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Some Applications of the Theory of Distributions

Laurent Schwartz

In this paper we give a summary of the main results of the theory of distributions, and some selected applications.

SUMMARY OF THE MAIN ELEMENTARY RESULTS*

Definition [36]. Let $\mathcal{E}(\mathbf{R}^n)$ be the space of complex-valued infinitely differentiable functions on $\mathbf{R}^n$, equipped with the topology of uniform convergence of every derivative on every compact subset of $\mathbf{R}^n$. We adopt the following notations: $p = (p_1, p_2, \ldots, p_n)$, p_i positive or zero integers, will be a multi-index of differentiation of order $|p| = p_1 + p_2 + \cdots + p_n$, so that, for $\phi \in \mathcal{E}(\mathbf{R}^n)$, $\phi^{(p)}$ means the derivative $\left(\frac{\partial}{\partial x_1}\right)^{p_1} \left(\frac{\partial}{\partial x_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n} \phi$. We shall also introduce (for the applications to Fourier transform) the symbol $D^p = \left(\frac{1}{2i\pi}\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{1}{2i\pi}\frac{\partial}{\partial x_n}\right)^{p_n}$. If we adopt also the convention that $p!$ will be $p_1!\,p_2! \cdots p_n!$, and that, for $x \in \mathbf{R}^n$,

* References are given here to the individual chapters of our book on the subject [36–41]. It will also be very useful, for this whole section, to consult [17], which is a complete exposition of the theory of generalized functions, with many examples and applications.

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$x = (x_1, x_2, \ldots, x_n)$, x^p will be $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, then the usual Taylor formula is written in $\mathbf{R}^n$ as it is in $\mathbf{R}$:

$$\phi(x) = \sum_p \frac{\phi^p(0)}{p!} x^p \tag{1}$$

Among the multi-indices we introduce an order relation by putting $p \leq q$ if $p_1 \leq q_1, p_2 \leq q_2, \ldots, p_n \leq q_n$, and an addition, with $p + q = (p_1 + q_1, p_2 + q_2, \ldots, p_n + q_n)$. The Leibnitz formula, for the derivative of a product, is written

$$(uv)^{(p)} = \sum_{q \leq p} \frac{p!}{q!(p-q)!} u^{(q)} v^{(p-q)} \tag{2}$$

The topology of $\mathcal{E}(\mathbf{R}^n)$ may be defined by the family of seminorms $P_{m,K}$ [2]:

$$P_{m,K}(\phi) = \max_{\substack{x \in K \\ |p| \leq m}} |\phi^{(p)}(x)| \tag{3}$$

The support of a continuous function ϕ being the smallest closed subset of $\mathbf{R}^n$ outside which ϕ is zero, $\mathfrak{D}_K(\mathbf{R}^n)$ will be the subspace of $\mathcal{E}(\mathbf{R}^n)$ formed by the functions having their support in the compact set K of $\mathbf{R}^n$; it will be endowed with the topology induced by $\mathcal{E}(\mathbf{R}^n)$. The space $\mathfrak{D}(\mathbf{R}^n)$ of infinitely differentiable functions with compact support is the union of the $\mathfrak{D}_K(\mathbf{R}^n)$, K compact $\subset \mathbf{R}^n$. One puts on it a topology called *inductive limit* of the topologies of the $\mathfrak{D}_K$: a subset of $\mathfrak{D}(\mathbf{R}^n)$ will be said to be a neighborhood of 0 in this topology if and only if it contains a convex set, intersecting every $\mathfrak{D}_K$ according to a neighborhood of 0 in $\mathfrak{D}_K$. One may prove that a linear form on $\mathfrak{D}$ is continuous for this topology if and only if its restriction to every $\mathfrak{D}_K$ is continuous. Such a continuous linear form is called a distribution T on $\mathbf{R}^n$ (or generalized function); it is simply a linear form on $\mathfrak{D}$ denoted by $\phi \rightsquigarrow \langle T, \phi \rangle$, and having the following property: if functions ϕ_j of $\mathfrak{D}$ converge uniformly to 0 on $\mathbf{R}^n$ for j going to infinity as well as their derivatives of any order, and if they vanish outside the same compact K of $\mathbf{R}^n$, then $\langle T, \phi_j \rangle$ converge to 0 for j infinite. The space of distributions is called $\mathfrak{D}'(\mathbf{R}^n)$, dual of $\mathfrak{D}(\mathbf{R}^n)$.

A function f on $\mathbf{R}^n$, locally integrable, defines a distribution T_f by

$$\langle T_f, \phi \rangle = \int_{\mathbf{R}^n} f(x)\phi(x)\, dx \tag{4}$$

dx being the Lebesgue measure. Two functions define the same distribution if and only if they coincide almost everywhere. Therefore the Lebesgue classes (where two functions belong to the same class if they are almost everywhere equal) of locally integrable functions form a subspace of $\mathfrak{D}'$. We shall identify f and T_f, and write $\langle f, \phi \rangle$ instead of $\langle T_f, \phi \rangle$; a distribution defined by a function f will have the privilege that both $\langle f, \phi \rangle$ and $f(x)$ (for almost all x) will have a meaning; for a general distribution T, which will not be a function, only $\langle T, \phi \rangle$ has a meaning. Dirac's distribution δ is defined by

$$\langle \delta, \phi \rangle = \phi(0) \tag{5}$$

The general Dirac distribution relative to a point a of $\mathbf{R}^n$ is $\delta_{(a)}$ defined by

$$\langle \delta_{(a)}, \phi \rangle = \phi(a) \tag{6}$$

Supports. If Ω is any open subset of $\mathbf{R}^n$, one can define in the same way the space $\mathcal{E}(\Omega)$ of infinitely differentiable functions on Ω, the subspace $\mathfrak{D}(\Omega)$ of functions having a compact support in Ω, its dual $\mathfrak{D}'(\Omega)$, space of distributions over Ω. A distribution T on Ω is said to be 0 on an open subset $\mathcal{U}$ of Ω if $\langle T, \phi \rangle = 0$ as soon as ϕ has its support in $\mathcal{U}$. Using the classical tool of partition of unity (saying that, given any covering of Ω by open sets Ω_i, $i \in I$ there exists a system of $\alpha_i \in \mathcal{E}(\Omega)$, $\alpha_i \geq 0$ having its support in Ω_i, such that on every compact subset of Ω only a finite number of α_i are not identically 0, and $\sum_{i \in I} \alpha_i$ is identical to 1), one can see that, for every $T \in \mathfrak{D}'(\Omega)$, there exists a smallest closed subset F of Ω in the complement of which T is zero; F is called the support of T; if T is a continuous function, its support is the same as with the previous definition (page 24). The dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ may be identified with the subspace of $\mathfrak{D}'(\Omega)$ formed by the distributions with compact support. Moreover, one can extend the definition of $\langle T, \phi \rangle$ to the case where neither T nor ϕ have a compact support, provided $\phi \in \mathcal{E}(\Omega)$, and the intersection of the supports of T and ϕ is compact, by putting $\langle T, \phi \rangle = \langle T, \alpha\phi \rangle$, $\alpha \in \mathfrak{D}$ being equal to 1 on a neighborhood of the intersection of the supports of T and ϕ.

Differentiation [37]. The derivative of a distribution is defined in such a way that if this distribution is a usual C^1 function, its derivative coincides with the usual derivative of this function. One is therefore led to put:

$$\langle T^{(p)}, \phi \rangle = (-1)^{|p|}\langle T, \phi^{(p)} \rangle \tag{7}$$

Every distribution (in particular, every locally integrable function) is infinitely differentiable in the sense of distributions. The computation of the derivative is much more complicated than that in classical analysis: it involves different kinds of integral formulae, such as Stokes, Green, and so on. For instance, the function $\frac{1}{|x|^{n-2}}$ (where $|x| = \sqrt{x_1{}^2 + \cdots + x_n{}^2}$) is known to be harmonic in the complement of the origin; it is not regular in the neighborhood of the origin. The classical computation of its laplacian gives the simple answer: $\Delta \frac{1}{|x|^{n-2}} = 0$ in $\complement\, 0$. But $\frac{1}{|x|^{n-2}}$ is locally integrable in the whole of $\mathbf{R}^n$, and therefore is a distribution over $\mathbf{R}^n$; this distribution must have a laplacian over $\mathbf{R}^n$, and this is

$$\Delta\left(\frac{1}{|x|^{n-2}}\right) = -(n-2)S_n\delta, \tag{8}$$

where S_n is the area of the unit sphere in $\mathbf{R}^n$. This result will contain Poisson's integral formula, and play a central role in the theory of potentials and partial differential equations. In the so-called Heaviside symbolic calculus, a fundamental formula is

$$Y' = \delta \tag{9}$$

where Y is the Heaviside function over $\mathbf{R}$, equal to 0 for $x < 0$, to $+1$ for $x > 0$.

Multiplication [39]. Multiplication of two distributions cannot be defined. One can define easily the product of a distribution T by a function α of $\mathcal{E}$ in such a way that, if T is a locally integrable function f, we just get the usual product αf. One puts

$$\langle \alpha T, \phi \rangle = \langle T, \alpha\phi \rangle \qquad T \in \mathfrak{D}', \phi \in \mathfrak{D}, \alpha \in \mathcal{E} \tag{10}$$

This multiplication has the usual properties, in particular bilinearity, associativity with multiplication in $\mathcal{E}$ (that is $(\alpha\beta)T = \alpha(\beta T)$), and Leibnitz's formula for the derivative of a product:

$$(11) \qquad (uT)^{(p)} = \sum_{q \leq p} \frac{p!}{q!(p-q)!} u^{(q)} T^{(p-q)}$$

Many people have tried to extend this very restricted multiplication, but without any great success. Of course, formula 10 holds if α is only m times continuously differentiable, $\alpha \in \mathcal{E}^m$, provided T is only a distribution of order $\leq m$ (here $\mathfrak{D}^m$ is the subspace of $\mathcal{E}^m$ formed by the functions of compact support; it is equipped with the inductive limit of the topologies of the $\mathfrak{D}_K{}^m$; and a distribution T is of the order $\leq m$ if it can be extended as a continuous linear form on $\mathfrak{D}^m$). But it can be easily proved that no product can be defined for two arbitrary distributions so that it possesses reasonable properties (as associativity, Leibnitz formula). It appears more and more that some of the greatest mathematical difficulties in theoretical physics, for instance, in quantum field theory, proceed precisely from this impossibility of multiplication.

