$ . There exist $u_h \to u$ in $L^p$ such that

$$\begin{aligned}
&\Gamma^-(s\text{-}L^p) \varliminf_{v \to u} [\|v\|^p + F_h^1(v,\omega) + F_h^2(v,\omega)] \\
&\quad = \varliminf [\|u_h\|^p + F_h^1(u_h,\omega) + F_h^2(u_h,\omega)] \\
&\quad = \varliminf [\|u_h \wedge \psi\|^p + \|u_h \vee \psi\|^p - \|\psi\|^p + F_h^1(u_h \wedge \psi,\omega) + F_h^2(u_h \vee \psi,\omega)] \\
&\quad \geq \varliminf [\|u_h \wedge \psi\|^p + F_h^1(u_h \wedge \psi,\omega)] + \varliminf [\|u_h \vee \psi\|^p + F_h^2(u_h \vee \psi,\omega)] - \|\psi\|^p \\
&\quad \geq \|u \wedge \psi\|^p + F^1(u \wedge \psi,\omega) + \|u \vee \psi\|^p + F^2(u \vee \psi,\omega) - \|\psi\|^p \\
&\quad \geq \|u\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega) \,.
\end{aligned}$$

Conversely, let $u \in V$ . There exist $u_h^1 \to u \wedge \psi$ in $L^p$ and $u_h^2 \to u \vee \psi$ in $L^p$ such that

$$\|u\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega)$$

$$= - \|\psi\|^p + \|u \wedge \psi\|^p + \|u \vee \psi\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega)$$

$$= - \|\psi\|^p + \lim \left[\|u_h^1\|^p + F_h^1(u_h^1,\omega) + \|u_h^2\|^p + F_h^2(u_h^2,\omega)\right]$$

$$= - \|\psi\|^p + \lim \left[\|u_h^1\|^p + F_h^1(u_h^1 \wedge \psi,\omega) + \|u_h^2\|^p + F_h^2(u_h^2 \vee \psi,\omega)\right]$$

$$\geqslant - \|\psi\|^p + \overline{\lim} \left[\|u_h^1\|^p + F_h^1((u_h^1 \wedge \psi) \vee (u \wedge \psi),\omega) + \|u_h^2\|^p\right.$$

$$\left. + F_h^2((u_h^2 \vee \psi) \wedge (u \vee \psi),\omega)\right]$$

$$\geqslant - \|\psi\|^p + \overline{\lim} \left[\|v_h^1\|^p + F_h^1(v_h^1,\omega) + \|v_h^2\|^p + F_h^2(v_h^2,\omega)\right]$$

$$+ \underline{\lim} \left[\|u_h^1\|^p - \|v_h^1\|^p\right] + \underline{\lim} \left[\|u_h^2\|^p - \|v_h^2\|^p\right]$$

where $v_h^1 = (u_h^1 \wedge \psi) \vee (u \wedge \psi)$ and $v_h^2 = (u_h^2 \vee \psi) \wedge (u \vee \psi)$ .

But $\underline{\lim} \left[\|u_h^1\|^p - \|v_h^1\|^p\right]$

$$= \underline{\lim} \left[\|u_h^1 \wedge \psi\|^p + \|u_h^1 \vee \psi\|^p - \|\psi\|^p - \|v_h^1\|^p\right]$$

$$= \underline{\lim} \left[\|(u_h^1 \wedge \psi) \wedge (u \wedge \psi)\|^p + \|v_h^1\|^p - \|u \wedge \psi\|^p + \|(u_h^1 \vee \psi) \wedge (u \wedge \psi)\|^p\right.$$

$$\left. + \|(u_h^1 \vee \psi) \vee (u \wedge \psi)\|^p - \|u \wedge \psi\|^p - \|\psi\|^p - \|v_h^1\|^p\right]$$

$$\geqslant \underline{\lim} \left[\|(u_h^1 \wedge \psi) \wedge (u \wedge \psi)\|^p - \|u \wedge \psi\|^p\right] + \underline{\lim} \left[\|(u_h^1 \vee \psi) \wedge (u \wedge \psi)\|^p - \|u \wedge \psi\|^p\right]$$

$$+ \underline{\lim} \left[\|(u_h^1 \vee \psi) \vee (u \wedge \psi)\|^p - \|\psi\|^p\right]$$

$$\geqslant 0 .$$

With a similar decomposition we obtain

$$\underline{\lim}\ [\|u_h^2\|^p - \|v_h^2\|^p] \geqslant 0 .$$

Hence we have

$$\|u\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega)$$

$$\geqslant -\|\psi\|^p + \overline{\lim}\ [\|v_h^1\|^p + F_h^1(v_h^1,\omega) + \|v_h^2\|^p + F_h^2(v_h^2,\ \omega)] \quad .$$