Convolution [41]. Differentiation and multiplication are local operations which have a meaning on any open set Ω of $\mathbf{R}^n$. On the contrary, convolution is a global operation related to the group structure of $\mathbf{R}^n$. Let S and T be two distributions on $\mathbf{R}^n$. One can form $S_\xi T_\eta$, "tensor product" of S and T, a distribution on the $2n$-dimensional space $\mathbf{R}^n \times \mathbf{R}^n$. Then, if some conditions on supports are verified, one may compute $\langle S_\xi T_\eta,\ \phi(\xi + \eta)\rangle$ for $\phi \in \mathfrak{D}(\mathbf{R}^n)$. If S and T are such that this has a meaning for every $\phi \in \mathfrak{D}(\mathbf{R}^n)$, one says it is the product of convolution $S * T$ of S and T; thus we have

$$(12) \qquad \langle S * T,\ \phi\rangle = \langle S_\xi T_\eta,\ \phi(\xi + \eta)\rangle$$

$S,\ T \in \mathfrak{D}'(\mathbf{R}^n)$, $\phi \in \mathfrak{D}(\mathbf{R}^n)$; the right side is computed between $\mathfrak{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ and $\mathcal{E}(\mathbf{R}^n \times \mathbf{R}^n)$. Formula 12 surely defines a distribution $S * T \in \mathfrak{D}'(\mathbf{R}^n)$ if, for every $\phi \in \mathfrak{D}(\mathbf{R}^n)$, the supports of $S_\xi T_\eta$ and $\phi(\xi + \eta)$ have a compact intersection in $\mathbf{R}^n \times \mathbf{R}^n$ (see page 25). This will happen if the supports A and B of S and T verify the following condition:

(13) $(\xi, \eta) \rightsquigarrow \xi + \eta$ (addition) is a *proper* map from $A \times B$ into $\mathbf{R}^n$ (that is, the pre-image of any compact subset of $\mathbf{R}^n$ is compact in $A \times B$).

Condition 13 is satisfied if one of the supports is compact, so that convolution is a bilinear map from $\mathcal{E}'(\mathbf{R}^n) \times \mathfrak{D}'(\mathbf{R}^n)$ into $\mathfrak{D}'(\mathbf{R}^n)$. Many other cases are important; for instance, for $n = 1$, if $\mathfrak{D}_+'(\mathbf{R}^n)$ is the space of distributions whose support is bounded below, then convolution is a bilinear map from $\mathfrak{D}_+'(\mathbf{R}^n) \times \mathfrak{D}_+'(\mathbf{R}^n)$ into $\mathfrak{D}_+'(\mathbf{R}^n)$. The *support* of $S * T$ is always contained in $\overline{A + B}$, $A + B$ being the set of the $\xi + \eta$'s, $\xi \in A$, $\eta \in B$. But, of course, it may be much smaller: formula 18 will show that $\delta' * 1 = 0$, so that here $A = \{0\}$, $B = \mathbf{R}$, $A + B = \mathbf{R}$, and the support of $\delta' * 1$ is empty. However, Lions [23] proved that, if A and B are compact, the convex hull of the support of $S * T$ is *exactly* the sum $A' + B'$ of the convex hulls A' and B' of the supports A and B of S and T. Convolution is commutative, associative, provided some conditions on the supports are satisfied. If f and g are two functions whose supports verify condition 13, their convolution is a function h defined almost everywhere by

(14) $$h(x) = \int_{\mathbf{R}^n} f(x - \xi) g(\xi)\, d\xi$$

One sees immediately that

(15) $$\delta * T = T$$

δ is the unit of convolution. In the same way:

(16) $$\delta_{(a)} * T = \tau_a T$$

transform of T by the translation τ_a: $x \rightsquigarrow x + a$, of $\mathbf{R}^n$. Therefore convolution commutes with translation:

(17) $$\tau_a(S * T) = \tau_a S * T = S * \tau_a T$$

Convolution has also remarkable properties with respect to differentiation, which is the most significant reason for its application:

(18) $$\delta^{(p)} * T = T^{(p)}$$

and therefore convolution commutes with differentiation:

(19) $$\begin{aligned} (S * T)^{(p)} &= S^{(p)} * T = S * T^{(p)} \\ &= S^{(q)} * T^{(p-q)} \qquad q \leq p \end{aligned}$$

An immediate application is given by Poisson's formula for potentials. The potential of a distribution T is defined by

$$U_T = T * \frac{1}{|x|^{n-2}} \tag{20}$$

(Generalization of the usual formula for the potential of a function; (16) has a meaning if T has a compact support.) From (8) and (19) we deduce

$$\Delta U_T = T * \Delta\left(\frac{1}{|x|^{n-2}}\right) \tag{21}$$

$$= -(n-2)S_n(\delta * T) = -(n-2)S_nT$$

which is Poisson's formula. It is exactly this kind of formula we shall discuss more carefully in the following paragraphs on applications.

Topologies [38]. A lot of topological problems about topological vector spaces arose from the theory of distributions. A large part of the research done in the direction of going beyond the Banach spaces into locally convex spaces found their origin or their applications here. First of all, the notion of inductive limit [1] of topologies introduced to define $\mathfrak{D}$. Now one has to define topologies on $\mathfrak{D}'$. There are two natural ones, the weak and the strong ones. The weak one is defined by the seminorms of pointwise convergence [2]: $P_\phi(T) = |\langle T, \phi\rangle|$, for every ϕ of $\mathfrak{D}$. The strong one is defined by the seminorms P_B of uniform convergence over every bounded subset of $\mathfrak{D}$: $P_B(T) = \sup_{\phi \in B} |\langle T, \phi\rangle|$, B bounded subset of $\mathfrak{D}$. It occurs because of very particular properties of $\mathfrak{D}$ ($\mathfrak{D}$ is a "Montel" space) [5] that any sequence of $\mathfrak{D}'$, which is weakly convergent, is also strongly convergent. $\mathfrak{D}$ and $\mathfrak{D}'$ (as $\mathcal{E}$ and $\mathcal{E}'$) are reflexive spaces [4]: each of them is the dual of the other. Moreover, the usual properties of topological vectors spaces, as Hahn-Banach theorem [3], enable us to make a deep and fruitful study about the structure of distributions and about the bounded sets and convergent sequences of distributions (for instance, any distribution in Ω is, in any relatively compact open subset $\mathcal{U}$ of Ω, a finite derivative of a continuous function).

Fourier Transform [41]. The Fourier transform, which is also a global one, cannot be defined for all distributions. One has to introduce a condition of slow increase at infinity. Let $\mathcal{S}(\mathbf{R}^n)$ be the space of infinitely differentiable functions ϕ having the following property: for every p and q, $x^q D^p \phi$ is bounded on $\mathbf{R}^n$. One can say that ϕ is rapidly decreasing at infinity (more than any power of $\frac{1}{|x|}$), as well as each of its derivatives. $\mathcal{S}(\mathbf{R}^n)$ is endowed with the topology defined by the family of seminorms $P_{q,p}$:

$$(22) \qquad P_{q,p}(\phi) = \sup_{x \in \mathbf{R}^n} |x^q D^p \phi(x)|$$

A distribution T on $\mathbf{R}^n$ is said to be tempered if it can be extended as a continuous linear form on $\mathcal{S}(\mathbf{R}^n)$. Tempered distributions form a subspace of $\mathcal{D}'(\mathbf{R}^n)$, the dual $\mathcal{S}'(\mathbf{R}^n)$ of $\mathcal{S}(\mathbf{R}^n)$; $\mathcal{S}'$ may be equipped with the weak or strong topology, and here again $\mathcal{S}$ and $\mathcal{S}'$ are reflexive spaces, and a weakly convergent sequence in $\mathcal{S}'$ is strongly convergent. A continuous function, bounded by a polynomial (or "slowly increasing at infinity") is tempered. This slow order of increase is the origin of the name "tempered." Conversely, it can be proved that any tempered distribution is a finite derivative of a slowly increasing continuous function.

$\mathcal{S}'$ is the natural domain of Fourier transform and harmonic analysis. The usual Fourier transform being defined for functions by

$$(23) \qquad g = \mathfrak{F}f, \text{ or } \hat{f}, \; g(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2i\pi\langle \xi, x\rangle}\, dx$$

where $\langle \xi, x\rangle = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$, one sees that $\mathfrak{F}$ is a continuous linear map of $\mathcal{S}$ into $\mathcal{S}$. Then we will define the Fourier transform of a distribution in such a way that, if this distribution is an integrable function, its transform is the bounded function given by (23). We now write

$$(24) \qquad \langle \mathfrak{F}T, \phi\rangle = \langle T, \mathfrak{F}\phi\rangle$$

a formula which has a meaning for every $\phi \in \mathcal{S}$ if $T \in \mathcal{S}'$. Formula 24 defines a Fourier transform for all tempered distributions, and $\mathfrak{F}$ is a continuous linear map from $\mathcal{S}'$ into $\mathcal{S}'$.

Simultaneously with $\mathfrak{F}$, one can define the operation $\bar{\mathfrak{F}}$, changing $-2i\pi\langle \xi, x\rangle$ in (23) into $+2i\pi\langle \xi, x\rangle$. $\bar{\mathfrak{F}}$ has the same properties. Moreover, the Fourier reciprocity formula expresses that $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ are inverse of each other: $\bar{\mathfrak{F}}\mathfrak{F} = \mathfrak{F}\bar{\mathfrak{F}} =$ identity, for tempered distributions as for functions.

One has the famous Parseval-Plancherel formula: $\mathfrak{F}$ is a unitary operation from L^2 onto L^2.

Finally, the Fourier transform exchanges multiplication and convolution; but one has to give very precise conditions. Let us call $\mathcal{O}_M$ (operators of multiplication) the space of infinitely differentiable functions ϕ having the following property: for every p, $\mathfrak{D}^p\phi$ is bounded by a polynomial. $\mathcal{O}_M$ can be called the space of functions which are slowly increasing at infinity, as well as each of their derivatives. One can prove easily that, if $\alpha \in \mathcal{O}_M$ and $T \in \mathcal{S}'$, the multiplication product αT still belongs to $\mathcal{S}'$. Now let us call $\mathcal{O}_C{}'$ (operators of convolution) the space of distributions T having the following property: for every q, $x^q T$ is a "bounded distribution," or finite sum of derivatives of continuous, bounded functions. One can say that $\mathcal{O}_C{}'$ is the space of distributions which "decrease rapidly at infinity." Then one can prove that $S * T$ has a meaning and belongs to $\mathcal{S}'$ (although the conditions on supports (13) are not satisfied), for $S \in \mathcal{O}_C{}'$ and $T \in \mathcal{S}'$. Now $\mathfrak{F}$ exchanges $\mathcal{O}_M$ and $\mathcal{O}_C{}'$; and, if $\alpha \in \mathcal{O}_M$, $T \in \mathcal{S}'$, $S \in \mathcal{O}_C{}'$, one has

(25)
$$\begin{aligned} \mathfrak{F}(\alpha T) &= \mathfrak{F}\alpha * \mathfrak{F}T, \\ \mathfrak{F}(S * T) &= (\mathfrak{F}S)(\mathfrak{F}T) \end{aligned}$$

In particular, one has

(26)
$$\mathfrak{F}\delta = 1, \ \mathfrak{F}1 = \delta$$

(exchange between the unit δ of convolution and the unit 1 of multiplication),

(27)
$$\begin{aligned} \mathfrak{F}(D^p\delta) &= \xi^p \\ \mathfrak{F}((-x)^q) &= D^q\delta \end{aligned}$$

From it results that $\mathfrak{F}$ exchanges differentiation and multiplication by monomials:

(28)
$$\begin{aligned} \mathfrak{F}(D^pT) &= \mathfrak{F}(D^p\delta * T| \\ &= \mathfrak{F}(D^p\delta)\mathfrak{F}T = \xi^p\mathfrak{F}T \\ \mathfrak{F}((-x^q)T) &= \mathfrak{F}((-x)^q) * \mathfrak{F}T \\ &= D^q\delta * \mathfrak{F}T = D^q\mathfrak{F}T \end{aligned}$$

These formulae (28) are the basic ones in the applications of Fourier transform to partial differential equations with constant coefficients.

Let us finally give the generalized Paley-Wiener theorem. If T is a distribution with compact support, its Fourier transform $\hat{T}$ is a function, defined for every ξ by the formula:

$$\hat{T}(\xi) = \langle T_x, e^{-2i\pi\langle \xi, x\rangle}\rangle \tag{29}$$

Moreover, this formula is still meaningful if we replace ξ by $\xi + i\eta$; T can be extended as a holomorphic function on the complex space $\mathbf{C}^n$, and this entire function is of exponential type, that is, there exists a constant C such that

$$\left|\hat{T}(\xi + i\eta)\right| \leq Ce^{(|\xi|+|\eta|)} \tag{30}$$

Conversely, one can prove that, if a tempered distribution on $\mathbf{R}^n$ is a function which can be extended as a holomorphic function on $\mathbf{C}^n$ of exponential type, it is the Fourier transform of a distribution with compact support [41].

FUNDAMENTAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Comparison of Operators [18, 43]. A polynomial P on $\mathbf{R}^n$, of degree $\leq m$, may be written

$$P(\xi) = \sum_{|p|\leq m} a_p \xi^p \tag{31}$$

It defines a differential operator with constant coefficients, of order $\leq m$,

$$P(D) = \sum_{|p|\leq m} a_p D^p \tag{32}$$

and conversely. Therefore all the properties of such a differential operator must be visible on the associated polynomial. For such operators, Hörmander generalized Leibnitz's formula in a very useful way:

$$P(D)(uv) = \sum_q \frac{D^q u}{q!} P^{(q)}(D)v \tag{33}$$

(for $P(D) = D^p$, it is formula 2; for P arbitrary, one gets it by addition).

A very powerful tool in the study of partial differential operators with constant coefficients is the comparison of operators (or polynomials), introduced by Hörmander* in the following way.

Assume that for some open bounded (nonempty) set Ω of $\mathbf{R}^n$ there exists a constant c such that, for all $\phi \in D(\Omega)$ one has:

$$\|Q(D)\phi\| \leq c\|P(D)\phi\| \tag{34}$$

$\| \quad \|$ being the L^2- norm. Then one proves that an analogous relation holds for all such Ω's, the constant c, of course, depending on Ω. In this case we shall say that P is stronger than Q, and write $P \geq Q$; or $Q \leq P$; if $P \geq Q$ and $Q \geq P$, Q and P will be said equally strong. Moreover, Hörmander proved that the necessary and sufficient condition for P to be stronger than Q is the existence of a constant C such that, for all $\xi \in \mathbf{R}^n$:

$$\tilde{Q}(\xi) \leq C\tilde{P}(\xi) \tag{34a}$$

where

$$\tilde{P}(\xi) = \sum_q |P^{(q)}(\xi)| \tag{35}$$

(Note that at least one derivative of P is a constant $\neq 0$. Therefore $\tilde{P}(\xi)$ is greater than some > 0 constant; and the inequality (34a) only involves the large values of $|\xi|$.)

There are different proofs of this theorem. An indispensable step in the proof of the sufficiency is the proof that, for every q, the derivative $P^{(q)}$ is weaker than P; after that one has only to apply Fourier transform. But this particular case may be improved by considering, instead of Ω (which always had to be bounded), $\mathbf{R}^n$ itself. Of course, one cannot obtain a relation of the type (34) because $\mathbf{R}^n$ is unbounded; but, for every polynomial P and every $\alpha > 0$, there exists a constant C (depending on P and α) such that, for every $\phi \in D(\mathbf{R}^n)$ and every q [19, 25, 43]

$$\|P^{(q)}(D)\phi\| \leq C\|e^{\alpha|x|}P(D)\phi\| \tag{35a}$$

If Ω is bounded, $e^{\alpha|x|}$ is bounded on Ω, and may be canceled with

* A book just published by Hörmander (*Linear Partial Differential Operators*, New York (1963) Academic Press) contains many new and known results on partial differential equations.

a modification of the constant, so as to obtain (34). The presence of this factor $e^{\alpha|x|}$ introduces some dissymmetry between the two sides of the inequality. Trèves [31, 42] has given a more symmetric one proving that, for every P and every $k > 0$, there exists a constant C (depending on P and k) such that, for every $\phi \in D(\mathbf{R}^n)$ and every q

$$\|e^{k|x|^2}P^{(q)}(D)\phi\| \leq C\|e^{k|x|^2}P(D)\phi\| \tag{36}$$

from which one still obtains an inequality of the type in (34) for a bounded set Ω, $e^{k|x|^2}$ being bounded above and below in Ω. As a case of particular interest, assume that P is *elliptic;* it means that, if P_m is the homogeneous part of highest degree m of P, $P_m(\xi)$ never vanishes for $\xi \in \mathbf{R}^n$, except $\xi = 0$. Then one sees immediately that $\tilde{P}(\xi)$ is equivalent to constant times $|\xi|^m$ for $|\xi|$ infinite, so that every polynomial of degree $\leq m$ is weaker than P.

If now P is *of principal type*, which means that there is no $\xi \in \mathbf{R}^n$ (except $\xi = 0$) for which all the first derivatives of P_m vanish simultaneously, then $\tilde{P}(\xi)$ is $\geq$ constant times $|\xi|^{m-1}$ for large $|\xi|$; so that every polynomial of degree $\leq m - 1$ is weaker than P.

Fundamental Solution. A fundamental solution of the differential operator $P(D)$ is a distribution E such that

$$P(D)E = \delta \tag{37}$$

This definition, using distributions, replaces now the old ones in which E was defined as a solution of the homogeneous equation $P(D)E = 0$, having however some singularities at the origin or on the characteristic cone. Moreover, for instance, for a hyperbolic operator of the second order (say, the wave operator) in $\mathbf{R}^n$, $n \geq 4$, a fundamental solution cannot be a (locally integrable) function. Finally, the main interest of a fundamental solution is that it allows one to solve nonhomogeneous equations where the nonhomogeneous term has compact support. If $A \in \mathcal{E}'$, the distribution $x = E * A$ (which has meaning as a convolution between $\mathcal{E}'$ and $\mathcal{D}'$) is trivially a solution of

$$P(D)X = A \tag{38}$$

because of (19): $P(D)(E * A) = P(D)E * A = \delta * A = A$. This procedure generalizes the classical one in potential theory, namely Poisson's formula: for $P(D) =$ laplacian Δ, one may take

$E = -\frac{1}{(n-2)S_n}\frac{1}{|x|^{n-2}}$ (at least for $n \neq 2$), and a solution of $\Delta X = A$ is given (for $A \in \mathcal{E}'$) by the potential $E * A = -\frac{1}{(n-2)S_n} U_A$ (formulae 20 and 21).

Of course, a fundamental solution is never unique (except for P = constant) because one can add to such a distribution any solution of the homogeneous equation.

A natural way to look for a fundamental solution is to make a Fourier transform. If E is tempered, and thus has a Fourier transform $\mathfrak{F}E = \hat{E}$, this has to verify

$$P\hat{E} = 1 \tag{39}$$

So that one is led to take for $\hat{E}$ the function $\hat{E}(\xi)$:

$$\hat{E}(\xi) = \frac{1}{P(\xi)} \tag{40}$$

But P vanishes generally on an algebraic variety of $\mathbf{R}^n$, and $1/P$ has no reason to be locally integrable and tempered. *If* it is, then the problem is solved and $\mathfrak{F}(1/P)$ will be a tempered fundamental solution. It is just the case for $P(D) = \Delta$, or $P(\xi) = -4\pi^2|\xi|^2$, in $\mathbf{R}^n$ for $n \geq 3$. Then $-\frac{1}{4\pi^2|\xi|^2}$ is locally integrable, and trivially tempered (tends to 0 at infinity!), and $\mathfrak{F}\left(\frac{-1}{4\pi^2|\xi|^2}\right)$ is just the distribution $-\frac{1}{(n-2)S_n}\frac{1}{|x|^{n-2}}$ previously found. On the contrary, this method fails for $n = 1, 2$, and, although being an exceptionally simple case, it gives a good indication of the difficulty of the problem. Of course, if we take now for $n = 4$ the wave operator $\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$, then $P(\xi) = -4\pi^2(\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2)$, and $1/P$ vanishes on the light cone $P(\xi) = 0$, and is surely not locally integrable; even more will this be the case for the iterates $\Box^m$.

It happened that the research on fundamental solutions produced two essential progresses in the theory of distributions, as well as in the theory of partial differential operators. One is the solution of the problem of division (see page 37), and the other is the collection of L^2 inequalities, in the way of the comparison (34) between polynomials.

Existence of Fundamental Solutions [11, 19, 25, 35, 43, 44]. It was first proved by Malgrange and Ehrenpreis that every $P(D)$ (except 0) has a fundamental solution. Although different proofs may be given now, the best seems to use inequalities of the type (35a). If we take $|q| = m$, degree of P, one of the derivatives $P^{(q)}$ is a constant. Therefore, for every $\alpha > 0$, there exists a constant C such that, for every $\phi \in D(\mathbf{R}^n)$:

$$\|\phi\| \leq C\|e^{\alpha|x|}P(-D)\phi\|^* \tag{41}$$

On the other hand, Fourier transform and Schwarz inequality prove immediately that there exists a constant C' such that, for every $a \in \mathbf{R}^n$ and $\phi \in \mathfrak{D}$:

$$|\phi(a)| \leq C'\Big(\sum_{|p|\leq N}\|D^p\phi\|^2\Big)^{1/2} \tag{42}$$

where $N = \left[\frac{n}{2}\right] + 1$, $\left[\frac{n}{2}\right]$ = integer part of $\frac{n}{2}$ (this number N cannot be improved). Taking $a = 0$, we get

$$|\phi(0)| \leq CC'\Big(\sum_{|p|\leq N}\|e^{\alpha|x|}D^pP(-D)\phi\|^2\Big)^{1/2} \tag{43}$$

But $\psi \rightsquigarrow \Big(\sum_{|p|\leq N}\|e^{\alpha|x|}D^p\psi\|^2\Big)^{1/2}$ is a continuous seminorm on $\mathfrak{D}$. Therefore the linear form $P(-D)\phi \rightsquigarrow \phi(0)$, defined on the linear subspace $P(-D)\mathfrak{D}$ of $\mathfrak{D}$, is continuous; by the Hahn-Banach theorem [3] it can be extended on the whole of $\mathfrak{D}$, that is, as a distribution E. Thus E verifies

$$\langle E, P(-D)\phi\rangle = \phi(0) \tag{44}$$

Applying formula 7, the left side is $\langle P(D)E, \phi\rangle$; therefore E is a fundamental solution.