Let us define $v_n$ by

$$v_n - \psi = v_h^1 - \psi + v_h^2 - \psi .$$

Since $v_h^1 - \psi = -\ (u_h^1-\psi)^- \wedge (u-\psi)^- \leqslant 0$

$v_h^2 - \psi = \ (u_h^2-\psi)^+ \wedge (u-\psi)^+ \geqslant 0$

and $\inf ((u_h^1-\psi)^- \wedge (u-\psi)^-\ ,\ (u_h^2-\psi)^+ \wedge (u-\psi)^+) = 0$

we have $(v_n-\psi)^- = -\ (v_h^1-\psi)$ and $(v_h-\psi)^+ = v_h^2 - \psi$ ,

that is $v_h^1 = \psi - (v_h-\psi)^- = v_h \wedge \psi$ and $v_h^2 = \psi + (v_n-\psi)^+ = v_h \vee \psi$ .

In addition $v_h \to u$ in $L^p$ .

It follows

$$\|u\|^p + F^1(u \wedge \psi,\omega) + F^2(u \vee \psi,\omega)$$

$$\geqslant -\ \|\psi\|^p + \overline{\lim}\ [\|v_n \wedge \psi\|^p + F_h^1(v_n \wedge \psi,\omega) + \|v_n \vee \psi\|^p + F_h^2(v_n \vee \psi,\omega)]$$

$$\geqslant \overline{\lim}\ [\|v_h\|^p + F_h^1(v_h,\omega) + F_h^2(v_h,\omega)]$$

$$\geqslant \Gamma^-(s\text{-}L^p)\ \overline{\lim_{v\to u}}\ [\|v\|^p + F_h^1(v,\omega) + F_h^2(v,\omega)]$$

In conclusion

$$\|u\|^p + F^1(u \wedge \psi, \omega) + F^2(u \vee \psi, \omega) = \Gamma^-(s\text{-}L^p) \lim_{v \to u} [\|v\|^p + F_h^1(v,\omega) + F_h^2(v,\omega)] \ .$$

Example - Problem with holes

Let $S^i_{a_\varepsilon}$ be a hole as in paragraph IV and $\psi \in H^1_0(\Omega)$ .
Let $F_h^1$ and $F_h^2$ be defined by $(\varepsilon = \frac{1}{h})$

$$F_h^1(u,\omega) = \begin{cases} 0 & \text{if } u \geqslant \psi \text{ on } \omega \cap (\cup S^i_{a_\varepsilon}) \\ +\infty & \text{elsewhere} \end{cases}$$

$$F_h^2(u,\omega) = \begin{cases} 0 & \text{if } u \leqslant \psi \text{ on } \omega \cap (\cup S^i_{a_\varepsilon}) \\ +\infty & \text{elsewhere.} \end{cases}$$

By Theorem V.1 and result of IV, we obtain

$$\int_\Omega |Du|^2 + C_1 \int_\omega ((u-\psi)^-)^2 + C_2 \int_\omega ((u-\psi)^+)^2$$

$$= \Gamma^-(s\text{-}L^2) \lim_{v \to u} [\int_\Omega |Dv|^2 + F_h^1(u,\omega) + F_h^2(v,\omega)]$$

and in particular

$$\text{Min } \{\int_\Omega |Du|^2 \ ; u \in H^1_0(\Omega) \ , \quad u = \psi \text{ on } \omega \cap (\cup S^i_{a_\varepsilon})$$

$$\longrightarrow \text{Min } \{\int_\Omega |Du|^2 + C_1 \int_\omega ((u-\psi)^-)^2 + C_2 \int_\omega ((u-\psi)^+)^2 \ ; u \in H^1_0(\Omega)\} \ .$$

If $\psi = 0$ we obtain, since $C_1 = C_2 = C$ ,

$$\text{Min } \{\int_\Omega |Du|^2 \ ; u \in H^1_0(\Omega) \ , \ u = 0 \text{ on } \omega \cap (\cup S^i_{a_\varepsilon})$$

$$\longrightarrow \text{Min } \{\int_\Omega |Du|^2 + C \int_\omega |u|^2 \ ; u \in H^1_0(\Omega)\} \ .$$

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$$K(\psi_1,\psi_2) = \{v \in H_0^1(\Omega) \ / \ \psi_1 \leq v \leq \psi_2 \ \text{ pp. dans } \ \Omega\}$$

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