But one gets more: an estimate on E (according to the Hahn-Banach theorem). Let us reason a little differently. Consider the product of Hilbert spaces $(L^2)^M$, where M is the number of

* We use the polynomial $P(-\xi)$ instead of $P(\xi)$ for reasons which will appear later.

multi-indices p of order $|p| \leq N$. An element of this space is a system $(\psi_p)_{|p|\leq N}$. If $\phi \in \mathfrak{D}$, it defines an element of this space, namely $(e^{\alpha|x|}D^p\phi)_{|p|\leq N}$. Then we have a linear form on the subspace of the corresponding elements defined by the $P(-D)\phi$, namely

$$(e^{\alpha|x|}D^pP(-D)\phi)_{|p|\leq N} \rightsquigarrow \phi(0) \tag{45}$$

and (43) proves that this linear form is continuous. Therefore it can be extended on the whole of $(L^2)^M$ (Hahn-Banach theorem for a Hilbert space!). Besides, we know that on a Hilbert space every continuous linear form is defined by the scalar product with an element of the space. Therefore, there exists a system of functions e_p of L^2, $|p| \leq N$ with:

$$\begin{aligned}\phi(0) &= \sum_{|p|\leq N} \langle e_p, e^{\alpha|x|}D^pP(-D)\phi\rangle \\ &= \langle P(D) \sum_{|p|\leq N} (-1)^{|p|}D^p(e^{\alpha|x|}e_p(x)), \phi\rangle\end{aligned} \tag{46}$$

which proves that $P(D)E = \delta$, with

$$E = \sum_{|p|\leq N} (-1)^{|p|}D^pe^{\alpha|x|}e_p, \qquad e_p \in L^2 \tag{47}$$

Properties of Fundamental Solutions. The result is very remarkable from two points of view: (1) E is of exponential increase at infinity; α is arbitrarily small (of course, the e_p's change if α changes), but cannot be taken equal to 0. This could suggest there does not exist any tempered fundamental solution such as we looked for (page 35). In fact, it was not proved that there exist tempered fundamental solutions until recently when the problem of division was solved (see page 43). Thus the present method does not give as much as division; but the E we find here has additional properties, see (2), and Hörmander proved that the tempered solutions do not satisfy, in general, these properties. (2) E is the sum of derivatives of order $\leq N$ of functions which are locally L^2. One can say, in the sense of L^2, E is of order $\leq N$. This L^2 order is not the same as the order defined on page 27. Let us make this more precise.

Call H^s, for $s \geq 0$ integer, the space of functions which are L^2, as well as their derivatives (in the sense of distributions) of order

$\leq s$. H^s is a Hilbert space, for the scalar product

$$(48) \qquad (f|g)_s = \sum_{|p| \leq s} \int_{\mathbf{R}^n} D^p f \overline{D^p g}\, dx$$

A function of H^s is "s times differentiable, in the sense of L^2." Now define $H^{-s'}$ as the dual of $H^{s'}$ with the dual norm. It is still a Hilbert space. Moreover, the Hahn-Banach theorem, in the way we applied it twice already, proves that a distribution T belongs to $H^{-s'}$ if and only if it is the sum of derivatives of order $\leq s'$ of L^2 functions. Such a distribution is "of order $\leq s'$, in the sense of L^2." For the following, it will be useful to introduce also, for any real s, the subspace H^s_{comp} of H^s formed by the distributions with compact support, and the space H^s_{loc} of distributions which, in every open, relatively compact set of $\mathbf{R}^n$, coincide with a distribution of H^s. One must remark that, even for $s > 0$, a function of H^s_{loc} need not be continuous. One can only say, using the method of inequality (42), that any function of H^N_{loc}, $N = \left[\frac{n}{2}\right] + 1$, is continuous; and this cannot be improved. In the same way, δ is not of order 0, as it is in the sense of page 27; δ belongs to H^{-N}_{comp}, and not to H^{-N+1}_{comp}.

Now the expression (47) proves that, in $\mathbf{R}^n$, every differential operator with constant coefficients (except the 0 operator) has at least a fundamental solution belonging to H^{-N}_{loc}. This is a remarkable fact (already seen by Malgrange): this order $-N$ does not depend on the operator but only on the dimension n of the space! When one observes the facts which were known before, it is not as surprising as at first glance. All the specialists already noted that the equations which have the "worst" fundamental solutions in the sense of local regularity are the equations of low order. For instance, a fundamental solution of Δ^k is proportional to $\frac{1}{|x|^{n-2k}}$ or eventually $\frac{\log |x|}{|x|^{n-2k}}$. For k large, $n - 2k$ becomes < 0, and the fundamental solution becomes continuous and more and more differentiable for increasing k. Thus it is rather normal that we can find a common bound N for the L^2 order of E. The worst possible case is just given by the operator of order 0 which is the identity: $P(\xi) = 1$, $P(D) =$ identity, $E = \delta \in H^{-N}_{\text{comp}}$.

One can also see, in another way, the improvement of the local regularity of E when the order m of P increases. Hörmander proved that one can find a *proper fundamental solution* [19, 43, 44] E having the following property: for every $A \in H^s_{\text{comp}}$, s real, the convolution $E * A$ (which is a solution of the equation where the nonhomogeneous term has compact support, $P(D)X = A$, see formula 38) belongs to H^s_{loc} and, for any polynomial Q weaker than P in the sense of (34), $Q(D)(E * A)$ also belongs to H^s_{loc}. Moreover, such a proper fundamental solution may be built *explicitly* without the Hahn-Banach theorem [44]. This gives important information about local behavior of solutions of nonhomogeneous equations. Taking $A = \delta \in H^{-N}_{\text{comp}}$, we see in particular that not only $E \in H^{-N}_{\text{loc}}$, but also $Q(D)E$ for every Q weaker than P belongs to H^{-N}_{loc}. For instance, for P elliptic (see page 34), one has $D^pE \in H^{-N}_{\text{loc}}$ for $|p| \leq m$, therefore $E \in H^{-N+m}_{\text{loc}}$ which confirms the result observed with the powers of Δ; for P of principal type, $D^pE \in H^{-N}_{\text{loc}}$ for $|p| \leq m - 1$, therefore $E \in H^{-N+m-1}_{\text{loc}}$. For m large, E is a continuous function and becomes more and more differentiable as m increases. (Note that it is only true, in this simple way, for simple types of equations, as elliptic or principal ones. Note also that not all the fundamental solutions have these properties; except for the hypoelliptic case.)

The previous results, of course, do not exhaust what is known or what can be looked for about fundamental solutions. Let us point out a recent result of Trèves [44]: it is possible to choose a fundamental solution E for all the $P \not\equiv 0$ of degree $\leq m$, in such a way that, as a distribution, it depends C^∞ on the coefficients of P. This result is surprising, especially in the neighborhood of 0 values of all the coefficients of degree m of P.

INHOMOGENEOUS EQUATIONS

Now consider an inhomogeneous equation

$$(49) \qquad P(D)X = A \qquad A \in \mathcal{D}'$$

If A has a compact support, a solution was obtained in the previous section by convolution with a fundamental solution. It does not give any indication at all for the case where A has an arbitrary support. Many results had been given for many years in particular cases: elliptic or hyperbolic equations, for instance. It seems that the first general result, valid for all partial differential equations with constant coefficients, was given by Malgrange [26]: if $A \in \mathcal{E}(\mathbf{R}^n)$ (infinitely differentiable function), there exists a solution $X \in \mathcal{E}(\mathbf{R}^n)$, for every P (not identically 0); in other words, $P(D)(\mathcal{E}(\mathbf{R}^n)) = \mathcal{E}(\mathbf{R}^n)$. Of course, except for special cases such as P elliptic, not all the solutions are in $\mathcal{E}$. Surprisingly enough, this problem has, a priori, nothing to do with distributions because everything involves only infinitely differentiable functions; however, the first solution given by Malgrange [27] uses a lot concerning distributions. The space $\mathcal{E}$ is a Frechet space (topological vector space, Hausdorff, locally convex, with a countable fundamental system of neighborhoods of 0, and complete). A famous theorem of Banach expresses that, in order that a continuous linear map (here $P(D)$) of a Frechet space $\mathcal{E}$ into itself be onto, it is necessary and sufficient that its transposed map (here $P(-D)$, operating from $\mathcal{E}'$ into $\mathcal{E}'$) be one to one and have a weakly closed image [6, 7]. That $P(-D)$ is one to one in $\mathcal{E}'$ is trivial by Fourier transform because the Fourier image of a distribution of $\mathcal{E}'$ is an analytic function, therefore its product by $P(-\xi)$ can be 0 only if it is already 0 itself. Now, why is $P(-D)\mathcal{E}'$ weakly closed in $\mathcal{E}'$? Here, still one uses a theorem of Banach saying that a vector subspace of the dual $\mathcal{E}'$ of a Frechet space $\mathcal{E}$ is surely weakly closed if its intersection with any bounded set of $\mathcal{E}'$ is weakly closed in this bounded set [7, 9]. But a bounded set of $\mathcal{E}'$ is contained in an $\mathcal{E}_K'$, K compact of $\mathbf{R}^n(\mathcal{E}_K'$ is the set of distributions having their support in K) so that finally it remains to be proved that $P(-D)\mathcal{E}' \cap \mathcal{E}_K'$ is closed. Imagine that distributions of this subset, therefore of the form $P(-D)S_j$, $S_j \in \mathcal{E}'$, $P(-D)S_j \in \mathcal{E}_K'$, have a limit T in $\mathcal{E}'$. Lions's theorem of supports says that the support of $P(-D)S_j$ has a convex hull which is exactly equal to the convex hull of the support of $S_j(P(-D)\delta$ having as support the origin) (see page 28). Therefore the support of S_j is contained in $\tilde{K}$, convex hull of K. Now S_j must be equal to $E * P(-D)S_j$, E being a fundamental solution of $P(-D)$, for S_j has a compact support. Therefore, when $P(-D)S_j$ converges to T, with support in K, S_j converges to $E * T$ by the continuity of convolution; $E * T$ has its support in $\tilde{K}$ as limit of the S_j's, and $T = P(-D)(E * T)$ belongs to $P(-D)\mathcal{E}' \cap \mathcal{E}_K'$, which therefore is closed, Q.E.D. [11, 12, 26, 27, 46].

Improving this method, one can see that, for any $A \in H^s_{\text{loc}}(\mathbf{R}^n)$, there exists a solution $X \in H^s_{\text{loc}}(\mathbf{R}^n)$. Consequently, if A is a distribution of finite order, there exists a solution X of finite order. It has been much more difficult to prove that, if A is any distribution (locally, A is always of finite order but not in the whole of $\mathbf{R}^n$), there still exists a solution X; it was proved by Ehrenpreis [13].

These results are equivalent (by duality and an extension of Mittag-Loeffler's method to build a meromorphic function with given singularities in $\mathbf{C}$) to some approximation theorem. It is well known that any harmonic function f ($\Delta f = 0$) in a convex open set $\mathcal{U}$ of $\mathbf{R}^n$ can be approximated, uniformly on every compact subset of $\mathcal{U}$ [or also for the topology of $\mathcal{E}(\mathcal{U})$] by functions harmonic in the whole space $\mathbf{R}^n$. The same result is obtained for holomorphic functions in a convex open set $\mathcal{U}$ of $\mathbf{C}$ (a holomorphic function is simply a solution of the Cauchy-Riemann equation $\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$). Now, the previous results are exactly equivalent to the following one (which, in the case of harmonic or holomorphic functions, had nothing to do with the elliptic character of the corresponding operators Δ or $\frac{\partial}{\partial \bar{z}}$): every $T \in H^s_{\text{loc}}(\mathcal{U})$, $\mathcal{U}$ open convex set of $\mathbf{R}^n$, satisfying $P(D)T = 0$, is the limit, in $H^s_{\text{loc}}(\mathcal{U})$, of distributions of $H^s_{\text{loc}}(\mathbf{R}^n)$, verifying the same equation in the whole of $\mathbf{R}^n$.

Now these properties can be extended to an open set Ω of $\mathbf{R}^n$, but of course the result is no longer true for any Ω. One can prove that, for a given Ω and a given P, the following properties are equivalent (all true or all false) [11, 12, 25, 26, 27, 46]:

(1) For every $A \in \mathcal{E}(\Omega)$, there exists $X \in \mathcal{E}(\Omega)$ such that $P(D)X = A$.
(2) For every $A \in H^s_{\text{loc}}(\Omega)$, there exists $X \in H^s_{\text{loc}}(\Omega)$ such that $P(D)X = A$.
(3) $P(-D)(\mathcal{E}'(\Omega))$ is closed in $\mathcal{E}'(\Omega)$.
(4) For every compact subset K of Ω, there exists another compact subset H of Ω such that $S \in \mathcal{E}'(\Omega)$, $P(-D)S \in \mathcal{E}_K'(\Omega)$, implies $S \in \mathcal{E}_H'(\Omega)$.
(5) For every relatively compact open subset V of Ω, there exists a relatively compact open subset $U \supset V$ of Ω, such that every $f \in H^s_{\text{loc}}(U)$, solution of $P(D)f = 0$, can be approximated, in $H^s_{\text{loc}}(V)$ by functions of $H^s_{\text{loc}}(\Omega)$, solutions of the same homogeneous equation in the whole of Ω.

These equivalent properties are relative to the pair (P, Ω). If they are satisfied, one says that Ω is P convex. For instance, they are verified for every P, if Ω is convex (in this case, the H of (4) is the convex hull of K; the U of (5) is the convex hull of V). But if P is elliptic, Ω may be arbitrary (here, an argument of analyticity of the solutions of a homogeneous elliptic equation shows that the H of the fourth question is the union of K and the relatively compact connected components of $\complement K$ in Ω). Many conditions can be given for the P convexity of Ω using the relationships between the frontier $\dot{\Omega}$ and the characteristic lines of P.

The previous properties are not sufficient to insure that, for $A \in \mathcal{D}'(\Omega)$ (of infinite order), there will be a solution X in $\mathcal{D}'(\Omega)$. For this, Hörmander gave a necessary and sufficient condition [21, 47]: besides the previous properties, for any compact K of Ω, there must exist a compact H of Ω, such that $S \in \mathcal{E}'(\Omega)$, $P(-D)S \in \mathcal{E}(\complement K \cap \Omega)$ implies $S \in \mathcal{E}(\complement H \cap \Omega)$.

If this condition is satisfied, Ω is said to be strongly P convex.

Let us also point out some properties in relation with the theory of mean-periodic distributions. Let us call exponential polynomial on $\mathbf{R}^n$ a function of the form $x \rightsquigarrow e^{\langle a,x \rangle}Q(x)$, $a \in \mathbf{R}^n$, Q polynomial. The first result in the theory of ordinary homogeneous differential equations with constant coefficients (dimension $n = 1$) is that the exponential polynomials, solutions of the equation, generate all the solutions. This is true for more general equations in dimension 1, namely the homogeneous convolution equations $C * X = 0$, $C \in \mathcal{E}'(\mathbf{R})$ (a differential equation with constant coefficients $P(D)X = 0$ is a convolution equation with $P(D)\delta = C$), or even the systems of convolution equations, in the sense that every solution of the system is a *limit* of sums of exponential polynomials which are solutions [22, 35]. This is true for every dimension n, for one convolution equation, as was proven by Malgrange [28]; we shall see later that it is still true for systems of partial differential equations with constant coefficients (see page 53); it is still unknown for systems of convolution equations.

On the other hand, the necessary and sufficient condition for the equation $P(D)X = A$ to have a solution X in $\mathcal{E}'(\mathbf{R}^n)$, for $A \in \mathcal{E}'$, is that the Fourier transform $\hat{A}$ (which is an entire function on $\mathbf{C}^n$, see Paley-Wiener theorem, page 32) be divisible by $\hat{P}$ in the ring of entire functions on $\mathbf{C}^n$[26].

DIVISION OF DISTRIBUTIONS

Position of the Problem [39]. Let A be a distribution on an open set Ω of $\mathbf{R}^n$, and α a function of $\mathcal{E}(\Omega)$. Does there exist a distribution X such that

$$\alpha X = A \text{ on } \Omega? \tag{50}$$

If it exists, one can say that X is a quotient of A by α; for this reason, this problem is called problem of division of distributions.

The problem is trivial if α never vanishes; thus one has one and only one solution

$$X = \frac{1}{\alpha} \tag{51}$$

Thus it becomes interesting and difficult if α, without being identically zero, has a null set. One sees trivially that the problem is purely local: if Ω is a union of open subsets Ω_i, if in each Ω_i we know a distribution X_i such that $\alpha X_i = \Omega_i$, then it will be sufficient to use a partition of unity β_i (see page 25) relative to the Ω_i's and $X = \sum_i \beta_i X_i$ will be a solution of (50) in Ω.

The best way to see the main features of the problem is to treat completely division by $\alpha = x$ on $\mathbf{R}$; α has a simple zero at the origin.

In this case, a solution X of (50) has to verify

$$\langle X, x\psi\rangle = \langle A, \psi\rangle \text{ for every } \psi \in \mathfrak{D} \tag{52}$$

Thus we know $\langle X, \chi\rangle$ for every χ of the form $x\psi$, $\psi \in \mathfrak{D}$ by

$$\langle X, \chi\rangle = \left\langle A, \frac{\chi}{x}\right\rangle \tag{53}$$

But a function χ of $\mathfrak{D}$ is divisible by x in $\mathfrak{D}$ if and only if $\chi(0) = 0$. Let us choose, once and for all, a function $\theta \in \mathfrak{D}$ such that $\theta(0) = 1$. Now every $\phi \in \mathfrak{D}$ has a unique decomposition

$$\phi = \lambda\theta + \chi,\ \chi \in \mathfrak{D},\ \chi(0) = 0 \tag{54}$$

namely with $\lambda = \phi(0)$, $\psi = \phi - \phi(0)\theta$.

Therefore, (50) is now equivalent to

$$\langle X, \phi\rangle = \phi(0)\langle X, \theta\rangle + \left\langle A, \frac{\phi - \phi(0)\theta}{x}\right\rangle \tag{55}$$

Conversely, it is easily seen that, for an *arbitrary choice of* $\langle X, \theta \rangle$, this formula defines X as a distribution (continuous linear form on $\mathfrak{D}$). Therefore the problem always has solutions. Moreover, one can choose $\langle X, \theta \rangle$ arbitrarily, for one function $\theta \in \mathfrak{D}$ such that $\theta(0) \neq 0$. The difference between two solutions is therefore of the form $\phi \rightsquigarrow C\phi(0)$; it is $C\delta$, proportional to δ. In other words, all the solutions of the homogeneous equation $xX = 0$, are the $C\delta$'s. Thus, for $\alpha = x$ on $\mathbf{R}$, the problem is completely solved. For $\alpha = x^k$, it is easy to have a solution by k successive divisions by x; and the solutions of the homogeneous equation $x^kX = 0$ are of the form $C_0\delta + C_1\delta' + \cdots + C_k\delta^{(k)}$. The result is the same if α is a function of $\mathcal{E}$, having the origin as the only zero with multiplicity of order k: for $\dfrac{x^k}{\alpha}$ never vanishes, and (50) is equivalent to

$$x^kX = \frac{x^k}{\alpha} A \tag{56}$$

If α has an isolated 0 at a point a of $\mathbf{R}$, of order k, one has an analogous result, δ being replaced by $\delta_{(a)}$. And finally, using the local character of division, one sees that the division by α is always possible on $\mathbf{R}$, provided α has only isolated zeros of finite order; moreover, the degree of indeterminacy is completely known. On the contrary, if α has nonisolated zeros or a zero of infinite order, the division by α ceases being possible for every righthand side A.

The problem on $\mathbf{R}^n$, $n \geq 2$, is considerably more difficult. If the null set $\{x;\ \alpha(x) = 0\}$ is a C^∞-manifold of dimension $n - 1$, on each point of which α has at least a nonzero derivative, then the problem can be solved as easily as for $n = 1$; for it is purely local and, locally, a manifold can be written $x_n = 0$ by changing the coordinates, and α can be replaced by ${x_n}^k$ as in (56). But, even if α is an analytic function, the set of zeros which is an analytic variety may be highly singular, and new methods become necessary.

Łojasiewicz Inequality. The problem was solved for α polynomial on $\mathbf{R}^n$ by Hörmander [20, 48] and Łojasiewicz [24, 30] quite recently, and even for α analytic by Łojasiewicz. Hörmander's method is simpler; but it does not seem to go as far as Łojasiewicz's. Therefore we shall briefly expose the latter's method.

Let Ω be an open subset of $\mathbf{R}^n$. A distribution T on Ω is *continuable* if it is the restriction to Ω of a distribution on $\mathbf{R}^n$. For that, there is a necessary and sufficient condition: for every point a of $\dot{\Omega}$, there must exist an open neighborhood $\mathcal{V}_a$ of a, a constant C, and an integer m such that, for every $\phi \in \mathcal{D}(\mathcal{V}_a \cap \Omega)$:

$$|\langle T, \phi\rangle| \leq C \sup_{|p| \leq m} |D^p\phi| \tag{57}$$

Now, call $\mathcal{O}_M(\Omega)$ the subset of $\mathcal{E}(\Omega)$ formed by the functions which are slowly increasing at the border of Ω, as well as each of their derivatives (slowly increasing means: bounded by some power of the inverse of the distance to $\dot{\Omega}$). Then one sees that, if T is a continuable distribution on Ω, and if $\alpha \in \mathcal{O}_M(\Omega)$, then αT is still continuable. The first step of Łojasiewicz's proof is Łojasiewicz's inequality: if α is an analytic function on $\mathbf{R}^n$, if V is its variety of zeros, then, for every point a of V, there exist constants ρ, C, such that

$$|\alpha(x)| \geq C(d(x, V))^\rho \tag{58}$$

for x close enough to a. The proof of this inequality is of considerable difficulty; it involves a double induction, and all the structure of analytic varieties. It is equivalent to say that $\frac{1}{\alpha}$, as defined on $\complement V$, belongs to $\mathcal{O}_M$ $(\complement V)$.

Let us now introduce a new notion. Two closed sets A, B of $\mathbf{R}^n$ are said to be "regularly separated" if, for every $a \in A \cap B$, there exist constants C, ρ, such that

$$d(x, A) + d(x, B) \geq C(d(x, A \cap B))^\rho \tag{59}$$

for x close enough to a. (For instance if A and B are two C^∞ curves in $\mathbf{R}^2$, tangent at the origin as isolated common point, (59) is verified at the origin if and only if the curves have only a contact of finite order). Then one can prove that A and B are regularly separated if and only if there exists a function on $\complement(A \cap B)$ belonging to $\mathcal{O}_M$ $(\complement(A \cap B))$ equal to 0 in a neighborhood of A and to 1 in a neighborhood of B in $\complement(A \cap B)$. Łojasiewicz's inequality (58) proves thus that *two analytic varieties of* $\mathbf{R}^n$ *are always regularly separated* (as indicated by intuition, one never has contacts of infinite order with analytic functions!).

Now let T be a distribution carried by the union $A \cup B$. Is it always expressible as a sum of two distributions, respectively carried by A and B? A priori, one should hope so. But an easy counter example can be given. And precisely the necessary and sufficient condition for this to be possible for all T is that A and B be regularly separated, so that it is always possible for T carried by $A \cup B$ if A and B are two analytic varieties. Moreover, if one studies the local structure of an irreducible (real) analytic variety by the "canonical representation," one sees that the different branches (or sheets) of this variety are mutually regularly separated, so that the same conclusion holds for them.

Solution of the Problem. Łojasiewicz's proof of division is based on an induction on the dimension of analytic varieties of $\mathbf{R}^n$. We cannot give the details of it, but we shall try to sketch it. α and A being given, $A \in \mathfrak{D}'(\mathbf{R}^n)$, α analytic (not identically zero), let V be the variety of zeros of α. Then the division is easy in $\complement V$, and unique by putting $X = \frac{1}{\alpha} A$. But A being defined on $\mathbf{R}^n$ is a continuable distribution on $\complement V$, and we saw that $\frac{1}{\alpha}$ belongs to $\mathcal{O}_M$ $(\complement V)$. Therefore $\frac{1}{\alpha} A$ is continuable; let $\tilde{X}$ be an arbitrary continuation. We have $\alpha\tilde{X} = A$ on V; if we put $B = A - \alpha\tilde{X}$, B is a distribution carried by V. If we find a distribution Y such that $\alpha Y = B$, then $X = \tilde{X} + Y$ is a solution of $\alpha X = \alpha\tilde{X} + B = A$. We are, by this first step, led back to solve a division problem $\alpha Y = B$, where B is carried by V, analytic variety of dimension $\leq n - 1$. At first glance, it may seem that it is a very illusory first step because, α being zero on V, the division by α of a distribution carried by V will be very difficult. That is not the case. Remember the case of division by x on $\mathbf{R}$. We completely solved the problem in general; but, if B is a distribution carried by the origin, the solution was immediate because B can be written as a finite sum:

$$B = \sum_p c_p \delta^{(p)}, \tag{60}$$

and the solution of division by x is explicitly given by

$$Y = -\sum_p c_p \frac{\delta^{(p+1)}}{p+1} + c\delta \tag{61}$$

c arbitrary.

And again this procedure is valid in $\mathbf{R}^n$ for a regular manifold.

Put $V = V_0 \cup V_1'$, V_0 being the regular part of V (nonclosed analytic variety), V_1' the set of singular points of V, closed analytic variety of dimension $\leq n - 2$. Then the division is possible on $\complement V_1'$ by a generalization of (61); and Y is obtained as a distribution carried by V_0, by multiplying B by functions involving $\frac{1}{D^r\alpha}$, where $D^r\alpha$ is a derivative of α of sufficiently high order to be nonidentically zero on V_0. Let V_1 be the union of V_1' and the set of zeros of $D^r\alpha$ on V_0. V_1 is an analytic set of dimension $\leq n - 2$. And, by this multiplication by $\frac{1}{D^r\alpha}$ on the regular part V_0, the solution of $\alpha Y = B$ is found in $\complement V_1$. But because of Łojasiewicz's inequality (58) applied to $D^r\alpha$, $\frac{1}{D^r\alpha}$ belongs to $\mathcal{O}_M$ $(\complement V_1)$, and the Y we find is continuable. Let $\tilde{Y}$ be an arbitrary continuation. Thus $\alpha\tilde{Y} = B$ on $\complement V_1$. If we put $C = B - \alpha\tilde{Y}$, C is carried by V_1; if we solve the problem $\alpha Z = C$, then $Y = \tilde{Y} + Z$ will be a solution of $\alpha Y = \alpha\tilde{Y} + C = B$, and $X = \tilde{X} + \tilde{Y} + Z$ a solution of $\alpha X = A$. Then we are led back to the solution of a problem of division $\alpha Z = C$, where C is carried by V_1, analytic variety of dimension $\leq n - 2$. And the induction proceeds, by separating the regular and singular parts of V_1, so that, finally, we have to solve the problem for a distribution carried by only one point, and still by the same procedure, the problem is solved.

The problem of division $\alpha X = A$ always has a solution, for every A and every analytic nonidentically zero α. This method is not only a solution of the problem of division; it gives a lot of new properties of analytic functions and analytic varieties, and Łojasiewicz is proceeding to enrich this part of mathematics. On the other hand, this method gives some information about the indeterminacy of the problem; but not in an easy and practical way.

Malgrange [30] obtained a considerable generalization of these results, using a refinement of Łojasiewicz's method. Consider a $(p, 1)$ matrix distribution

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_p \end{pmatrix},$$

and a (p, q) matrix $\alpha = (\alpha_{i,j})$, analytic. Is it possible to find a $(q, 1)$ matrix distribution

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_q \end{pmatrix}$$

such that

$$(62) \qquad \alpha X = A, \text{ or } \sum_{j=1}^{q} \alpha_{i,j} X_j = A_i, \qquad i = 1, 2, \ldots, p$$

Certainly, some trivial conditions of possibility are necessary. Indeed, there are analytic relations between the rows of α. Such an analytic relation is, by definition, a system of analytic functions $\beta_1, \beta_2, \ldots, \beta_p$, such that $\sum_{i=1}^{p} \beta_i \alpha_{ij} = 0$ for $j = 1, 2, \ldots, q$; or $\beta\alpha = 0$ if we call β the $(1, p)$ matrix $(\beta_1, \beta_2, \ldots, \beta_p)$. And, if there exists a solution X of (62) we must have

$$(63) \qquad \beta A = \beta \alpha X = 0, \text{ or } \sum_{i=1}^{p} \beta_i A_i = 0$$

The relations β between the rows of α form a submodule $\mathfrak{R}$ of the module $\mathfrak{A}^p$, $\mathfrak{A} = \mathfrak{A}(\mathbf{R}^n)$ being the ring of analytic functions; a necessary condition for (62) to have a solution is that $\beta A = 0$ for every β of $\mathfrak{R}$. Malgrange proved that this condition is also sufficient. The proof, of course, uses Oka's theorem about the relations between analytic functions and, although based on the same principle as division by scalar analytic functions, is much more difficult.

Of course, everything we said before is valid on any real analytic manifold V of dimension n instead of $\mathbf{R}^n$; a distribution on V is understood as a continuous linear form on $\mathfrak{D}(V)$; if A is a matrix distribution on Y, α an analytic matrix on V, the necessary and sufficient condition for the existence of a matrix distribution X on V, such that $\alpha X = A$, is that $\beta A = 0$ for every analytic matrix β such that $\beta\alpha = 0$. This generalization of the $\mathbf{R}^n$ results to V is trivial, the problem of division being purely local (page 43) and V being locally isomorphic to an open set of $\mathbf{R}^n$. (One could see a difficulty in the fact that analytic functions on V, namely the β's, are not purely local objects! One avoids it, either by localizing the conditions "$\beta\alpha = 0 \Rightarrow \beta A = 0$" in the previous statement on $\mathbf{R}^n$, or by using Remmert's theorem, according to which any real-analytic manifold V, being immersible in a Euclidean space, is a Stein manifold).

The problem of division of distributions is interesting in itself. But it arose as a way of solving partial differential equations with constant coefficients, and its main applications are still there.

Let P be a polynomial of degree m on $\mathbf{R}^n$. If A is a tempered distribution, does there exist a tempered distribution X such that

$$(64) \qquad P(D)X = A?$$

As we saw on page 35, we may try to make a Fourier transform. If we call $\hat{X}$, $\hat{A}$ the Fourier transforms of X and A, we get the equivalent problem:

$$(65) \qquad P(\xi)\hat{X} = \hat{A} \qquad [30, 41, 48]$$

which is now a problem of division. For $\hat{A}$ given in $\mathcal{S}'$, we know, by the previous results, that there exists $\hat{X} \in \mathcal{D}'$ satisfying (65). But, of course, we need $\hat{X} \in \mathcal{S}'$. We have to improve the results about division to be sure that, if P is a polynomial, and $\hat{A}$ tempered, the division of $\hat{A}$ by P is possible in such a way that we get $\hat{X}$ tempered.

For that, we need the following property of tempered distributions. Let $\overline{\mathbf{R}^n}$ be the compactification of $\mathbf{R}^n$ by adding a point ∞ at infinity; $\overline{\mathbf{R}^n}$ will be identified, as a real analytic manifold, to a n-dimensional sphere. $\mathbf{R}^n$ can be covered by two open sets, each analytically isomorphic to $\mathbf{R}^n$; one is $\Omega_0 = \mathbf{R}^n = \complement_{\overline{\mathbf{R}^n}}\infty$, the second one is $\Omega_\infty = \complement_{\overline{\mathbf{R}^n}}0$, isomorphic to $\mathbf{R}^n$ by inversion $\xi \rightsquigarrow \dfrac{\xi}{|\xi|^2}$; moreover, this inversion is an analytic automorphism of $\overline{\mathbf{R}^n}$ (exchanging 0 and ∞). Now one can prove [41] that *a distribution $\hat{A}$ on $\mathbf{R}^n$ is tempered if and only if it is continuable to a distribution $\tilde{\hat{A}}$ on $\overline{\mathbf{R}^n}$.* (Moreover, the set $\mathcal{O}_M(\mathbf{R}^n)$ relative to the open subset $\mathbf{R}^n$ of $\overline{\mathbf{R}^n}$, in the sense of page 45, is exactly the space $\mathcal{O}_M(\mathbf{R}^n)$ defined for Fourier transform on page 31). But there is still a difficulty; we cannot solve $P\tilde{\hat{X}} = \tilde{\hat{A}}$ on $\overline{\mathbf{R}^n}$ because a polynomial P is not an analytic function on $\overline{\mathbf{R}^n}$! But we may consider the equation

$$\frac{P(\xi)}{(1+|\xi|^2)^m}\,\hat{\tilde{X}} = \frac{1}{(1+|\xi|^2)^m}\,\hat{\tilde{A}}$$

m being the degree of P; for the function $\xi \to \dfrac{P(\xi)}{(1+|\xi|^2)^m}$, extended by the value 0 at the point ∞, is analytic on $\overline{\mathbf{R}^n}$. If $\hat{X}$ is a solution of (65) on $\overline{\mathbf{R}^n}$, its restriction $\hat{X}$ on $\mathbf{R}^n$ is a solution of (65). Finally, *every inhomogeneous partial differential equation with constant coefficients and a tempered inhomogeneous term has a tempered solution.*

Division of distributions is the only known way to prove this theorem. By taking $A = \delta$, we get a tempered fundamental solution. But, as we saw on page 35, such a tempered fundamental solution, having the best possible properties as to the behavior at infinity, has not the best local behavior, it need not be a proper fundamental solution, in the sense of page 39.

If now $P(D)$ is a (p, q)-matricial partial differential operator with constant coefficients, and A is a $(p, 1)$ matrix distribution which is tempered, then the necessary and sufficient condition for the system

$$(66)\quad P(D)X = A, \text{ or } \sum_{j=1}^{q} P_{ij}(D)X_j = A_i \qquad i = 1, 2, \ldots, p$$

to have a tempered $(q, 1)$-matrix solution X, is that

(67) $Q(D)A = 0$, for every $(1, p)$-matricial polynomial Q such that $QP = 0$

The method uses the result of page 48. In particular, if there exists an $X \in (\mathfrak{D}'(\mathbf{R}^n))^q$ verifying (66), this condition is surely satisfied. Therefore: if, for A tempered, (66) has a solution in $(\mathfrak{D}'(\mathbf{R}^n))^q$, it also has a tempered solution. This result proves that the image $P(D)((\mathcal{S}'(\mathbf{R}^n))^q)$ is closed in $(\mathcal{S}'(\mathbf{R}^n))^p$; the map $P(D)$ is a homomorphism of $(\mathcal{S}'(\mathbf{R}^n))^q$ into $(\mathcal{S}'(\mathbf{R}^n))^p$, in the sense of the theory of topological vector spaces [6, 8] (a homomorphism onto or epimorphism if $p = q = 1$); besides, by transposition, replacing D by $-D$ and the matrix P by ${}^tP$, we see that, according to a well-known theorem of Banach, $P(D)$ is also a homomorphism of

$(\mathcal{S}(\mathbf{R}^n))^q$ into $(\mathcal{S}(\mathbf{R}^n))^p$ (a monomorphism or one-to-one homomorphism if $p = q = 1$), and $P(D)((\mathcal{S}(\mathbf{R}^n))^q)$ is closed in $(\mathcal{S}(\mathbf{R}^n))^p$.

We see how division gives a new method for the approach to the inhomogeneous systems, and will in fact enable us to extend some of the results of pages 39–42. However, it will not be possible to use in this way any L^2 inequality, nor any fundamental solution, because they do not exist or are not known for systems. In the following, $\mathcal{H}$ will be one of the spaces $\mathcal{D}(\Omega)$, $\mathcal{E}(\Omega)$, $\mathcal{E}'(\Omega)$, Ω convex open set of $\mathbf{R}^n$, or $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{S}'(\mathbf{R}^n)$. P will be a (p, q)-matricial polynomial, so that $P(D)$ operates from $\mathcal{H}^q$ into $\mathcal{H}^p$. It would be nice to extend the following results also to $\mathcal{H} = \mathcal{D}'(\Omega)$, $\mathcal{O}_M(\mathbf{R}^n)$, $\mathcal{O}_C'(\mathbf{R}^n)$, but it has not yet been done (the last results of Malgrange [40] are from June 1962!). (As to all the results relative to $\mathcal{S}$ and $\mathcal{S}'$, division and Fourier transform are in the nature of the problem. On the contrary, for $\mathcal{E}(\Omega)$, $\mathcal{D}(\Omega)$, $\mathcal{E}'(\Omega)$, all the corresponding results have been proved by Malgrange and Ehrenpreis, before any theorem about division was found, in the case of one equation ($p = q = 1$); why should the division be more indispensable for the systems? On the other hand, Ehrenpreis announced the results on systems and the fundamental principle (see page 53) several years ago, also before the solution of the problem of division; the proofs have not yet been published, therefore I give here Malgrange's statements using division and recently published with complete proofs) [10, 14, 15, 16, 32, 33, 34].

1. $P(D)\mathcal{H}^q$ is closed in $\mathcal{H}^p$; except perhaps for $\mathcal{H} = \mathcal{D}(\Omega)$, $P(D)$ is a homomophism of $\mathcal{H}^q$ into $\mathcal{H}^p$. This had been seen on page 39 for one equation ($p = q = 1$), but not for $\mathcal{H} = \mathcal{S}$, $\mathcal{S}'$; for $\mathcal{H} = \mathcal{S}$, $\mathcal{S}'$, we just saw it, even for systems, in the previous pages.

2. For $\mathcal{H} = \mathcal{E}(\Omega)$, or $\mathcal{S}'(\mathbf{R}^n)$, the necessary and sufficient condition for (66) to have a solution in $\mathcal{H}^q$ for A given in $\mathcal{H}^p$, is the condition (67) already given for $\mathcal{H} = \mathcal{S}'$. In particular, for A in $\mathcal{H}$, if (66) has a solution in $\mathcal{D}'$, it has a solution in $\mathcal{H}$, too. This result is probabily valid for $\mathcal{H} = \mathcal{O}_M(\mathbf{R}^n)$, too.

3. For $\mathcal{H} = \mathcal{S}(\mathbf{R}^n)$, (67) is, of course, still necessary, but no longer sufficient. It insures, for A given in $\mathcal{S}^p$, the existence of a solution X of (66) in $\mathcal{S}'^q$; but X need not belong to $\mathcal{S}^q$. A trivial example will clarify the situation. Take $n = p = q = 1$, $P(\xi) = \xi$, $P(D) = \frac{1}{2i\pi}\frac{d}{dx}$. Here (67) is satisfied because we have only one equation:

Q is always 0. In fact, for every A of $\mathcal{S}$, we can find a function such that $\frac{1}{2i\pi}\frac{dX}{dx} = A$, namely $X(x) = \text{constant} + 2i\pi \int_{-\infty}^{x} A(t)\,dt$. But for x converging to $+\infty$ or $-\infty$, $X(x)$ does not converge in both cases to 0 and therefore does not belong to $\mathcal{S}$, except if $\int_{-\infty}^{+\infty} A(t)\,dt = 0$. If this particular condition is satisfied, then $X(x) = 2i\pi \int_{-\infty}^{x} A(t)\,dt = -2i\pi \int_{x}^{+\infty} A(t)\,dt$ lies in $\mathcal{S}$, and (66) has a solution (and here only one) in $\mathcal{S}$. Instead of (67), which is always a condition of local and differential character, we have here a condition of global and integral character; it comes from the fact that $\mathcal{F}\mathcal{S} = \mathcal{S}$ is a space of differentiable functions, and we shall see that the condition of solvability of (66) for $\mathcal{K} = \mathcal{S}$ involves local and differential conditions for the Fourier transform $\hat{A}$ of A which gives global and integral conditions for A itself.

By Fourier transform (66) becomes

$$P(\xi)\hat{X}(\xi) = \hat{A}(\xi) \tag{68}$$

We have seen in (1), page 51, that $P\mathcal{S}^q$ is closed in $\mathcal{S}^p$. But $\mathcal{S}^p$ is a module over $\mathcal{S}$ for multiplication, and $P(D)\mathcal{S}^q$ is trivially a submodule $\mathfrak{M}$. And there is precisely a theorem by Whitney [49] about the closed $\mathcal{S}$ submodules of $\mathcal{S}^p$ (or closed $\mathfrak{D}$ submodules of $\mathfrak{D}^p$, or closed $\mathcal{E}$ submodulus of $\mathcal{E}^p$). Let a be a point of $\mathbf{R}^n$. Every function $\theta \in \mathcal{E}^p$ has a formal Taylor expansion at the point a which is:

$$\theta_a \simeq \sum_r \frac{\theta^{(r)}(a)}{r!}(\xi - a)^r \tag{69}$$

Let us call S_a the ring of formal power series at the point a; $S_a{}^p$ is an S_a module. Then θ_a is the image of $\theta \in \mathcal{E}^p$ into $S_a{}^p$. If $\mathfrak{M}$ is a submodule, it has an image $\mathfrak{M}_a$ in $S_a{}^p$. This theorem of Whitney says that, if $\mathfrak{M}$ is *a closed submodule* in $\mathfrak{D}^p$, $\mathcal{S}^p$, or $\mathcal{E}^p$, then the trivial necessary condition: "$\forall_a \in \mathbf{R}^n$, $\theta_a \in \mathfrak{M}_a$" for θ to belong to $\mathfrak{M}$, is also sufficient. Hence, the trivial necessary condition

(70) $\forall a \in \mathbf{R}^n$, there exists a formal power series $u_a \in S_a{}^q$ such that $P_a u_a = \hat{A}_a$,

for the existence of a solution $\hat{X} \in \mathcal{S}^q$ of (68) or for the existence of a solution $X \in \mathcal{S}^q$ of (66), is also sufficient.

In the example on page 51, $p = q = n = 1$, $P(D) = \frac{1}{2i\pi}\frac{d}{dx}$, (68) is $\xi\hat{X}(\xi) = \hat{A}(\xi)$. At every point $a \neq 0$, $\mathfrak{M}_a$ is equal to S_a because $\xi = a + (\xi - a)$ is invertible in S_a, and the condition $A_a \in \mathfrak{M}_a$ is always satisfied. For $a = 0$, $\mathfrak{M}_0$ is the ideal of multiples of ξ in S_0; $\hat{A}_0$ belongs to $\mathfrak{M}_0$ if, and only if, $\hat{A}(0) = 0$, which means exactly $\int_{-\infty}^{+\infty} A(t)\,dt = 0$ by the Fourier reciprocity formula.

Although the analogous result is not known for $\mathcal{H} = \mathcal{O}_C'$, it is very likely the same because $\mathcal{F}(\mathcal{O}_C') = \mathcal{O}_M$ is also a space of C^∞ functions.

4. Take now $\mathcal{H} = \mathcal{D}(\Omega)$, $\mathcal{E}'(\Omega)$. Here the Fourier transform $\hat{A} = \mathcal{F}A$ is not only a function of $\mathcal{S}^p$, but also a holomorphic function of exponential type on $\mathbf{C}^n$ (Paley-Wiener theorem, see page 32). Therefore, there is a new trivial necessary condition, stronger than (70), for (66) to have a solution in $\mathcal{H}^q$ for A given in $\mathcal{H}^p$, namely (see page 42):

(71) For every $a \in \mathbf{C}^n$, there exists a formal power series $u_a \in S_a{}^q$ such that $P_a u_a = \hat{A}_a$.

The condition being the same for $\mathcal{D}$ and $\mathcal{E}'$, we see that if, for $A \in \mathcal{D}(\Omega)$, (66) has a solution in $\mathcal{E}'(\Omega)$, it has also a solution in $\mathcal{D}(\Omega)$.

5. Every solution of the homogeneous system of $P(D)X = 0$ in $\mathcal{E}(\Omega)$ is a limit, in $\mathcal{E}(\Omega)$, of solutions of the same system in $\mathcal{E}(\mathbf{R}^n)$. And these are themselves limits of combinations of the exponential polynomials which are solutions (see page 42).

6. It seems that a very important property in all these problems is the "integral representation" or "fundamental principle" of Ehrenpreis [10, 15, 16, 33]. Let Γ be a convex compact set of $\mathbf{R}^n$. If $\phi \in \mathcal{D}(\Gamma)$, its Fourier image $\hat{\phi}$ can be extended as an analytic function on $\mathbf{C}^n$ (Paley-Wiener theorem), and one sees easily that it has the following property of decreasing at infinity: if

$$\hat{\Gamma}(\eta) = \int_\Gamma e^{2\pi\langle x,\eta\rangle}\,dx \tag{72}$$

then $(\hat{\Gamma}(\eta))^{-1}\hat{\phi}(\xi + i\eta)$ belongs to $\mathcal{S}_{\xi,\eta} = \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n) = \mathcal{S}(\mathbf{C}^n)$. Let us call $\mathcal{S}_{\xi,\eta}{}^\Gamma$ or simply $\mathcal{S}^\Gamma$ the space of functions θ (not necessarily holomorphic) on $\mathbf{R}^n \times \mathbf{R}^n$, such that $\hat{\Gamma}^{-1}\theta \in \mathcal{S}_{\xi,\eta}$: thus $\hat{\phi} \in \mathcal{S}^\Gamma$ if $\phi \in \mathcal{D}(\Gamma)$. Conversely, if θ is a holomorphic function of $\mathcal{S}^\Gamma$, one sees [23] that its restriction to $\mathbf{R}^n$ is the Fourier image $\hat{\phi}$ of a function ϕ of $\mathcal{D}(\Gamma)$. Finally, $\mathcal{F}(\mathcal{D}(\Gamma))$ can be identified with the subspace of holomorphic functions of $\mathcal{S}_{\xi,\eta}{}^{\Gamma}$. There is on $\mathcal{S}_{\xi,\eta}{}^{\Gamma}$ a natural topology, the weakest one for which $\theta \rightsquigarrow \hat{\Gamma}^{-1}\,\theta$ is continuous from $S_{\xi,\eta}{}^{\Gamma}$ into $S_{\xi,\eta}$. And this topology induces on the subspace of holomorphic functions just the topology transferred by $\mathfrak{F}$ from that of $\mathfrak{D}(\Gamma)$.

Now let us take a distribution T on $\mathbf{R}^n$. It defines a continuous linear form on $\mathfrak{D}(\Gamma)$, therefore on the subspace of holomorphic functions of $\mathcal{S}_{\xi,\eta}{}^{\Gamma}$. By the Hahn-Banach theorem, this last one can be extended as a continuous linear form on $\mathcal{S}_{\xi,\eta}{}^{\Gamma}$, or an element of its dual $(S_{\xi,\eta}{}^{\Gamma})'$. This dual is the space of distributions on $\mathbf{R}^n \times \mathbf{R}^n$ whose product by $\hat{\Gamma}(\eta)$ lies in $\mathcal{S}_{\xi,\eta}{}'$. Therefore we assign to T and Γ a distribution on $\mathbf{R}^n \times \mathbf{R}^n$; call $\hat{T}_\Gamma$ the symmetric of this distribution with respect to the origin (this has to be done because $\mathfrak{F}$ does not preserve the scalar product, but one has $\langle \hat{U}_\xi, \hat{\phi}(-\xi)\rangle = \langle U, \hat{\phi}\rangle$).

Finally, $\hat{T}_\Gamma$ is defined as a distribution on $\mathbf{R}^n \times \mathbf{R}^n$, belonging to $(S_{\xi,\eta}{}^{\Gamma})'$, such that, for every $\phi \in \mathfrak{D}(\Gamma)$:

$$\langle T_x, \phi\rangle = \langle (\hat{T}_\Gamma)_{\xi,\eta}, \hat{\phi}(-\xi - i\eta)\rangle \tag{73}$$

One could consider $\hat{T}_\Gamma$ as a kind of Fourier transform of T. In fact, if T is tempered, one can take for $\hat{T}_\Gamma$ the distribution $T_\xi\delta_\eta$ because

$$\langle \hat{T}_\xi\delta_\eta, \hat{\phi}(-\xi - i\eta)\rangle = \langle \hat{T}_\xi, \hat{\phi}(-\xi)\rangle = \langle T, \phi\rangle \tag{74}$$

But one must remark that:

(a) $\hat{T}_\Gamma$ belongs to $\mathfrak{D}'(\mathbf{R}^n \times \mathbf{R}^n)$, not to $\mathfrak{D}'(\mathbf{R}^n)$.

(b) $\hat{T}_\Gamma$ is not unique because it is obtained by the Hahn-Banach theorem; and it depends on Γ. In fact, for every T and Γ, we get a family of possible distributions $\hat{T}_\Gamma$, and the family of these distributions becomes smaller if Γ becomes larger, T remaining fixed.

Take now $X \in (\mathfrak{D}'(\mathbf{R}^n))^q$, solution of the homogeneous system $P(D)X = 0$. For every Γ, the linear form $\hat{X}_\Gamma$, restricted to the subspace of holomorphic functions of $(\mathcal{S}_{\xi,\eta}{}^{\Gamma})^q$, surely verifies $P(\xi + i\eta)X_\Gamma = 0$. The remarkable fact, called fundamental principle, is that $\hat{X}_\Gamma$, as defined by the Hahn-Banach theorem as a distribution on $\mathbf{R}^n \times \mathbf{R}^n$, may be chosen in such a way that $P(\xi + i\eta)\hat{X}_\Gamma$ still is zero. In particular, in case of one equation, $p = q = 1$, $\hat{T}_\Gamma$ is carried by the complex manifold $\{\zeta; P(\zeta) = 0\}$, which extends the results of dimension 1 about mean-periodic distributions [35].

To end this article, let us give a very nice theorem. We shall say that the (p, q)-matricial polynomial P has the continuation property if every $(q, 1)$ distribution X, defined on the complement of a compact set of $\mathbf{R}^n$, and solution of the homogeneous system $P(D)X = 0$, admits a "continuation" $\tilde{X}$ (equal to X in the complement of a compact set of $\mathbf{R}^n$; we do not intend necessarily the same compact set), defined in the whole of $\mathbf{R}^n$ and solution of the homogeneous system $P(D)\tilde{X} = 0$ in the whole of $\mathbf{R}^n$. Then the necessary and sufficient condition for P to have the continuation property is the following: if a $(p, 1)$ polynomial R verifies $QR = 0$ for all the $(1, p)$ polynomials Q such that $QP = 0$, then there exists a $(q, 1)$ polynomial R' such that $R = PR'$ [10, 33]. This property of P can be checked directly in the ring of polynomials. It is never verified in case of one nontrivial equation ($p = q = 1$); for in this case all the Q's are 0, then R is arbitrary, and an arbitrary polynomial is not a multiple of P; and, indeed, a fundamental solution of $P(D)$ is a solution of the homogeneous equation in $\complement\, 0$, and cannot be extended as a solution of the homogeneous equation in the whole of $\mathbf{R}^n$. On the contrary, if P_1, P_2 are two scalar polynomials without common factor, then the matrix $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ has the previous property; for one has $Q_1P_1 + Q_2P_2 = 0$ if and only if (Q_1, Q_2) is multiple (by a polynomial) of $(P_2, -P_1)$, then one has $Q_1R_1 + Q_2R_2 = 0$ for all (Q_1, Q_2) if and only if $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$ is multiple of $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. Thus every scalar distribution X, solution of $P_1(D)X = 0$, $P_2(D)X = 0$, in the complement of a compact set of $\mathbf{R}^n$, is continuable as a solution of the same system in the whole of $\mathbf{R}^n$.

Take now, on $\mathbf{C}^n$, the matrix $\begin{pmatrix} \zeta_1 \\ \cdot \\ \cdot \\ \cdot \\ \zeta_n \end{pmatrix}$. Among the relations $Q_1\zeta_1 + Q_2\zeta_2 + \cdots + Q_n\zeta_n = 0$, we have that for which (for $i \neq j$) $Q_i = \zeta_j$, $Q_j = -\zeta_i$, $Q_k = 0$ for $k \neq i$ or j. They generate all the relations (over the ring of polynomials), but this is not important here. Now an $\begin{pmatrix} R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{pmatrix}$ must be such that $R_i\zeta_j - R_j\zeta_i = 0$, for all pairs i, j; which implies the existence of a scalar polynomial R' such that $R_i = \zeta_i R'$. This result is only valid for $n \geq 2$ (for $n = 1$, $Q = 0$, and R is arbitrary, therefore not necessarily multiple of ζ_1). The ζ_j's correspond to the differential operators $\frac{1}{i\pi}\frac{\partial}{\partial \bar{z}_j}$, and the system $\frac{\partial X}{\partial \bar{z}_j} = 0$, $j = 1, 2, \ldots, n$, is verified if and only if X is a holomorphic function (Cauchy-Riemann equations). Thus we get, as a particular case of this general theory of partial differential systems with constant coefficients, the well-known theorem of Hartog: a holomorphic function of $n \geq 2$ complex variables, defined in the complement of a compact subset of $\mathbf{C}^n$, is continuable to the whole space.

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